

## ANALYSIS I—EXAMPLES 3

(updated 6 March 2026)

**Exercises** [discuss with your supervisor if you should hand these in]

- 1a.** Let  $f: X \subset \mathbb{C} \rightarrow \mathbb{C}$  be differentiable. Suppose  $\exists a, b \in X$  such that the line segment  $[a, b] := \{(1-t)a + tb : t \in [0, 1]\}$  is contained in  $X$ . Use the mean value theorem in  $\mathbb{R}$  to show that  $|f(b) - f(a)| \leq C|b - a| \sup_{x \in [a, b]} |f'(x)|$  for  $C > 0$  a constant you should specify. [Hint: Consider  $g: [0, 1] \rightarrow \mathbb{C}$  given by  $g(t) = f((1-t)a + tb)$  and note  $g(t) = u(t) + iv(t)$  for suitable  $u, v: [0, 1] \rightarrow \mathbb{R}$ .]
- 1b.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and continuously differentiable on  $(a, b)$ . Suppose there exists  $x_0 \in (a, b)$  such that  $f'(x_0) \neq 0$ . Use the inverse function theorem from lectures to show that there exists  $\varepsilon > 0$  with  $[x_0 - \varepsilon, x_0 + \varepsilon] \subset (a, b)$  such that the restriction

$$f: (x_0 - \varepsilon, x_0 + \varepsilon) \longrightarrow f((x_0 - \varepsilon, x_0 + \varepsilon))$$

is bijective, and  $f^{-1}$  is differentiable with derivative

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

for all  $y \in f((x_0 - \varepsilon, x_0 + \varepsilon))$ . This is called the *local* inverse function theorem.

- 1c.** Show directly from the definition of an integral that  $\int_0^a x^2 = a^3/3$  for  $a > 0$ .
- 1d.** Let  $f: [a, b] \rightarrow \mathbb{R}$ . For a partition  $\mathcal{P} = \{a = x_0 < \dots < x_n = b\}$  of  $[a, b]$ , let  $\mathcal{U}_{\mathcal{P}}$  denote all functions  $g$  which are linear on each  $[x_{j-1}, x_j]$  and satisfy  $g(x) \geq f(x)$  and  $\mathcal{L}_{\mathcal{P}}$  denote all functions  $h$  which are linear on each  $[x_{j-1}, x_j]$  and satisfy  $h(x) \leq f(x)$ . Define upper/lower trapezoidal sums by

$$U^T(f, \mathcal{P}) = \inf_{g \in \mathcal{U}_{\mathcal{P}}} \sum_{j=1}^n \frac{g(x_{j-1}) + g(x_j)}{2} (x_j - x_{j-1}) = \inf_{g \in \mathcal{U}_{\mathcal{P}}} I(g),$$
$$L^T(f, \mathcal{P}) = \sup_{h \in \mathcal{L}_{\mathcal{P}}} \sum_{j=1}^n \frac{h(x_{j-1}) + h(x_j)}{2} (x_j - x_{j-1}) = \sup_{h \in \mathcal{L}_{\mathcal{P}}} I(h).$$

Show that if, for each  $\varepsilon > 0$  there is a partition  $\mathcal{P}$  of  $[a, b]$  such that  $U^T(f, \mathcal{P}) - L^T(f, \mathcal{P}) < \varepsilon$ , then  $f$  is integrable.

## Problems

- 2.** Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $|f(x) - f(y)| \leq |x - y|^2$  for all  $x, y \in \mathbb{R}$ . Show that  $f$  is constant.
- 3.** “Let  $f'$  exist on  $(a, b)$  and let  $c \in (a, b)$ . If  $c+h \in (a, b)$  then  $(f(c+h) - f(c))/h = f'(c+\theta h)$ . Let  $h \rightarrow 0$ ; then  $f'(c + \theta h) \rightarrow f'(c)$ . Thus  $f'$  is continuous at  $c$ .” Explain why this argument is false; at what point does it fail?
- 4.** This question guides you through the proof and application of l'Hôpital's rule. Suppose that  $f, g: [a, b] \rightarrow \mathbb{R}$  are continuous and differentiable on  $(a, b)$ . Suppose that  $f(a) = g(a) = 0$ , that  $g'(x)$  does not vanish near  $a$  and  $f'(x)/g'(x) \rightarrow \ell$  as  $x \rightarrow a$ .
- (a) Show that  $f(x)/g(x) \rightarrow \ell$  as  $x \rightarrow a$ . This is l'Hôpital's rule.
- (b) Use the rule with  $g(x) = x - a$  to show that if  $f'(x) \rightarrow \ell$  as  $x \rightarrow a$ , then  $f$  is differentiable at  $a$  with  $f'(a) = \ell$ .

