Part II

Waves

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Paper 1, Section II

40C Waves

(a) Starting from the equations governing sound waves linearized about a state with density ρ_0 and sound speed c_0, derive the acoustic energy equation, giving expressions for the kinetic energy density K, the potential energy density due to compression W and the wave-energy flux $\mathbf{I}$.

(b) The radius $R(t)$ of a sphere oscillates according to

$$R(t) = a + \mathrm{Re}(\epsilon e^{i\omega t}),$$

where ϵ and ω are real, with $0 < \epsilon \ll a$.

(i) Find an expression for the velocity potential $\phi(r,t)$ in the region outside the sphere.

(ii) Show that for an appropriate time-average, which you should define carefully, the time-averaged rate of working by the surface of the sphere is

$$2\pi a^2 \rho_0 \omega^2 \epsilon^2 c_0 \frac{\omega^2 a^2}{c_0^2 + \omega^2 a^2}.$$

(iii) Calculate the value at $r = a$ of the dimensionless ratio $c_0\langle K + W\rangle / |\langle \mathbf{I}\rangle|$, where angle brackets denote the time average used above.

(iv) Comment briefly on the limits $c_0 \ll \omega a$ and $c_0 \gg \omega a$, explaining their physical meaning and considering the relative magnitudes of the time-averaged kinetic energy, potential energy and acoustic energy flux.

Paper 2, Section II

40C Waves

(a) A uniform elastic solid with wave speeds c_P and c_S (using the usual notation) occupies the region $z < 0$. An SV-wave with unit amplitude displacement

$$\mathbf{u}_I = \mathrm{Re}\{(\cos\theta, 0, -\sin\theta) e^{ik_I(x\sin\theta + z\cos\theta) - i\omega t}\}$$

is incident from $z < 0$ on a rigid boundary at $z = 0$. Find the form and amplitudes of the reflected waves.

(b) Derive a condition on the incident angle θ for the reflected P-wave to be evanescent. Show by explicit calculation that if the P-wave is evanescent:

(i) the reflected SV-wave also has unit amplitude and

(ii) the P-wave has zero acoustic energy flux in the z-direction if time-averaged in an appropriate way, which you should specify carefully.

Paper 3, Section II

39C Waves

(a) The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + \frac{1}{9}\frac{\partial^9 \phi}{\partial x^9} = 0,$$

where $U > 0$ is a constant.

(i) Find the dispersion relation for waves of frequency ω and wavenumber k.

(ii) Sketch both the phase velocity c_p and the group velocity c_g as functions of k.

(iii) Do wave crests move faster or slower than a wave packet?

(b) Suppose that $\phi(x,0)$ is real and given by a Fourier transform as

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk.$$

(i) Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $V > U$ and large t.

(ii) If the initial condition is now restricted further to be even, so that $\phi(x,0) = \phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t) = 0$.

(iii) What can be said about $\phi(Vt,t)$ if $V < U$? [Detailed calculation is not required in this case.]

[*Hint: You may assume that* $\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\frac{\pi}{a}}$ *for* $\mathrm{Re}(a) \geqslant 0, a \neq 0$.]

Paper 4, Section II

39C Waves

For adiabatic motion of an ideal gas, the pressure p is given in terms of the density ρ by a relation of the form

$$p(\rho) = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad (\dagger)$$

where p_0, ρ_0 and γ are positive constants, with $\gamma > 1$. For such a gas, you are given that the compressive internal energy per unit volume W can be expressed as

$$W(\rho) = \frac{p(\rho)}{\gamma - 1}.$$

(a) For one-dimensional motion with speed u, write down expressions for the mass flux and the momentum flux. Using the expressions for the energy flux $u(p + W + \frac{1}{2}\rho u^2)$ and the mass flux, deduce that if the motion is steady then

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}u^2 = C, \qquad (\star)$$

for some constant C.

(b) A one-dimensional shock wave propagates at constant speed along a tube containing the gas. Upstream of the shock the gas is at rest with pressure p_0 and density ρ_0. Downstream of the shock the pressure is maintained at the constant value $p_1 = (1 + \beta)p_0$ with $\beta > 0$. Show that

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma + 1)\beta}{2\gamma + (\gamma - 1)\beta}, \qquad (\ddagger)$$

assuming that $(\star)$ holds throughout the flow.

(c) For small β, show that the density ratio $(\ddagger)$ from part (b) satisfies approximately the adiabatic relation $(\dagger)$, correct to $\mathcal{O}(\beta^2)$.

Paper 1, Section II
40C Waves

(a) Starting from the equations for mass and momentum conservation and a suitable equation of state, derive the linearised wave equation for perturbation pressure $\tilde{p}(\mathbf{x},t)$ for 3-dimensional sound waves in a compressible gas with sound speed c_0 and density ρ_0.

(b) For a 1-dimensional wave of given frequency ω propagating in the x-direction, the perturbation pressure $\tilde{p}(x,t)$ may be written in the form $\Re(\hat{p}(x)e^{i\omega t})$. What is the form of $\hat{p}$ for a harmonic plane wave of frequency ω propagating in the positive x-direction? Express the perturbation fluid speed $\tilde{u}(x,t)$ in terms of $\tilde{p}(x,t)$.

(c) The gas occupies the region $x < L$, with a rigid boundary at $x = L$. A thin flexible membrane of mass m per unit area is located within the gas at equilibrium position $x = 0$. A plane wave of unit amplitude of the form specified in part (b) is incident from $x = -\infty$. The combined effects of the membrane and the rigid boundary result in a reflected wave of complex amplitude R, where R is the ratio between the individual complex amplitudes at $x = 0^-$ of the reflected and incident waves.

(i) Show that

$$R = \frac{\cos\beta + (\alpha - i)\sin\beta}{\cos\beta + (\alpha + i)\sin\beta} \quad \text{where } \alpha = \frac{\omega m}{\rho_0 c_0} \quad \text{and} \quad \beta = \frac{\omega L}{c_0}\,.$$

Deduce that $|R| = 1$ in general and briefly discuss this result physically.

(ii) Identify a condition on β so that the membrane is stationary and there is non-trivial pressure perturbation in $0 < x < L$. Briefly discuss this result physically.

(iii) Identify and interpret a limit for α in which the pressure perturbation in $0 < x < L$ becomes very small relative to that in $x < 0$.

Paper 2, Section II
40C Waves

Infinitesimal displacements $\boldsymbol{u}(\boldsymbol{x},t)$ in a uniform, linear isotropic elastic solid with density ρ_0 and Lamé moduli λ and μ satisfy the linearised Cauchy momentum equation:

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu)\, \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u}.$$

(a) Show that the dilatation $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ and the rotation $\boldsymbol{\nabla} \times \boldsymbol{u}$ satisfy wave equations, and find the wave-speeds c_P and c_S.

(b) A plane harmonic P-wave with wavevector $\boldsymbol{k}$ lying in the (x,z) plane is incident from $z<0$ at an oblique angle on the planar horizontal interface $z=0$ between two elastic solids with different densities and elastic moduli. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves is evanescent. State the boundary conditions on components of $\boldsymbol{u}$ and the stress tensor $\boldsymbol{\sigma}$ and explain why these are sufficient to determine the amplitudes. (You do not need to calculate the directions or amplitudes explicitly.)

(c) Now consider a plane harmonic P-wave of unit amplitude, with $\boldsymbol{k} = k(\sin\theta, 0, \cos\theta)$, incident from $z<0$ on the interface $z=0$ between two elastic (and inviscid) liquids with modulus λ, density ρ and wave-speed c_P in $z<0$ and modulus λ', density ρ' and wave-speed c'_P in $z>0$, with $\rho' \neq \rho$.

(i) Under what conditions is there a propagating transmitted wave in $z>0$?

(ii) Assume from here on that these conditions are met. Obtain solutions for the reflected and transmitted waves.

(iii) Show that the amplitude of the reflected wave is

$$R = \frac{\lambda' \sin 2\theta - \lambda \sin 2\theta'}{\lambda' \sin 2\theta + \lambda \sin 2\theta'},$$

where θ' is the angle the wave vector of the transmitted wave makes with the vertical.

(iv) Hence obtain an expression for θ in terms of the wave-speeds and densities of the two liquids that implies no reflection (i.e. $R=0$).

Paper 3, Section II
39C Waves

Waves propagating in a slowly-varying medium satisfy the local dispersion relation $\omega = \Omega(\boldsymbol{k}; \boldsymbol{x}, t)$ in the standard notation.

(a) Derive the ray-tracing equations:

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t},$$

governing the evolution of a wave packet specified by

$$\phi(\boldsymbol{x}, t) = A(\boldsymbol{x}, t; \varepsilon) \exp\left(\frac{i\theta(\boldsymbol{x}, t)}{\varepsilon}\right),$$

where $0 < \varepsilon \ll 1$. A rigorous derivation is not required, but assumptions should be clearly stated and the meaning of the d/dt notation should be carefully explained.

(b) The dispersion relation for two-dimensional, small amplitude, internal gravity waves of wavenumber vector $\boldsymbol{k} = (k, 0, m)$, relative to Cartesian coordinates (x, y, z) with z vertical, propagating in an inviscid, incompressible, stratified fluid with a slowly-varying mean flow $\boldsymbol{U}$ is

$$\omega = \frac{Nk}{\sqrt{k^2 + m^2}} + \boldsymbol{k} \cdot \boldsymbol{U},$$

where N is the buoyancy frequency. Consider the specific flow $\boldsymbol{U} = \gamma(x, 0, -z)$. N and γ are positive constants.

(i) Calculate $k(t)$ and $m(t)$, applying the initial conditions $k(0) = k_0 > 0$, $m(0) = m_0$.

(ii) Consider a wave packet with initial wave vector $(k_0, 0, m_0)$, released from $(x_0, 0, z_0)$ where $x_0 > 0$ and $z_0 > 0$. Show that the wave packet can initially propagate upwards provided $z_0 < z_m$, where z_m is a function of k_0 and m_0.

(iii) Demonstrate that such a wave packet eventually approaches $z = 0$, but takes an infinite amount of time to do so. [*Hint: It is not essential to solve for an explicit expression for the position of the wave packet at arbitrary time t.*]

Paper 4, Section II
39C Waves

Consider finite amplitude, one-dimensional sound waves in a perfect gas with ratio of specific heats γ.

(a) Show that the fluid speed u and local sound speed c satisfy

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right) R_{\pm} = 0\,,$$

where the *Riemann invariants* $R_{\pm}(x,t)$ should be defined carefully. Write down parametric equations for the paths on which these quantities are actually invariant.

(b) At time $t = 0$ the gas occupies the region $x > 0$. It is at rest and has uniform density ρ_0, pressure p_0 and sound speed c_0. A piston initially at $x = 0$ starts moving backwards at time $t = 0$ with displacement $x = -\varepsilon t(1-t)$, where $\varepsilon > 0$ is constant.

(i) Show that prior to any shock forming $c = c_0 + \frac{1}{2}(\gamma - 1)u$.

(ii) For small ε, derive an expression for the relative pressure fluctuation $\delta p/p_0 = p/p_0 - 1$ to second order in the relative sound speed fluctuation $\delta c/c_0 = c/c_0 - 1$.

(iii) Calculate the time average over the interval $0 \leqslant t \leqslant 1$ of the relative pressure fluctuation, measured on the piston, and briefly discuss your result physically.

Paper 1, Section II
40A Waves

Compressible fluid of equilibrium density ρ_0, pressure p_0 and sound speed c_0 is contained in the region between an inner rigid sphere of radius R and an outer elastic sphere of equilibrium radius $2R$. The elastic sphere is made to oscillate radially in such a way that it exerts a spherically symmetric, perturbation pressure $\tilde{p} = \epsilon p_0 \cos \omega t$ on the fluid at $r = 2R$, where $\epsilon \ll 1$ and the frequency ω is sufficiently small that

$$\alpha \equiv \frac{\omega R}{c_0} \leqslant \frac{\pi}{2} \, .$$

You may assume that the acoustic velocity potential satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi \, .$$

(a) Derive an expression for $\phi(r, t)$.

