ANALYSIS I EXAMPLES 4

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1. Show directly from the definition of an integral that $\int_0^a x^2 = a^3/3$ for $a > 0$.

2. Let $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Does $\int_0^1 f$ exist?

3. Give an example of a continuous function $f : [0, \infty) \to [0, \infty)$, such that $\int_0^\infty f$ exists but f is unbounded.

4. Give an example of an integrable function $f : [0, 1] \to \mathbb{R}$ with $f \geqslant 0$, $\int_0^1 f = 0$, and $f(x) > 0$ for some value of x. Show that this cannot happen if f is continuous.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that $\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$ is countable. Let x_n, $n \geqslant 1$ be a sequence of distinct points in $(0, 1]$. Let $f_n(x) = 0$ if $0 \leqslant x < x_n$ and $f_n(x) = 1$ if $x_n \leqslant x \leqslant 1$. Let $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x)$. Show that this series converges for every $x \in [0, 1]$. Show that f is increasing (and so is integrable). Show that f is discontinuous at every x_n.

6. Let $f(x) = \log(1 - x^2)$. Use the mean value theorem to show that $|f(x)| \leqslant 8x^2/3$ for $0 \leqslant x \leqslant 1/2$. Now let $I_n = \int_{n-1/2}^{n+1/2} \log x \, dx - \log n$ for $n \in \mathbb{N}$. Show that $I_n = \int_0^{1/2} f(t/n)\, dt$ and hence that $|I_n| \leqslant 1/9n^2$. By considering $\sum_{j=1}^n I_j$, deduce that $n!/n^{n+1/2}e^{-n} \to \ell$ for some constant ℓ.
[The bounds $8x^2/3$ and $1/9n^2$ are not best possible; they are merely good enough for the conclusion.]

7. Let $I_n = \int_0^{\pi/2} \cos^n x$. Prove that $nI_n = (n-1)I_{n-2}$, and hence $\frac{2n}{2n+1} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$. Deduce Wallis's Product:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)} = \lim_{n\to\infty} \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-2}.$$

By taking note of the previous exercise, prove that $n!/n^{n+1/2}e^{-n} \to \sqrt{2\pi}$ (Stirling's formula).

8. Do these improper integrals converge? (i) $\int_1^\infty \sin^2(1/x) dx$, (ii) $\int_0^\infty x^p \exp(-x^q) dx$ where $p, q > 0$.

9. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$ as $n \to \infty$, and find $\lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{(-1)^{n-1}}{2n}$.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $\int_a^b f(x)g(x)\, dx = 0$ for every continuous function $g : [a, b] \to \mathbb{R}$ which vanishes near a and b. Must f vanish identically?
[We say g vanishes near a and b if there exists $\epsilon > 0$ such that $g(x) = 0$ for $x \notin (a + \epsilon, b - \epsilon)$.]

11. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Let $G(x, t) = t(x - 1)$ for $t \leqslant x$ and $G(x, t) = x(t - 1)$ for $t \geqslant x$. Let $g(x) = \int_0^1 f(t)G(x, t)dt$. Show that $g''(x)$ exists for $x \in (0, 1)$ and equals $f(x)$.

12. Let $I_n(\theta) = \int_{-1}^{1}(1-x^2)^n \cos(\theta x)dx$. Prove that $\theta^2 I_n = 2n(2n - 1)I_{n-1} - 4n(n - 1)I_{n-2}$ for $n \geqslant 2$, and hence that $\theta^{2n+1} I_n(\theta) = n!(P_n(\theta) \sin\theta + Q_n(\theta) \cos\theta)$, where P_n and Q_n are polynomials of degree at most $2n$ with integer coefficients. Deduce that π is irrational.

13. Let $f_1, f_2 : [-1, 1] \to \mathbb{R}$ be increasing and $g = f_1 - f_2$. Show that there exists K such that, for any dissection $\mathcal{D} = x_0 < \cdots < x_n$ of $[-1, 1]$, $\sum_{j=1}^{n} |g(x_j)-g(x_{j-1})| \leqslant K$. Now let $g(x) = x \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$. Show that g is integrable but is not the difference of two increasing functions.

14$^+$. Show that if $f : [0, 1] \to \mathbb{R}$ is integrable then f has a point of continuity. Deduce that f must have infinitely many points of continuity.