- (c) Find a pair of functions  $f$  and  $g$  as above for which  $\lim_{x \rightarrow a} f(x)/g(x)$  exists, but  $\lim_{x \rightarrow a} f'(x)/g'(x)$  does not.
- (d) Investigate the limit as  $x \rightarrow 1$  of

$$\frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

5. Prove Darboux's theorem: if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable then  $f'$  has the intermediate value property, i.e.  $f'$  maps intervals to intervals. (That is to say, if  $a < b$  and  $f'(a) < z < f'(b)$  then there exists  $c$ ,  $a < c < b$ , with  $f'(c) = z$ .)
6. Given  $\alpha \in \mathbb{R}$ , define  $f_\alpha : [-1, 1] \rightarrow \mathbb{R}$  by  $f_\alpha(x) = |x|^\alpha \sin(1/x)$  for  $x \neq 0$  and  $f_\alpha(0) = 0$ . Is  $f_0$  continuous? Is  $f_1$  differentiable? Draw a table, with 9 columns labelled  $\alpha = -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$  and with 6 rows labelled " $f_\alpha$  bounded", " $f_\alpha$  continuous", " $f_\alpha$  differentiable", " $f'_\alpha$  bounded", " $f'_\alpha$  continuous", " $f'_\alpha$  differentiable". Place ticks and crosses at appropriate places in the table.
7. Let  $f$  be continuous on  $[-1, 1]$  and twice differentiable on  $(-1, 1)$ . Let

$$\phi(x) = \begin{cases} \frac{f(x) - f(0)}{x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases}$$

- (a) Show that  $\phi$  is continuous on  $[-1, 1]$  and differentiable on  $(-1, 1)$ .
- (b) Using a second order mean value theorem for  $f$ , show that  $\phi'(x) = f''(\theta x)/2$  for some  $\theta \in (0, 1)$ . Hence prove that there exists  $c \in (-1, 1)$  with  $f''(c) = f(-1) + f(1) - 2f(0)$ .
8. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \exp(-1/x^2)$  for  $x \neq 0$  and  $f(0) = 0$ . Show that  $f$  is continuous and differentiable. Show that  $f$  is twice differentiable. Indeed, show that  $f$  is infinitely differentiable, and that  $f^{(n)}(0) = 0$  for all  $n \in \mathbb{N}$ . Comment, in the light of what you know about Taylor series. [You may assume standard properties of the exponential function.]
9. Let  $f(x) = \sin(1/x)$  for  $x \neq 0$  and  $f(0) = 0$ . Does  $\int_0^1 f$  exist?
10. Give an example of an integrable function  $f : [0, 1] \rightarrow \mathbb{R}$  with  $f \geq 0$ ,  $\int_0^1 f = 0$ , and  $f(x) > 0$  for some value of  $x$ . Show that this cannot happen if  $f$  is continuous.
11. Let  $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$  be increasing and  $g = f_1 - f_2$ .

- (a) Show that there exists  $K$  such that, for any partition  $\mathcal{P} = x_0 < \dots < x_n$  of  $[a, b]$ ,  $\sum_{j=1}^n |g(x_j) - g(x_{j-1})| \leq K$ .
- (b) Deduce that  $g$  is integrable.
- (c) Now let  $h(x) = x \sin(1/x)$  for  $x \neq 0$  and  $h(0) = 0$ . Show that  $h$  is integrable. Show that  $h$  is not the difference of two increasing functions.

12. This question exemplifies how badly behaved integrable functions can be.

- (a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be monotonic. Show that  $\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$  is countable.
- (b) Let  $x_n, n \geq 1$  be a sequence of distinct points in  $(0, 1]$ . Let

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x), \quad f_n(x) = \begin{cases} 0, & 0 \leq x < x_n \\ 1, & x_n \leq x \leq 1 \end{cases}.$$

Show that this series converges for every  $x \in [0, 1]$ . Show that  $f$  is increasing (and so is integrable). Show that  $f$  is discontinuous at every  $x_n$ .

