

*Starred questions are optional.*

### Conformal maps

1. (i) Let  $f(z) = \frac{az+b}{cz+d}$ , with  $ad-bc \neq 0$ . Where in  $\mathbb{C}$  is  $f$  conformal?
- (ii) Let  $f(z) = \frac{z+1}{z-1}$ . What are the images of the real axis, the imaginary axis, and the unit circle? What are the images of the unit disc and the quadrant  $\{x+iy : x, y > 0\}$ ?
- (iii) Let  $C_1$  be the unit circle, and  $C_2$  the circle  $|z - (1+i)| = 1$ . Find a Möbius map which simultaneously sends  $C_1$  to the real axis and  $C_2$  to the imaginary axis. The circles divide the plane into four regions. Where does your map send each region?

\* (iv) Prove that a Möbius map sends the unit disc onto itself if and only if it has the form

$$f(z) = e^{i\theta} \left( \frac{z + \alpha}{\bar{\alpha}z + 1} \right)$$

for some  $\theta \in \mathbb{R}$  and some  $\alpha \in \mathbb{C}$  with  $|\alpha| < 1$ .

2. Find the images of the following maps, using the principal branches in (ii) and (iv). *If you haven't met branches yet then have a go anyway, as though I hadn't mentioned them.*
  - (i)  $f(z) = z^2$  on the half-disc  $\{z : |z| < 1, \operatorname{Re}(z) > 0\}$
  - (ii)  $f(z) = z^{1/3}$  on the 'cut' plane  $\mathbb{C} \setminus \{x+iy : x \leq 0, y = 0\}$
  - (iii)  $f(z) = \exp z$  on the half-strip  $\{x+iy : x > 0, 0 < y < \frac{\pi}{2}\}$
  - (iv)  $f(z) = \log z$  on the half-disc  $\{z : |z| < 1, \operatorname{Re}(z) > 0\}$
3. For each of the following regions, construct a bijective conformal map from the region to the unit disc. *If you give a composition of several functions, it would be helpful to provide a sketch to illustrate each step.*
  - (i) the open quarter-disc  $\{z : |z| < 1, \arg z \in (0, \frac{\pi}{2})\}$
  - (ii) the half-strip  $\{x+iy : -1 < x < 1, y > 0\}$
  - (iii) the open region enclosed between the circles  $|z-1| = 1$  and  $|z-2| = 2$ .

4. Use the decomposition

$$\frac{z}{(z-1)^2} = \left( \frac{1}{1-z} - \frac{1}{2} \right)^2 - \frac{1}{4}$$

to show that  $f(z) = z/(z-1)^2$  is a bijective conformal map from the disc  $|z| < 1$  to the domain  $\mathbb{C} \setminus \{x+iy : x \leq -\frac{1}{4}, y = 0\}$ .

- \*5. Let  $f$  be analytic and  $z_0 \in \mathbb{C}$  be such that  $f'(z_0) = 0$  and  $f''(z_0) \neq 0$ . By considering the Taylor expansion of  $f(z+h)$  about  $z_0$ , prove that  $f$  doubles angles between curves intersecting at  $z_0$ . What happens if  $f'(z_0) = f''(z_0) = 0$  and  $f'''(z_0) \neq 0$ ?
6. Consider the map  $f(z) = \frac{1}{2}(z + \frac{1}{z})$ . Find the points where this map is not conformal and determine how the angles between two curves at those points change at their image. *You can quote the results of the previous question here, whether you did it or not.*
- Show that  $f$  takes concentric circles with radius  $r > 1$  centered at the origin to cofocal ellipses. What is the image of the unit circle?
- By considering angles at certain points, sketch the image of the circle  $|z-1| = 2$ .
- \* Sketch the images of the circles  $|z-(1+i)| = \sqrt{5}$  and  $|z-i| = \sqrt{2}$ .