(b) Hence show that the net radial component of the acoustic intensity (wave-energy flux) $\mathbf{I} = \tilde{p}\mathbf{u}$ is zero when averaged appropriately in a way you should define. Interpret this result physically.

(c) Briefly discuss the possible behaviour of the system if the forcing frequency ω is allowed to increase to larger values.

$\left[\textit{For a spherically symmetric variable } \psi(r,t), \; \nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\psi) \, . \right]$

Paper 2, Section II

40A Waves

A semi-infinite elastic medium with shear modulus μ and shear-wave speed c_s lies in $z \leqslant 0$. Above it, there is a layer $0 \leqslant z \leqslant h$ of a second elastic medium with shear modulus $\overline{\mu}$ and shear-wave speed $\overline{c}_s < c_s$. The top boundary is stress-free. Consider a monochromatic SH-wave propagating in the x-direction at speed c with wavenumber $k > 0$.

(a) Derive the dispersion relation

$$\tan\left[kh\sqrt{c^2/\overline{c}_s^2 - 1}\right] = \frac{\mu}{\overline{\mu}}\frac{\sqrt{1 - c^2/c_s^2}}{\sqrt{c^2/\overline{c}_s^2 - 1}}$$

for trapped modes with no disturbance as $z \to -\infty$.

(b) Show graphically that there is always a zeroth mode, and show that the other modes have cut-off frequencies

$$\omega_c^{(n)} = \frac{n\pi\overline{c}_s c_s}{h\sqrt{c_s^2 - \overline{c}_s^2}},$$

where n is a positive integer. Sketch a graph of frequency ω against k for the $n = 1$ mode showing the behaviour near cut-off and for large k.

Paper 3, Section II

39A Waves

Consider a two-dimensional stratified fluid of sufficiently slowly varying background density $\rho_b(z)$ that small-amplitude vertical-velocity perturbations $w(x,z,t)$ can be assumed to satisfy the linear equation

$$\nabla^2\left(\frac{\partial^2 w}{\partial t^2}\right) + N^2(z)\frac{\partial^2 w}{\partial x^2} = 0, \qquad \text{where } N^2 = \frac{-g}{\rho_0}\frac{d\rho_b}{dz}$$

and ρ_0 is a constant. The background density profile is such that N^2 is piecewise constant with $N^2 = N_0^2 > 0$ for $|z| > L$ and with $N^2 = 0$ in a layer $|z| < L$ of uniform density ρ_0.

A monochromatic internal wave of amplitude A_I is incident on the intermediate layer from $z = -\infty$, and produces velocity perturbations of the form

$$w(x,z,t) = \widehat{w}(z)e^{i(kx-\omega t)},$$

where $k > 0$ and $0 < \omega < N_0$.

(a) Show that the vertical variations have the form

$$\widehat{w}(z) = \begin{cases} A_I \exp\big[-im\,(z+L)\big] + A_R \exp\big[im\,\big(z+L\big)\big] & \text{for } z < -L\,, \\ B_C \cosh kz + B_S \sinh kz & \text{for } |z| < L\,, \\ A_T \exp\big[-im\,(z-L)\big] & \text{for } z > L\,, \end{cases}$$

where A_R, B_C, B_S and A_T are (in general) complex amplitudes and

$$m = k\sqrt{\frac{N_0^2}{\omega^2} - 1}\,.$$

In particular, you should justify the choice of signs for the coefficients involving m.

(b) What are the appropriate boundary conditions to impose on $\widehat{w}$ at $z = \pm L$ to determine the unknown amplitudes?

(c) Apply these boundary conditions to show that

$$\frac{A_T}{A_I} = \frac{2imk}{2imk\cosh 2\alpha + (k^2 - m^2)\sinh 2\alpha}\,,$$

where $\alpha = kL$.

(d) Hence show that

$$\left|\frac{A_T}{A_I}\right|^2 = \left[1 + \left(\frac{\sinh 2\alpha}{\sin 2\psi}\right)^2\right]^{-1},$$

where ψ is the angle between the incident wavevector and the downward vertical.

Paper 4, Section II
39A Waves

A plane shock is moving with speed U into a perfect gas. Ahead of the shock the gas is at rest with pressure p_1 and density ρ_1, while behind the shock the velocity, pressure and density of the gas are u_2, p_2 and ρ_2 respectively.

(a) Write down the Rankine–Hugoniot relations across the shock, briefly explaining how they arise.

(b) Show that

$$\frac{\rho_1}{\rho_2} = \frac{2c_1^2 + (\gamma - 1)U^2}{(\gamma + 1)U^2},$$

where $c_1^2 = \gamma p_1/\rho_1$ and γ is the ratio of the specific heats of the gas.

(c) Now consider a change of frame such that the shock is stationary and the gas has a component of velocity U parallel to the shock on both sides. Deduce that a stationary shock inclined at a 45 degree angle to an incoming stream of Mach number $M = \sqrt{2}U/c_1$ deflects the flow by an angle δ given by

$$\tan \delta = \frac{M^2 - 2}{\gamma M^2 + 2}.$$

$\left[\textit{Note that } \tan(\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.\right]$

Paper 1, Section II

40B Waves

(a) Write down the linearised equations governing motion of an inviscid compressible fluid at uniform entropy. Assuming that the velocity is irrotational, show that the velocity potential $\phi(\mathbf{x}, t)$ satisfies the wave equation and identify the wave speed c_0. Obtain from these linearised equations the energy-conservation equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{I} = 0,$$

and give expressions for the acoustic-energy density E and the acoustic-energy flux, or intensity, $\mathbf{I}$.

(b) Inviscid compressible fluid with density ρ_0 and sound speed c_0 occupies the regions $y < 0$ and $y > 0$, which are separated by a thin elastic membrane at an undisturbed position $y = 0$. The membrane has mass per unit area m and is under a constant tension T. Small displacements of the membrane to $y = \eta(x, t)$ are coupled to small acoustic disturbances in the fluid with velocity potential $\phi(x, y, t)$.

(i) Write down the (linearised) kinematic and dynamic boundary conditions at the membrane. [*Hint: The elastic restoring force on the membrane is like that on a stretched string.*]

(ii) Show that the dispersion relation for waves proportional to $\cos(kx - \omega t)$ propagating along the membrane with $|\phi| \to 0$ as $y \to \pm\infty$ is given by

$$\left\{ m + \frac{2\rho_0}{\left(k^2 - \omega^2/c_0^2\right)^{1/2}} \right\} \omega^2 = Tk^2\,.$$

Interpret this equation by explaining physically why all disturbances propagate with phase speed c less than $(T/m)^{1/2}$ and why $c(k) \to 0$ as $k \to 0$.

(iii) Show that in such a wave the component $\langle I_y \rangle$ of mean acoustic intensity perpendicular to the membrane is zero.

Paper 2, Section II

39B Waves

Small displacements $\mathbf{u}(\mathbf{x}, t)$ in a homogeneous elastic medium are governed by the equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u})\,,$$

where ρ is the density, and λ and μ are the Lamé constants.

(a) Show that the equation supports two types of harmonic plane-wave solutions, $\mathbf{u} = \mathbf{A}\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, with distinct wavespeeds, c_P and c_S, and distinct polarizations. Write down the direction of the displacement vector $\mathbf{A}$ for a P-wave, an SV-wave and an SH-wave, in each case for the wavevector $(k, 0, m)$.

(b) Given k and c, with $c > c_P\ (> c_S)$, explain how to construct a superposition of P-waves with wavenumbers $(k, 0, m_P)$ and $(k, 0, -m_P)$, such that

$$\mathbf{u}(x, z, t) = e^{ik(x-ct)}\big(f_1(z), 0, i f_3(z)\big)\,, \qquad (*)$$

where $f_1(z)$ is an even function, and f_1 and f_3 are both real functions, to be determined. Similarly, find a superposition of SV-waves with $\mathbf{u}$ again in the form $(*)$.

(c) An elastic waveguide consists of an elastic medium in $-H < z < H$ with rigid boundaries at $z = \pm H$. Using your answers to part (b), show that the waveguide supports propagating eigenmodes that are a mixture of P- and SV-waves, and have dispersion relation $c(k)$ given by

$$a\tan(akH) = -\frac{\tan(bkH)}{b}, \quad \text{where} \quad a = \left(\frac{c^2}{c_P^2} - 1\right)^{1/2} \quad \text{and} \quad b = \left(\frac{c^2}{c_S^2} - 1\right)^{1/2}.$$

Sketch the two sides of the dispersion relationship as functions of c. Explain briefly why there are infinitely many solutions.

Paper 3, Section II

39B Waves

The dispersion relation for capillary waves on the surface of deep water is

$$\omega^2 = S^2|k|^3\,,$$

where $S = (T/\rho)^{1/2}$, ρ is the density and T is the coefficient of surface tension. The free surface $z = \eta(x,t)$ is undisturbed for $t < 0$, when it is suddenly impacted by an object, giving the initial conditions at time $t = 0$:

$$\eta = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \begin{cases} -W\,, & |x| < \epsilon\,, \\ 0\,, & |x| > \epsilon\,, \end{cases}$$

where W is a constant.

(i) Use Fourier analysis to find an integral expression for $\eta(x,t)$ when $t > 0$.

(ii) Use the method of stationary phase to find the asymptotic behaviour of $\eta(Vt,t)$ for fixed $V > 0$ as $t \to \infty$, for the case $V \ll \epsilon^{-1/2}S$. Show that the result can be written in the form

$$\eta(x,t) \sim \frac{W\epsilon\, S\, t^2}{x^{5/2}}\, F\left(\frac{x^3}{S^2t^2}\right),$$

and determine the function F.

(iii) Give a brief physical interpretation of the link between the condition $\epsilon V^2/S^2 \ll 1$ and the simple dependence on the product $W\epsilon$.

[You are given that $\displaystyle\int_{-\infty}^{\infty} e^{\pm iau^2}\,du = (\pi/a)^{1/2}\,e^{\pm i\pi/4}$ for $a > 0$.]

Paper 4, Section II

39B Waves

(a) Show that the equations for one-dimensional unsteady flow of an inviscid compressible fluid at constant entropy can be put in the form

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)R_{\pm} = 0\,,$$

where u and c are the fluid velocity and the local sound speed, respectively, and the Riemann invariants $R_{\pm}$ are to be defined.

Such a fluid occupies a long narrow tube along the x-axis. For times $t < 0$ it is at rest with uniform pressure p_0, density ρ_0 and sound speed c_0. At $t = 0$ a finite segment, $0 \leqslant x \leqslant L$, is disturbed so that $u = U(x)$ and $c = c_0 + C(x)$, with $U = C = 0$ for $x \leqslant 0$ and $x \geqslant L$. Explain, with the aid of a carefully labelled sketch, how two independent simple waves emerge after some time. You may assume that no shock waves form.

(b) A fluid has the adiabatic equation of state

$$p(\rho) = A - \frac{B^2}{\rho}\,,$$

where A and B are positive constants and $\rho > B^2/A$.

(i) Calculate the Riemann invariants for this fluid, and express $u \pm c$ in terms of $R_{\pm}$ and c_0. Deduce that in a simple wave with $R_- = 0$ the velocity field translates, without any nonlinear distortion, at the equilibrium sound speed c_0.

(ii) At $t = 0$ this fluid occupies $x > 0$ and is at rest with uniform pressure, density and sound speed. For $t > 0$ a piston initially at $x = 0$ executes simple harmonic motion with position $x(t) = a \sin \omega t$, where $a\omega < c_0$. Show that $u(x,t) = U(\phi)$, where $\phi = \omega(t - x/c_0)$, for some function U that is zero for $\phi < 0$ and is 2π-periodic, but not simple harmonic, for $\phi > 0$. By approximately inverting the relationship between ϕ and the time τ that a characteristic leaves the piston for the case $\epsilon = a\omega/c_0 \ll 1$, show that

$$U(\phi) = a\omega\Big(\cos\phi - \epsilon \sin^2\phi - \tfrac{3}{2}\epsilon^2 \sin^2\phi \cos\phi + O(\epsilon^3)\Big) \quad \text{for} \quad \phi > 0\,.$$

Paper 4, Section II

38A Waves

(a) Assuming a slowly-varying two-dimensional wave pattern of the form

$$\varphi(\mathbf{x},t) = A(\mathbf{x},t;\varepsilon)\exp\left[\frac{i}{\varepsilon}\theta(\mathbf{x},t)\right],$$

where $0 < \varepsilon \ll 1$, and a local dispersion relation $\omega = \Omega(\mathbf{k};\mathbf{x},t)$, derive the ray tracing equations,

$$\frac{dx_i}{dt} = \frac{\partial\Omega}{\partial k_i}, \quad \frac{d\omega}{dt} = \frac{\partial\Omega}{\partial t}, \quad \frac{dk_i}{dt} = -\frac{\partial\Omega}{\partial x_i}, \quad \frac{1}{\varepsilon}\frac{d\theta}{dt} = -\omega + k_j\frac{\partial\Omega}{\partial k_j},$$

for $i, j = 1, 2$, explaining carefully the meaning of the notation used.