13. Let  $f : [a, b] \rightarrow \mathbb{R}$  with  $a \neq b$ . **We will study how discontinuities affect integrability.**

- (a) For an interval  $I \subset [a, b]$ , define the *oscillation* of  $f$  in  $I$  by  $\text{osc}_I(f) = \sup_I f - \inf_I f$ . Show that if  $I' \subset I$  is an interval, then  $\text{osc}_{I'}(f) \leq \text{osc}_I(f)$ . Show that  $U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{j=1}^n |I_j| \text{osc}_{I_j}(f)$  for  $I_j = [x_{j-1}, x_j]$  and  $\mathcal{P} = x_0 < \dots < x_n$ . **Bonus:** Show that  $f$  is continuous at  $x$  iff  $\omega(x) = \inf_{\delta > 0} \text{osc}_{(x-\delta, x+\delta) \cap [a, b]}(f) = 0$ . Hence, check  $f$  discontinuous at  $x$  iff  $\omega(x) > 0$  i.e. if  $\exists \eta > 0$  s.t.  $\forall I = [c, d] \subset [a, b]$  with  $x \in (c, d)$  we have  $\text{osc}_I(f) \geq \eta$ .
- (b) Show that, if  $f$  is bounded and has discontinuities only in the set  $\{d_1, \dots, d_k\} \subset (a, b)$ , for some finite  $k \in \mathbb{N}$ , then  $f$  is integrable.

We've seen enough examples of this style of proof to try to write a more abstract result.

**Lemma.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. If, for all  $\varepsilon > 0$  and all  $\eta > 0$  we can find a partition  $\mathcal{P} = \mathcal{P}(\varepsilon, \eta)$  of  $[a, b]$  such that  $\sum_{I \in \mathcal{P}: \text{osc}_I(f) \geq \eta} |I| < \varepsilon$ , then  $f$  is integrable.

**Proof.** We estimate

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{I \in \mathcal{P}: \text{osc}_I(f) \geq \eta} |I| \text{osc}_I(f) + \sum_{I \in \mathcal{P}: \text{osc}_I(f) < \eta} |I| \text{osc}_I(f) \\ &< 2 \sup_{[a, b]} |f| \varepsilon + \eta(b - a), \end{aligned}$$

which we can make as small as we want by shrinking  $\varepsilon, \eta$ . Note  $\{I \in \mathcal{P}: \text{osc}_I(f) \geq \eta\}$  are those subintervals of  $\mathcal{P}$  where  $f$  exhibits ‘‘large’’ oscillation (hence where ‘‘bad’’ discontinuities lie, see comments to (a)), the bad sets where the best we can say about the oscillation is that it is bounded since  $f$  is bounded.  $\square$

Convince yourself that in the above problem, in the proof that if  $f$  differs from integrable  $g$  on finitely many points it is integrable, and in the proof that Thomae's function is integrable, we were relying on this abstract lemma i.e. always building the partitions  $\mathcal{P}(\varepsilon, \eta)$  -- sometimes quite challenging!

- (c) (★) Show that, if  $f$  is integrable, then  $f$  has a point of continuity. Deduce that  $f$  must have infinitely many points of continuity.
- (d) (★★) [OPTIONAL, UNOFFICIAL, MAY NOT BE MARKED] This takes you through a proof of the Lebesgue integrability criterion, which asserts  $f: [a, b] \rightarrow \mathbb{R}$  bounded is Riemann integrable iff  $f$  its set of discontinuities has measure zero. We say  $X \subset \mathbb{R}$  has measure zero if for all  $\varepsilon > 0$  we can cover it by a countable collection of open intervals with total length below  $\varepsilon$ , i.e.  $X \subset \cup_{j=1}^{\infty} I_j$  and  $\sum_{j=1}^{\infty} |I_j| < \varepsilon$ . **Warning:** this is a challenging Analysis II problem (sheet 2, Q6c).
- Show that, if  $X \subset [a, b]$  is a closed set with measure zero, then it can be covered by finitely many small intervals, i.e. there is  $N \in \mathbb{N}$  such that  $X \subset \cup_{j=1}^N I_j$  and  $\sum_{j=1}^N |I_j| < \varepsilon$ . [*Hint: Argue by contradiction and use Bolzano-Weierstrass.*]
  - Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. Show that, if  $I \subset [a, b]$  is a closed interval such that  $\text{osc}_J(f) \geq \eta$  on every closed subinterval  $J \subset I$ , then there exists  $x \in I$  such that  $\omega(x) \geq \eta$ . [*Hint: Use the nested interval property and the idea behind its proof.*]
  - Using i–ii, and the abstract lemma from the comments to part (a), deduce that if  $f: [a, b] \rightarrow \mathbb{R}$  is bounded and its set of discontinuities has measure zero, then  $f$  is Riemann integrable.
  - Show that the converse to (iii) holds: if  $f: [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, then its set of discontinuities has measure zero.
  - Deduce that the Dirichlet function is not integrable, Thomae's function is integrable, and the indicator function for the Cantor set is integrable as well.