## Cauchy-Riemann equations ; harmonic functions

7. (i) Let  $u(x, y)$  and  $v(x, y)$  satisfy the Cauchy-Riemann equations, and define

$$g(z, \bar{z}) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Use the chain rule to show that  $\partial g / \partial \bar{z} = 0$ . Explain the significance of this result.

- (ii) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$\operatorname{Im} z ; \quad |z|^2 ; \quad \operatorname{sech} z .$$

- (iii) Let  $f(z) = z^5/|z|^4$  for  $z \neq 0$ , and  $f(0) = 0$ . Show that the real and imaginary parts of  $f$  satisfy the Cauchy-Riemann equations at  $z = 0$ , but that  $f$  is not differentiable there. *To calculate, say,  $u_x(0, 0)$ , set  $y = 0$  before differentiating with respect to  $x$ .*

8. Let  $f$  be analytic on an open set  $D \subset \mathbb{C}$ . Show that if any of  $\operatorname{Re} f$ ,  $\operatorname{Im} f$ ,  $|f|$  or  $\arg f$  is constant on  $D$ , then  $f$  is constant on  $D$ .

9. Find complex analytic functions  $f(z)$  whose real parts  $u(x, y)$  are the following:

$$(i) \ xy \quad (ii) \ \sin x \cosh y \quad (iii) \ e^x(x \cos y - y \sin y) \quad (iv) \ \frac{y}{(x+1)^2 + y^2}$$

*You should give your answers as functions of  $z$ , rather than of  $x$  and  $y$ .*

- \* Notice that in (ii) the expression  $u(z, 0)$  agrees with  $f(z)$ . When is this true in general?

10. (i) Let  $u(x, y) = \tan^{-1}(y/x)$ . Show that  $u$  is harmonic on  $\{(x, y) : x > 0\}$  by finding a complex analytic function defined on  $\{x+iy : x > 0\}$  whose real (or imaginary) part is  $u$ . Show similarly that  $u$  is harmonic on  $\{(x, y) : x < 0\}$ .
- (ii) Let  $u(x, y) = \tan^{-1}\left(\frac{2x}{x^2 + y^2 - 1}\right)$ . Where on  $\mathbb{R}^2$  is  $u$  defined? By considering  $w(z) = \frac{z+i}{z-i}$  and using two cases as in (i), show that  $u$  is harmonic where defined.

11. Show that  $g(z) = \exp z$  maps  $\{x + iy : 0 < y < \pi\}$  onto  $\{x + iy : y > 0\}$ .

Show that  $h(z) = \sin z$  maps  $\{x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0\}$  onto  $\{x + iy : y > 0\}$ .

Hence find a conformal map of  $\{x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0\}$  onto  $\{x + iy : 0 < y < \pi\}$ .

Find a function  $v(x, y)$  which is harmonic on the strip  $\{-\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0\}$ , with limiting values on the boundaries given by:  $v = 0$  on the two parts of the boundary in the left half-plane, and  $v = 1$  on the two parts of the boundary in the right half-plane.

Is your function  $v$  unique?

*You should give  $v$  as a function of  $x$  and  $y$ , rather than of  $z$ .*

- \*12. Find a function  $v(x, y)$  which is harmonic on the unit disc, with limiting values as follows:  $v = 1$  on the part of the boundary in the first quadrant,  $v = -1$  on the part in the third quadrant, and  $v = 0$  on the parts in the second and fourth quadrants.

## Branches

13. Explain how the principal branch of  $\log z$  can be used to define a branch of  $z^i$  which is single-valued on the set  $D = \mathbb{C} \setminus \{x + iy : x \leq 0, y = 0\}$ .

What is  $i^i$  for this branch? Does the identity  $(zw)^i = z^i w^i$  hold?

Using polar coordinates, show that the branch of  $z^i$  defined above maps  $D$  onto an annulus which is covered infinitely often.

How would your answers change for a different branch (with the same cut)?