(b) For a homogeneous, time-independent (but not necessarily isotropic) medium, show that all rays are straight lines. When the waves have zero frequency, deduce that if the point $\mathbf{x}$ lies on a ray emanating from the origin in the direction given by a unit vector $\widehat{\mathbf{c}}_{\mathbf{g}}$, then

$$\theta(\mathbf{x}) = \theta(\mathbf{0}) + \widehat{\mathbf{c}}_{\mathbf{g}}\cdot\mathbf{k}\,|\mathbf{x}|\,.$$

(c) Consider a stationary obstacle in a steadily moving homogeneous medium which has the dispersion relation

$$\Omega = \alpha\left(k_1^2 + k_2^2\right)^{1/4} - Vk_1\,,$$

where $(V,0)$ is the velocity of the medium and $\alpha > 0$ is a constant. The obstacle generates a steady wave system. Writing $(k_1,k_2) = \kappa(\cos\phi,\sin\phi)$, with $\kappa > 0$, show that the wave satisfies

$$\kappa = \frac{\alpha^2}{V^2\cos^2\phi}, \quad \widehat{\mathbf{c}}_{\mathbf{g}} = (\cos\psi,\sin\psi)\,,$$

where ψ is defined by

$$\tan\psi = -\frac{\tan\phi}{1+2\tan^2\phi}$$

with $\frac{1}{2}\pi < \psi < \frac{3}{2}\pi$ and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. Deduce that the wave pattern occupies a wedge of semi-angle $\tan^{-1}\left(2^{-3/2}\right)$, extending in the negative x_1-direction.

Paper 2, Section II
38A Waves

The linearised equation of motion governing small disturbances in a homogeneous elastic medium of density ρ is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} \,,$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement, and λ and μ are the Lamé moduli.

(a) The medium occupies the region between a rigid plane boundary at $y = 0$ and a free surface at $y = h$. Show that SH waves can propagate in the x-direction within this region, and find the dispersion relation for such waves.

(b) For each mode, deduce the cutoff frequency, the phase velocity and the group velocity. Plot the latter two velocities as a function of wavenumber.

(c) Verify that in an average sense (to be made precise), the wave energy flux is equal to the wave energy density multiplied by the group velocity.

[*You may assume that the elastic energy per unit volume is given by*

$$E_p = \tfrac{1}{2} \lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij} \,.]$$

Paper 3, Section II
39A Waves

(a) Derive the wave equation for perturbation pressure for linearised sound waves in a compressible gas.

(b) For a single plane wave show that the perturbation pressure and the velocity are linearly proportional and find the constant of proportionality, i.e. the acoustic impedance.

(c) Gas occupies a tube lying parallel to the x-axis. In the regions $x < 0$ and $x > L$ the gas has uniform density ρ_0 and sound speed c_0. For $0 < x < L$ the temperature of the gas has been adjusted so that it has uniform density ρ_1 and sound speed c_1. A harmonic plane wave with frequency ω and unit amplitude is incident from $x = -\infty$. If T is the (in general complex) amplitude of the wave transmitted into $x > L$, show that

$$|T| = \left(\cos^2 k_1 L + \tfrac{1}{4} \left(\lambda + \lambda^{-1} \right)^2 \sin^2 k_1 L \right)^{-\frac{1}{2}} \,,$$

where $\lambda = \rho_1 c_1 / \rho_0 c_0$ and $k_1 = \omega / c_1$. Discuss both of the limits $\lambda \ll 1$ and $\lambda \gg 1$.

Paper 1, Section II

39A Waves

The equation of state relating pressure p to density ρ for a perfect gas is given by

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma,$$

where p_0 and ρ_0 are constants, and $\gamma > 1$ is the specific heat ratio.

(a) Starting from the equations for one-dimensional unsteady flow of a perfect gas of uniform entropy, show that the Riemann invariants,

$$R_\pm = u \pm \frac{2}{\gamma - 1}(c - c_0)$$

are constant on characteristics $C_\pm$ given by

$$\frac{dx}{dt} = u \pm c,$$

where $u(x,t)$ is the velocity of the gas, $c(x,t)$ is the local speed of sound, and c_0 is a constant.

(b) Such an ideal gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas and the piston are initially at rest. At time $t = 0$ the piston starts moving to the left with path given by

$$x = X_p(t), \quad \text{with } X_p(0) = 0.$$

(i) Solve for $u(x,t)$ and $\rho(x,t)$ in the region $x > X_p(t)$ under the assumptions that $-\frac{2c_0}{\gamma-1} < \dot{X}_p < 0$ and that $|\dot{X}_p|$ is monotonically increasing, where dot indicates a time derivative.

[It is sufficient to leave the solution in implicit form, i.e. for given x, t you should not attempt to solve the C_+ characteristic equation explicitly.]

(ii) Briefly outline the behaviour of u and ρ for times $t > t_c$, where t_c is the solution to $\dot{X}_p(t_c) = -\frac{2c_0}{\gamma-1}$.

(iii) Now suppose,

$$X_p(t) = -\frac{t^{1+\alpha}}{1+\alpha},$$

where $\alpha \geqslant 0$. For $0 < \alpha \ll 1$, find a leading-order approximation to the solution of the C_+ characteristic equation when $x = c_0 t - at$, $0 < a < \frac{1}{2}(\gamma + 1)$ and $t = O(1)$.

[*Hint: You may find it useful to consider the structure of the characteristics in the limiting case when $\alpha = 0$.*]

Paper 4, Section II

39C Waves

A physical system permits one-dimensional wave propagation in the x-direction according to the equation

$$\left(1 - 2\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^4 \varphi}{\partial x^4} = 0\,.$$

Derive the corresponding dispersion relation and sketch graphs of frequency, phase velocity and group velocity as functions of the wavenumber. Waves of what wavenumber are at the front of a dispersing wave train arising from a localised initial disturbance? For waves of what wavenumbers do wave crests move faster or slower than a packet of waves?

Find the solution of the above equation for the initial disturbance given by

$$\varphi(x,0) = \int_{-\infty}^{\infty} 2A(k)e^{ikx}dk\,, \quad \frac{\partial \varphi}{\partial t}(x,0) = 0\,,$$

where $A^*(-k) = A(k)$, and A^* is the complex conjugate of A. Let $V = x/t$ be held fixed. Use the method of stationary phase to obtain a leading-order approximation to this solution for large t when $0 < V < V_m = (3\sqrt{3})/8$, where the solutions for the stationary points should be left in implicit form.

Very briefly discuss the nature of the solutions for $-V_m < V < 0$ and $|V| > V_m$.

[*Hint: You may quote the result that the large time behaviour of*

$$\Phi(x,t) = \int_{-\infty}^{\infty} A(k)e^{ikx - i\omega(k)t}dk\,,$$

due to a stationary point $k = \alpha$, is given by

$$\Phi(x,t) \sim \left(\frac{2\pi}{|\omega''(\alpha)|\, t}\right)^{\frac{1}{2}} A(\alpha)\, e^{i\alpha x - i\omega(\alpha)t + i\sigma\pi/4}\,,$$

where $\sigma = -\mathrm{sgn}(\omega''(\alpha))$.]

Paper 2, Section II

39C Waves

A perfect gas occupies the region $x > 0$ of a tube that lies parallel to the x-axis. The gas is initially at rest, with density ρ_1, pressure p_1, speed of sound c_1 and specific heat ratio γ. For times $t > 0$ a piston, initially at $x = 0$, is pushed into the gas at a constant speed V. A shock wave propagates at constant speed U into the undisturbed gas ahead of the piston. Show that the excess pressure in the gas next to the piston, $p_2 - p_1 \equiv \beta p_1$, is given implicitly by the expression

$$V^2 = \frac{2\beta^2}{2\gamma + (\gamma+1)\beta}\,\frac{p_1}{\rho_1}\,.$$

Show also that

$$\frac{U^2}{c_1^2} = 1 + \frac{\gamma+1}{2\gamma}\beta\,,$$

and interpret this result.

[*Hint: You may assume for a perfect gas that the speed of sound is given by*

$$c^2 = \frac{\gamma p}{\rho}\,,$$

and that the internal energy per unit mass is given by

$$e = \frac{1}{\gamma-1}\frac{p}{\rho}\,. \quad]$$

Paper 1, Section II

39C Waves

Derive the wave equation governing the velocity potential for linearised sound waves in a perfect gas. How is the pressure disturbance related to the velocity potential?

A high pressure gas with unperturbed density ρ_0 is contained within a thin metal spherical shell which makes small amplitude spherically symmetric vibrations. Let the metal shell have radius a, mass m per unit surface area, and an elastic stiffness which tries to restore the radius to its equilibrium value a_0 with a force $\kappa(a-a_0)$ per unit surface area. Assume that there is a vacuum outside the spherical shell. Show that the frequencies ω of vibration satisfy

$$\theta^2\left(1+\frac{\alpha}{\theta\cot\theta-1}\right)=\frac{\kappa a_0^2}{mc_0^2}\,,$$

where $\theta=\omega a_0/c_0$, $\alpha=\rho_0 a_0/m$, and c_0 is the speed of sound in the undisturbed gas. Briefly comment on the existence of solutions.

[*Hint: In terms of spherical polar coordinates you may assume that for a function* $\psi\equiv\psi(r)$,

$$\nabla^2\psi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\,.\quad]$$

Paper 3, Section II

40C Waves

Derive the ray-tracing equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t},$$

for wave propagation through a slowly-varying medium with local dispersion relation $\omega = \Omega(\mathbf{k}; \mathbf{x}, t)$, where ω and $\mathbf{k}$ are the frequency and wavevector respectively, t is time and $\mathbf{x} = (x, y, z)$ are spatial coordinates. The meaning of the notation d/dt should be carefully explained.

A slowly-varying medium has a dispersion relation $\Omega(\mathbf{k}; \mathbf{x}, t) = kc(z)$, where $k = |\mathbf{k}|$. State and prove Snell's law relating the angle ψ between a ray and the z-axis to c.

Consider the case of a medium with wavespeed $c = c_0(1 + \beta^2 z^2)$, where β and c_0 are positive constants. Show that a ray that passes through the origin with wavevector $k(\cos\phi, 0, \sin\phi)$, remains in the region

$$|z| \leqslant z_m \equiv \frac{1}{\beta}\left[\frac{1}{|\cos\phi|} - 1\right]^{1/2}.$$

By considering an approximation to the equation for a ray in the region $|z_m - z| \ll \beta^{-1}$, or otherwise, determine the path of a ray near z_m, and hence sketch rays passing through the origin for a few sample values of ϕ in the range $0 < \phi < \pi/2$.

Paper 2, Section II

37B Waves

Show that, for a one-dimensional flow of a perfect gas (with $\gamma > 1$) at constant entropy, the Riemann invariants $R_{\pm} = u \pm 2(c - c_0)/(\gamma - 1)$ are constant along characteristics $dx/dt = u \pm c$.

