Part II

Topics in Analysis

Paper 1, Section I

2F Topics In Analysis

(a) State and prove the theorem of Liouville on approximation of algebraic numbers.

(b) If u, v are coprime positive integers and p, q are coprime positive integers with $q > v$, show that

$$\left|\frac{p}{q} - \frac{u}{v}\right| > \frac{1}{q^2}.$$

(c) Show that, if $a_j \in \mathbb{Q}$, $a_j > 0$ and $\sum_{j=1}^{\infty} a_j$ converges, then we can find a strictly increasing sequence of positive integers $n(j)$ such that $\sum_{j=1}^{\infty} a_{n(j)}$ is transcendental.

Paper 2, Section I

2F Topics In Analysis

In this question we consider Γ, the collection of closed paths γ not passing through 0, that is to say, continuous functions $\gamma : [0,1] \to \mathbb{C} \setminus 0$ with $\gamma(0) = \gamma(1)$.

Define the *winding number* $w(\gamma, 0)$ of $\gamma \in \Gamma$. If $\gamma \in \Gamma$, $\phi : [0,1] \to \mathbb{C}$ is continuous with $\phi(0) = \phi(1)$ and $|\gamma(t)| > |\phi(t)|$ for all $t \in [0,1]$, what can we say about $w(\gamma + \phi, 0)$?

Explain what it means to say that γ_0, γ_1 are *homotopic by paths in* Γ.

State a theorem on the winding number of homotopic paths and use it to prove the fundamental theorem of algebra and the non-existence of retractions for discs.

Paper 3, Section I

2F Topics In Analysis

State Runge's theorem on polynomial approximation.

Which of the following statements are true and which false? Give reasons.

(i) Let $E = \{x + iy : x, y \geqslant 0\}$ and Ω be an open set containing E. Then, if $f : \Omega \to \mathbb{C}$ is analytic, we can find a sequence of polynomials converging uniformly on E to f.

(ii) Let $E = \{x + iy : x, y \geqslant 0\}$ and Ω be an open set containing E. Then, if $f : \Omega \to \mathbb{C}$ is analytic, we can find a sequence of polynomials converging pointwise on E to f.

(iii) Suppose Ω is open, K_1, K_2 are compact subsets of Ω, $f : \Omega \to \mathbb{C}$ is analytic and there exist polynomials $P_{j,n}$ with $P_{j,n} \to f$ uniformly on K_j. Then there exist polynomials P_n with $P_n \to f$ uniformly on $K_1 \cup K_2$.

(iv) Let $I = \{x + iy : 1 \geqslant x \geqslant 0, y = 0\}$. If $f : I \to \mathbb{C}$ is continuous, then we can find polynomials P_n such that $P_n \to f$ uniformly on I.

Paper 4, Section I

2F Topics In Analysis

We say that a function $f : X \to X$ has a fixed point if there exists an $x \in X$ with $f(x) = x$.

(i) Use the intermediate value theorem to show that, if $f : [0,1] \to [0,1]$ is continuous, it has a fixed point. Show also that, if $0, 1 \in f([0,1])$, then f is surjective.

(ii) Suppose that A and B are homeomorphic subsets of $\mathbb{R}^2$. Show that, if every continuous function $g : A \to A$ has a fixed point, then so does every continuous function $f : B \to B$.

(iii) State Brouwer's fixed point theorem for the closed unit disc $\bar{D}$.

(iv) Show that the closed unit disc is not homeomorphic to the annulus

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leqslant x^2 + y^2 \leqslant 2\}.$$

(v) Suppose that B is a subset of $\mathbb{R}^2$ containing at least two points. If every continuous function $g : B \to B$ has a fixed point, does it follow that B is homeomorphic to the closed unit disc? Give reasons.

Paper 2, Section II

11F Topics In Analysis

(a) State and prove the Baire Category Theorem.

(b) Consider the set $C^\infty([0,1])$ of infinitely differentiable functions on $[0,1]$. Show that

$$d(f, g) = \sum_{r=0}^{\infty} 2^{-r} \min\{1, \|f^{(r)} - g^{(r)}\|_\infty\}$$

is a well defined metric on $C^\infty([0,1])$ and that it is complete.

(c) Show that, if we use this metric, then there is a set E of first category for which the following is true. If $f \notin E$, $q \in (0,1)$ is rational and M is a positive integer, then there exists an $m \geqslant M$ such that

$$|f^{(m)}(q)| > m! \times m^m.$$

(d) If $f \notin E$, show that the Taylor series for f has radius of convergence 0 at every rational point $q \in (0,1)$. Explain briefly why this means that, for any point $x \in [0,1]$, there is no Taylor series which converges to f in a neighbourhood of that point.

Paper 4, Section II

12F Topics In Analysis

Let $C([0,1])$ denote the space of continuous real functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|_\infty$.

(a) Consider $\mathbb{R}^{n+1}$ with the standard Euclidean norm $\|\cdot\|$, and let T be the map $T:\mathbb{R}^{n+1}\to C([0,1])$ given by $T(\mathbf{a})=\sum_{r=0}^{n} a_r t^r$. Let S be the map $S:\mathbb{R}^{n+1}\to\mathbb{R}$ given by $S(\mathbf{a})=\|T\mathbf{a}\|_\infty$. Show that there exists a $\delta>0$ such that

$$|S(\mathbf{a})|\geqslant\delta \text{ whenever } \|\mathbf{a}\|=1.$$

Conclude that $\|T(\mathbf{a})\|_\infty\to\infty$ as $\|\mathbf{a}\|\to\infty$.

(b) If $f\in C([0,1])$ and $n\geqslant 0$, show that there exists a (not necessarily unique) 'best fit' polynomial P of degree at most n such that

$$\|P-f\|_\infty\leqslant\|Q-f\|_\infty \text{ whenever } Q \text{ is a polynomial of degree at most } n.$$

(c) State Chebychev's equiripple criterion and show that it is a sufficient condition for a polynomial to be best fit.

(d) Let $g\in C([0,1])$, $M=\|g\|_\infty$ and suppose that

$$0=u_0<v_0<u_1<v_1<\ldots<v_{m-1}<u_m<v_m=1$$

are such that

$$\begin{array}{lll} M\geqslant g(t)>-M & \text{for } t\in[u_{2j},v_{2j}], & (2j\leqslant m)\\ -M\leqslant g(t)<M & \text{for } t\in[u_{2j+1},v_{2j+1}], & (2j+1\leqslant m)\\ -M<g(t)<M & \text{for } t\in[v_{j-1},u_j], & (j\leqslant m). \end{array}$$

Let $w_j=(v_{j-1}+u_j)/2$ and set $Q(t)=(-1)^{m-1}\prod_{j=1}^{m-1}(t-w_j)$. Show that, if $\eta>0$ is sufficiently small, we have

$$\|\eta Q-g\|_\infty<M.$$

Deduce that Chebychev's criterion is also a necessary condition for a polynomial to be best fit.

Paper 1, Section I

2G Topics in Analysis

Show that if a, A, B, C, D are non-negative integers and $AD - BC = 1$, then

$$a + \frac{At + B}{Ct + D} = \frac{\alpha t + \beta}{\gamma t + \delta}$$

for some α, β, γ, δ non-negative integers with $\alpha\delta - \beta\gamma = 1$.

If $N, a_1, a_2, \ldots$ are strictly positive integers with $a_{N+k} = a_k$ for all k and

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

show that x is a root of a quadratic (or linear) equation with integer coefficients.

Give the quadratic equation explicitly in the case when $N = 2$, $a_1 = a$, $a_2 = b$. Explain how you know which root gives the continued fraction.

Paper 2, Section I
2G Topics in Analysis

In this question you should work in $\mathbb{R}^n$ with the usual Euclidean distance.

Define a set of *first Baire category.*

For each of the following statements, say whether it is true or false and give an appropriate proof or counterexample.

(i) The countable union of sets of first category is of first category.

(ii) If A is of first category in $\mathbb{R}^2$ and $y \in \mathbb{R}$, then

$$C_y = \{x \,:\, (x, y) \in A\}$$

is of first category in $\mathbb{R}$.

(iii) If C is of first category in $\mathbb{R}$, then

$$A = \{(x, y) \,:\, x \in C,\, y \in \mathbb{R}\}$$

is of first category in $\mathbb{R}^2$.

(iv) If A and B are sets of first category in $\mathbb{R}^2$, then

$$A + B = \{\mathbf{a} + \mathbf{b} \,:\, \mathbf{a} \in A,\, \mathbf{b} \in B\}$$

is of first category.

[*You may use results about complete metric spaces provided you state them precisely.*]

Paper 3, Section I
2G Topics in Analysis

Let Ω be a non-empty bounded open subset of $\mathbb{R}^2$ with closure $\mathrm{Cl}\,\Omega$ and boundary $\partial\Omega$. We take $\phi : \mathrm{Cl}\,\Omega \to \mathbb{R}$ to be a continuous function which is twice differentiable on Ω.

If $\nabla^2\phi > 0$ on Ω show that ϕ attains a maximum on $\partial\Omega$.

By giving proofs or counterexamples establish which of the following are true and which are false.

(i) If $\nabla^2\phi = 0$ on Ω, then ϕ attains a maximum on $\partial\Omega$.

(ii) If $\nabla^2\phi = 0$ on Ω, then ϕ attains a minimum on $\partial\Omega$.

(iii) If $\nabla^2\phi = f$ on Ω for some continuous function $f : \mathrm{Cl}\,\Omega \to \mathbb{R}$, then ϕ attains a maximum on $\partial\Omega$.

Paper 4, Section I
2G Topics in Analysis

Consider the continuous map $f : [0,1] \to \mathbb{C}$ given by $f(t) = t - 1/2$. Show that there does not exist a continuous function $\phi : [0,1] \to \mathbb{R}$ with $f(t) = |f(t)| \exp(i\phi(t))$.

Show that, if $g : [0,1] \to \mathbb{C} \setminus \{0\}$ is continuous, there exists a continuous function $\theta : [0,1] \to \mathbb{R}$ with $g(t) = |g(t)| \exp(i\theta(t))$. [You may assume that this result holds in the special case when $\Re g(t) > 0$ for all $t \in [0,1]$.]

Show that $r(g) = \theta(1) - \theta(0)$ is uniquely defined.

If $u(t) = g(t^2)$ and $v(t) = g(t)^2$, find $r(u)$ and $r(v)$ in terms of $r(g)$.