14. Find all branch points of  $f(z) = [z(z+1)]^{1/3}$ , and justify why they are branch points. Draw some possible branch cuts in the complex plane.

Repeat with  $f(z) = [z(z+1)(z+2)]^{1/3}$ .

\* Repeat with  $f(z) = [z(z+1)(z+2)(z+3)]^{1/n}$  for  $n = 2$  and  $n = 3$ .

15. Let  $f(z) = (z^2 - 1)^{1/2}$ . Consider the following two branches of this function.

$$f_1(z) : \text{branch cut } [-1, 1], \quad f_1(x) = +\sqrt{x^2 - 1} \quad \text{for real } x > 1$$

$$f_2(z) : \text{branch cut } \mathbb{R} \setminus (-1, 1), \quad f_2(x) = +i\sqrt{1 - x^2} \quad \text{for real } x \in (-1, 1).$$

Find the limiting values of  $f_1$  and  $f_2$  above and below their respective branch cuts.

Prove that  $f_1$  is an odd function and  $f_2$  is even, i.e.,  $f_1(-z) = -f_1(z)$  and  $f_2(-z) = f_2(z)$ .

- \*16. Let  $f(z)$  be a polynomial of even degree. Explain why there is an analytic function  $g(z)$ , defined on the region  $|z| > R$  for some suitable  $R$ , such that  $g(z)^2 = f(z)$ . When is there such a function defined on the region  $|z| < r$  for some  $r$ ?

## Series ; singularities

17. Use partial fractions to find the Laurent series of  $1/((z-a)(z-b))$  about  $z=0$ , where  $|b| > |a| > 0$ , in each of the regions  $|z| < |a|$ ,  $|a| < |z| < |b|$  and  $|z| > |b|$ .
18. Find the first two non-zero coefficients in the Taylor series about the origin of the following functions, assuming principal branches for (i), (ii) and (iii).

$$(i) \ z/\log(1+z) ; \quad (ii) \ \sqrt{\cos z} - 1 ; \quad (iii) \ \log(1+e^z) ; \quad (iv) \ e^{e^z} .$$

Find the radius of convergence of each series.

How would the series change for different branches (with the same cut)?

19. Find and classify the singularities in the (finite) complex plane of the following functions:

$$(i) \ \frac{1}{z^3(z-1)^2} \quad (ii) \ \frac{e^z - e}{(z-1)^3} \quad (iii) \ \frac{z}{\sinh z} \quad (iv) \ \frac{z}{\log z} \\ (v) \ \tan z \quad (vi) \ \exp(\tan z) \quad (vii) \ \log(\tan z) \quad (viii) \ \tan(z^{-1}) .$$

20. Find the first three terms of the Laurent series of  $f(z) = \operatorname{cosec}^2 z$  valid for  $0 < |z| < \pi$ .

Show that the function

$$g(z) = \operatorname{cosec}^2 z - \frac{1}{z^2} - \frac{1}{(z+\pi)^2} - \frac{1}{(z-\pi)^2} ,$$

has only removable singularities in  $|z| < 2\pi$ . Use this to show that, in the Laurent series of  $f(z)$  valid for  $\pi < |z| < 2\pi$ , the central three non-zero terms are

$$\cdots + \frac{3}{z^2} + \left(\frac{1}{3} - \frac{2}{\pi^2}\right) + \left(\frac{1}{15} - \frac{6}{\pi^4}\right)z^2 + \cdots$$

\* What are the corresponding terms in the series for  $f(z)$  valid for  $n\pi < |z| < (n+1)\pi$ ?

## Integration

*Some of the integrals in this section can be evaluated by real methods, but please do them all by contour integral methods.*

21. By parametrising the curves (and not using Cauchy's theorem or the residue theorem), evaluate  $\oint_C \bar{z} dz$  and  $\oint_C z^{1/2} dz$  (using the principal branch of  $z^{1/2}$ ) in the cases:
- (i)  $C$  is the circle  $|z| = 1$ , and (ii)  $C$  is the circle  $|z-1| = 1$ .
22. Using Cauchy's theorem or Cauchy's integral formula (but not the residue theorem), evaluate  $\oint_C \frac{1}{1+z^2} dz$  in the cases where  $C$  is:
- (i) the circle  $|z-1| = 1$ , (ii) the circle  $|z-i| = 1$ , and (iii) the circle  $|z| = 2$ .