Define a *simple wave*. Show that in a right-propagating simple wave

$$\frac{\partial u}{\partial t} + \left(c_0 + \tfrac{1}{2}(\gamma+1)u\right)\frac{\partial u}{\partial x} = 0\,.$$

In some circumstances, dissipative effects may be modelled by

$$\frac{\partial u}{\partial t} + \left(c_0 + \tfrac{1}{2}(\gamma+1)u\right)\frac{\partial u}{\partial x} = -\alpha u\,,$$

where α is a positive constant. Suppose also that u is prescribed at $t = 0$ for all x, say $u(x,0) = u_0(x)$. Demonstrate that, unless a shock develops, a solution of the form

$$u(x,t) = u_0(\xi)e^{-\alpha t}$$

can be found, where, for each x and t, ξ is determined implicitly as the solution of the equation

$$x - c_0 t = \xi + \frac{\gamma+1}{2\alpha}\left(1 - e^{-\alpha t}\right)u_0(\xi)\,.$$

Deduce that, despite the presence of dissipative effects, a shock will still form at some (x,t) unless $\alpha > \alpha_c$, where

$$\alpha_c = \tfrac{1}{2}(\gamma+1)\max_{u_0'<0}|u_0'(\xi)|\,.$$

Paper 1, Section II

38B Waves

Derive the wave equation governing the pressure disturbance $\tilde{p}$, for linearised, constant entropy sound waves in a compressible inviscid fluid of density ρ_0 and sound speed c_0, which is otherwise at rest.

Consider a harmonic acoustic plane wave with wavevector $\mathbf{k}_I = k_I(\sin\theta, \cos\theta, 0)$ and unit-amplitude pressure disturbance. Determine the resulting velocity field $\mathbf{u}$.

Consider such an acoustic wave incident from $y < 0$ on a thin elastic plate at $y = 0$. The regions $y < 0$ and $y > 0$ are occupied by gases with densities ρ_1 and ρ_2, respectively, and sound speeds c_1 and c_2, respectively. The kinematic boundary conditions at the plate are those appropriate for an inviscid fluid, and the (linearised) dynamic boundary condition is

$$m\frac{\partial^2\eta}{\partial t^2} + B\frac{\partial^4\eta}{\partial x^4} + [\tilde{p}(x,0,t)]_-^+ = 0\,,$$

where m and B are the mass and bending moment per unit area of the plate, and $y = \eta(x,t)$ (with $|\mathbf{k}_I\eta| \ll 1$) is its perturbed position. Find the amplitudes of the reflected and transmitted pressure perturbations, expressing your answers in terms of the dimensionless parameter

$$\beta = \frac{k_I\cos\theta(mc_1^2 - Bk_I^2\sin^4\theta)}{\rho_1 c_1^2}\,.$$

(i) If $\rho_1 = \rho_2 = \rho_0$ and $c_1 = c_2 = c_0$, under what condition is the incident wave perfectly transmitted?

(ii) If $\rho_1 c_1 \gg \rho_2 c_2$, comment on the reflection coefficient, and show that waves incident at a sufficiently large angle are reflected as if from a pressure-release surface (i.e. an interface where $\tilde{p} = 0$), no matter how large the plate mass and bending moment may be.

Paper 3, Section II

38B Waves

Waves propagating in a slowly-varying medium satisfy the local dispersion relation $\omega = \Omega(\mathbf{k}; \mathbf{x}, t)$ in the standard notation. Derive the ray-tracing equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}$$

governing the evolution of a wave packet specified by $\varphi(\mathbf{x}, t) = A(\mathbf{x}, t; \varepsilon)e^{i\theta(\mathbf{x},t)/\varepsilon}$, where $0 < \varepsilon \ll 1$. A formal justification is not required, but the meaning of the d/dt notation should be carefully explained.

The dispersion relation for two-dimensional, small amplitude, internal waves of wavenumber $\mathbf{k} = (k, 0, m)$, relative to Cartesian coordinates (x, y, z) with z vertical, propagating in an inviscid, incompressible, stratified fluid that would otherwise be at rest, is given by

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2},$$

where N is the Brunt–Väisälä frequency and where you may assume that $k > 0$ and $\omega > 0$. Derive the modified dispersion relation if the fluid is not at rest, and instead has a slowly-varying mean flow $(U(z), 0, 0)$.

In the case that $U'(z) > 0, U(0) = 0$ and N is constant, show that a disturbance with wavenumber $\mathbf{k} = (k, 0, 0)$ generated at $z = 0$ will propagate upwards but cannot go higher than a critical level $z = z_c$, where $U(z_c)$ is equal to the apparent wave speed in the x-direction. Find expressions for the vertical wave number m as $z \to z_c$ from below, and show that it takes an infinite time for the wave to reach the critical level.

Paper 4, Section II

38B Waves

Consider the Rossby-wave equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \ell^2\right)\varphi + \beta\frac{\partial \varphi}{\partial x} = 0\,,$$

where $\ell > 0$ and $\beta > 0$ are real constants. Find and sketch the dispersion relation for waves with wavenumber k and frequency $\omega(k)$. Find and sketch the phase velocity $c(k)$ and the group velocity $c_g(k)$, and identify in which direction(s) the wave crests travel, and the corresponding direction(s) of the group velocity.

Write down the solution with initial value

$$\varphi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk\,,$$

where $A(k)$ is real and $A(-k) = A(k)$. Use the method of stationary phase to obtain leading-order approximations to $\varphi(x,t)$ for large t, with x/t having the constant value V, for

(i) $0 < V < \beta/8\ell^2$,

(ii) $-\beta/\ell^2 < V \leqslant 0$,

where the solutions for the stationary points should be left in implicit form. [It is helpful to note that $\omega(-k) = -\omega(k)$.]

Briefly discuss the nature of the solution for $V > \beta/8\ell^2$ and $V < -\beta/\ell^2$. [Detailed calculations are not required.]

[*Hint: You may assume that*

$$\int_{-\infty}^{\infty} e^{\pm i\gamma u^2}du = \left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}} e^{\pm i\pi/4}$$

for $\gamma > 0$.]

Paper 4, Section II
37D Waves

A duck swims at a constant velocity $(-V, 0)$, where $V > 0$, on the surface of infinitely deep water. Surface tension can be neglected, and the dispersion relation for the linear surface water waves (relative to fluid at rest) is $\omega^2 = g|\mathbf{k}|$. Show that the wavevector $\mathbf{k}$ of a plane harmonic wave that is steady in the duck's frame, i.e. of the form

$$\mathrm{Re}\left[A\, e^{i(k_1 x' + k_2 y)}\right] ,$$

where $x' = x + Vt$ and y are horizontal coordinates relative to the duck, satisfies

$$(k_1, k_2) = \frac{g}{V^2}\sqrt{p^2+1}\,(1, p) ,$$

where $\hat{\mathbf{k}} = (\cos\phi, \sin\phi)$ and $p = \tan\phi$. [You may assume that $|\phi| < \pi/2$.]

Assume that the wave pattern behind the duck can be regarded as a Fourier superposition of such steady waves, i.e., the surface elevation η at $(x', y) = R(\cos\theta, \sin\theta)$ has the form

$$\eta = \mathrm{Re}\int_{-\infty}^{\infty} A(p)\, e^{i\lambda h(p;\theta)}\, \mathrm{d}p \quad \text{for } |\theta| < \tfrac{1}{2}\pi ,$$

where

$$\lambda = \frac{gR}{V^2} , \quad h(p;\theta) = \sqrt{p^2+1}\,(\cos\theta + p\sin\theta) .$$

Show that, in the limit $\lambda \to \infty$ at fixed θ with $0 < \theta < \cot^{-1}(2\sqrt{2})$,

$$\eta \sim \sqrt{\frac{2\pi}{\lambda}}\ \mathrm{Re}\left\{ \frac{A(p_+)}{\sqrt{h_{pp}(p_+;\theta)}}\, e^{i\left(\lambda h(p_+;\theta) + \frac{1}{4}\pi\right)} + \frac{A(p_-)}{\sqrt{-h_{pp}(p_-;\theta)}}\, e^{i\left(\lambda h(p_-;\theta) - \frac{1}{4}\pi\right)} \right\} ,$$

where

$$p_\pm = -\tfrac{1}{4}\cot\theta \pm \tfrac{1}{4}\sqrt{\cot^2\theta - 8}$$

and h_{pp} denotes $\partial^2 h/\partial p^2$. Briefly interpret this result in terms of what is seen.

Without doing detailed calculations, briefly explain what is seen as $\lambda \to \infty$ at fixed θ with $\cot^{-1}(2\sqrt{2}) < \theta < \pi/2$. Very briefly comment on the case $\theta = \cot^{-1}(2\sqrt{2})$.

[*Hint: You may find the following results useful.*

$$h_p = \left\{p\cos\theta + (2p^2+1)\sin\theta\right\}(p^2+1)^{-1/2} ,$$

$$h_{pp} = (\cos\theta + 4p\sin\theta)(p^2+1)^{-1/2} - \left\{p\cos\theta + (2p^2+1)\sin\theta\right\}p(p^2+1)^{-3/2} . \Big]$$

Paper 2, Section II

37D Waves

Starting from the equations for one-dimensional unsteady flow of a perfect gas at constant entropy, show that the Riemann invariants

$$R_{\pm} = u \pm \frac{2(c - c_0)}{\gamma - 1}$$

are constant on characteristics $C_{\pm}$ given by $dx/dt = u \pm c$, where $u(x,t)$ is the speed of the gas, $c(x,t)$ is the local speed of sound, c_0 is a constant and $\gamma > 1$ is the exponent in the adiabatic equation of state for $p(\rho)$.

At time $t = 0$ the gas occupies $x > 0$ and is at rest at uniform density ρ_0, pressure p_0 and sound speed c_0. For $t > 0$, a piston initially at $x = 0$ has position $x = X(t)$, where

$$X(t) = -U_0\, t\Big(1 - \frac{t}{2t_0}\Big)$$

and U_0 and t_0 are positive constants. For the case $0 < U_0 < 2c_0/(\gamma - 1)$, sketch the piston path $x = X(t)$ and the C_+ characteristics in $x \geqslant X(t)$ in the (x,t)-plane, and find the time and place at which a shock first forms in the gas.

Do likewise for the case $U_0 > 2c_0/(\gamma - 1)$.

Paper 1, Section II

37D Waves

Write down the linearised equations governing motion of an inviscid compressible fluid at uniform entropy. Assuming that the velocity is irrotational, show that it may be derived from a velocity potential $\phi(\mathbf{x}, t)$ satisfying the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi \,,$$

and identify the wave speed c_0. Obtain from these linearised equations the energy-conservation equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{I} = 0 \,,$$

and give expressions for the acoustic-energy density E and the acoustic-energy flux $\mathbf{I}$ in terms of ϕ.

Such a fluid occupies a semi-infinite waveguide $x > 0$ of square cross-section $0 < y < a$, $0 < z < a$ bounded by rigid walls. An impenetrable membrane closing the end $x = 0$ makes prescribed small displacements to

$$x = X(y, z, t) \equiv \mathrm{Re}\left[e^{-i\omega t} A(y, z)\right] \,,$$

where $\omega > 0$ and $|A| \ll a, c_0/\omega$. Show that the velocity potential is given by

$$\phi = \mathrm{Re}\left[e^{-i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\left(\frac{m\pi y}{a}\right) \cos\left(\frac{n\pi z}{a}\right) f_{mn}(x)\right] \,,$$

where the functions $f_{mn}(x)$, including their amplitudes, are to be determined, with the sign of any square roots specified clearly.

If $0 < \omega < \pi c_0/a$, what is the asymptotic behaviour of ϕ as $x \to +\infty$? Using this behaviour and the energy-conservation equation averaged over both time and the cross-section, or otherwise, determine the double-averaged energy flux along the waveguide,

$$\left\langle \overline{I_x} \right\rangle (x) \equiv \frac{\omega}{2\pi a^2} \int_0^{2\pi/\omega} \int_0^a \int_0^a I_x(x, y, z, t) \; \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}t \,,$$

explaining why this is independent of x.

Paper 3, Section II

37D Waves

Small disturbances in a homogeneous elastic solid with density ρ and Lamé moduli λ and μ are governed by the equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}),$$

where $\mathbf{u}(\mathbf{x},t)$ is the displacement. Show that a harmonic plane-wave solution

$$\mathbf{u} = \mathrm{Re}\left[\mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$

must satisfy

$$\omega^2\mathbf{A} = c_P^2\ \mathbf{k}\ (\mathbf{k}\cdot\mathbf{A}) - c_S^2\ \mathbf{k}\times(\mathbf{k}\times\mathbf{A})\ ,$$

where the wavespeeds c_P and c_S are to be identified. Describe mathematically how such plane-wave solutions can be classified into longitudinal P-waves and transverse SV- and SH-waves (taking the y-direction as the vertical direction).