Give an example with $g_1, g_2 : [0,1] \to \mathbb{C} \setminus \{0\}$ continuous such that $g_1(0) = g_2(0)$ and $g_1(1) = g_2(1)$, but $r(g_1) \neq r(g_2)$.

Paper 2, Section II
11G Topics in Analysis

Suppose $f : [0,1]^2 \to \mathbb{R}$ is continuous. Show, quoting carefully any theorems that you use, that

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} f(j/n, k/n) t^j (1-t)^{n-j} s^k (1-s)^{n-k} \to f(t,s)$$

uniformly on $[0,1]^2$ as $n \to \infty$.

Deduce that

$$\int_0^1 \left(\int_0^1 f(s,t)\, ds \right) dt = \int_0^1 \left(\int_0^1 f(s,t)\, dt \right) ds$$

whenever $f : [0,1]^2 \to \mathbb{R}$ is continuous.

By giving proofs or counterexamples establish which of the following statements are true and which are false. You may not use the Stone–Weierstrass theorem without proof.

(i) If $f : [0,1]^2 \to \mathbb{R}$ is continuous and $\int_0^1 \left(\int_0^1 s^n t^m f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(ii) Suppose $a < b$. If $f : [a,b]^2 \to \mathbb{R}$ is continuous and $\int_a^b \left(\int_a^b s^n t^m f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(iii) If $f : [-1,1]^2 \to \mathbb{R}$ is continuous and $\int_{-1}^1 \left(\int_{-1}^1 s^{2n} t^{2m} f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(iv) If $f : [0,1]^2 \to \mathbb{R}$ is continuous and $\int_0^1 \left(\int_0^1 s^{2n} t^{2m} f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

Paper 4, Section II
12G Topics in Analysis

(a) State Brouwer's fixed point theorem for the closed unit disc D. For which of the following $E \subset \mathbb{R}^2$ is it the case that every continuous function $f : E \to E$ has a fixed point? Give a proof or a counterexample.

(i) E is the union of two disjoint closed discs.

(ii) $E = \{(x, 0) \,:\, 0 < x < 1\}$.

(iii) $E = \{(x, 0) \,:\, 0 \leqslant x \leqslant 1\}$.

(iv) $E = \{\mathbf{x} \in \mathbb{R}^2 \,:\, 1 \leqslant |\mathbf{x}| \leqslant 2\}$.

(b) Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function with the property that $|f(\mathbf{x})| \leqslant 1$ whenever $|\mathbf{x}| = 1$, then f has a fixed point.

[*Hint: Consider $T \circ f$ where for $\mathbf{x} \in \mathbb{R}^2$, $T\mathbf{x}$ is the element of D closest to $\mathbf{x}$.*]

(c) Let

$$E = \{(p_1, p_2, q_1, q_2) \,:\, 0 \leqslant p_i, q_i \leqslant 1 \text{ and } p_1 + p_2 = 1, q_1 + q_2 = 1\}$$

and suppose A, $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are given by

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} p_i q_j \text{ and } B(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{2} \sum_{j=1}^{2} b_{ij} p_i q_j$$

with a_{ij} and b_{ij} constant. Let

$$u_1(\mathbf{p}, \mathbf{q}) = \max\{0, A((1,0), \mathbf{q}) - A(\mathbf{p}, \mathbf{q})\}, \; u_2(\mathbf{p}, \mathbf{q}) = \max\{0, A((0,1), \mathbf{q}) - A(\mathbf{p}, \mathbf{q})\}.$$

By considering $(\mathbf{p}', \mathbf{q}')$ with

$$\mathbf{p}' = \frac{\mathbf{p} + \mathbf{u}(\mathbf{p}, \mathbf{q})}{1 + u_1(\mathbf{p}, \mathbf{q}) + u_2(\mathbf{p}, \mathbf{q})}$$

and $\mathbf{q}'$ defined appropriately, show that we can find a $(\mathbf{p}^*, \mathbf{q}^*) \in E$ with

$$\forall (\mathbf{p}, \mathbf{q}) \in E, \quad A\big(\mathbf{p}^*, \mathbf{q}^*\big) \geqslant A\big(\mathbf{p}, \mathbf{q}^*\big) \text{ and } B\big(\mathbf{p}^*, \mathbf{q}^*\big) \geqslant B\big(\mathbf{p}^*, \mathbf{q}\big).$$

Carefully explain the result in terms of a two-person game.

Paper 1, Section I

2H Topics in Analysis

Write

$$P = \{\mathbf{x} \in \mathbb{R}^n : x_j \geqslant 0 \text{ for all } 1 \leqslant j \leqslant n\}$$

and suppose that K is a non-empty, closed, convex and bounded subset of $\mathbb{R}^n$ with $K \cap \operatorname{Int} P \neq \emptyset$. By taking logarithms, or otherwise, show that there is a unique $\mathbf{x}^* \in K \cap P$ such that

$$\prod_{j=1}^{n} x_j \leqslant \prod_{j=1}^{n} x_j^*$$

for all $\mathbf{x} \in K \cap P$.

Show that $\displaystyle\sum_{j=1}^{n} \frac{x_j}{x_j^*} \leqslant n$ for all $\mathbf{x} \in K \cap P$.

Identify the point $\mathbf{x}^*$ in the case that K has the property

$$(x_1, x_2, \ldots, x_{n-1}, x_n) \in K \Rightarrow (x_2, x_3, \ldots, x_n, x_1) \in K\,,$$

and justify your answer.

Show that, given any $\mathbf{a} \in \operatorname{Int} P$, we can find a set K, as above, with $\mathbf{x}^* = \mathbf{a}$.

Paper 2, Section I

2H Topics in Analysis

Let Ω be a non-empty bounded open set in $\mathbb{R}^2$ with closure $\overline{\Omega}$ and boundary $\partial\Omega$ and let $\phi : \overline{\Omega} \to \mathbb{R}$ be a continuous function. Give a proof or a counterexample for each of the following assertions.

(i) If ϕ is twice differentiable on Ω with $\nabla^2\phi(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

(ii) If ϕ is twice differentiable on Ω with $\nabla^2\phi(\mathbf{x}) < 0$ for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

(iii) If ϕ is four times differentiable on Ω with

$$\frac{\partial^4\phi}{\partial x^4}(\mathbf{x}) + \frac{\partial^4\phi}{\partial y^4}(\mathbf{x}) > 0$$

for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

(iv) If ϕ is twice differentiable on Ω with $\nabla^2\phi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

Paper 3, Section I

2H Topics in Analysis

State Runge's theorem on the approximation of analytic functions by polynomials.

Let $\Omega = \{z \in \mathbb{C},\ \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$. Establish whether the following statements are true or false by giving a proof or a counterexample in each case.

(i) If $f : \Omega \to \mathbb{C}$ is the uniform limit of a sequence of polynomials P_n, then f is a polynomial.

(ii) If $f : \Omega \to \mathbb{C}$ is analytic, then there exists a sequence of polynomials P_n such that for each integer $r \geqslant 0$ and each $z \in \Omega$ we have $P_n^{(r)}(z) \to f^{(r)}(z)$.

Paper 4, Section I

2H Topics in Analysis

(a) State Brouwer's fixed-point theorem in 2 dimensions.

(b) State an equivalent theorem on retraction and explain (without detailed calculations) why it is equivalent.

(c) Suppose that A is a 3×3 real matrix with strictly positive entries. By defining an appropriate function $f : \triangle \to \triangle$, where

$$\triangle = \{\mathbf{x} \in \mathbb{R}^3 \,:\, x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geqslant 0\},$$

show that A has a strictly positive eigenvalue.

Paper 2, Section II
11H Topics in Analysis

Let $r : [-1,1] \to \mathbb{R}$ be a continuous function with $r(x) > 0$ for all but finitely many values of x.

(a) Show that

$$\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)r(x)\,dx \qquad (*)$$

defines an inner product on $C([-1,1])$.

(b) Show that for each n there exists a polynomial P_n of degree exactly n which is orthogonal, with respect to the inner product $(*)$, to all polynomials of lower degree.

(c) Show that P_n has n simple zeros $\omega_1(n)$, $\omega_2(n)$, $\ldots$, $\omega_n(n)$ on $[-1,1]$.

(d) Show that for each n there exist unique real numbers $A_j(n)$, $1 \leqslant j \leqslant n$, such that whenever Q is a polynomial of degree at most $2n-1$,

$$\int_{-1}^{1} Q(x)r(x)\,dx = \sum_{j=1}^{n} A_j(n)Q\big(\omega_j(n)\big).$$

(e) Show that

$$\sum_{j=1}^{n} A_j(n)f\big(\omega_j(n)\big) \to \int_{-1}^{1} f(x)r(x)\,dx$$

as $n \to \infty$ for all $f \in C([-1,1])$.

(f) If $R > 1$, $K > 0$, a_m is real with $|a_m| \leqslant KR^{-m}$ and $f(x) = \sum_{m=1}^{\infty} a_m x^m$, show that

$$\left|\int_{-1}^{1} f(x)r(x)\,dx - \sum_{j=1}^{n} A_j(n)f\big(\omega_j(n)\big)\right| \leqslant \frac{2KR^{-2n+1}}{R-1}\int_{-1}^{1} r(x)\,dx.$$

(g) If $r(x) = (1-x^2)^{1/2}$ and $P_n(0) = 1$, identify P_n (giving brief reasons) and the $\omega_j(n)$. [*Hint: A change of variable may be useful.*]

Paper 4, Section II

12H Topics in Analysis

Let x be irrational with nth continued fraction convergent

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}}}\,.$$

Show that

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

and deduce that

$$\left| \frac{p_n}{q_n} - x \right| \leqslant \frac{1}{q_n q_{n+1}}\,.$$

[You may quote the result that x lies between p_n/q_n and p_{n+1}/q_{n+1}.]

We say that y is a *quadratic irrational* if it is an irrational root of a quadratic equation with integer coefficients. Show that if y is a quadratic irrational, we can find an $M > 0$ such that

$$\left| \frac{p}{q} - y \right| \geqslant \frac{M}{q^2}$$

for all integers p and q with $q > 0$.

Using the hypotheses and notation of the first paragraph, show that if the sequence (a_n) is unbounded, x cannot be a quadratic irrational.

Paper 1, Section I

2H Topics in Analysis

Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous map never taking the value 0 and satisfying $\gamma(0) = \gamma(1)$. Define the *degree* (or *winding number*) $w(\gamma; 0)$ of γ about 0. Prove the following.

