

# Complex Methods: Example Sheet 3

Part IB, Lent Term 2022

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

## Fourier transforms

1. By using the relationship between the Fourier transform and its inverse, show that for real  $a$  and  $b$  with  $a > 0$ ,

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} e^{i\omega t} d\omega = \frac{\pi}{a} e^{-a|t|} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2} e^{i\omega t} d\omega = 2\pi e^{-at} \sin bt H(t)$$

where  $H(t)$  is the Heaviside step function. What are the values of the integrals when  $a < 0$ ? What happens when  $a = 0$ ?

2. Show that the convolution of the function  $e^{-|x|}$  with itself is given by  $f(x) = (1 + |x|)e^{-|x|}$ . Use the convolution theorem for Fourier transforms to show that

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1 + k^2)^2} dk$$

and verify this result by contour integration.

3. Let

$$f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a, \\ 0 & \text{otherwise;} \end{cases} \quad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\tilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \tilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.$$

What is the convolution of  $f$  with itself? Use Parseval's identity to evaluate  $\int_{-\infty}^{\infty} (\sin^2 x)/x^2 dx$ . Verify by contour integration the inversion formula for  $f(x)$  for all values of  $x$  except  $\pm \frac{1}{2}a$ .

\* Verify the inversion formula also at  $x = \pm \frac{1}{2}a$ .

- \* 4. The displacement  $x(t)$  of a damped harmonic oscillator obeys the the equation

$$\ddot{x} + 2\gamma\dot{x} + q^2x = f(t), \quad \text{where } \gamma > 0.$$

Assuming that the Fourier transforms  $\tilde{x}(\omega)$  and  $\tilde{f}(\omega)$  exist, show that

$$x(t) = \int_{-\infty}^{\infty} G(t - t')f(t')dt', \quad \text{where } G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{q^2 + 2i\gamma\omega - \omega^2} d\omega.$$

Show, by differentiation under the integral sign, that

$$\frac{d^2}{dt^2}G(t) + 2\gamma\frac{d}{dt}G(t) + q^2G(t) = \delta(t).$$

Show that for  $0 < \gamma < q$ ,

$$G(t) = \frac{1}{p} e^{-\gamma t} \sin(pt) H(t), \quad \text{where } p = \sqrt{q^2 - \gamma^2}.$$

[You may use here the results from question 1.]

## Laplace transforms

5. Starting from the Laplace transform of 1 (namely  $s^{-1}$ ), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions: (i)  $e^{-2t}$ ; (ii)  $t^3 e^{-3t}$ ; (iii)  $e^{3t} \sin 4t$ ; (iv)  $e^{-4t} \cosh 4t$ ; (v)  $e^{-t} H(t-1)$ , where  $H$  is the Heaviside step function.
6. Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of  $F(s) = (s+3)/\{(s-2)(s^2+1)\}$ . Verify this result using the Bromwich inversion formula.
7. Use Laplace transforms to solve the differential equation

$$\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^t$$

with initial conditions  $y(0) = 1$ ,  $\dot{y}(0) = 0$ ,  $\ddot{y}(0) = -2$ .

8. Consider a linear system obeying the differential equation

$$\ddot{y} - 3\dot{y} + 2y = u(t), \quad \dot{y}(0) = y(0) = 0.$$

Use Laplace transforms to determine the response of the system to the signal  $u(t) = t$ . Determine also the response  $y(t)$  to a signal  $u(t) = \delta(t)$ .

[For  $\delta(t)$ , take the Laplace transform to be  $F(s) = \int_0^\infty f(t)e^{-st} dt$ , i.e. start "just left of 0".]

9. Solve the integral equation  $f(t) + 4 \int_0^t (t-\tau)f(\tau) d\tau = t$  for the unknown function  $f$ . Verify your solution.

- \* 10. The zeroth order Bessel function  $J_0(x)$  satisfies the differential equation

$$xJ_0'' + J_0' + xJ_0 = 0$$

for  $x > 0$ , with  $J_0(0) = 1$  (and  $J_0'(0) = 0$  from the equation). Find the Laplace transform of  $J_0$  and deduce that  $\int_0^\infty J_0(x) dx = 1$ . Find the convolution of  $J_0$  with itself.

11. Use Laplace transforms to solve the heat equation  $\partial T / \partial t = \partial^2 T / \partial x^2$  with boundary conditions  $T(x, 0) = \sin^3 \pi x$  ( $0 < x < 1$ ),  $T(0, t) = T(1, t) = 0$  ( $t > 0$ ). [Hint:  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ .]

12. Using the equality  $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$ , find the Laplace transform of  $f(t) = t^{-1/2}$ . By integrating around a Bromwich keyhole contour, verify the inversion formula for  $f(t)$ . What is the Laplace transform of  $t^{1/2}$ ?

- \* 13. The gamma and beta functions are defined for  $z, w \in \mathbb{C}$  by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{and} \quad B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt$$

when  $\text{Re}(z), \text{Re}(w) > 0$ . Show that  $\Gamma(z+1) = z\Gamma(z)$  and hence that  $\Gamma(n+1) = n!$  if  $n$  is a non-negative integer. Using the previous question, write down the value of  $\Gamma(\frac{1}{2})$ .

For a fixed value of  $z$ , find the Laplace transform of  $f(t) = t^{z-1}$  in terms of  $\Gamma(z)$ . Find the Laplace transform of the convolution  $t^{z-1} * t^{w-1}$ . Hence establish that

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \quad (*)$$

The domain of  $\Gamma$  and  $B$  can be extended to the whole of  $\mathbb{C}$ , apart from isolated singularities, by analytic continuation. Does the relation (\*) still hold?