The half-space $y<0$ is filled with the elastic solid described above, while the slab $0<y<h$ is filled with a homogeneous elastic solid with Lamé moduli $\overline{\lambda}$ and $\overline{\mu}$, and wavespeeds $\overline{c}_P$ and $\overline{c}_S$. There is a rigid boundary at $y=h$. A harmonic plane SH-wave propagates from $y<0$ towards the interface $y=0$, with displacement

$$\mathrm{Re}\left[Ae^{i(\ell x+my-\omega t)}\right](0,0,1)\,. \qquad (*)$$

How are ℓ, m and ω related? The total displacement in $y<0$ is the sum of $(*)$ and that of the reflected SH-wave,

$$\mathrm{Re}\left[RAe^{i(\ell x-my-\omega t)}\right](0,0,1)\ .$$

Write down the form of the displacement in $0<y<h$, and determine the (complex) reflection coefficient R. Verify that $|R|=1$ regardless of the parameter values, and explain this physically.

Paper 4, Section II

36B Waves

The shallow-water equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

describe one-dimensional flow over a horizontal boundary with depth $h(x,t)$ and velocity $u(x,t)$, where g is the acceleration due to gravity.

Show that the Riemann invariants $u \pm 2(c - c_0)$ are constant along characteristics $C_\pm$ satisfying $dx/dt = u \pm c$, where $c(h)$ is the linear wave speed and c_0 denotes a reference state.

An initially stationary pool of fluid of depth h_0 is held between a stationary wall at $x = a > 0$ and a removable barrier at $x = 0$. At $t = 0$ the barrier is instantaneously removed allowing the fluid to flow into the region $x < 0$.

For $0 \leqslant t \leqslant a/c_0$, find $u(x,t)$ and $c(x,t)$ in each of the regions

$$\begin{array}{ll}\text{(i)} & c_0 t \leqslant x \leqslant a \\ \text{(ii)} & -2c_0 t \leqslant x \leqslant c_0 t\end{array}$$

explaining your argument carefully with a sketch of the characteristics in the (x,t) plane.

For $t \geqslant a/c_0$, show that the solution in region (ii) above continues to hold in the region $-2c_0 t \leqslant x \leqslant 3a(c_0 t/a)^{1/3} - 2c_0 t$. Explain why this solution does not hold in $3a(c_0 t/a)^{1/3} - 2c_0 t < x < a$.

Paper 2, Section II

36B Waves

A uniform elastic solid with density ρ and Lamé moduli λ and μ occupies the region between rigid plane boundaries $z = 0$ and $z = h$. Starting with the linear elastic wave equation, show that SH waves can propagate in the x-direction within this waveguide, and find the dispersion relation $\omega(k)$ for the various modes.

State the cut-off frequency for each mode. Find the corresponding phase velocity $c(k)$ and group velocity $c_g(k)$, and sketch these functions for $k, \omega > 0$.

Define the time and cross-sectional average appropriate for a mode with frequency ω. Show that for each mode the average kinetic energy is equal to the average elastic energy. [You may assume that the elastic energy per unit volume is $\frac{1}{2}(\lambda e_{kk}^2 + 2\mu e_{ij}e_{ij})$.]

An elastic displacement of the form $\mathbf{u} = (0, f(x,z), 0)$ is created in a region near $x = 0$, and then released at $t = 0$. Explain briefly how the amplitude of the resulting disturbance varies with time as $t \to \infty$ at the moving position $x = Vt$ for each of the cases $0 < V^2 < \mu/\rho$ and $V^2 > \mu/\rho$. [You may quote without proof any generic results from the method of stationary phase.]

Paper 3, Section II

37B Waves

Derive the ray-tracing equations for the quantities $\mathrm{d}k_i/\mathrm{d}t$, $\mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}x_i/\mathrm{d}t$ during wave propagation through a slowly varying medium with local dispersion relation $\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$, explaining the meaning of the notation $\mathrm{d}/\mathrm{d}t$.

The dispersion relation for water waves is $\Omega^2 = g\kappa \tanh(\kappa h)$, where h is the water depth, $\kappa^2 = k^2 + l^2$, and k and l are the components of $\mathbf{k}$ in the horizontal x and y directions. Water waves are incident from an ocean occupying $x > 0$, $-\infty < y < \infty$ onto a beach at $x = 0$. The undisturbed water depth is $h(x) = \alpha x^p$, where α, p are positive constants and α is sufficiently small that the depth can be assumed to be slowly varying. Far from the beach, the waves are planar with frequency ω_∞ and with crests making an acute angle θ_∞ with the shoreline.

Obtain a differential equation (with k defined implicitly) for a ray $y = y(x)$ and show that near the shore the ray satisfies

$$y - y_0 \sim Ax^q$$

where A and q should be found. Sketch the shape of the wavecrests near the shoreline for the case $p < 2$.

Paper 1, Section II

37B Waves

An acoustic plane wave (not necessarily harmonic) travels at speed c_0 in the direction $\hat{\mathbf{k}}$, where $|\hat{\mathbf{k}}| = 1$, through an inviscid, compressible fluid of unperturbed density ρ_0. Show that the velocity $\tilde{\mathbf{u}}$ is proportional to the perturbation pressure $\tilde{p}$, and find $\tilde{\mathbf{u}}/\tilde{p}$. Define the *acoustic intensity* $\mathbf{I}$.

A harmonic acoustic plane wave with wavevector $\mathbf{k} = k(\cos\theta, \sin\theta, 0)$ and unit-amplitude perturbation pressure is incident from $x < 0$ on a thin elastic membrane at unperturbed position $x = 0$. The regions $x < 0$ and $x > 0$ are both occupied by gas with density ρ_0 and sound speed c_0. The kinematic boundary conditions at the membrane are those appropriate for an inviscid fluid, and the (linearized) dynamic boundary condition is

$$m\frac{\partial^2 X}{\partial t^2} - T\frac{\partial^2 X}{\partial y^2} + \big[\,\tilde{p}(0, y, t)\,\big]_-^+ = 0$$

where T and m are the tension and mass per unit area of the membrane, and $x = X(y, t)$ (with $|kX| \ll 1$) is its perturbed position. Find the amplitudes of the reflected and transmitted pressure perturbations, expressing your answers in terms of the dimensionless parameter

$$\alpha = \frac{\rho_0 c_0^2}{k\cos\theta(mc_0^2 - T\sin^2\theta)}\,.$$

Hence show that the time-averaged energy flux in the x-direction is conserved across the membrane.

Paper 4, Section II

38C Waves

A one-dimensional shock wave propagates at a constant speed along a tube aligned with the x-axis and containing a perfect gas. In the reference frame where the shock is at rest at $x = 0$, the gas has speed U_0, density ρ_0 and pressure p_0 in the region $x < 0$ and speed U_1, density ρ_1 and pressure p_1 in the region $x > 0$.

Write down equations of conservation of mass, momentum and energy across the shock. Show that

$$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_0}{\rho_0}\right) = \frac{p_1 - p_0}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right) ,$$

where γ is the ratio of specific heats.

From now on, assume $\gamma = 2$ and let $P = p_1/p_0$. Show that $\frac{1}{3} < \rho_1/\rho_0 < 3$.

The increase in entropy from $x < 0$ to $x > 0$ is given by $\Delta S = C_V \log(p_1\rho_0^2/p_0\rho_1^2)$, where C_V is a positive constant. Show that ΔS is a monotonic function of P.

If $\Delta S > 0$, deduce that $P > 1$, $\rho_1/\rho_0 > 1$, $(U_0/c_0)^2 > 1$ and $(U_1/c_1)^2 < 1$, where c_0 and c_1 are the sound speeds in $x < 0$ and $x > 0$, respectively. Given that ΔS must have the same sign as U_0 and U_1, interpret these inequalities physically in terms of the properties of the flow upstream and downstream of the shock.

Paper 2, Section II

38C Waves

The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^4\phi}{\partial x^2\partial t^2}\,.$$

Derive the dispersion relation, and sketch graphs of frequency, phase velocity and group velocity as functions of the wavenumber. In the case of a localised initial disturbance, will it be the shortest or the longest waves that are to be found at the front of a dispersing wave packet? Do the wave crests move faster or slower than the wave packet?

Give the solution to the initial-value problem for which at $t=0$

$$\phi = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk \quad \text{and} \quad \frac{\partial\phi}{\partial t} = 0\,,$$

and $\phi(x,0)$ is real. Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $0<V<1$ and large t. If, in addition, $\phi(x,0)=\phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t)=0$.

You are given that $\phi(t,t)$ decreases like $t^{-1/4}$ for large t. Give a brief physical explanation why this rate of decay is slower than for $0<V<1$. What can be said about $\phi(Vt,t)$ for large t if $V>1$? [Detailed calculation is not required in these cases.]

[*You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{-au^2}du = \sqrt{\frac{\pi}{a}}$ *for* $\mathrm{Re}(a) \geqslant 0,\ a \neq 0$.]

Paper 3, Section II

39C Waves

The equations describing small-amplitude motions in a stably stratified, incompressible, inviscid fluid are

$$\frac{\partial \tilde{\rho}}{\partial t} + w\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = 0\,, \qquad \rho_0\frac{\partial \mathbf{u}}{\partial t} = \tilde{\rho}\mathbf{g} - \boldsymbol{\nabla}\tilde{p}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0\,,$$

where $\rho_0(z)$ is the background stratification, $\tilde{\rho}(\mathbf{x},t)$ and $\tilde{p}(\mathbf{x},t)$ are the perturbations about an undisturbed hydrostatic state, $\mathbf{u}(\mathbf{x},t) = (u,v,w)$ is the velocity, and $\mathbf{g} = (0,0,-g)$.

Show that

$$\left[\frac{\partial^2}{\partial t^2}\nabla^2 + N^2\Big(\nabla^2 - \frac{\partial^2}{\partial z^2}\Big)\right] w = 0\,,$$

stating any approximation made, and define the Brunt–Väisälä frequency N.

Deduce the dispersion relation for plane harmonic waves with wavevector $\mathbf{k} = (k,0,m)$. Calculate the group velocity and verify that it is perpendicular to $\mathbf{k}$.

Such a stably stratified fluid with a uniform value of N occupies the region $z > h(x,t)$ above a moving lower boundary $z = h(x,t)$. Find the velocity field $w(x,z,t)$ generated by the boundary motion for the case $h = \epsilon\sin[k(x-Ut)]$, where $0 < \epsilon k \ll 1$ and $U > 0$ is a constant.

For the case $k^2 < N^2/U^2$, sketch the orientation of the wave crests, the direction of propagation of the crests, and the direction of the group velocity.

Paper 1, Section II

39C Waves

State the equations that relate strain to displacement and stress to strain in a uniform, linear, isotropic elastic solid with Lamé moduli λ and μ. In the absence of body forces, the Cauchy momentum equation for the infinitesimal displacements $\mathbf{u}(\mathbf{x}, t)$ is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \ ,$$

where ρ is the density and $\boldsymbol{\sigma}$ the stress tensor. Show that both the dilatation $\boldsymbol{\nabla} \cdot \mathbf{u}$ and the rotation $\boldsymbol{\nabla} \wedge \mathbf{u}$ satisfy wave equations, and find the wave-speeds c_P and c_S.

A plane harmonic P-wave with wavevector $\mathbf{k}$ lying in the (x, z) plane is incident from $z < 0$ at an oblique angle on the planar interface $z = 0$ between two elastic solids with different densities and elastic moduli. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves are evanescent. State the boundary conditions on components of $\mathbf{u}$ and $\boldsymbol{\sigma}$ that would, in principle, determine the amplitudes.

