

Part II

Stochastic Financial Models

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Paper 1, Section II**30K Stochastic Financial Models**

Let U be a smooth, increasing and concave function on \mathbb{R} .

(a) Given a vector space \mathcal{X} of random variables, define a function F by

$$F(y) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X + y)],$$

and suppose that for all constants $y \in \mathbb{R}$ the supremum is achieved by some $X_y \in \mathcal{X}$. Show that F is increasing and concave.

(b) Given a constant m and a random variable Z such that $\mathbb{E}(Z) = 0$, define a function G by

$$G(s) = \mathbb{E}[U(m + sZ)].$$

Show that G is concave. Show that G is decreasing on $[0, \infty)$.

Now consider a market with interest rate $r \geq 0$ and d risky assets, such that the vector of time- n prices S_n evolves as

$$S_n = (1 + r)S_{n-1} + \xi_n.$$

Assume that the sequence $(\xi_n)_{n \geq 1}$ of \mathbb{R}^d -valued random vectors is IID and generates the filtration. Given initial wealth X_0 , an investor's time- n wealth X_n evolves as

$$X_n = (1 + r)X_{n-1} + \theta_n^\top \xi_n$$

for $n \geq 1$, where the trading strategy $(\theta_n)_{n \geq 1}$ is previsible. Given a time horizon $N \geq 1$, the investor seeks a trading strategy to maximise $\mathbb{E}[U(X_N)]$.

(c) Write down the Bellman equation for the investor's value function V . Assuming that an optimal portfolio exists at each time-step, show that $V(n, \cdot)$ is increasing and concave for all $0 \leq n \leq N$.

(d) Now assume that for all $n \geq 1$, ξ_n has the multi-variate Gaussian $N(b, \Sigma)$ distribution, where $b \neq 0$ and Σ is positive definite. Assuming the existence of a unique optimal time- n portfolio θ_n^* for each $1 \leq n \leq N$, show that there exist non-negative random variables λ_n such that $\theta_n^* = \lambda_n \Sigma^{-1} b$.

[You may use the mutual fund theorem or the Gaussian integration-by-parts formula without proof.]

Paper 2, Section II**30K Stochastic Financial Models**

Let $(M_n)_{n \geq 0}$ be a martingale with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$.

(a) Given an \mathcal{F}_0 -measurable X_0 and a previsible process $(A_n)_{n \geq 1}$, let

$$X_n = X_0 + \sum_{k=1}^n A_k (M_k - M_{k-1})$$

for $n \geq 1$. Assuming that X_n is integrable for each $n \geq 0$, show that $(X_n)_{n \geq 0}$ is a martingale.

(b) Let T be a stopping time and let

$$Y_n = M_{\min\{n, T\}}$$

for $n \geq 0$. Show that $(Y_n)_{n \geq 0}$ is a martingale.

(c) Let $(X_n)_{n \geq 0}$ be as in part (a) and suppose that both X_n and M_n are square-integrable for each $n \geq 0$. Suppose that $\text{Var}(M_n | \mathcal{F}_{n-1}) > 0$ almost surely for each $n \geq 1$. Show that

$$A_n = \frac{\text{Cov}(X_n, M_n | \mathcal{F}_{n-1})}{\text{Var}(M_n | \mathcal{F}_{n-1})}$$

for all $n \geq 1$.

Let $(\xi_n)_{n \geq 1}$ be a sequence of independent random variables such that for all $n \geq 1$, $\mathbb{P}(\xi_n = +1) = \frac{1}{2} = \mathbb{P}(\xi_n = -1)$. Suppose that the filtration $(\mathcal{F}_n)_{n \geq 0}$ is generated by $(\xi_n)_{n \geq 1}$.

(d) Show that there exists a previsible process $(B_n)_{n \geq 1}$ such that

$$M_n = M_0 + \sum_{k=1}^n B_k \xi_k$$

for all $n \geq 0$.

(e) Now show for any bounded stopping time T that

$$\mathbb{E}[M_T^2] = M_0^2 + \mathbb{E} \left[\sum_{k=1}^T B_k^2 \right].$$

Paper 3, Section II**29K Stochastic Financial Models**

Let $(W_t)_{t \geq 0}$ be a Brownian motion, and let a and b be positive constants.