(i) If $\delta : [0,1] \to \mathbb{C}\setminus\{0\}$ is a continuous map satisfying $\delta(0) = \delta(1)$, then the winding number of the product $\gamma\delta$ is given by $w(\gamma\delta; 0) = w(\gamma; 0) + w(\delta; 0)$.

(ii) If $\sigma : [0,1] \to \mathbb{C}$ is continuous, $\sigma(0) = \sigma(1)$ and $|\sigma(t)| < |\gamma(t)|$ for each $0 \leqslant t \leqslant 1$, then $w(\gamma + \sigma; 0) = w(\gamma; 0)$.

(iii) Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and let $f : D \to \mathbb{C}$ be a continuous function with $f(z) \neq 0$ whenever $|z| = 1$. Define $\alpha : [0,1] \to \mathbb{C}$ by $\alpha(t) = f(e^{2\pi i t})$. Then if $w(\alpha; 0) \neq 0$, there must exist some $z \in D$, such that $f(z) = 0$. [It may help to define $F(s,t) := f(se^{2\pi i t})$. Homotopy invariance of the winding number may be assumed.]

Paper 2, Section I

2H Topics in Analysis

Show that every Legendre polynomial p_n has n distinct roots in $[-1,1]$, where n is the degree of p_n.

Let $x_1, \ldots, x_n$ be distinct numbers in $[-1,1]$. Show that there are unique real numbers $A_1, \ldots, A_n$ such that the formula

$$\int_{-1}^{1} P(t)dt = \sum_{i=1}^{n} A_i P(x_i)$$

holds for every polynomial P of degree less than n.

Now suppose that the above formula in fact holds for every polynomial P of degree less than $2n$. Show that then $x_1, \ldots, x_n$ are the roots of p_n. Show also that $\sum_{i=1}^{n} A_i = 2$ and that all A_i are positive.

Paper 3, Section I

2H Topics in Analysis

State Runge's theorem about the uniform approximation of holomorphic functions by polynomials.

Explicitly construct, with a brief justification, a sequence of polynomials which converges uniformly to $1/z$ on the semicircle $\{z : |z| = 1,\ \mathrm{Re}(z) \leqslant 0\}$.

Does there exist a sequence of polynomials converging uniformly to $1/z$ on $\{z : |z| = 1,\ z \neq 1\}$? Give a justification.

Paper 4, Section I

2H Topics in Analysis

Define what is meant by a *nowhere dense* set in a metric space. State a version of the Baire Category theorem.

Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function such that $f(nx) \to 0$ as $n \to \infty$ for every fixed $x \geqslant 1$. Show that $f(t) \to 0$ as $t \to \infty$.

Paper 2, Section II

11H Topics in Analysis

Let T be a (closed) triangle in $\mathbb{R}^2$ with edges I, J, K. Let A, B, C, be closed subsets of T, such that $I \subset A$, $J \subset B$, $K \subset C$ and $T = A \cup B \cup C$. Prove that $A \cap B \cap C$ is non-empty.

Deduce that there is no continuous map $f : D \to \partial D$ such that $f(p) = p$ for all $p \in \partial D$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ is the closed unit disc and $\partial D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is its boundary.

Let now α, β, $\gamma \subset \partial D$ be three closed arcs, each arc making an angle of $2\pi/3$ (in radians) in ∂D and $\alpha \cup \beta \cup \gamma = \partial D$. Let P, Q and R be open subsets of D, such that $\alpha \subset P$, $\beta \subset Q$ and $\gamma \subset R$. Suppose that $P \cup Q \cup R = D$. Show that $P \cap Q \cap R$ is non-empty. [You may assume that for each closed bounded subset $K \subset \mathbb{R}^2$, $d(x, K) = \min\{\|x - y\| : y \in K\}$ defines a continuous function on $\mathbb{R}^2$.]

Paper 4, Section II

12H Topics in Analysis

(a) State Liouville's theorem on the approximation of algebraic numbers by rationals.

(b) Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive integers and let

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

be the value of the associated continued fraction.

(i) Prove that the nth convergent p_n/q_n satisfies

$$\left|\alpha - \frac{p_n}{q_n}\right| \leqslant \left|\alpha - \frac{p}{q}\right|$$

for all the rational numbers $\dfrac{p}{q}$ such that $0 < q \leqslant q_n$.

(ii) Show that if the sequence (a_n) is bounded, then one can choose $c > 0$ (depending only on α), so that for every rational number $\dfrac{a}{b}$,

$$\left|\alpha - \frac{a}{b}\right| > \frac{c}{b^2}.$$

(iii) Show that if the sequence (a_n) is unbounded, then for each $c > 0$ there exist infinitely many rational numbers $\dfrac{a}{b}$ such that

$$\left|\alpha - \frac{a}{b}\right| < \frac{c}{b^2}.$$

[*You may assume without proof the relation*

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n+1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}, \quad n = 1, 2, \dots .]$$

Paper 4, Section I

2H Topics in Analysis

Show that π is irrational. [*Hint: consider the functions* $f_n : [0,\pi] \to \mathbb{R}$ *given by* $f_n(x) = x^n(\pi - x)^n \sin x$.]

Paper 3, Section I

2H Topics in Analysis

State *Nash's theorem* for a non zero-sum game in the case of two players with two choices.

The role playing game Tixerb involves two players. Before the game begins, each player i chooses a p_i with $0 \leqslant p_i \leqslant 1$ which they announce. They may change their choice as many times as they wish, but, once the game begins, no further changes are allowed. When the game starts, player i becomes a Dark Lord with probability p_i and a harmless peasant with probability $1 - p_i$. If one player is a Dark Lord and the other a peasant the Lord gets 2 points and the peasant -2. If both are peasants they get 1 point each, if both Lords they get $-U$ each. Show that there exists a U_0, to be found, such that, if $U > U_0$ there will be three choices of (p_1, p_2) for which neither player can increase the expected value of their outcome by changing their choice unilaterally, but, if $U_0 > U$, there will only be one. Find the appropriate (p_1, p_2) in each case.

Paper 2, Section I

2H Topics in Analysis

Let $\mathcal{K}$ be the collection of non-empty closed bounded subsets of $\mathbb{R}^n$.

(a) Show that, if A, $B \in \mathcal{K}$ and we write

$$A + B = \{a + b \, : \, a \in A, \, b \in B\},$$

then $A + B \in \mathcal{K}$.

(b) Show that, if $K_n \in \mathcal{K}$, and

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$$

then $K := \bigcap_{n=1}^{\infty} K_n \in \mathcal{K}$.

(c) Assuming the result that

$$\rho(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b| + \sup_{b \in B} \inf_{a \in A} |a - b|$$

defines a metric on $\mathcal{K}$ (the Hausdorff metric), show that if K_n and K are as in part (b), then $\rho(K_n, K) \to 0$ as $n \to \infty$.

Paper 1, Section I

2H Topics in Analysis

Let T_n be the nth Chebychev polynomial. Suppose that $\gamma_i > 0$ for all i and that $\sum_{i=1}^{\infty} \gamma_i$ converges. Explain why $f = \sum_{i=1}^{\infty} \gamma_i T_{3^i}$ is a well defined continuous function on $[-1, 1]$.

Show that, if we take $P_n = \sum_{i=1}^{n} \gamma_i T_{3^i}$, we can find points x_k with

$$-1 \leqslant x_0 < x_1 < \ldots < x_{3^{n+1}} \leqslant 1$$

such that $f(x_k) - P_n(x_k) = (-1)^{k+1} \sum_{i=n+1}^{\infty} \gamma_i$ for each $k = 0, 1, \ldots, 3^{n+1}$.

Suppose that δ_n is a decreasing sequence of positive numbers and that $\delta_n \to 0$ as $n \to \infty$. Stating clearly any theorem that you use, show that there exists a continuous function f with

$$\sup_{t \in [-1,1]} |f(t) - P(t)| \geqslant \delta_n$$

for all polynomials P of degree at most n and all $n \geqslant 1$.

Paper 2, Section II

11H Topics in Analysis

Throughout this question I denotes the closed interval $[-1,1]$.

(a) For $n \in \mathbb{N}$, consider the $2n+1$ points $r/n \in I$ with $r \in \mathbb{Z}$ and $-n \leqslant r \leqslant n$. Show that, if we colour them red or green in such a way that -1 and 1 are coloured differently, there must be two neighbouring points of different colours.

(b) Deduce from part (a) that, if $I = A \cup B$ with A and B closed, $-1 \in A$ and $1 \in B$, then $A \cap B \neq \emptyset$.

(c) Deduce from part (b) that there does not exist a continuous function $f : I \to \mathbb{R}$ with $f(t) \in \{-1, 1\}$ for all $t \in I$ and $f(-1) = -1$, $f(1) = 1$.

(d) Deduce from part (c) that if $f : I \to I$ is continuous then there exists an $x \in I$ with $f(x) = x$.

(e) Deduce the conclusion of part (c) from the conclusion of part (d).

(f) Deduce the conclusion of part (b) from the conclusion of part (c).

(g) Suppose that we replace I wherever it occurs by the unit circle

$$C = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Which of the conclusions of parts (b), (c) and (d) remain true? Give reasons.

Paper 4, Section II

12H Topics in Analysis

(a) Suppose that $K \subset \mathbb{C}$ is a non-empty subset of the square $\{x+iy : x, y \in (-1,1)\}$ and f is analytic in the larger square $\{x+iy : x, y \in (-1-\delta, 1+\delta)\}$ for some $\delta > 0$. Show that f can be uniformly approximated on K by polynomials.

(b) Let K be a closed non-empty proper subset of $\mathbb{C}$. Let Λ be the set of $\lambda \in \mathbb{C} \setminus K$ such that $g_\lambda(z) = (z-\lambda)^{-1}$ can be approximated uniformly on K by polynomials and let $\Gamma = \mathbb{C} \setminus (K \cup \Lambda)$. Show that Λ and Γ are open. Is it always true that Λ is non-empty? Is it always true that, if K is bounded, then Γ is empty? Give reasons.

[*No form of Runge's theorem may be used without proof.*]

Paper 1, Section I

2F Topics in Analysis

State and prove Sperner's lemma concerning colourings of points in a triangular grid.