Now consider a plane harmonic P-wave of unit amplitude incident with $\mathbf{k} = k(\sin\theta, 0, \cos\theta)$ on the interface $z = 0$ between two elastic (and inviscid) *liquids* with wave-speed c_P and modulus λ in $z < 0$ and wave-speed c'_P and modulus λ' in $z > 0$. Obtain solutions for the reflected and transmitted waves. Show that the amplitude of the reflected wave is zero if

$$\sin^2\theta = \frac{Z'^2 - Z^2}{Z'^2 - (c'_P Z/c_P)^2} \ ,$$

where $Z = \lambda/c_P$ and $Z' = \lambda'/c'_P$.

Paper 4, Section II
38C Waves

A wave disturbance satisfies the equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} + c^2 \psi = 0\,,$$

where c is a positive constant. Find the dispersion relation, and write down the solution to the initial-value problem for which $\partial\psi/\partial t(x,0) = 0$ for all x, and $\psi(x,0)$ is given in the form

$$\psi(x,0) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk\ ,$$

where $A(k)$ is a real function with $A(k) = A(-k)$, so that $\psi(x,0)$ is real and even.

Use the method of stationary phase to obtain an approximation to $\psi(x,t)$ for large t, with x/t taking the constant value V, and $0 \leqslant V < c$. Explain briefly why your answer is inappropriate if $V > c$.

[You are given that

$$\int_{-\infty}^{\infty} \exp(iu^2)\, du = \pi^{1/2} e^{i\pi/4}\,.]$$

Paper 2, Section II
38C Waves

Show that the equations governing linear elasticity have plane-wave solutions, distinguishing between P, SV and SH waves.

A semi-infinite elastic medium in $y < 0$ (where y is the vertical coordinate) with density ρ and Lamé moduli λ and μ is overlaid by a layer of thickness h (in $0 < y < h$) of a second elastic medium with density ρ' and Lamé moduli λ' and μ'. The top surface at $y = h$ is free, that is, the surface tractions vanish there. The speed of the S-waves is lower in the layer, that is, ${c'_S}^2 = \mu'/\rho' < \mu/\rho = {c_S}^2$. For a time-harmonic SH-wave with horizontal wavenumber k and frequency ω, which oscillates in the slow top layer and decays exponentially into the fast semi-infinite medium, derive the dispersion relation for the apparent horizontal wave speed $c(k) = \omega/k$:

$$\tan\left(kh\sqrt{(c^2/{c'_S}^2) - 1}\right) = \frac{\mu\sqrt{1 - (c^2/c_S^2)}}{\mu'\sqrt{(c^2/{c'_S}^2) - 1}}\,. \qquad (*)$$

Show graphically that for a given value of k there is always at least one real value of c which satisfies equation $(*)$. Show further that there are one or more higher modes if $\sqrt{c_S^2/{c'_S}^2 - 1} > \pi/kh$.

Paper 3, Section II

39C Waves

The dispersion relation for sound waves of frequency ω in a stationary homogeneous gas is $\omega = c_0|\mathbf{k}|$, where c_0 is the speed of sound and $\mathbf{k}$ is the wavenumber. Derive the dispersion relation for sound waves of frequency ω in a uniform flow with velocity $\mathbf{U}$.

For a slowly-varying medium with local dispersion relation $\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$, derive the ray-tracing equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}\,, \qquad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}\,, \qquad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}\,,$$

explaining carefully the meaning of the notation used.

Suppose that two-dimensional sound waves with initial wavenumber $(k_0, l_0, 0)$ are generated at the origin in a gas occupying the half-space $y > 0$. If the gas has a slowly-varying mean velocity $(\gamma y, 0, 0)$, where $\gamma > 0$, show:

(a) that if $k_0 > 0$ and $l_0 > 0$ the waves reach a maximum height (which should be identified), and then return to the level $y = 0$ in a finite time;

(b) that if $k_0 < 0$ and $l_0 > 0$ then there is no bound on the height to which the waves propagate.

Comment *briefly* on the existence, or otherwise, of a quiet zone.

Paper 1, Section II

39C Waves

Starting from the equations for the one-dimensional unsteady flow of a perfect gas of uniform entropy, show that the Riemann invariants

$$R_{\pm} = u \pm \frac{2}{\gamma - 1}(c - c_0)$$

are constant on characteristics $C_{\pm}$ given by $dx/dt = u \pm c$, where $u(x,t)$ is the velocity of the gas, $c(x,t)$ is the local speed of sound, c_0 is a constant and γ is the ratio of specific heats.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas and the piston are initially at rest with $c = c_0$. At time $t = 0$ the piston starts moving to the left at a constant velocity V. Find $u(x,t)$ and $c(x,t)$ in the three regions

(i) $c_0 t \leqslant x\,,$
(ii) $at \leqslant x \leqslant c_0 t\,,$
(iii) $-Vt \leqslant x \leqslant at\,,$

where $a = c_0 - \frac{1}{2}(\gamma+1)V$. What is the largest value of V for which c is positive throughout region (iii)? What happens if V exceeds this value?

Paper 4, Section II

38D Waves

The shallow-water equations

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0, \qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}=0$$

describe one-dimensional flow in a channel with depth $h(x,t)$ and velocity $u(x,t)$, where g is the acceleration due to gravity.

(i) Find the speed $c(h)$ of linearized waves on fluid at rest and of uniform depth.

(ii) Show that the Riemann invariants $u \pm 2c$ are constant on characteristic curves $C_{\pm}$ of slope $u \pm c$ in the (x,t)-plane.

(iii) Use the shallow-water equations to derive the equation of momentum conservation

$$\frac{\partial (hu)}{\partial t}+\frac{\partial I}{\partial x}=0\ ,$$

and identify the horizontal momentum flux I.

(iv) A hydraulic jump propagates at constant speed along a straight constant-width channel. Ahead of the jump the fluid is at rest with uniform depth h_0. Behind the jump the fluid has uniform depth $h_1=h_0(1+\beta)$, with $\beta>0$. Determine both the speed V of the jump and the fluid velocity u_1 behind the jump.

Express $V/c(h_0)$ and $(V-u_1)/c(h_1)$ as functions of β. Hence sketch the pattern of characteristics in the frame of reference of the jump.

Paper 2, Section II

38D Waves

Derive the ray-tracing equations

$$\frac{dx_i}{dt}=\frac{\partial \Omega}{\partial k_i}, \qquad \frac{dk_i}{dt}=-\frac{\partial \Omega}{\partial x_i}, \qquad \frac{d\omega}{dt}=\frac{\partial \Omega}{\partial t},$$

for wave propagation through a slowly-varying medium with local dispersion relation $\omega=\Omega(\mathbf{k},\mathbf{x},t)$. The meaning of the notation d/dt should be carefully explained.

A non-dispersive slowly varying medium has a local wave speed c that depends only on the z coordinate. State and prove Snell's Law relating the angle ψ between a ray and the z-axis to c.

Consider the case of a medium with wavespeed $c=A\cosh\beta z$, where A and β are positive constants. Find the equation of the ray that passes through the origin with wavevector $(k_0,0,m_0)$, and show that it remains in the region $\beta|z|\leqslant \sinh^{-1}(m_0/k_0)$. Sketch several rays passing through the origin.

Paper 3, Section II

39D Waves

The function $\phi(x,t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + \frac{1}{5}\frac{\partial^5 \phi}{\partial x^5} = 0\,,$$

where $U > 0$ is a constant. Find the dispersion relation for waves of frequency ω and wavenumber k. Sketch a graph showing both the phase velocity $c(k)$ and the group velocity $c_g(k)$, and state whether wave crests move faster or slower than a wave packet.

Suppose that $\phi(x,0)$ is real and given by a Fourier transform as

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk\,.$$

Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $V > U$ and large t. If, in addition, $\phi(x,0) = \phi(-x,0)$, deduce an approximation for the sequence of times at which $\phi(Vt,t) = 0$.

What can be said about $\phi(Vt,t)$ if $V < U$? [Detailed calculation is **not** required in this case.]

[You may assume that $\displaystyle\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\frac{\pi}{a}}$ for $\mathrm{Re}(a) \geqslant 0$, $a \neq 0$.]

Paper 1, Section II

39D Waves

Write down the linearized equations governing motion in an inviscid compressible fluid and, assuming an adiabatic relationship $p = p(\rho)$, derive the wave equation for the velocity potential $\phi(\mathbf{x}, t)$. Obtain from these linearized equations the energy equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{I} = 0\,,$$

and give expressions for the acoustic energy density E and the acoustic intensity, or energy-flux vector, $\mathbf{I}$.

An inviscid compressible fluid occupies the half-space $y > 0$, and is bounded by a very thin flexible membrane of negligible mass at an undisturbed position $y = 0$. Small acoustic disturbances with velocity potential $\phi(x, y, t)$ in the fluid cause the membrane to be deflected to $y = \eta(x, t)$. The membrane is supported by springs that, in the deflected state, exert a restoring force $K\eta\,\delta x$ on an element δx of the membrane. Show that the dispersion relation for waves proportional to $\exp(ikx - i\omega t)$ propagating freely along the membrane is

$$\left(k^2 - \frac{\omega^2}{c_0^2}\right)^{1/2} - \frac{\rho_0 \omega^2}{K} = 0\,,$$

where ρ_0 is the density of the fluid and c_0 is the sound speed. Show that in such a wave the component $\langle I_y \rangle$ of mean acoustic intensity perpendicular to the membrane is zero.

Paper 1, Section II

39B Waves

An inviscid fluid with sound speed c_0 occupies the region $0 < y < \pi\alpha$, $0 < z < \pi\beta$ enclosed by the rigid boundaries of a rectangular waveguide. Starting with the acoustic wave equation, find the dispersion relation $\omega(k)$ for the propagation of sound waves in the x-direction.

Hence find the phase speed $c(k)$ and the group velocity $c_g(k)$ of both the dispersive modes and the nondispersive mode, and sketch the form of the results for $k, \omega > 0$.

Define the time and cross-sectional average appropriate for a mode with frequency ω. For each dispersive mode, show that the average kinetic energy is equal to the average compressive energy.

A general multimode acoustic disturbance is created within the waveguide at $t = 0$ in a region around $x = 0$. Explain briefly how the amplitude of the disturbance varies with time as $t \to \infty$ at the moving position $x = Vt$ for each of the cases $0 < V < c_0$, $V = c_0$ and $V > c_0$. [You may quote without proof any generic results from the method of stationary phase.]

Paper 2, Section II

38B Waves

A uniform elastic solid with wavespeeds c_P and c_S occupies the region $z < 0$. An S-wave with displacement

$$\mathbf{u} = (\cos\theta, 0, -\sin\theta)\, \mathrm{e}^{\mathrm{i}k(x\sin\theta + z\cos\theta) - \mathrm{i}\omega t}$$

is incident from $z < 0$ on a rigid boundary at $z = 0$. Find the form and amplitudes of the reflected waves.

When is the reflected P-wave evanescent? Show that if the P-wave is evanescent then the amplitude of the reflected S-wave has the same magnitude as the incident wave, and interpret this result physically.

Paper 3, Section II

38B Waves

The dispersion relation in a stationary medium is given by $\omega = \Omega_0(\mathbf{k})$, where Ω_0 is a known function. Show that, in the frame of reference where the medium has a uniform velocity $-\mathbf{U}$, the dispersion relation is given by $\omega = \Omega_0(\mathbf{k}) - \mathbf{U} \cdot \mathbf{k}$.

An aircraft flies in a straight line with constant speed Mc_0 through air with sound speed c_0. If $M > 1$ show that, in the reference frame of the aircraft, the steady waves lie behind it on a cone of semi-angle $\sin^{-1}(1/M)$. Show further that the unsteady waves are confined to the interior of the cone.

A small insect swims with constant velocity $\mathbf{U} = (U, 0)$ over the surface of a pool of water. The resultant capillary waves have dispersion relation $\omega^2 = T|\mathbf{k}|^3/\rho$ on stationary water, where T and ρ are constants. Show that, in the reference frame of the insect, steady waves have group velocity

$$\mathbf{c}_g = U(\tfrac{3}{2}\cos^2\beta - 1, \tfrac{3}{2}\cos\beta\sin\beta)\,,$$

where $\mathbf{k} \propto (\cos\beta, \sin\beta)$. Deduce that the steady wavefield extends in all directions around the insect.

