Complex Methods: Example Sheet 3

Part IB, Lent Term 2020

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Fourier transforms

1. By using the relationship between the Fourier transform and its inverse, show that for real a and b with $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2}\, e^{i\omega t}\, \mathrm{d}\omega = \frac{\pi}{a} e^{-a|t|} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{b}{(i\omega + a)^2 + b^2}\, e^{i\omega t}\, \mathrm{d}\omega = 2\pi e^{-at} \sin bt\, H(t)$$

where $H(t)$ is the Heaviside step function. What are the values of the integrals when $a < 0$? What happens when $a = 0$?

2. Show that the convolution of the function $e^{-|x|}$ with itself is given by $f(x) = (1 + |x|)e^{-|x|}$. Use the convolution theorem for Fourier transforms to show that

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(1 + k^2)^2}\, \mathrm{d}k$$

and verify this result by contour integration.

3. Let

$$f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a, \\ 0 & \text{otherwise;} \end{cases} \qquad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\widetilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \widetilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.$$

What is the convolution of f with itself? Use Parseval's identity to evaluate $\int_{-\infty}^{\infty} (\sin^2 x)/x^2\, \mathrm{d}x$. Verify by contour integration the inversion formula for $f(x)$ for all values of x except $\pm\frac{1}{2}a$.

∗ Verify the inversion formula also at $x = \pm\frac{1}{2}a$.

∗ **4.** Suppose that $f(x)$ has period 2π. Let $F(k) = \int_0^{2\pi} f(x)e^{-ikx}\, \mathrm{d}x$ and $g(x) = f(x)e^{-a|x|}$ where $a > 0$. Show that the Fourier transform of g is given by

$$\widetilde{g}(k) = \frac{F(k - ia)}{1 - e^{-2\pi i(k - ia)}} - \frac{F(k + ia)}{1 - e^{-2\pi i(k + ia)}}.$$

Assuming that F is analytic, sketch the locations of the singularities of $\widetilde{g}$ in the complex k-plane. Use a suitable contour to show that

$$g(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n)e^{(in-a)x}$$

for $x > 0$ and derive a similar result when $x < 0$. Deduce that

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n)e^{inx}.$$

Justify briefly your choices of contour. (You may assume that F grows sufficiently slowly at infinity. In fact it is sufficient that $F(k)e^{2\pi ik} \to 0$ as $|k| \to \infty$ in the upper half-plane, and $F(k) \to 0$ in the lower half-plane, but you are not expected to carry out detailed calculations.)

[*This shows how the Fourier transform representation of a periodic function reduces to a Fourier series.*]

Laplace transforms

5. Starting from the Laplace transform of 1 (namely s^{-1}), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions: (i) e^{-2t}; (ii) t^3e^{-3t}; (iii) $e^{3t}\sin 4t$; (iv) $e^{-4t}\cosh 4t$; (v) $e^{-t}H(t-1)$, where H is the Heaviside step function.

6. Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of $\widehat{f}(s) = (s+3)/\{(s-2)(s^2+1)\}$. Verify this result using the Bromwich inversion formula.

7. Use Laplace transforms to solve the differential equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} - 3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} - y = t^2e^t$$

with initial conditions $y(0)=1$, $\dot{y}(0)=0$, $\ddot{y}(0)=-2$.

8. A damped simple harmonic oscillator $y(t)$ is at rest for $t<0$ but receives a positive unit impulse at $t=0$ and, subsequently, a negative one at $t=t_0>0$. It obeys the differential equation

$$\ddot{y} + 2\dot{y} + 2y = \delta(t) - \delta(t-t_0).$$

Find the Laplace transform of y and, without inverting it, show that $y\to 0$ as $t\to\infty$ by considering the locations of the singularities in the complex plane. Now use the Bromwich inversion formula to find $y(t)$ for all t.

9. Solve the *integral equation* $f(t) + 4\displaystyle\int_0^t (t-\tau)f(\tau)\,\mathrm{d}\tau = t$ for the unknown function *f*. Verify your solution.

∗ **10.** The zeroth order Bessel function $J_0(x)$ satisfies the differential equation

$$xJ_0'' + J_0' + xJ_0 = 0$$

for $x>0$, with $J_0(0)=1$ (and $J_0'(0)=0$ from the equation). Find the Laplace transform of J_0 and deduce that $\int_0^\infty J_0(x)\,\mathrm{d}x = 1$. Find the convolution of J_0 with itself.

11. Use Laplace transforms to solve the heat equation $\partial T/\partial t = \partial^2 T/\partial x^2$ with boundary conditions $T(x,0)=\sin^3\pi x$ $(0<x<1)$, $T(0,t)=T(1,t)=0$ $(t>0)$. [*Hint:* $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.]

12. Using the equality $\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\pi}$, find the Laplace transform of $f(t)=t^{-1/2}$. By integrating around a Bromwich keyhole contour, verify the inversion formula for $f(t)$. What is the Laplace transform of $t^{1/2}$?

∗ **13.** The gamma and beta functions are defined for $z,w\in\mathbb{C}$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t \quad\text{and}\quad \mathrm{B}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}t$$

when $\mathrm{Re}\,z$, $\mathrm{Re}\,w>0$. Show that $\Gamma(z+1)=z\Gamma(z)$ and hence that $\Gamma(n+1)=n!$ if n is a non-negative integer. Using the previous question, write down the value of $\Gamma(\frac{1}{2})$.

For a fixed value of z, find the Laplace transform of $f(t)=t^{z-1}$ in terms of $\Gamma(z)$. Find the Laplace transform of the convolution $t^{z-1}*t^{w-1}$. Hence establish that

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \qquad (*)$$

The domain of Γ and B can be extended to the whole of $\mathbb{C}$, apart from isolated singularities, by analytic continuation. Does the relation $(*)$ still hold?