Suppose that $\triangle$ is a non-degenerate closed triangle with closed edges α_1, α_2 and α_3. Show that we cannot find closed sets A_j with $A_j \supseteq \alpha_j$, for $j = 1, 2, 3$, such that

$$\bigcup_{j=1}^{3} A_j = \triangle, \text{ but } \bigcap_{j=1}^{3} A_j = \emptyset.$$

Paper 2, Section I

2F Topics in Analysis

For $\mathbf{x} \in \mathbb{R}^n$ we write $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Define

$$P := \{\mathbf{x} \in \mathbb{R}^n : x_j \geqslant 0 \text{ for } 1 \leqslant j \leqslant n\}.$$

(a) Suppose that L is a convex subset of P, that $(1, 1, \ldots, 1) \in L$ and that $\prod_{j=1}^{n} x_j \leqslant 1$ for all $\mathbf{x} \in L$. Show that $\sum_{j=1}^{n} x_j \leqslant n$ for all $\mathbf{x} \in L$.

(b) Suppose that K is a non-empty closed bounded convex subset of P. Show that there is a $\mathbf{u} \in K$ such that $\prod_{j=1}^{n} x_j \leqslant \prod_{j=1}^{n} u_j$ for all $\mathbf{x} \in K$. If $u_j \neq 0$ for each j with $1 \leqslant j \leqslant n$, show that

$$\sum_{j=1}^{n} \frac{x_j}{u_j} \leqslant n,$$

for all $\mathbf{x} \in K$, and that $\mathbf{u}$ is unique.

Paper 3, Section I

2F Topics in Analysis

State a version of the Baire category theorem and use it to prove the following result:

If $f : \mathbb{C} \to \mathbb{C}$ is analytic, but not a polynomial, then there exists a point $z_0 \in \mathbb{C}$ such that each coefficient of the Taylor series of f at z_0 is non-zero.

Paper 4, Section I

2F Topics in Analysis

Let $0 \leqslant \alpha < 1$ and $A > 0$. If we have an infinite sequence of integers m_n with $1 \leqslant m_n \leqslant An^{\alpha}$, show that

$$\sum_{n=1}^{\infty} \frac{m_n}{n!}$$

is irrational.

Does the result remain true if the m_n are not restricted to integer values? Justify your answer.

Paper 2, Section II

11F Topics in Analysis

(a) Give Bernstein's probabilistic proof of Weierstrass's theorem.

(b) Are the following statements true or false? Justify your answer in each case.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then there exists a sequence of polynomials P_n converging pointwise to f on $\mathbb{R}$.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then there exists a sequence of polynomials P_n converging uniformly to f on $\mathbb{R}$.

(iii) If $f : (0,1] \to \mathbb{R}$ is continuous and bounded, then there exists a sequence of polynomials P_n converging uniformly to f on $(0,1]$.

(iv) If $f : [0,1] \to \mathbb{R}$ is continuous and $x_1, x_2, \ldots, x_m$ are distinct points in $[0,1]$, then there exists a sequence of polynomials P_n with $P_n(x_j) = f(x_j)$, for $j = 1, \ldots, m$, converging uniformly to f on $[0,1]$.

(v) If $f : [0,1] \to \mathbb{R}$ is m times continuously differentiable, then there exists a sequence of polynomials P_n such that $P_n^{(r)} \to f^{(r)}$ uniformly on $[0,1]$ for each $r = 0, \ldots, m$.

Paper 4, Section II

12F Topics in Analysis

We work in $\mathbb{C}$. Consider

$$K = \{z : |z-2| \leqslant 1\} \cup \{z : |z+2| \leqslant 1\}$$

and

$$\Omega = \{z : |z-2| < 3/2\} \cup \{z : |z+2| < 3/2\}.$$

Show that if $f : \Omega \to \mathbb{C}$ is analytic, then there is a sequence of polynomials p_n such that $p_n(z) \to f(z)$ uniformly on K.

Show that there is a sequence of polynomials P_n such that $P_n(z) \to 0$ uniformly for $|z-2| \leqslant 1$ and $P_n(z) \to 1$ uniformly for $|z+2| \leqslant 1$.

Give two disjoint non-empty bounded closed sets K_1 and K_2 such that there does not exist a sequence of polynomials Q_n with $Q_n(z) \to 0$ uniformly on K_1 and $Q_n(z) \to 1$ uniformly on K_2. Justify your answer.

Paper 2, Section I

2F Topics In Analysis

Are the following statements true or false? Give reasons, quoting any theorems that you need.

(i) There is a sequence of polynomials P_n with $P_n(t) \to \sin t$ uniformly on $\mathbb{R}$ as $n \to \infty$.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then there is a sequence of polynomials Q_n with $Q_n(t) \to f(t)$ for each $t \in \mathbb{R}$ as $n \to \infty$.

(iii) If $g : [1, \infty) \to \mathbb{R}$ is continuous with $g(t) \to 0$ as $t \to \infty$, then there is a sequence of polynomials R_n with $R_n(1/t) \to g(t)$ uniformly on $[1, \infty)$ as $n \to \infty$.

Paper 4, Section I

2F Topics In Analysis

If $x \in (0, 1]$, set

$$x = \frac{1}{N(x) + T(x)}\,,$$

where $N(x)$ is an integer and $1 > T(x) \geqslant 0$. Let $N(0) = T(0) = 0$.

If x is also irrational, write down the continued fraction expansion in terms of $NT^j(x)$ (where $NT^0(x) = N(x)$).

Let X be a random variable taking values in $[0, 1]$ with probability density function

$$f(x) = \frac{1}{(\log 2)(1 + x)}.$$

Show that $T(X)$ has the same distribution as X.

Paper 1, Section I

2F Topics In Analysis

State Liouville's theorem on the approximation of algebraic numbers by rationals.

Suppose that we have a sequence ζ_n with $\zeta_n \in \{0, 1\}$. State and prove a necessary and sufficient condition on the ζ_n for

$$\sum_{n=0}^{\infty} \zeta_n \, 10^{-n!}$$

to be transcendental.

Paper 3, Section I
2F Topics In Analysis

(a) Suppose that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function such that there exists a $K > 0$ with $\|g(\mathbf{x}) - \mathbf{x}\| \leqslant K$ for all $\mathbf{x} \in \mathbb{R}^2$. By constructing a suitable map f from the closed unit disc into itself, show that there exists a $\mathbf{t} \in \mathbb{R}^2$ with $g(\mathbf{t}) = \mathbf{0}$.

(b) Show that g is surjective.

(c) Show that the result of part (b) may be false if we drop the condition that g is continuous.

Paper 2, Section II
10F Topics In Analysis

State and prove Baire's category theorem for complete metric spaces. Give an example to show that it may fail if the metric space is not complete.

Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n(x)$ converges for all $x \in [0,1]$. Show that if $\epsilon > 0$ is fixed we can find an $N \geqslant 0$ and a non-empty open interval $J \subseteq [0,1]$ such that $|f_n(x) - f_m(x)| \leqslant \epsilon$ for all $x \in J$ and all n, $m \geqslant N$.

Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that we cannot find continuous functions $g_n : [0,1] \to \mathbb{R}$ with $g_n(x) \to g(x)$ for each $x \in [0,1]$ as $n \to \infty$.

Define a sequence of continuous functions $h_n : [0,1] \to \mathbb{R}$ and a discontinuous function $h : [0,1] \to \mathbb{R}$ with $h_n(x) \to h(x)$ for each $x \in [0,1]$ as $n \to \infty$.

Paper 4, Section II

11F Topics In Analysis

(a) Suppose that $\gamma : [0,1] \to \mathbb{C}$ is continuous with $\gamma(0) = \gamma(1)$ and $\gamma(t) \neq 0$ for all $t \in [0,1]$. Show that if $\gamma(0) = |\gamma(0)| \exp(i\theta_0)$ (with θ_0 real) we can define a continuous function $\theta : [0,1] \to \mathbb{R}$ such that $\theta(0) = \theta_0$ and $\gamma(t) = |\gamma(t)| \exp\big(i\theta(t)\big)$. Hence define the *winding number* $w(\gamma) = w(0,\gamma)$ of γ around 0.

(b) Show that $w(\gamma)$ can take any integer value.

(c) If γ_1 and γ_2 satisfy the requirements of the definition, and $(\gamma_1 \times \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$, show that

$$w(\gamma_1 \times \gamma_2) = w(\gamma_1) + w(\gamma_2).$$

(d) If γ_1 and γ_2 satisfy the requirements of the definition and $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for all $t \in [0,1]$, show that

$$w(\gamma_1) = w(\gamma_2).$$

(e) State and prove a theorem that says that winding number is unchanged under an appropriate homotopy.

Paper 1, Section I

2H Topics in Analysis

By considering the function $\mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$R(a_0, \ldots, a_n) = \sup_{t \in [-1,1]} \left| \sum_{j=0}^{n} a_j t^j \right|,$$

or otherwise, show that there exist $K_n > 0$ and $\delta_n > 0$ such that

$$K_n \sum_{j=0}^{n} |a_j| \geqslant \sup_{t \in [-1,1]} \left| \sum_{j=0}^{n} a_j t^j \right| \geqslant \delta_n \sum_{j=0}^{n} |a_j|$$

for all $a_j \in \mathbb{R}$, $0 \leqslant j \leqslant n$.

Show, quoting carefully any theorems you use, that we must have $\delta_n \to 0$ as $n \to \infty$.

Paper 2, Section I

2H Topics in Analysis

Define what it means for a subset E of $\mathbb{R}^n$ to be *convex*. Which of the following statements about a convex set E in $\mathbb{R}^n$ (with the usual norm) are always true, and which are sometimes false? Give proofs or counterexamples as appropriate.

(i) The closure of E is convex.

(ii) The interior of E is convex.

(iii) If $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then $\alpha(E)$ is convex.

(iv) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then $f(E)$ is convex.

Paper 3, Section I

2H Topics in Analysis

In the game of 'Chicken', A and B drive fast cars directly at each other. If they both swerve, they both lose 10 status points; if neither swerves, they both lose 100 status points. If one swerves and the other does not, the swerver loses 20 status points and the non-swerver gains 40 status points. Find all the pairs of probabilistic strategies such that, if one driver maintains their strategy, it is not in the interest of the other to change theirs.

Paper 4, Section I

2H Topics in Analysis

Let a_0, a_1, $a_2, \ldots$ be integers such that there exists an M with $M \geqslant |a_n|$ for all n. Show that, if infinitely many of the a_n are non-zero, then $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n!}$ is an irrational number.