(a) Let $B_0 = 0$ and $B_t = tW_{1/t}$ for $t > 0$. Show that $(B_t)_{t \geq 0}$ is a Brownian motion. [You may use without proof a characterisation of Brownian motion as a Gaussian process. You may also use without proof the fact that $W_t/t \rightarrow 0$ almost surely as $t \rightarrow \infty$.]

(b) Prove that $\mathbb{P}(\sup_{0 \leq s \leq t} (W_s - as) \leq b) = \mathbb{P}(\sup_{u \geq 1/t} (W_u - bu) \leq a)$.

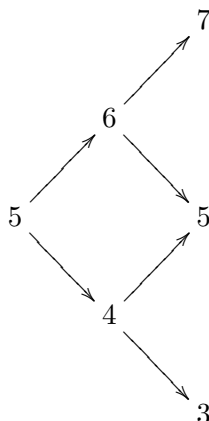
(c) Use the reflection principle and the Cameron–Martin theorem to show that

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} (W_s - as) \leq b\right) = \mathbb{P}(W_t - at \leq b) - e^{-2ab} \mathbb{P}(W_t - at \leq -b).$$

(d) Let $T = \sup\{t \geq 0 : W_t - at > b\}$ with the convention that $\sup \emptyset = 0$. Find $\mathbb{P}(T \leq t)$ in terms of the standard normal distribution function Φ .

Paper 4, Section II**29K Stochastic Financial Models**

Consider the following two-period market model. There is a single risky stock with prices $(S_n)_{n \in \{0,1,2\}}$ given by



where in each period the price is equally likely to go up as to go down.

(a) Suppose that the interest rate $r = 1/6$. Find an arbitrage $(\varphi_n)_{n \in \{1,2\}}$.

For the rest of the problem, suppose $r = 1/8$.

(b) Find the time-0 no-arbitrage price of a European put option maturing at $T = 2$ with strike $K = 5$. How many shares of the stock should be held in the first period to replicate the payout of the put?

(c) Find the time-0 no-arbitrage price of a European call option maturing at $T = 2$ with strike $K = 5$.

(d) Now find the time-0 no-arbitrage price of an American put option maturing at $T = 2$ with strike $K = 5$. What is an optimal exercise policy?

Paper 1, Section II**30K Stochastic Financial Models**

Fix a positive integer N and consider the problem of minimising

$$\mathbb{E} \left(X_N^2 + \sum_{n=1}^N u_n^2 \right),$$

where X_0 is given and

$$X_n = X_{n-1} + u_n + \xi_n$$

for $1 \leq n \leq N$. Here $(\xi_n)_{1 \leq n \leq N}$ is an IID sequence of random variables with $\mathbb{E}(\xi_1) = 0$ and $\text{Var}(\xi_1) = \sigma^2$, and the controls $(u_n)_{1 \leq n \leq N}$ are previsible with respect to the filtration generated by $(\xi_n)_{1 \leq n \leq N}$.

(a) Write down the Bellman equation for this problem.

(b) Show that the value function can be expressed as

$$V(n, x) = A_n + B_n x + C_n x^2$$

for constants $(A_n, B_n, C_n)_{0 \leq n \leq N}$ to be found.

(c) Show that the optimal control is

$$u_n^* = -\frac{X_0}{N+1} - \frac{\xi_1}{N} - \frac{\xi_2}{N-1} - \dots - \frac{\xi_{n-1}}{N-n+2}$$

for $1 \leq n \leq N$.

Paper 2, Section II**30K Stochastic Financial Models**

Consider a one-period market model with constant interest rate r and d risky assets. For $n \in \{0, 1\}$ let S_n denote the vector of time- n prices of the risky assets and let X_n be the time- n wealth of an investor. Let $\mu = \mathbb{E}(S_1)$ and $V = \text{Cov}(S_1)$. Assume $\mu \neq (1+r)S_0$.

(a) Suppose V is non-singular. Find, with proof, the minimum of $\text{Var}(X_1)$ subject to the constraints that $X_0 = x$ and $\mathbb{E}(X_1) = m$ for given constants x and m . Show that the optimal portfolio of risky assets is of the form $\theta^* = \lambda V^{-1}[\mu - (1+r)S_0]$ for a constant λ to be found. Now find the minimum of $\text{Var}(X_1)$ subject to $X_0 = x$ and $\mathbb{E}(X_1) \geq m$.

