ANALYSIS I EXAMPLES 1

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1. Suppose $(a_n), (b_n)$ are two sequences of real numbers. Prove that if $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$.

2. Sketch the graphs of $y = x$ and $y = (x^4+1)/3$, and thereby illustrate the behaviour of the real sequence a_n where $a_{n+1} = (a_n^4+1)/3$. For which of the three starting cases $a_1 = 0$, $a_1 = 1$ and $a_1 = 2$ does the sequence converge? Now prove your assertion.

3. Let $a_1 > b_1 > 0$ and let $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = 2a_nb_n/(a_n + b_n)$ for $n \geqslant 1$. Show that $a_n > a_{n+1} > b_{n+1} > b_n$ and deduce that the two sequences converge to a common limit. What limit?

4. The real sequence a_n is bounded but does not converge. Prove that it has two convergent subsequences with different limits.

5. Investigate the convergence of the following series. For those expressions containing the complex number z, find those z for which convergence occurs.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4+\sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n}$$

6. Show that $\sum \frac{1}{n(\log n)^\alpha}$ converges if $\alpha > 1$ and diverges otherwise.
Does $\sum 1/(n \log n \log\log n)$ converge?

7. Consider the two series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ and $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots$, having the same terms but taken in a different order. Let s_n and t_n be the corresponding partial sums to n terms. Show that $s_{2n} = h_{2n} - h_n$ and $t_{3n} = h_{4n} - \frac{1}{2}h_{2n} - \frac{1}{2}h_n$, where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$. Show that s_n converges to a limit $s > 0$ and that t_n converges to $3s/2$.

8. For $n \geqslant 1$, let

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}.$$

Show that each a_n is positive and that $\lim a_n = 0$. Show also that $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ diverges. [This shows that, in the alternating series test, it is essential that the moduli of the terms decrease as n increases.]

9. Let a_n and b_n be two sequences and let $S_n = \sum_{j=1}^n a_j$ and $S_0 = 0$. Show that for any $1 \leqslant m \leqslant n$ we have:

$$\sum_{j=m}^{n} a_j b_j = S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}).$$

Suppose now that b_n is a decreasing sequence of positive terms tending to zero. Moreover, suppose that S_n is a bounded sequence. Prove that $\sum_{j=1}^{\infty} a_j b_j$ converges. Deduce the alternating series test.

Does the series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$ converge or diverge?

10. Suppose that $\sum a_n$ diverges and $a_n > 0$. Show that there exist b_n with $b_n/a_n \to 0$ and $\sum b_n$ divergent.

11. Let $z \in \mathbb{C}$ such that $z^{2^j} \neq 1$ for any positive integer j. Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \frac{z^8}{1-z^{16}} + \cdots$$

converges to $z/(1-z)$ if $|z| < 1$, converges to $1/(1-z)$ if $|z| > 1$, and diverges if $|z| = 1$.

12. Prove that every real sequence has a monotonic subsequence. Deduce the Bolzano-Weierstrass theorem.

13. Let (x_n) and (y_n) be real sequences with $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$.

(i) Show that there is a sequence (ε_n) of signs (i.e. $\varepsilon_n \in \{-1, +1\}$ for all n) such that $\sum \varepsilon_n x_n$ is convergent.

(+)(ii) Must there be a sequence (ε_n) of signs such that $\sum \varepsilon_n x_n$ and $\sum \varepsilon_n y_n$ are both convergent?

14. Can we write the open interval (0,1) as a disjoint union of closed intervals of positive length?