Paper 2, Section II

10H Topics in Analysis

Prove Bernstein's theorem, which states that if $f : [0,1] \to \mathbb{R}$ is continuous and

$$f_m(t) = \sum_{r=0}^{m} \binom{m}{r} f(r/m) t^r (1-t)^{m-r}$$

then $f_m(t) \to f(t)$ uniformly on $[0,1]$. [Theorems from probability theory may be used without proof provided they are clearly stated.]

Deduce Weierstrass's theorem on polynomial approximation for any closed interval.

Proving any results on Chebyshev polynomials that you need, show that, if $g : [0,\pi] \to \mathbb{R}$ is continuous and $\epsilon > 0$, then we can find an $N \geqslant 0$ and $a_j \in \mathbb{R}$, for $0 \leqslant j \leqslant N$, such that

$$\left| g(t) - \sum_{j=0}^{N} a_j \cos jt \right| \leqslant \epsilon$$

for all $t \in [0,\pi]$. Deduce that $\int_0^\pi g(t) \cos nt \, dt \to 0$ as $n \to \infty$.

Paper 4, Section II

11H Topics in Analysis

Explain briefly how a positive irrational number x gives rise to a continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

with the a_j non-negative integers and $a_j \geqslant 1$ for $j \geqslant 1$.

Show that, if we write

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

for $n \geqslant 0$.

Use the observation [which need not be proved] that x lies between p_n/q_n and p_{n+1}/q_{n+1} to show that

$$|p_n/q_n - x| \leqslant 1/q_n q_{n+1} \, .$$

Show that $q_n \geqslant F_n$ where F_n is the nth Fibonacci number (thus $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$), and conclude that

$$\left| \frac{p_n}{q_n} - x \right| \leqslant \frac{1}{F_n F_{n+1}}.$$

Paper 4, Section I

2I Topics in Analysis

Let $\mathcal{K}$ be the set of all non-empty compact subsets of m-dimensional Euclidean space $\mathbb{R}^m$. Define the Hausdorff metric on $\mathcal{K}$, and prove that it is a metric.

Let $K_1 \supseteq K_2 \supseteq \ldots$ be a sequence in $\mathcal{K}$. Show that $K = \bigcap_{n=1}^{\infty} K_n$ is also in $\mathcal{K}$ and that $K_n \to K$ as $n \to \infty$ in the Hausdorff metric.

Paper 3, Section I

2I Topics in Analysis

Let K be a compact subset of $\mathbb{C}$ with path-connected complement. If $w \notin K$ and $\epsilon > 0$, show that there is a polynomial P such that

$$\left| P(z) - \frac{1}{w-z} \right| \leqslant \epsilon$$

for all $z \in K$.

Paper 2, Section I

2I Topics in Analysis

Let x_1, x_2, ..., x_n be the roots of the Legendre polynomial of degree n. Let A_1, A_2, ..., A_n be chosen so that

$$\int_{-1}^{1} p(t)\, dt = \sum_{j=1}^{n} A_j p(x_j)$$

for all polynomials p of degree $n-1$ or less. Assuming any results about Legendre polynomials that you need, prove the following results:

(i) $\displaystyle\int_{-1}^{1} p(t)\, dt = \sum_{j=1}^{n} A_j p(x_j)$ for all polynomials p of degree $2n-1$ or less;

(ii) $A_j \geqslant 0$ for all $1 \leqslant j \leqslant n$;

(iii) $\displaystyle\sum_{j=1}^{n} A_j = 2$.

Now consider $Q_n(f) = \sum_{j=1}^{n} A_j f(x_j)$. Show that

$$Q_n(f) \to \int_{-1}^{1} f(t)\, dt$$

as $n \to \infty$ for all continuous functions f.

Paper 1, Section I

2I Topics in Analysis

Let Ω be a non-empty bounded open subset of $\mathbb{R}^2$ with closure $\bar{\Omega}$ and boundary $\partial\Omega$. Let $\phi : \bar{\Omega} \to \mathbb{R}$ be continuous with ϕ twice differentiable on Ω.

(i) Why does ϕ have a maximum on $\bar{\Omega}$?

(ii) If $\epsilon > 0$ and $\nabla^2\phi \geqslant \epsilon$ on Ω, show that ϕ has a maximum on $\partial\Omega$.

(iii) If $\nabla^2\phi \geqslant 0$ on Ω, show that ϕ has a maximum on $\partial\Omega$.

(iv) If $\nabla^2\phi = 0$ on Ω and $\phi = 0$ on $\partial\Omega$, show that $\phi = 0$ on $\bar{\Omega}$.

Paper 2, Section II

9I Topics in Analysis

State and prove Sperner's lemma concerning the colouring of triangles.

Deduce a theorem, to be stated clearly, on retractions to the boundary of a disc.

State Brouwer's fixed point theorem for a disc and sketch a proof of it.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function such that for some $K > 0$ we have $\|g(x) - x\| \leqslant K$ for all $x \in \mathbb{R}^2$. Show that g is surjective.

Paper 3, Section II

10I Topics in Analysis

Let $\alpha > 0$. By considering the set E_m consisting of those $f \in C([0,1])$ for which there exists an $x \in [0,1]$ with $|f(x+h) - f(x)| \leqslant m|h|^\alpha$ for all $x+h \in [0,1]$, or otherwise, give a Baire category proof of the existence of continuous functions f on $[0,1]$ such that

$$\limsup_{h \to 0} |h|^{-\alpha}|f(x+h) - f(x)| = \infty$$

at each $x \in [0,1]$.

Are the following statements true? Give reasons.

(i) There exists an $f \in C([0,1])$ such that

$$\limsup_{h \to 0} |h|^{-\alpha}|f(x+h) - f(x)| = \infty$$

for each $x \in [0,1]$ and each $\alpha > 0$.

(ii) There exists an $f \in C([0,1])$ such that

$$\limsup_{h \to 0} |h|^{-\alpha}|f(x+h) - f(x)| = \infty$$

for each $x \in [0,1]$ and each $\alpha \geqslant 0$.

Paper 4, Section I

2G Topics in Analysis

State Liouville's theorem on approximation of algebraic numbers by rationals.

Prove that the number $\sum_{n=0}^{\infty} \frac{1}{2^{n^n}}$ is transcendental.

Paper 3, Section I

2G Topics in Analysis

State Runge's theorem about uniform approximation of holomorphic functions by polynomials.

Let $\mathbb{R}_+ \subset \mathbb{C}$ be the subset of non-negative real numbers and let

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Prove that there is a sequence of complex polynomials which converges to the function $1/z$ uniformly on each compact subset of $\Delta \setminus \mathbb{R}_+$.

Paper 2, Section I

2G Topics in Analysis

State Chebyshev's equal ripple criterion.

Let

$$h(t) = \prod_{\ell=1}^{n} \left(t - \cos \frac{(2\ell - 1)\pi}{2n} \right).$$

Show that if $q(t) = \sum_{j=0}^{n} a_j t^j$ where $a_0, \ldots, a_n$ are real constants with $|a_n| \geqslant 1$, then

$$\sup_{t \in [-1,1]} |h(t)| \leqslant \sup_{t \in [-1,1]} |q(t)|.$$

Paper 1, Section I

2G Topics in Analysis

(i) State Brouwer's fixed point theorem in the plane and an equivalent theorem concerning mapping a triangle T to its boundary ∂T.

(ii) Let A be a 3×3 matrix with positive real entries. Use the theorems you stated in (i) to prove that A has an eigenvector with positive entries.

Paper 2, Section II

11G Topics in Analysis

Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous map never taking the value 0 and satisfying $\gamma(0) = \gamma(1)$. Define the *degree* (or *winding number*) $w(\gamma; 0)$ of γ about 0. Prove the following:

(i) $w(1/\gamma;\, 0) = w(\gamma^-; 0)$, where $\gamma^-(t) = \gamma(1-t)$.

(ii) If $\sigma : [0,1] \to \mathbb{C}$ is continuous, $\sigma(0) = \sigma(1)$ and $|\sigma(t)| < |\gamma(t)|$ for each $0 \leqslant t \leqslant 1$, then $w(\gamma + \sigma; 0) = w(\gamma; 0)$.

(iii) If $\gamma_m : [0,1] \to \mathbb{C}$, $m = 1, 2, \ldots$, are continuous maps with $\gamma_m(0) = \gamma_m(1)$, which converge to γ uniformly on $[0,1]$ as $m \to \infty$, then $w(\gamma_m; 0) = w(\gamma; 0)$ for sufficiently large m.

Let $\alpha : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map such that $\alpha(0) = \alpha(1)$ and $|\alpha(t) - e^{2\pi i t}| \leqslant 1$ for each $t \in [0,1]$. Deduce from the results of (ii) and (iii) that $w(\alpha; 0) = 1$.

[You may not use homotopy invariance of the winding number without proof.]

Paper 3, Section II

12G Topics in Analysis

Define what is meant by a *nowhere dense* set in a metric space. State a version of the Baire Category Theorem. Show that any complete non-empty metric space without isolated points is uncountable.

Let A be the set of real numbers whose decimal expansion does *not* use the digit 6. (A terminating decimal representation is used when it exists.) Show that there exists a real number which cannot be written as $a + q$ with $a \in A$ and $q \in \mathbb{Q}$.

Paper 4, Section I

2F Topics in Analysis

State the Baire Category Theorem. A set $X \subseteq \mathbb{R}$ is said to be a G_δ-set if it is the intersection of countably many open sets. Show that the set $\mathbb{Q}$ of rationals is not a G_δ-set.

[You may assume that the rationals are countable and that $\mathbb{R}$ is complete.]

Paper 3, Section I

2F Topics in Analysis

State Brouwer's fixed point theorem. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function with the property that $|f(x) - x| \leqslant 1$ for all x. Show that f is surjective.

Paper 2, Section I

2F Topics in Analysis

(i) Show that for every $\epsilon > 0$ there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $|\frac{1}{x} - p(x)| \leqslant \epsilon$ for all $x \in \mathbb{R}$ satisfying $\frac{1}{2} \leqslant |x| \leqslant 2$.

[You may assume standard results provided they are stated clearly.]

(ii) Show that there is no polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $|\frac{1}{z} - p(z)| < 1$ for all $z \in \mathbb{C}$ satisfying $\frac{1}{2} \leqslant |z| \leqslant 2$.

Paper 1, Section I

2F Topics in Analysis

Show that $\sin(1)$ is irrational. [The angle is measured in radians.]