(b) Again suppose V is non-singular. Find, with proof, the maximum of the quantity

$$\frac{\mathbb{E}(X_1) - (1+r)X_0}{\sqrt{\text{Var}(X_1)}},$$

subject to $X_0 = x$. Show that all optimal portfolios are mean-variance efficient.

(c) Now suppose V is singular and that there exists *no* vector $\theta \in \mathbb{R}^d$ such that $V\theta = \mu - (1+r)S_0$. Show that for any m and x ,

$$\min\{\text{Var}(X_1) : \mathbb{E}(X_1) = m \text{ and } X_0 = x\} = 0.$$

Show that there exists an arbitrage in this market.

Paper 3, Section II**29K Stochastic Financial Models**

(a) Let W be a Brownian motion and c a constant. Let $M_t = e^{cW_t - c^2 t/2}$ for $t \geq 0$. Show that M is a martingale in the filtration generated by W .

For the rest of the question, consider the Black–Scholes model with constant interest rate r and time- t stock price $S_t = S_0 e^{\mu t + \sigma W_t}$ for $0 \leq t \leq T$, where μ, σ, T are constants with $\sigma > 0$.

(b) Show that there exists a risk-neutral measure for the Black–Scholes model. [You may use the Cameron–Martin theorem without proof if it is clearly stated.]

(c) Find the time-0 Black–Scholes price of a European claim with time- T payout $Y_0 = S_T^p$ where the exponent p is a constant.

(d) Consider two European claims with time- T payouts

$$Y_1 = \max_{0 \leq t \leq T} S_t \text{ and } Y_2 = \frac{S_0^{2-p} S_T^p}{\min_{0 \leq t \leq T} S_t}.$$

Find, with proof, the exponent p such that these two claims have the same time-0 Black–Scholes price.

Paper 4, Section II**29K Stochastic Financial Models**

Consider a discrete-time market with constant interest rate r and a stock with time- n price S_n for $0 \leq n \leq N$.

(a) Suppose a self-financing investor holds θ_n shares of the stock between times $n-1$ and n for $1 \leq n \leq N$. Explain why the investor's wealth process $(X_n)_{0 \leq n \leq N}$ evolves as

$$X_n = (1+r)X_{n-1} + \theta_n[S_n - (1+r)S_{n-1}] \quad \text{for } 1 \leq n \leq N.$$

For the rest of the question, suppose $S_n = S_{n-1}\xi_n$ where

$$\begin{aligned} \mathbb{P}(\xi_n = 1+b) &= p \\ \mathbb{P}(\xi_n = 1+a) &= 1-p \end{aligned}$$

for all $n \geq 1$, for given constants $0 < p < 1$ and $a < r < b$.

(b) Show that

$$\mathbb{Q}(S_N = S_0(1+b)^i(1+a)^{N-i}) = \binom{N}{i} q^i (1-q)^{N-i}$$

for all $0 \leq i \leq N$, where \mathbb{Q} is the unique risk-neutral measure and q is a constant which you should find.

(c) Now introduce a European contingent claim into this market with time- N payout $g(S_N)$ for a given function g . Find, with proof, the constant x and the previsible process $\theta = (\theta_n)_{1 \leq n \leq N}$ such that if an investor has time-0 wealth $X_0 = x$ and employs the trading strategy θ then the time- N wealth is $X_N = g(S_N)$ almost surely. Express your answer in terms of the function V defined by

$$V(n, s) = (1+r)^{-(N-n)} \mathbb{E}^{\mathbb{Q}}[g(S_N) | S_n = s] \quad \text{for } 0 \leq n \leq N, s > 0.$$

(d) Suppose the claim in part (c) is a European call option with strike K . Show that the corresponding initial cost x of the claim is of the form

$$S_0 \hat{\mathbb{Q}}(S_N > K) - K(1+r)^{-N} \mathbb{Q}(S_N > K),$$

where $\hat{\mathbb{Q}}$ is a probability measure such that

$$\hat{\mathbb{Q}}(S_N = S_0(1+b)^i(1+a)^{N-i}) = \binom{N}{i} \hat{q}^i (1-\hat{q})^{N-i}$$

for $0 \leq i \leq N$ and a constant \hat{q} which you should find.