23. By evaluating  $\oint_C \frac{1}{z} dz$ , where  $C$  is the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with  $a, b > 0$ , show that

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \frac{2\pi}{ab}.$$

24. Suppose that at  $z = z_0$ , the function  $f$  is non-zero and the function  $g$  has a simple zero. Prove that the residue of  $f/g$  at  $z_0$  is  $f(z_0)/g'(z_0)$ .

Suppose that at  $z = z_0$ , the function  $f$  has a pole of order  $N$ . Prove that residue of  $f$  at  $z_0$  is

$$\lim_{z \rightarrow z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \left[ (z - z_0)^N f(z) \right].$$

25. Evaluate  $\oint_C \frac{z^3 e^{1/z}}{1+z} dz$ , where  $C$  is the circle  $|z| = 2$ , using the following methods.

- (i) Integrate directly, using the residue theorem.
- (ii) Apply the substitution  $w = 1/z$ , then use the residue theorem.
- \* (iii) By considering the Laurent series valid on  $C$ , and integrating term by term.

26. (i) Evaluate  $\int_{-\infty}^{\infty} \frac{1}{1+x+x^2} dx$  by closing the contour in the upper half-plane.

How does the calculation differ if you close the contour in the lower half-plane?

- (ii) Evaluate  $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{x}{1+x+x^2} dx$  (without using real methods or part (i)).

Why is the limit necessary here, but not in (i)?

- (iii) Evaluate  $\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} dx$  for  $k > 0$  and for  $k < 0$ .

27. (i) By integrating around a keyhole contour, show that

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin \pi a} \quad (0 < a < 1).$$

- (ii) By integrating around a contour involving the line  $\arg z = \frac{2\pi}{n}$ , evaluate

$$\int_0^{\infty} \frac{1}{1+x^n} dx \quad (n \geq 2).$$

Check by change of variable that your answer agrees with that of part (i).

28. By evaluating each side using a suitable contour integral, show that

$$\int_0^{\pi} \sin^{2n} \theta d\theta = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx \quad (n \in \mathbb{N}).$$

Find a change of variables that explains why they are equal.

**29.** Establish the following:

$$(i) \int_0^\infty \frac{\cos x}{(1+x^2)^3} dx = \frac{7\pi}{16e}; \quad (ii) \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}; \quad (iii) \int_0^\infty \frac{\log x}{1+x^2} dx = 0.$$

Do (ii) by a contour integral, even though there are other ways – see question 34.

For (iii), you could integrate  $\frac{(\log z)^2}{1+z^2}$  around a keyhole contour, or integrate  $\frac{\log z}{1+z^2}$  around an arch-shaped contour, i.e. a semicircle with a bump at 0. (You could do both!) What goes wrong if you integrate  $\frac{\log z}{1+z^2}$  around a keyhole?

**30.** Using a (mostly) rectangular contour involving the line  $\text{Im } z = \pi$ , show that

$$\int_0^\infty \frac{\sin ax}{\sinh x} dx = \frac{\pi}{2} \tanh \frac{\pi a}{2}, \quad \text{for } a \in \mathbb{R}.$$

Deduce the value of  $\int_0^\infty \frac{x}{\sinh x} dx$ .

