

Complex Methods: Example Sheet 3

Part IB, Lent Term 2025

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Fourier transforms

1. By using the relationship between the Fourier transform and its inverse, show that for real a and b with $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} e^{i\omega t} d\omega = \frac{\pi}{a} e^{-a|t|} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2} e^{i\omega t} d\omega = 2\pi e^{-at} \sin bt H(t)$$

where $H(t)$ is the Heaviside step function. What are the values of the integrals when $a < 0$? What happens when $a = 0$?

2. Show that the convolution of the function $e^{-|x|}$ with itself is given by $f(x) = (1 + |x|)e^{-|x|}$. Use the convolution theorem for Fourier transforms to show that

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1 + k^2)^2} dk$$

and verify this result by contour integration.

3. Let

$$f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a, \\ 0 & \text{otherwise;} \end{cases} \quad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\tilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \tilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.$$

What is the convolution of f with itself? Use Parseval's identity to evaluate $\int_{-\infty}^{\infty} (\sin^2 x)/x^2 dx$. Verify by contour integration the inversion formula for $f(x)$ for all values of x except $\pm \frac{1}{2}a$.

* Verify the inversion formula also at $x = \pm \frac{1}{2}a$.

- * 4. The displacement $x(t)$ of a damped harmonic oscillator obeys the equation

$$\ddot{x} + 2\gamma\dot{x} + q^2x = f(t), \quad \text{where } \gamma > 0.$$

Assuming that the Fourier transforms $\tilde{x}(\omega)$ and $\tilde{f}(\omega)$ exist, show that

$$x(t) = \int_{-\infty}^{\infty} G(t - t') f(t') dt', \quad \text{where } G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{q^2 + 2i\gamma\omega - \omega^2} d\omega.$$

Show, by differentiation under the integral sign, that

$$\frac{d^2}{dt^2} G(t) + 2\gamma \frac{d}{dt} G(t) + q^2 G(t) = \delta(t).$$

Show that for $0 < \gamma < q$,

$$G(t) = \frac{1}{p} e^{-\gamma t} \sin(pt) H(t), \quad \text{where } p = \sqrt{q^2 - \gamma^2}.$$

[You may use here the results from question 1.]

Laplace transforms

- Starting from the Laplace transform of 1 (namely s^{-1}), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions: (i) e^{-2t} ; (ii) $t^3 e^{-3t}$; (iii) $e^{3t} \sin 4t$; (iv) $e^{-4t} \cosh 4t$; (v) $e^{-t} H(t-1)$, where H is the Heaviside step function.
- Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of $F(s) = (s+3)/\{(s-2)(s^2+1)\}$. Verify this result using the Bromwich inversion formula.
- Use Laplace transforms to solve the differential equation

$$\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^t$$

with initial conditions $y(0) = 1$, $\dot{y}(0) = 0$, $\ddot{y}(0) = -2$.

- Consider a linear system obeying the differential equation

$$\ddot{y} - 3\dot{y} + 2y = u(t), \quad \dot{y}(0) = y(0) = 0.$$

Use Laplace transforms to determine the response of the system to the signal $u(t) = t$. Determine also the response $y(t)$ to a signal $u(t) = \delta(t)$.

[For $\delta(t)$, take the Laplace transform to be $F(s) = \int_0^\infty f(t)e^{-st} dt$, i.e. start “just left of 0”.]

- Solve the integral equation $f(t) + 4 \int_0^t (t-\tau)f(\tau) d\tau = t$ for the unknown function f . Verify your solution.

- * 10. The zeroth order Bessel function $J_0(x)$ satisfies the differential equation

$$xJ_0'' + J_0' + xJ_0 = 0$$

for $x > 0$, with $J_0(0) = 1$ (and $J_0'(0) = 0$ from the equation). Find the Laplace transform of J_0 and deduce that $\int_0^\infty J_0(x) dx = 1$. Find the convolution of J_0 with itself.

- Use Laplace transforms to solve the heat equation $\partial T / \partial t = \partial^2 T / \partial x^2$ with boundary conditions $T(x, 0) = \sin^3 \pi x$ ($0 < x < 1$), $T(0, t) = T(1, t) = 0$ ($t > 0$). [Hint: $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.]
- Using the equality $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$, find the Laplace transform of $f(t) = t^{-1/2}$. By integrating around a Bromwich keyhole contour, verify the inversion formula for $f(t)$. What is the Laplace transform of $t^{1/2}$?

- * 13. The gamma and beta functions are defined for $z, w \in \mathbb{C}$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{and} \quad B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt$$

when $\operatorname{Re}(z), \operatorname{Re}(w) > 0$. Show that $\Gamma(z+1) = z\Gamma(z)$ and hence that $\Gamma(n+1) = n!$ if n is a non-negative integer. Using the previous question, write down the value of $\Gamma(\frac{1}{2})$.

For a fixed value of z , find the Laplace transform of $f(t) = t^{z-1}$ in terms of $\Gamma(z)$. Find the Laplace transform of the convolution $t^{z-1} * t^{w-1}$. Hence establish that

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \quad (*)$$

The domain of Γ and B can be extended to the whole of \mathbb{C} , apart from isolated singularities, by analytic continuation. Does the relation $(*)$ still hold?