Paper 1, Section II**30K Stochastic Financial Models**

(a) What does it mean to say that a stochastic process $(X_n)_{n \geq 0}$ is a *martingale* with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$?

(b) Let $(X_n)_{n \geq 0}$ be a martingale, and let $\xi_n = X_n - X_{n-1}$ for $n \geq 1$. Suppose ξ_n takes values in the set $\{-1, +1\}$ almost surely for all $n \geq 1$. Show that $(X_n)_{n \geq 0}$ is a simple symmetric random walk, i.e. that the sequence $(\xi_n)_{n \geq 1}$ is IID with $\mathbb{P}(\xi_1 = 1) = 1/2 = \mathbb{P}(\xi_1 = -1)$.

(c) Let $(X_n)_{n \geq 0}$ be a martingale and let the bounded process $(H_n)_{n \geq 1}$ be previsible. Let $\hat{X}_0 = 0$ and

$$\hat{X}_n = \sum_{k=1}^n H_k(X_k - X_{k-1}) \text{ for } n \geq 1.$$

Show that $(\hat{X}_n)_{n \geq 0}$ is a martingale.

(d) Let $(X_n)_{n \geq 0}$ be a simple symmetric random walk with $X_0 = 0$, and let

$$T_a = \inf\{n \geq 0 : X_n = a\},$$

where a is a positive integer. Let

$$\hat{X}_n = \begin{cases} X_n & \text{if } n \leq T_a \\ 2a - X_n & \text{if } n > T_a. \end{cases}$$

Show that $(\hat{X}_n)_{n \geq 0}$ is a simple symmetric random walk.

(e) Let $(X_n)_{n \geq 0}$ be a simple symmetric random walk with $X_0 = 0$, and let $M_n = \max_{0 \leq k \leq n} X_k$. Compute $\mathbb{P}(M_n = a)$ for a positive integer a .

Paper 2, Section II**30K Stochastic Financial Models**

Consider a one-period market model with d risky assets and one risk-free asset. Let S_t denote the vector of prices of the risky assets at time $t \in \{0, 1\}$ and let r be the interest rate.

(a) What does it mean to say a portfolio $\varphi \in \mathbb{R}^d$ is an *arbitrage* for this market?

(b) An investor wishes to maximise their expected utility of time-1 wealth X_1 attainable by investing in the market with their time-0 wealth $X_0 = x$. The investor's utility function U is increasing and concave. Show that, if there exists an optimal solution X_1^* to the investor's expected utility maximisation problem, then the market has no arbitrage. [Assume that $U(X_1)$ is integrable for any attainable time-1 wealth X_1 .]

(c) Now introduce a contingent claim with time-1 bounded payout Y . How does the investor in part (b) calculate an *indifference bid price* $\pi(Y)$ for the claim? Assuming each such claim has a unique indifference price, show that the map $Y \mapsto \pi(Y)$ is concave. [Assume that any relevant utility maximisation problem that you consider has an optimal solution.]

(d) Consider a contingent claim with time-1 bounded payout Y . Let $I \subseteq \mathbb{R}$ be the set of initial no-arbitrage prices for the claim; that is, the set I consists of all p such that the market augmented with the contingent claim with time-0 price p has no arbitrage. Show that $\pi(Y) \leq \sup\{p \in I\}$. [Assume that any relevant utility maximisation problem that you consider has an optimal solution. You may use results from lectures without proof, such as the fundamental theorem of asset pricing or the existence of marginal utility prices, as long as they are clearly stated.]

Paper 3, Section II**29K Stochastic Financial Models**

(a) Let $M = (M_n)_{n \geq 0}$ be a martingale and $\hat{M} = (\hat{M}_n)_{n \geq 0}$ a supermartingale. If $M_0 = \hat{M}_0$, show that $\mathbb{E}(M_T) \geq \mathbb{E}(\hat{M}_T)$ for any bounded stopping time T . [If you use a general result about supermartingales, you must prove it.]

(b) Consider a market with one stock with time- n price S_n and constant interest rate r . Explain why a self-financing investor's wealth process $(X_n)_{n \geq 0}$ satisfies

$$X_n = (1 + r)X_{n-1} + \theta_n[S_n - (1 + r)S_{n-1}],$$

where θ_n is the number of shares of the stock held during the n th period.