**31.** (i) By considering the integral of  $\frac{\cot z}{z^2 + \pi^2 a^2}$  around a suitable contour, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a, \quad \text{for } ia \notin \mathbb{Z}.$$

Using the Laurent expansion of  $\coth z$ , deduce the value of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

(ii) Show similarly that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}, \quad \text{for } a \notin \mathbb{Z}.$$

\* By choosing a suitable value of  $a$ , deduce the value of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

**\*32.** By integrating around a contour involving the line  $\arg z = \frac{\pi}{4}$ , evaluate

$$\int_0^\infty \cos x^2 dx \quad \text{and} \quad \int_0^\infty \sin x^2 dx.$$

You may quote that  $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$ .

**\*33.** Evaluate the following using suitable contour integrals, where  $0 < a < 1$  in (i) and (iii).

$$(i) \int_0^\infty \frac{\log(x+1)}{x^{a+1}} dx; \quad (ii) \int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx; \quad (iii) \int_0^{2\pi} \log(1+a \cos \theta) d\theta.$$

## Fourier Transforms

**34.** Let

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \quad \text{and} \quad g(x) = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}.$$

- (i) Show that  $\tilde{f}(k) = \frac{2}{k} \sin ak$  and verify the inversion formula by contour integration.
- (ii) Show that  $g(x) = \frac{1}{2}(f * f)(2x)$  and deduce that  $\tilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}$ .
- (iii) Use Parseval's Identity to show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{2\pi}{3}.$$

**35.** Use the Fourier inversion formula (not contour integration) to show that, for  $a > 0$ ,

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2 + k^2} dk = \frac{\pi}{a} e^{-a|x|} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{ikx}}{b^2 + (a + ik)^2} dk = \frac{2\pi}{b} H(x) e^{-ax} \sin bx,$$

where  $H(x)$  is the Heaviside function. What are the answers when  $a < 0$ ?

**36.** By considering the convolution of the function  $f(x) = e^{-|x|}$  with itself, show that

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{(1 + k^2)^2} dk = \frac{\pi}{2} (1 + |x|) e^{-|x|}.$$

Verify this result by contour integration.

**37.** *This question shows how the Fourier transform representation of a function reduces to a Fourier series if the function is periodic.*

Suppose that  $f(x)$  has period  $2\pi$ . Let  $F(k) = \int_0^{2\pi} f(x) e^{-ikx} dx$ , and let  $g(x) = f(x) e^{-a|x|}$ , where  $a > 0$ . Show that the Fourier transform of  $g(x)$  is given by

$$\tilde{g}(k) = \frac{F(k - ia)}{1 - e^{-2\pi i(k - ia)}} - \frac{F(k + ia)}{1 - e^{-2\pi i(k + ia)}}.$$

Assuming that  $F$  is analytic, sketch the locations of the singularities of  $\tilde{g}$  in the complex  $k$ -plane. Assuming further that the sequence  $\sup\{|\tilde{g}(k)| : |k| = n + \frac{1}{2}\}$  tends to 0 as  $n \rightarrow \infty$ , use the Fourier inversion theorem and a suitable contour to show that

$$g(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n) e^{(in-a)x}$$

for  $x > 0$ , and derive a similar result for  $x < 0$ .

Deduce that

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n) e^{inx}.$$

## Laplace Transforms

**38.** Use standard properties (translation, scaling, etc.) of the Laplace transform to find the Laplace transforms of the following functions: (i)  $t^3 e^{-3t}$ , (ii)  $2e^{3t} \sin 4t$ , (iii)  $e^{-4t} \cosh 2t$ .

**39.** The function  $f(t)$  has Laplace transform  $F(s) = \frac{1}{s^3(s^2 + 1)}$ . Find  $f(t)$  in three ways:

- (i) using partial fractions and standard transforms
- (ii) using the inversion formula and a contour integral
- (iii) using standard transforms and the convolution theorem.

**40.** Find the Laplace transforms of  $f(t) = t^{-1/2}$  and  $g(t) = t^{1/2}$ .

*You may quote that  $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$ .*

Verify, by integrating around a keyhole contour, that the inversion formula holds for  $f(t)$ .