Paper 2, Section II

11F Topics in Analysis

(i) Let $n \geqslant 4$ be an integer. Show that

$$1+\frac{1}{n+\frac{1}{1+\frac{1}{n+\ldots}}} \geqslant 1+\frac{1}{2n}\,.$$

(ii) Let us say that an irrational number α is *badly approximable* if there is some constant $c>0$ such that

$$\left|\alpha-\frac{p}{q}\right| \geqslant \frac{c}{q^2}$$

for all $q \geqslant 1$ and for all integers p. Show that if the integers a_n in the continued fraction expansion $\alpha=[a_0,a_1,a_2,\ldots]$ are bounded then α is badly approximable.

Give, with proof, an example of an irrational number which is not badly approximable.

[Standard facts about continued fractions may be used without proof provided they are stated clearly.]

Paper 3, Section II

12F Topics in Analysis

Suppose that $x_0,x_1,\ldots,x_n \in [-1,1]$ are distinct points. Let f be an infinitely differentiable real-valued function on an open interval containing $[-1,1]$. Let p be the unique polynomial of degree at most n such that $f(x_r)=p(x_r)$ for $r=0,1,\ldots,n$. Show that for each $x\in[-1,1]$ there is some $\xi\in(-1,1)$ such that

$$f(x)-p(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\ldots(x-x_n)\,.$$

Now take $x_r=\cos\frac{2r+1}{2n+2}\pi$. Show that

$$|f(x)-p(x)| \leqslant \frac{1}{2^n(n+1)!}\sup_{\xi\in[-1,1]}|f^{(n+1)}(\xi)|$$

for all $x\in[-1,1]$. Deduce that there is a polynomial p of degree at most n such that

$$\left|\frac{1}{3+x}-p(x)\right| \leqslant \frac{1}{4^{n+1}}$$

for all $x\in[-1,1]$.

Paper 4, Section I

2F Topics in Analysis

Let $A_1, A_2, \ldots, A_n$ be real numbers and suppose that $x_1, x_2, \ldots, x_n \in [-1,1]$ are distinct. Suppose that the formula

$$\int_{-1}^{1} p(x)\, dx = \sum_{j=1}^{n} A_j p(x_j)$$

is valid for every polynomial p of degree $\leqslant 2n-1$. Prove the following:

(i) $A_j > 0$ for each $j = 1, 2, \ldots, n$.

(ii) $\sum_{j=1}^{n} A_j = 2$.

(iii) $x_1, x_2, \ldots, x_n$ are the roots of the Legendre polynomial of degree n.

[*You may assume standard orthogonality properties of the Legendre polynomials.*]

Paper 3, Section I

2F Topics in Analysis

State and prove Liouville's theorem concerning approximation of algebraic numbers by rationals.

Paper 2, Section I

2F Topics in Analysis

(a) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map such that $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ of γ about the origin. State precisely a theorem about homotopy invariance of the winding number.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous map such that $z^{-10} f(z)$ is bounded as $|z| \to \infty$. Prove that there exists a complex number z_0 such that

$$f(z_0) = z_0^{11}\,.$$

Paper 1, Section I

2F Topics in Analysis

State a version of the Baire category theorem for a complete metric space. Let T be the set of real numbers x with the property that, for each positive integer n, there exist integers p and q with $q \geqslant 2$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n} .$$

Is T an open subset of $\mathbb{R}$? Is T a dense subset of $\mathbb{R}$? Justify your answers.

Paper 2, Section II

11F Topics in Analysis

(a) State Runge's theorem about uniform approximability of analytic functions by complex polynomials.

(b) Let K be a compact subset of the complex plane.

(i) Let Σ be an unbounded, connected subset of $\mathbb{C} \setminus K$. Prove that for each $\zeta \in \Sigma$, the function $f(z) = (z - \zeta)^{-1}$ is uniformly approximable on K by a sequence of complex polynomials.

[You may not use Runge's theorem without proof.]

(ii) Let Γ be a bounded, connected component of $\mathbb{C} \setminus K$. Prove that there is no point $\zeta \in \Gamma$ such that the function $f(z) = (z - \zeta)^{-1}$ is uniformly approximable on K by a sequence of complex polynomials.

Paper 3, Section II

12F Topics in Analysis

State Brouwer's fixed point theorem on the plane, and also an equivalent version of it concerning continuous retractions. Prove the equivalence of the two statements.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map with the property that $|f(x)| \leqslant 1$ whenever $|x| = 1$. Show that f has a fixed point. [*Hint. Compose f with the map that sends x to the nearest point to x inside the closed unit disc.*]

Paper 1, Section I

2F Topics in Analysis

(i) State the Baire Category Theorem for metric spaces in its closed sets version.

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex analytic function which is not a polynomial. Prove that there exists a point $z_0 \in \mathbb{C}$ such that each coefficient of the Taylor series of f at z_0 is non-zero.

Paper 2, Section I

2F Topics in Analysis

(i) Let $x_1, x_2, \ldots, x_n \in [-1,1]$ be any set of n distinct numbers. Show that there exist numbers A_1, A_2, $\ldots$, A_n such that the formula

$$\int_{-1}^{1} p(x)\, dx = \sum_{j=1}^{n} A_j p(x_j)$$

is valid for every polynomial p of degree $\leqslant n-1$.

(ii) For $n = 0, 1, 2, \ldots$, let p_n be the Legendre polynomial, over $[-1,1]$, of degree n. Suppose that $x_1, x_2, \ldots, x_n \in [-1,1]$ are the roots of p_n, and $A_1, A_2, \ldots, A_n$ are the numbers corresponding to $x_1, x_2, \ldots, x_n$ as in (i).

[You may assume without proof that for $n \geqslant 1$, p_n has n distinct roots in $[-1,1]$.]

Prove that the integration formula in (i) is now valid for any polynomial p of degree $\leqslant 2n-1$.

(iii) Is it possible to choose n distinct points $x_1, x_2, \ldots, x_n \in [-1,1]$ and corresponding numbers $A_1, A_2, \ldots, A_n$ such that the integration formula in (i) is valid for any polynomial p of degree $\leqslant 2n$? Justify your answer.

Paper 3, Section I

2F Topics in Analysis

Let $\Gamma = \{z \in \mathbb{C} : z \neq 1,\ |\mathrm{Re}(z)| + |\mathrm{Im}(z)| = 1\}$.

(i) Prove that, for any $\zeta \in \mathbb{C}$ with $|\mathrm{Re}(\zeta)| + |\mathrm{Im}(\zeta)| > 1$ and any $\epsilon > 0$, there exists a complex polynomial p such that

$$\sup_{z \in \Gamma} |p(z) - (z - \zeta)^{-1}| < \epsilon\,.$$

(ii) Does there exist a sequence of polynomials p_n such that $p_n(z) \to (z-1)^{-1}$ for every $z \in \Gamma$? Justify your answer.

Paper 4, Section I

2F Topics in Analysis

(a) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map such that $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ of γ about the origin. State precisely a theorem about homotopy invariance of the winding number.

(b) Let $B = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and let $f : B \to \mathbb{C}$ be a continuous map satisfying

$$|f(z) - z| \leqslant 1$$

for each $z \in \partial B$.

(i) For $0 \leqslant t \leqslant 1$, let $\gamma(t) = f(e^{2\pi i t})$. If $\gamma(t) \neq 0$ for each $t \in [0,1]$, prove that $w(\gamma; 0) = 1$.
[*Hint: Consider a suitable homotopy between* γ *and the map* $\gamma_1(t) = e^{2\pi i t}$, $0 \leqslant t \leqslant 1$.]

(ii) Deduce that $f(z) = 0$ for some $z \in B$.

Paper 2, Section II

11F Topics in Analysis

Let $C[0,1]$ be the space of real continuous functions on the interval $[0,1]$. A mapping $L : C[0,1] \to C[0,1]$ is said to be *positive* if $L(f) \geqslant 0$ for each $f \in C[0,1]$ with $f \geqslant 0$, and *linear* if $L(af+bg) = aL(f)+bL(g)$ for all functions $f, g \in C[0,1]$ and constants $a, b \in \mathbb{R}$.

(i) Let $L_n : C[0,1] \to C[0,1]$ be a sequence of positive, linear mappings such that $L_n(f) \to f$ uniformly on $[0,1]$ for the three functions $f(x) = 1, x, x^2$. Prove that $L_n(f) \to f$ uniformly on $[0,1]$ for every $f \in C[0,1]$.

(ii) Define $B_n : C[0,1] \to C[0,1]$ by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Using the result of part (i), or otherwise, prove that $B_n(f) \to f$ uniformly on $[0,1]$.

(iii) Let $f \in C[0,1]$ and suppose that

$$\int_0^1 f(x) x^{4n}\, dx = 0$$

for each $n = 0, 1, \ldots$. Prove that f must be the zero function.

[You should not use the Stone–Weierstrass theorem without proof.]

Paper 3, Section II

12F Topics in Analysis

Let $f : [0,1] \to \mathbb{R}$ be continuous and let n be a positive integer. For $g : [0,1] \to \mathbb{R}$ a continuous function, write $\|f-g\|_{L^\infty} = \sup_{x\in[0,1]} |f(x)-g(x)|$.

(i) Let p be a polynomial of degree at most n with the property that there are $(n+2)$ distinct points $x_1, x_2, \ldots, x_{n+2} \in [0,1]$ with $x_1 < x_2 < \ldots < x_{n+2}$ such that

$$f(x_j) - p(x_j) = (-1)^j \|f-p\|_{L^\infty}$$

for each $j = 1, 2, \ldots, n+2$. Prove that

$$\|f-p\|_{L^\infty} \leqslant \|f-q\|_{L^\infty}$$

for every polynomial q of degree at most n.

(ii) Prove that there exists a polynomial p of degree at most n such that

$$\|f-p\|_{L^\infty} \leqslant \|f-q\|_{L^\infty}$$

for every polynomial q of degree at most n.

[If you deduce this from a more general result about abstract normed spaces, you must prove that result.]

(iii) Let $Y = \{y_1, y_2, \ldots, y_{n+2}\}$ be any set of $(n+2)$ distinct points in $[0,1]$.