Paper 4, Section II

38B Waves

Show that, in the standard notation for one-dimensional flow of a perfect gas, the Riemann invariants $u \pm 2(c - c_0)/(\gamma - 1)$ are constant on characteristics $C_\pm$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c\,.$$

Such a gas occupies the region $x > X(t)$ in a semi-infinite tube to the right of a piston at $x = X(t)$. At time $t = 0$, the piston and the gas are at rest, $X = 0$, and the gas is uniform with $c = c_0$. For $t > 0$ the piston accelerates smoothly in the positive x-direction. Show that, prior to the formation of a shock, the motion of the gas is given parametrically by

$$u(x,t) = \dot{X}(\tau) \quad \text{on} \quad x = X(\tau) + \left[c_0 + \tfrac{1}{2}(\gamma+1)\dot{X}(\tau)\right](t - \tau)\,,$$

in a region that should be specified.

For the case $X(t) = \frac{2}{3}c_0 t^3/T^2$, where $T > 0$ is a constant, show that a shock first forms in the gas when

$$t = \frac{T}{\gamma+1}(3\gamma+1)^{1/2}\,.$$

Paper 1, Section II

38A Waves

Derive the wave equation governing the velocity potential ϕ for linearized sound waves in a compressible inviscid fluid. How is the pressure disturbance related to the velocity potential?

A semi-infinite straight tube of uniform cross-section is aligned along the positive x-axis with its end at $x = -L$. The tube is filled with fluid of density ρ_1 and sound speed c_1 in $-L < x < 0$ and with fluid of density ρ_2 and sound speed c_2 in $x > 0$. A piston at the end of the tube performs small oscillations such that its position is $x = -L + \epsilon\, e^{i\omega t}$, with $\epsilon \ll L$ and $\epsilon\,\omega \ll c_1,\, c_2$. Show that the complex amplitude of the velocity potential in $x > 0$ is

$$-\epsilon\, c_1 \left(\frac{c_1}{c_2} \cos\frac{\omega L}{c_1} + i\, \frac{\rho_2}{\rho_1} \sin\frac{\omega L}{c_1} \right)^{-1} .$$

Calculate the time-averaged acoustic energy flux in $x > 0$. Comment briefly on the variation of this result with L for the particular case $\rho_2 \ll \rho_1$ and $c_2 = \mathrm{O}(c_1)$.

Paper 2, Section II

38A Waves

The equation of motion for small displacements $\mathbf{u}(\mathbf{x}, t)$ in a homogeneous, isotropic, elastic medium of density ρ is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu \nabla^2 \mathbf{u}\,,$$

where λ and μ are the Lamé constants. Show that the dilatation $\boldsymbol{\nabla}\cdot\mathbf{u}$ and rotation $\boldsymbol{\nabla}\wedge\mathbf{u}$ each satisfy wave equations, and determine the corresponding wave speeds c_P and c_S.

Show also that a solution of the form $\mathbf{u} = \mathbf{A}\exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$ satisfies

$$\omega^2 \mathbf{A} = c_P^2\, \mathbf{k}\,(\mathbf{k}\cdot\mathbf{A}) - c_S^2\, \mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{A})\,.$$

Deduce the dispersion relation and the direction of polarization relative to $\mathbf{k}$ for plane harmonic P-waves and plane harmonic S-waves.

Now suppose the medium occupies the half-space $z \leqslant 0$ and that the boundary $z = 0$ is stress free. Show that it is possible to find a self-sustained combination of evanescent P-waves and SV-waves (i.e. a Rayleigh wave), proportional to $\exp\left[ik(x - ct)\right]$ and propagating along the boundary, provided the wavespeed c satisfies

$$\left(2 - \frac{c^2}{c_S^2}\right)^2 = 4\left(1 - \frac{c^2}{c_S^2}\right)^{1/2}\left(1 - \frac{c^2}{c_P^2}\right)^{1/2}.$$

[You are not required to show that this equation has a solution.]

Paper 3, Section II

38A Waves

Consider the equation

$$\frac{\partial^2 \phi}{\partial t \partial x} = -\alpha\, \phi\,,$$

where α is a positive constant. Find the dispersion relation for waves of frequency ω and wavenumber k. Sketch graphs of the phase velocity $c(k)$ and the group velocity $c_g(k)$.

A disturbance localized near $x = 0$ at $t = 0$ evolves into a dispersing wave packet. Will the wavelength and frequency of the waves passing a stationary observer located at a large positive value of x increase or decrease for $t > 0$? In which direction do the crests pass the observer?

Write down the solution $\phi(x, t)$ with initial value

$$\phi(x, 0) = \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk\,.$$

What can be said about $A(-k)$ if ϕ is real?

Use the method of stationary phase to obtain an approximation for $\phi(Vt, t)$ for fixed $V > 0$ and large t. What can be said about the solution at $x = -Vt$ for large t?

[You may assume that $\displaystyle\int_{-\infty}^{\infty} e^{-au^2}\, du = \sqrt{\frac{\pi}{a}}$ for $\mathrm{Re}(a) \geqslant 0$, $a \neq 0$.]

Paper 4, Section II

38A Waves

Starting from the equations for one-dimensional unsteady flow of an inviscid compressible fluid, show that it is possible to find Riemann invariants $u \pm Q$ that are constant on characteristics $C_\pm$ given by

$$\frac{dx}{dt} = u \pm c\,,$$

where $u(x,t)$ is the velocity of the fluid and $c(x,t)$ is the local speed of sound. Show that $Q = 2(c - c_0)/(\gamma - 1)$ for the case of a perfect gas with adiabatic equation of state $p = p_0(\rho/\rho_0)^\gamma$, where p_0, ρ_0 and γ are constants, $\gamma > 1$ and $c = c_0$ when $\rho = \rho_0$.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas is initially uniform and at rest with density ρ_0. At $t = 0$ the piston starts moving to the left at a constant speed V. Assuming that the gas keeps up with the piston, find $u(x,t)$ and $c(x,t)$ in each of the three distinct regions that are defined by families of C_+ characteristics.

Now assume that the gas does not keep up with the piston. Show that the gas particle at $x = x_0$ when $t = 0$ follows a trajectory given, for $t > x_0/c_0$, by

$$x(t) = \frac{\gamma+1}{\gamma-1}\left(\frac{c_0 t}{x_0}\right)^{2/(\gamma+1)} x_0 - \frac{2\,c_0 t}{\gamma-1}\,.$$

Deduce that the velocity of any given particle tends to $-2\,c_0/(\gamma-1)$ as $t \to \infty$.

Paper 1, Section II

38A Waves

The wave equation with spherical symmetry may be written

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{p}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{p} = 0\,.$$

Find the solution for the pressure disturbance $\tilde{p}$ in an outgoing wave, driven by a time-varying source with mass outflow rate $q(t)$ at the origin, in an infinite fluid.

A semi-infinite fluid of density ρ and sound speed c occupies the half space $x > 0$. The plane $x = 0$ is occupied by a rigid wall, apart from a small square element of side h that is centred on the point $(0, y', z')$ and oscillates in and out with displacement $f_0 e^{i\omega t}$. By modelling this element as a point source, show that the pressure field in $x > 0$ is given by

$$\tilde{p}(t, x, y, z) = -\frac{2\rho\omega^2 f_0 h^2}{4\pi R} e^{i\omega\left(t - \frac{R}{c}\right)},$$

where $R = [x^2 + (y - y')^2 + (z - z')^2]^{1/2}$, on the assumption that $R \gg c/\omega \gg f_0, h$. Explain the factor 2 in the above formula.

Now suppose that the plane $x = 0$ is occupied by a loudspeaker whose displacement is given by

$$x = f(y, z)e^{i\omega t}\,,$$

where $f(y, z) = 0$ for $|y|, |z| > L$. Write down an integral expression for the pressure in $x > 0$. In the far field where $r = (x^2 + y^2 + z^2)^{1/2} \gg L$, $\omega L^2/c$, c/ω, show that

$$\tilde{p}(t, x, y, z) \approx -\frac{\rho\omega^2}{2\pi r} e^{i\omega(t - r/c)} \hat{f}(m, n),$$

where $m = -\dfrac{\omega y}{rc}$, $n = -\dfrac{\omega z}{rc}$ and

$$\hat{f}(m, n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(y', z'\right) e^{-i(my' + nz')} dy' dz'.$$

Evaluate this integral when f is given by

$$f(y, z) = \begin{cases} 1, & -a < y < a, -b < z < b, \\ 0, & \text{otherwise,} \end{cases}$$

and discuss the result in the case $\omega b/c$ is small but $\omega a/c$ is of order unity.

Paper 2, Section II

38A Waves

An elastic solid of density ρ has Lamé moduli λ and μ. From the dynamic equation for the displacement vector $\mathbf{u}$, derive equations satisfied by the dilatational and shear potentials ϕ and $\boldsymbol{\psi}$. Show that two types of plane harmonic wave can propagate in the solid, and explain the relationship between the displacement vector and the propagation direction in each case.

A semi-infinite solid occupies the half-space $y < 0$ and is bounded by a traction-free surface at $y = 0$. A plane P-wave is incident on the plane $y = 0$ with angle of incidence θ. Describe the system of reflected waves, calculate the angles at which they propagate, and show that there is no reflected P-wave if

$$4\sigma(1-\sigma)^{1/2}(\beta-\sigma)^{1/2} = (1-2\sigma)^2,$$

where

$$\sigma = \beta \sin^2\theta \quad \text{and} \quad \beta = \frac{\mu}{\lambda + 2\mu}.$$

Paper 3, Section II

38A Waves

Starting from the equations of motion for an inviscid, incompressible, stratified fluid of density $\rho_0(z)$, where z is the vertical coordinate, derive the dispersion relation

$$\omega^2 = \frac{N^2\left(k^2 + \ell^2\right)}{(k^2 + \ell^2 + m^2)}$$

for small amplitude internal waves of wavenumber (k, ℓ, m), where N is the constant Brunt–Väisälä frequency (which should be defined), explaining any approximations you make. Describe the wave pattern that would be generated by a small body oscillating about the origin with small amplitude and frequency ω, the fluid being otherwise at rest.

The body continues to oscillate when the fluid has a slowly-varying velocity $[U(z), 0, 0]$, where $U'(z) > 0$. Show that a ray which has wavenumber $(k_0, 0, m_0)$ with $m_0 < 0$ at $z = 0$ will propagate upwards, but cannot go higher than $z = z_c$, where

$$U(z_c) - U(0) = N\left(k_0^2 + m_0^2\right)^{-1/2}.$$

Explain what happens to the disturbance as z approaches z_c.

Paper 4, Section II

38A Waves

A perfect gas occupies a tube that lies parallel to the x-axis. The gas is initially at rest, with density ρ_1, pressure p_1 and specific heat ratio γ, and occupies the region $x > 0$. For times $t > 0$ a piston, initially at $x = 0$, is pushed into the gas at a constant speed V. A shock wave propagates at constant speed U into the undisturbed gas ahead of the piston. Show that the pressure in the gas next to the piston, p_2, is given by the expression

$$V^2 = \frac{(p_2 - p_1)^2}{\rho_1 \left(\frac{\gamma + 1}{2} p_2 + \frac{\gamma - 1}{2} p_1 \right)}.$$

[You may assume that the internal energy per unit mass of perfect gas is given by

$$E = \frac{1}{\gamma - 1} \frac{p}{\rho}. \quad]$$

1/II/37B **Waves**

Show that in an acoustic plane wave the velocity and perturbation pressure are everywhere proportional and find the constant of proportionality.

Gas occupies a tube lying parallel to the x-axis. In the regions $x < 0$ and $x > L$ the gas has uniform density ρ_0 and sound speed c_0. For $0 < x < L$ the gas is cooled so that it has uniform density ρ_1 and sound speed c_1. A harmonic plane wave with frequency ω is incident from $x = -\infty$. Show that the amplitude of the wave transmitted into $x > L$ relative to that of the incident wave is

$$|T| = \left[\cos^2 k_1 L + \frac{1}{4}\left(\lambda + \lambda^{-1}\right)^2 \sin^2 k_1 L\right]^{-1/2},$$

where $\lambda = \rho_1 c_1 / \rho_0 c_0$ and $k_1 = \omega / c_1$.