**41.** (i) The *Gamma function* is defined for  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > 0$  by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt .$$

Show that  $\Gamma(z+1) = z\Gamma(z)$ , and deduce that  $\Gamma(n+1) = n!$  if  $n$  is a positive integer. Using the previous question, write down the value of  $\Gamma(\frac{1}{2})$ .

For fixed  $z$ , find the Laplace transform of  $f(t) = t^{z-1}$  in terms of  $\Gamma(z)$ .

(ii) The *Beta function* is defined for  $z, w \in \mathbb{C}$  with  $\operatorname{Re}(z), \operatorname{Re}(w) > 0$  by

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt .$$

Use the convolution theorem for Laplace transforms to show that

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} .$$

(iii) Using question 27, deduce that  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ .

For which range of  $z$  does this hold?

**\*42.** Using the relation  $\Gamma(z+1) = z\Gamma(z)$ , show that we may use analytic continuation to extend the definition of  $\Gamma(z)$  to the whole of  $\mathbb{C}$ , apart from isolated singularities. Find and classify these singularities and find the residues of  $\Gamma(z)$  at them.

Does the relation  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$  hold for the continuation?

**\*43.** *This isn't really to do with Complex Methods – it's just a nice use of the results above.*

Let  $I_{m,n} = \int_0^{\pi/2} \cos^m \theta \sin^n \theta d\theta$ , for  $m, n \in \mathbb{N}$ .

Find  $I_{m,n}$  in terms of the Gamma function. Deduce that  $I_{m,n}$  is a rational multiple of  $\pi$  if  $m$  and  $n$  are both even, and is rational otherwise.



## Laplace Transforms – differential equations

In questions 44–48, use Laplace transforms to solve the given equations for  $t \geq 0$ .

44. Solve the differential equation  $y''' - 3y'' + 3y' - y = t^2 e^t$ , with initial conditions  $y(0) = 1$ ,  $y'(0) = 0$ ,  $y''(0) = -2$ .

45. Solve the differential equation  $y'' - 3y' + 2y = \delta(t)$ , with initial conditions  $y(0) = 0$ ,  $y'(0) = 0$ , where  $\delta(t)$  is the Dirac delta function.

For  $\delta(t)$ , take the Laplace transform to be  $\int_0^\infty f(t)e^{-st} dt$ , i.e. start ‘just to the left of 0’.

46. Solve the system of differential equations  $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -5 & 10 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ , with  $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ .

47. Solve the integral equation  $f(t) + 4 \int_0^t (t - \tau) f(\tau) d\tau = t$ .

Verify that your solution for  $f(t)$  is correct.

48. Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 1 \quad \text{for } x \geq 0, t \geq 0,$$

with boundary conditions

$$y(0, t) = y(x, 0) = \frac{\partial y}{\partial t}(x, 0) = 0 \quad \text{and} \quad y(x, t) \rightarrow \frac{1}{2}t^2 \text{ as } x \rightarrow \infty.$$

49. By considering the Laplace transform of  $f'(t)$ , and assuming that  $\lim_{t \rightarrow \infty} f(t)$  exists for the second case, prove that

$$f(0) = \lim_{s \rightarrow \infty} sF(s) \quad \text{and} \quad \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

\* Show that these still hold if we use the variant Laplace transform given in question 45, providing we replace  $f(0)$  with  $\lim_{t \rightarrow 0^+} f(t)$ .

50. The zeroth-order Bessel function  $J_0(t)$  satisfies the differential equation

$$tJ_0'' + J_0' + tJ_0 = 0, \quad J_0(0) = 1. \quad (\dagger)$$

Find the Laplace transform of  $J_0(t)$ .

Find the convolution of  $J_0(t)$  with itself, and show that  $\int_0^\infty J_0(t) dt = 1$ .

\* By using the inversion formula and a suitable branch cut, show that

$$J_0(t) = \frac{1}{\pi} \int_0^\pi \cos(t \cos \theta) d\theta.$$

Verify that this does indeed solve  $(\dagger)$ .