(a) For $j = 1, 2, \ldots, n+2$, let

$$r_j(x) = \prod_{k=1,\, k\neq j}^{n+2} \frac{x-y_k}{y_j-y_k},$$

$t(x) = \sum_{j=1}^{n+2} f(y_j) r_j(x)$ and $r(x) = \sum_{j=1}^{n+2} (-1)^j r_j(x)$. Explain why there is a unique number $\lambda \in \mathbb{R}$ such that the degree of the polynomial $t - \lambda r$ is at most n.

(b) Let $\|f-g\|_{L^\infty(Y)} = \sup_{x\in Y} |f(x)-g(x)|$. Deduce from part (a) that there exists a polynomial p of degree at most n such that

$$\|f-p\|_{L^\infty(Y)} \leqslant \|f-q\|_{L^\infty(Y)}$$

for every polynomial q of degree at most n.

Paper 1, Section I

2F Topics in Analysis

Let (X, d) be a non-empty complete metric space with no isolated points, G an open dense subset of X and E a countable dense subset of X.

(i) Stating clearly any standard theorem you use, prove that $G \setminus E$ is a dense subset of X.

(ii) If G is only assumed to be uncountable and dense in X, does it still follow that $G \setminus E$ is dense in X? Justify your answer.

Paper 2, Section I

2F Topics in Analysis

(a) State the Weierstrass approximation theorem concerning continuous real functions on the closed interval $[0, 1]$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be continuous.

(i) If $\int_0^1 f(x)\, x^n\, dx = 0$ for each $n = 0, 1, 2, \ldots$, prove that f is the zero function.

(ii) If we only assume that $\int_0^1 f(x)\, x^{2n}\, dx = 0$ for each $n = 0, 1, 2, \ldots$, prove that it still follows that f is the zero function.

[If you use the Stone–Weierstrass theorem, you must prove it.]

(iii) If we only assume that $\int_0^1 f(x)\, x^{2n+1}\, dx = 0$ for each $n = 0, 1, 2, \ldots$, does it still follow that f is the zero function? Justify your answer.

Paper 3, Section I

2F Topics in Analysis

Let $A = \{z \in \mathbb{C} : 1/2 \leqslant |z| \leqslant 2\}$ and suppose that f is complex analytic on an open subset containing A.

(i) Give an example, with justification, to show that there need not exist a sequence of complex polynomials converging to f uniformly on A.

(ii) Let $R \subset \mathbb{C}$ be the positive real axis and $B = A \setminus R$. Prove that there exists a sequence of complex polynomials $p_1, p_2, p_3, \ldots$ such that $p_j \to f$ uniformly on each compact subset of B.

(iii) Let $p_1, p_2, p_3, \ldots$ be the sequence of polynomials in (ii). If this sequence converges uniformly on A, show that $\int_C f(z)\, dz = 0$, where $C = \{z \in \mathbb{C} : |z| = 1\}$.

Paper 4, Section I

2F Topics in Analysis

Find explicitly a polynomial p of degree $\leqslant 3$ such that

$$\sup_{x \in [-1,1]} |\, x^4 - p(x)| \;\leqslant\; \sup_{x \in [-1,1]} |\, x^4 - q(x)|$$

for every polynomial q of degree $\leqslant 3$. Justify your answer.

Paper 2, Section II

11F Topics in Analysis

Let

$$B_r(0) = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < r^2\}\,,$$

$B = B_1(0)$, and

$$C = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}\,.$$

Let $D = B \cup C$.

(i) State the Brouwer fixed point theorem on the plane.

(ii) Show that the Brouwer fixed point theorem on the plane is equivalent to the non-existence of a continuous map $F \,:\, D \to C$ such that $F(p) = p$ for each $p \in C$.

(iii) Let $G \,:\, D \to \mathbb{R}^2$ be continuous, $0 < \epsilon < 1$ and suppose that

$$|p - G(p)| \,<\, \epsilon$$

for each $p \in C$. Using the Brouwer fixed point theorem or otherwise, prove that

$$B_{1-\epsilon}(0) \,\subseteq\, G(B)\,.$$

[*Hint: argue by contradiction.*]

(iv) Let $q \in B$. Does there exist a continuous map $H \,:\, D \to \mathbb{R}^2 \setminus \{q\}$ such that $H(p) = p$ for each $p \in C$? Justify your answer.

Paper 3, Section II

12F Topics in Analysis

(i) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map with $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ of γ about the origin.

(ii) For $j = 0, 1$, let $\gamma_j : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous with $\gamma_j(0) = \gamma_j(1)$. Make the following statement precise, and prove it: *if γ_0 can be continuously deformed into γ_1 through a family of continuous curves missing the origin, then $w(\gamma_0; 0) = w(\gamma_1; 0)$.*

[You may use without proof the following fact: if $\gamma, \delta : [0,1] \to \mathbb{C} \setminus \{0\}$ are continuous with $\gamma(0) = \gamma(1)$, $\delta(0) = \delta(1)$ and if $|\gamma(t)| < |\delta(t)|$ for each $t \in [0,1]$, then $w(\gamma + \delta; 0) = w(\delta; 0)$.]

(iii) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be continuous with $\gamma(0) = \gamma(1)$. If $\gamma(t)$ is not equal to a negative real number for each $t \in [0,1]$, prove that $w(\gamma; 0) = 0$.

(iv) Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. If $g : D \to C$ is continuous, prove that for each non-zero integer n, there is at least one point $z \in C$ such that $z^n + g(z) = 0$.

Paper 1, Section I

2F Topics in Analysis

(i) Let $n \geqslant 1$ and let $x_1, \ldots, x_n$ be distinct points in $[-1, 1]$. Show that there exist numbers $A_1, \ldots, A_n$ such that

$$\int_{-1}^{1} P(x)\, dx = \sum_{j=1}^{n} A_j P(x_j) \qquad (*)$$

for every polynomial P of degree $\leqslant n-1$.

(ii) Explain, without proof, how one can choose the points $x_1, \ldots, x_n$ and the numbers $A_1, \ldots, A_n$ such that $(*)$ holds for all polynomials P of degree $\leqslant 2n-1$.

Paper 2, Section I

2F Topics in Analysis

(a) State *Chebychev's Equal Ripple Criterion.*

(b) Let n be a positive integer, $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ and

$$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

Use Chebychev's Equal Ripple Criterion to prove that

$$\sup_{x \in [-1,1]} |p(x)| \geqslant 2^{1-n}.$$

[You may use without proof that there is a polynomial $T_n(x)$ in x of degree n, with the coefficient of x^n equal to 2^{n-1}, such that $T_n(\cos\theta) = \cos n\theta$ for all $\theta \in \mathbb{R}$.]

Paper 3, Section I

2F Topics in Analysis

(a) If $f : (0,1) \to \mathbb{R}$ is continuous, prove that there exists a sequence of polynomials P_n such that $P_n \to f$ uniformly on compact subsets of $(0,1)$.

(b) If $f : (0,1) \to \mathbb{R}$ is continuous and bounded, prove that there exists a sequence of polynomials Q_n such that Q_n are uniformly bounded on $(0,1)$ and $Q_n \to f$ uniformly on compact subsets of $(0,1)$.

Paper 4, Section I

2F Topics in Analysis

State Liouville's theorem on approximation of algebraic numbers by rationals, and use it to prove that the number

$$\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$$

is transcendental.

Paper 2, Section II

11F Topics in Analysis

(a) State Brouwer's fixed point theorem in the plane.

(b) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors in $\mathbb{R}^2$ making 120^o angles with one another. Let T be the triangle with vertices given by the points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and let I, J, K be the three sides of T. Prove that the following two statements are equivalent:

(1) There exists no continuous function $f : T \to \partial T$ with $f(I) \subseteq I$, $f(J) \subseteq J$ and $f(K) \subseteq K$.

(2) If A, B, C are closed subsets of $\mathbb{R}^2$ such that $T \subseteq A \cup B \cup C$, $I \subseteq A$, $J \subseteq B$ and $K \subseteq C$, then $A \cap B \cap C \neq \emptyset$.

(c) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be continuous positive functions. Show that the system of equations

$$(1-x^2)f^2(x,y) - x^2 g^2(x,y) = 0$$
$$(1-y^2)g^2(x,y) - y^2 f^2(x,y) = 0$$

has four distinct solutions on the unit circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$.

Paper 3, Section II

12F Topics in Analysis

(a) State Runge's theorem on uniform approximation of analytic functions by polynomials.

(b) Let Ω be an unbounded, connected, proper open subset of $\mathbb{C}$. For any given compact set $K \subset \mathbb{C} \setminus \Omega$ and any $\zeta \in \Omega$, show that there exists a sequence of complex polynomials converging uniformly on K to the function $f(z) = (z-\zeta)^{-1}$.

(c) Give an example, with justification, of a connected open subset Ω of $\mathbb{C}$, a compact subset K of $\mathbb{C} \setminus \Omega$ and a point $\zeta \in \Omega$ such that there is no sequence of complex polynomials converging uniformly on K to the function $f(z) = (z-\zeta)^{-1}$.

1/I/2F Topics in Analysis

Let $P_0, P_1, P_2, \ldots$ be non-zero orthogonal polynomials on an interval $[a,b]$ such that the degree of P_j is equal to j for every $j = 0, 1, 2, \ldots$, where the orthogonality is with respect to the inner product $< f, g > = \int_a^b fg$. If f is any continuous function on $[a,b]$ orthogonal to $P_0, P_1, \ldots, P_{n-1}$ and not identically zero, prove that f must have at least n distinct zeros in (a,b).

2/II/11F Topics in Analysis

Let $L : C([0,1]) \to C([0,1])$ be an operator satisfying the conditions

(i) $Lf \geqslant 0$ for any $f \in C([0,1])$ with $f \geqslant 0$,

(ii) $L(af + bg) = aLf + bLg$ for any $f, g \in C([0,1])$ and $a, b \in \mathbf{R}$ and

(iii) $Z_f \subseteq Z_{Lf}$ for any $f \in C([0,1])$, where Z_f denotes the set of zeros of f.

Prove that there exists a function $h \in C([0,1])$ with $h \geqslant 0$ such that $Lf = hf$ for every $f \in C([0,1])$.

2/I/2F Topics in Analysis

(a) State Brouwer's fixed point theorem in the plane and prove that the statement is equivalent to non-existence of a continuous retraction of the closed disk D to its boundary ∂D.

(b) Use Brouwer's fixed point theorem to prove that there is a complex number z in the closed unit disc such that $z^6 - z^5 + 2z^2 + 6z + 1 = 0$.