What are the implications of this result if $\lambda \gg 1$?

2/II/37B **Waves**

Show that, in one-dimensional flow of a perfect gas at constant entropy, the Riemann invariants $u \pm 2(c - c_0)/(\gamma - 1)$ are constant along characteristics $dx/dt = u \pm c$.

A perfect gas occupies a tube that lies parallel to the x-axis. The gas is initially at rest and is in $x > 0$. For times $t > 0$ a piston is pulled out of the gas so that its position at time t is

$$x = X(t) = -\frac{1}{2} f t^2,$$

where $f > 0$ is a constant. Sketch the characteristics of the resulting motion in the (x, t) plane and explain why no shock forms in the gas.

Calculate the pressure exerted by the gas on the piston for times $t > 0$, and show that at a finite time t_v a vacuum forms. What is the speed of the piston at $t = t_v$?

3/II/37B **Waves**

The real function $\phi(x,t)$ satisfies the Klein–Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} - \phi\,, \quad -\infty < x < \infty,\ t \geqslant 0\,.$$

Find the dispersion relation for disturbances of wavenumber k and deduce their phase and group velocities.

Suppose that at $t = 0$

$$\phi(x,0) = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial t}(x,0) = e^{-|x|}\,.$$

Use Fourier transforms to find an integral expression for $\phi(x,t)$ when $t > 0$.

Use the method of stationary phase to find $\phi(Vt,t)$ for $t \to \infty$ for fixed $0 < V < 1$. What can be said if $V > 1$?

[*Hint: you may assume that*

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}\,, \quad \mathrm{Re}(a) > 0\,.]$$

4/II/38B **Waves**

A layer of rock of shear modulus $\bar{\mu}$ and shear wave speed $\bar{c}_s$ occupies the region $0 \leqslant y \leqslant h$ with a free surface at $y = h$. A second rock having shear modulus μ and shear wave speed $c_s > \bar{c}_s$ occupies $y \leqslant 0$. Show that elastic SH waves of wavenumber k and phase speed c can propagate in the layer with zero disturbance at $y = -\infty$ if $\bar{c}_s < c < c_s$ and c satisfies the dispersion relation

$$\tan\left[kh\sqrt{c^2/\bar{c}_s^2 - 1}\right] = \frac{\mu}{\bar{\mu}}\,\frac{\sqrt{1 - c^2/c_s^2}}{\sqrt{c^2/\bar{c}_s^2 - 1}}\,.$$

Show graphically, or otherwise, that this equation has at least one real solution for any value of kh, and determine the smallest value of kh for which the equation has at least two real solutions.

1/II/37C **Waves**

A uniform elastic solid with density ρ and Lamé moduli λ and μ occupies the region between rigid plane boundaries $y = 0$ and $y = h$. Show that SH waves can propagate in the x direction within this layer, and find the dispersion relation for such waves.

Deduce for each mode (a) the cutoff frequency, (b) the phase velocity, and (c) the group velocity.

Show also that for each mode the kinetic energy and elastic energy are equal in an average sense to be made precise.

[*You may assume that the elastic energy per unit volume* $W = \frac{1}{2}(\lambda e_{kk}^2 + 2\mu e_{ij}e_{ij})$.]

2/II/37C **Waves**

Show that for a one-dimensional flow of a perfect gas at constant entropy the Riemann invariants $u \pm 2(c - c_0)/(\gamma - 1)$ are constant along characteristics $dx/dt = u \pm c$.

Define a simple wave. Show that in a right-propagating simple wave

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = 0 .$$

Now suppose instead that, owing to dissipative effects,

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma+1}{2}u\right)\frac{\partial u}{\partial x} = -\alpha u$$

where α is a positive constant. Suppose also that u is prescribed at $t = 0$ for all x, say $u(x, 0) = v(x)$. Demonstrate that, unless a shock forms,

$$u(x,t) = v(x_0)\, e^{-\alpha t}$$

where, for each x and t, x_0 is determined implicitly as the solution of the equation

$$x - c_0 t = x_0 + \frac{\gamma+1}{2}\left(\frac{1-e^{-\alpha t}}{\alpha}\right) v(x_0) .$$

Deduce that a shock will not form at any (x, t) if

$$\alpha > \frac{\gamma+1}{2} \max_{v'<0} |v'(x_0)| .$$

3/II/37C **Waves**

Waves propagating in a slowly-varying medium satisfy the local dispersion relation

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$$

in the standard notation. Give a brief derivation of the ray-tracing equations for such waves; a formal justification is *not* required.

An ocean occupies the region $x > 0$, $-\infty < y < \infty$. Water waves are incident on a beach near $x = 0$. The undisturbed water depth is

$$h(x) = \alpha\, x^p$$

with α a small positive constant and p positive. The local dispersion relation is

$$\Omega^2 = g\kappa \tanh(\kappa h) \qquad \text{where} \qquad \kappa^2 = k_1^2 + k_2^2$$

and where k_1, k_2 are the wavenumber components in the x, y directions. Far from the beach, the waves are planar with frequency ω_∞ and crests making an acute angle θ_∞ with the shoreline $x = 0$. Obtain a differential equation (in implicit form) for a ray $y = y(x)$, and show that near the shore the ray satisfies

$$y - y_0 \sim A\, x^q$$

where A and q should be found. Sketch the appearance of the wavecrests near the shoreline.

4/II/38C **Waves**

Show that, for a plane acoustic wave, the acoustic intensity $\tilde{p}\,\mathbf{u}$ may be written as $\rho_0 c_0 |\mathbf{u}|^2 \hat{\mathbf{k}}$ in the standard notation.

Derive the general spherically-symmetric solution of the wave equation. Use it to find the velocity potential $\phi(r,t)$ for waves radiated into an unbounded fluid by a pulsating sphere of radius

$$a\,(1 + \varepsilon\, e^{i\omega t}) \qquad (\varepsilon \ll 1)\ .$$

By considering the far field, or otherwise, find the time-average rate at which energy is radiated by the sphere.

$$\left[\ \textit{You may assume that}\ \ \nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right).\ \right]$$

1/II/37C Waves

An elastic solid occupies the region $y < 0$. The wave speeds in the solid are c_{p} and c_{s}. A P-wave with dilatational potential

$$\phi = \exp\{ik(x\sin\theta + y\cos\theta - c_{\mathrm{p}}t)\}$$

is incident from $y < 0$ on a rigid barrier at $y = 0$. Obtain the reflected waves.

Are there circumstances where the reflected S-wave is evanescent? Give reasons for your answer.

2/II/37C Waves

The dispersion relation for waves in deep water is

$$\omega^2 = g|k| \ .$$

At time $t = 0$ the water is at rest and the elevation of its free surface is $\zeta = \zeta_0 \exp(-|x|/b)$ where b is a positive constant. Use Fourier analysis to find an integral expression for $\zeta(x,t)$ when $t > 0$.

Use the method of stationary phase to find $\zeta(Vt,t)$ for fixed $V > 0$ and $t \to \infty$.

$$\left[\int_{-\infty}^{\infty} \exp\left(ikx - \frac{|x|}{b}\right)\, dx \ = \ \frac{2b}{1+k^2b^2} \ ; \quad \int_{-\infty}^{\infty} \exp(-ax^2)\, dx \ = \ \sqrt{\frac{\pi}{a}} \quad (\operatorname{Re} a \geqslant 0) \ .\right]$$

3/II/37C Waves

An acoustic waveguide consists of a long straight tube $z > 0$ with square cross-section $0 < x < a, \ \ 0 < y < a$ bounded by rigid walls. The sound speed of the gas in the tube is c_0. Find the dispersion relation for the propagation of sound waves along the tube. Show that for every dispersive mode there is a cut-off frequency, and determine the lowest cut-off frequency $\omega_{\min}$.

An acoustic disturbance is excited at $z = 0$ with a prescribed pressure perturbation $\tilde{p}(x,y,0,t) = \tilde{P}(x,y)\exp(-i\omega t)$ with $\omega = \frac{1}{2}\omega_{\min}$. Find the pressure perturbation $\tilde{p}(x,y,z,t)$ at distances $z \gg a$ along the tube.

4/II/38C **Waves**

Obtain an expression for the compressive energy $W(\rho)$ per unit volume for adiabatic motion of a perfect gas, for which the pressure p is given in terms of the density ρ by a relation of the form

$$p = p_0(\rho/\rho_0)^{\gamma} \ , \qquad (*)$$

where p_0, ρ_0 and γ are positive constants.

For one-dimensional motion with speed u write down expressions for the mass flux and the momentum flux. Deduce from the energy flux $u\big(p + W + \frac{1}{2}\rho u^2\big)$ together with the mass flux that if the motion is steady then

$$\frac{\gamma}{\gamma - 1}\,\frac{p}{\rho} \ + \ \tfrac{1}{2}u^2 \ = \ \text{constant.} \qquad (\dagger)$$

A one-dimensional shock wave propagates at constant speed along a tube containing the gas. Ahead of the shock the gas is at rest with pressure p_0 and density ρ_0. Behind the shock the pressure is maintained at the constant value $(1+\beta)p_0$ with $\beta > 0$. Determine the density ρ_1 behind the shock, assuming that ($\dagger$) holds throughout the flow.

For small β show that the changes in pressure and density across the shock satisfy the adiabatic relation ($*$) approximately, correeect to order β^2.

1/II/37E **Waves**

An elastic solid with density ρ has Lamé moduli λ and μ. Write down equations satisfied by the dilational and shear potentials ϕ and $\boldsymbol{\psi}$.

For a two-dimensional disturbance give expressions for the displacement field $\mathbf{u} = (u_x, u_y, 0)$ in terms of $\phi(x, y; t)$ and $\boldsymbol{\psi} = (0, 0, \psi(x, y; t))$.

Suppose the solid occupies the region $y < 0$ and that the surface $y = 0$ is free of traction. Find a combination of solutions for ϕ and ψ that represent a propagating surface wave (a Rayleigh wave) near $y = 0$. Show that the wave is non-dispersive and obtain an equation for the speed c. [*You may assume without proof that this equation has a unique positive root.*]

2/II/37E **Waves**

Show that, in the standard notation for a one-dimensional flow of a perfect gas at constant entropy, the quantity $u + 2(c - c_0)/(\gamma - 1)$ remains constant along characteristics $dx/dt = u + c$.

A perfect gas is initially at rest and occupies a tube in $x > 0$. A piston is pushed into the gas so that its position at time t is $x(t) = \frac{1}{2} f t^2$, where $f > 0$ is a constant. Find the time and position at which a shock first forms in the gas.

3/II/37E **Waves**

The real function $\phi(x, t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = \frac{\partial^3 \phi}{\partial x^3},$$

where $U > 0$ is a constant. Find the dispersion relation for waves of wavenumber k and deduce whether wave crests move faster or slower than a wave packet.

Suppose that $\phi(x, 0)$ is given by a Fourier transform as

$$\phi(x, 0) = \int_{-\infty}^{\infty} A(k) e^{ikx} dk.$$

Use the method of stationary phase to find $\phi(Vt, t)$ as $t \to \infty$ for fixed $V > U$.

[*You may use the result that* $\int_{-\infty}^{\infty} e^{-a\xi^2} d\xi = (\pi/a)^{1/2}$ *if* $\mathrm{Re}(a) \geqslant 0$.]

What can be said if $V < U$? [*Detailed calculation is* **not** *required in this case.*]

4/II/38E **Waves**

Starting from the equations of conservation of mass and momentum for an inviscid compressible fluid, show that for small perturbations about a state of rest and uniform density the velocity is irrotational and the velocity potential satisfies the wave equation. Identify the sound speed c_0.

Define the acoustic energy density and acoustic energy flux, and derive the equation for conservation of acoustic energy.

Show that in any (not necessarily harmonic) acoustic plane wave of wavenumber $\mathbf{k}$ the kinetic and potential energy densities are equal and that the acoustic energy is transported with velocity $c_0\widehat{\mathbf{k}}$.

Calculate the kinetic and potential energy densities for a spherically symmetric outgoing wave. Are they equal?