3/II/12F **Topics in Analysis**

(a) State Liouville's theorem on approximation of algebraic numbers by rationals.

(b) Consider the continued fraction expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

in which the coefficients a_n are all positive integers forming an unbounded set. Let $\frac{p_n}{q_n}$ be the nth convergent. Prove that

$$\left| x - \frac{p_n}{q_n} \right| \leqslant \frac{1}{q_n q_{n+1}}$$

and use this inequality together with Liouville's theorem to deduce that x^2 is irrational.

[You may assume without proof that, for $n = 1, 2, 3, \ldots$,

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}.]$$

3/I/2F **Topics in Analysis**

(a) State the Baire category theorem in its closed sets version.

(b) Let $f_n : \mathbf{R} \to \mathbf{R}$ be a continuous function for each $n = 1, 2, 3, \ldots$ and suppose that there is a function $f : \mathbf{R} \to \mathbf{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbf{R}$. Prove that for each $\epsilon > 0$, there exists an integer N_0 and a non-empty open interval $I \subset \mathbf{R}$ such that $|f_n(x) - f(x)| \leqslant \epsilon$ for all $n \geqslant N_0$ and $x \in I$.

[*Hint: consider, for* $N = 1, 2, 3, \ldots$, *the sets*

$$Q_N = \{x \in \mathbf{R} : |f_n(x) - f_m(x)| \leqslant \epsilon : \forall n, m \geqslant N\}.]$$

4/I/2F **Topics in Analysis**

(a) State Runge's theorem on uniform approximation of analytic functions by polynomials.

(b) Suppose f is analytic on

$$\Omega = \{z \in \mathbf{C} : |z| < 1\} \setminus \{z \in \mathbf{C} : \mathrm{Im}(z) = 0,\ \mathrm{Re}(z) \leqslant 0\}.$$

Prove that there exists a sequence of polynomials which converges to f uniformly on compact subsets of Ω.

1/I/2F **Topics in Analysis**

Let n be an integer with $n \geqslant 1$. Are the following statements true or false? Give proofs.

(i) There exists a real polynomial T_n of degree n such that

$$T_n(\cos t) = \cos nt$$

for all real t.

(ii) There exists a real polynomial R_n of degree n such that

$$R_n(\cosh t) = \cosh nt$$

for all real t.

(iii) There exists a real polynomial S_n of degree n such that

$$S_n(\cos t) = \sin nt$$

for all real t.

2/II/12F **Topics in Analysis**

(i) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove the theorem of Bernstein which states that, if we write

$$f_m(t) = \sum_{r=0}^{m} \binom{m}{r} f(r/m) t^r (1-t)^{m-r},$$

for $0 \leqslant t \leqslant 1$, then $f_m \to f$ uniformly as $m \to \infty$.

(ii) Let $n \geqslant 1$, $a_{1,n}, a_{2,n}, \ldots, a_{n,n} \in \mathbb{R}$ and let $x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ be distinct points in $[0,1]$. We write

$$I_n(g) = \sum_{j=1}^{n} a_{j,n} g(x_{j,n})$$

for every continuous function $g : [0,1] \to \mathbb{R}$. Show that, if

$$I_n(P) = \int_0^1 P(t)\,dt\,,$$

for all polynomials P of degree $2n-1$ or less, then $a_{j,n} \geqslant 0$ for all $1 \leqslant j \leqslant n$ and $\sum_{j=1}^{n} a_{j,n} = 1$.

(iii) If I_n satisfies the conditions set out in (ii), show that

$$I_n(f) \to \int_0^1 f(t)\,dt$$

as $n \to \infty$ whenever $f : [0,1] \to \mathbb{R}$ is continuous.

2/I/2F **Topics in Analysis**

Write

$$P^+ = \{(x,y) \in \mathbb{R}^2 \,:\, x,\, y > 0\}.$$

Suppose that K is a convex, compact subset of $\mathbb{R}^2$ with $K \cap P^+ \neq \emptyset$. Show that there is a unique point $(x_0, y_0) \in K \cap P^+$ such that

$$xy \leqslant x_0 y_0$$

for all $(x,y) \in K \cap P^+$.

3/II/12F **Topics in Analysis**

(i) State and prove Liouville's theorem on approximation of algebraic numbers by rationals.

(ii) Consider the continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}}$$

where the a_j are strictly positive integers. You may assume the following algebraic facts about the nth convergent p_n/q_n.

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Show that

$$\left|\frac{p_n}{q_n} - x\right| \leqslant \frac{1}{q_n q_{n+1}}.$$

Give explicit values for a_n so that x is transcendental and prove that you have done so.

3/I/2F **Topics in Analysis**

State a version of Runge's theorem and use it to prove the following theorem:

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and define $f : D \to \mathbb{C}$ by the condition

$$f(re^{i\theta}) = r^{3/2} e^{3i\theta/2}$$

for all $0 \leqslant r < 1$ and all $0 \leqslant \theta < 2\pi$. (We take $r^{1/2}$ to be the positive square root.) Then there exists a sequence of analytic functions $f_n : D \to \mathbb{C}$ such that $f_n(z) \to f(z)$ for each $z \in D$ as $n \to \infty$.

4/I/2F **Topics in Analysis**

State Brouwer's fixed point theorem for a triangle in two dimensions.

Let $A = (a_{ij})$ be a 3×3 matrix with real positive entries and such that all its columns are non-zero vectors. Show that A has an eigenvector with positive entries.

1/I/2G **Topics in Analysis**

State Brouwer's fixed-point theorem, and also an equivalent version of the theorem that concerns retractions of the disc. Prove that these two versions are equivalent.

1/II/11G **Topics in Analysis**

Let $\mathbb{T} = \{z : |z| = 1\}$ be the unit circle in $\mathbb{C}$, and let $\phi : \mathbb{T} \to \mathbb{C}$ be a continuous function that never takes the value 0. Define the *degree* (or *winding number*) of ϕ about 0. [*You need not prove that the degree is well-defined.*]

Denote the degree of ϕ about 0 by $w(\phi)$. Prove the following facts.

(i) If ϕ_1 and ϕ_2 are two functions with the properties of ϕ above, then $w(\phi_1.\phi_2) = w(\phi_1) + w(\phi_2)$.

(ii) If ψ is any continuous function such that $|\psi(z)| < |\phi(z)|$ for every $z \in \mathbb{T}$, then $w(\phi + \psi) = w(\phi)$.

Using these facts, calculate the degree $w(\phi)$ when ϕ is given by the formula $\phi(z) = (3z-2)(z-3)(2z+1)+1$.

2/I/2G **Topics in Analysis**

(a) State Chebyshev's equal ripple criterion.

(b) Let $f : [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \cos 4\pi x \ ,$$

and let g be a polynomial of degree 7. Prove that there exists an $x \in [-1,1]$ such that $|f(x) - g(x)| \geqslant 1$.

2/II/11G **Topics in Analysis**

(a) Let K be a closed subset of the unit disc in $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function with all its poles of modulus strictly greater than 1. Explain why f can be approximated uniformly on K by polynomials.
[*Standard results from complex analysis may be assumed.*]

(b) With K as above, define Λ to be the set of all $\lambda \in \mathbb{C} \setminus K$ such that the function $(z-\lambda)^{-1}$ can be uniformly approximated on K by polynomials. If $\lambda \in \Lambda$, prove that there is some $\delta > 0$ such that $\mu \in \Lambda$ whenever $|\lambda - \mu| < \delta$.

3/I/2G **Topics in Analysis**

Let a_0, a_1, a_2, ... be positive integers and, for each n, let

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}\,,$$

with $(p_n, q_n) = 1$.

Obtain an expression for the matrix $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ and use it to show that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}$.

4/I/2G **Topics in Analysis**

(a) State the Baire category theorem, in its closed-sets version.

(b) For every $n \in \mathbb{N}$ let f_n be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and let $g(x) = 1$ when x is rational and 0 otherwise. For each $N \in \mathbb{N}$, let

$$F_N = \left\{x \in \mathbb{R} : \forall n \geqslant N \quad f_n(x) \leqslant \tfrac{1}{3} \quad \text{or} \quad f_n(x) \geqslant \tfrac{2}{3}\right\}.$$

By applying the Baire category theorem, prove that the functions f_n cannot converge pointwise to g. (That is, it is not the case that $f_n(x) \to g(x)$ for every $x \in \mathbb{R}$.)

1/I/2F **Topics in Analysis**

Prove that $\cosh(1/2)$ is irrational.

1/II/11F **Topics in Analysis**

State and prove a discrete form of Brouwer's theorem, concerning colourings of points in triangular grids. Use it to deduce that there is no continuous retraction from a disc to its boundary.

2/I/2F **Topics in Analysis**

(i) Let α be an algebraic number and let p and q be integers with $q \neq 0$. What does Liouville's theorem say about α and p/q?

(ii) Let p and q be integers with $q \neq 0$. Prove that

$$\left|\sqrt{2}-\frac{p}{q}\right| \geqslant \frac{1}{4q^2}\ .$$

[*In (ii), you may not use Liouville's theorem unless you prove it.*]

2/II/11F **Topics in Analysis**

(i) State the Baire category theorem. Deduce from it a statement about nowhere dense sets.

(ii) Let X be the set of all real numbers with decimal expansions consisting of the digits 4 and 5 only. Prove that there is a real number t that cannot be written in the form $x+y$ with $x \in X$ and y rational.

3/I/2F **Topics in Analysis**

Let $-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n} a_i p(x_i)$$

for every polynomial p of degree less than $2n$. Prove the following three facts.

(i) $a_i > 0$ for every i.

(ii) $\sum_{i=1}^{n} a_i = 2$.

(iii) The numbers $x_1, x_2, \ldots, x_n$ are the roots of the Legendre polynomial of degree n.

[*You may assume standard orthogonality properties of the Legendre polynomials.*]

4/I/2F **Topics in Analysis**

(i) Let $D \subset \mathbb{C}$ be a domain, let $f : D \to \mathbb{C}$ be an analytic function and let $z_0 \in D$. What does Taylor's theorem say about z_0, f and D?

(ii) Let K be the square consisting of all complex numbers z such that

$$-1 \leqslant \mathrm{Re}(z) \leqslant 1 \text{ and } -1 \leqslant \mathrm{Im}(z) \leqslant 1\,,$$

and let w be a complex number not belonging to K. Prove that the function $f(z) = (z - w)^{-1}$ can be uniformly approximated on K by polynomials.