(c) Given an initial wealth X_0 , an investor seeks to maximize $\mathbb{E}[U(X_N)]$, where U is a given utility function. Suppose the stock price is such that $S_n = S_{n-1}\xi_n$, where $(\xi_n)_{n \geq 1}$ is a sequence of independent copies of a random variable ξ . Let V be defined inductively by

$$V(n-1, x) = \sup_{t \in \mathbb{R}} \mathbb{E}[V(n, (1+r)x + t(1+r-\xi))],$$

with terminal condition $V(N, x) = U(x)$ for all $x \in \mathbb{R}$.

Show that the process $(V(n, X_n))_{0 \leq n \leq N}$ is a supermartingale for any trading strategy $(\theta_n)_{1 \leq n \leq N}$. Suppose that the trading strategy $(\theta_n^*)_{1 \leq n \leq N}$ with corresponding wealth process $(X_n^*)_{0 \leq n \leq N}$ are such that the process $(V(n, X_n^*))_{0 \leq n \leq N}$ is a martingale. Show that $(\theta_n^*)_{1 \leq n \leq N}$ is optimal.

Paper 4, Section II**29K Stochastic Financial Models**

(a) What does it mean to say that a stochastic process is a *Brownian motion*? Show that, if $(W_t)_{t \geq 0}$ is a continuous Gaussian process such that $\mathbb{E}(W_t) = 0$ and $\mathbb{E}(W_s W_t) = s$ for all $0 \leq s \leq t$, then $(W_t)_{t \geq 0}$ is a Brownian motion.

For the rest of the question, let $(W_t)_{t \geq 0}$ be a Brownian motion.

(b) Let $\widehat{W}_0 = 0$ and $\widehat{W}_t = tW_{1/t}$ for $t > 0$. Show that $(\widehat{W}_t)_{t \geq 0}$ is a Brownian motion. [You may use without proof the Brownian strong law of large numbers: $W_t/t \rightarrow 0$ almost surely as $t \rightarrow \infty$.]

(c) Fix constants $c \in \mathbb{R}$ and $T > 0$. Show that

$$\mathbb{E}[f((W_t + ct)_{0 \leq t \leq T})] = \mathbb{E}[\exp(cW_T - \frac{1}{2}c^2T) f((W_t)_{0 \leq t \leq T})],$$

for any bounded function $f : C[0, T] \rightarrow \mathbb{R}$ of the form

$$f(\omega) = g(\omega(t_1), \dots, \omega(t_n)),$$

for some fixed g and fixed $0 < t_1 < \dots < t_n = T$, where $C[0, T]$ is the space of continuous functions on $[0, T]$. [If you use a general theorem from the lectures, you should prove it.]

(d) Fix constants $x \in \mathbb{R}$ and $T > 0$. Show that

$$\mathbb{E}[f((W_t + x)_{t \geq T})] = \mathbb{E}[\exp((x/T)W_T - \frac{1}{2}(x^2/T)) f((W_t)_{t \geq T})],$$

for any bounded function $f : C[T, \infty) \rightarrow \mathbb{R}$. [In this part you may use the Cameron–Martin theorem without proof.]

Paper 1, Section II**30K Stochastic Financial Models**

Consider a single-period asset price model (\bar{S}_0, \bar{S}_1) in \mathbb{R}^{d+1} where, for $n = 0, 1$,

$$\bar{S}_n = (S_n^0, S_n) = (S_n^0, S_n^1, \dots, S_n^d)$$

with S_0 a non-random vector in \mathbb{R}^d and

$$S_0^0 = 1, \quad S_1^0 = 1 + r, \quad S_1 \sim N(\mu, V).$$

Assume that V is invertible. An investor has initial wealth w_0 which is invested in the market at time 0, to hold θ^0 units of the riskless asset S^0 and θ^i units of risky asset i , for $i = 1, \dots, d$.

- (a) Show that in order to minimize the variance of the wealth $\bar{\theta} \cdot \bar{S}_1$ held at time 1, subject to the constraint

$$\mathbb{E}(\bar{\theta} \cdot \bar{S}_1) = w_1$$

with w_1 given, the investor should choose a portfolio of the form

$$\theta = \lambda \theta_m, \quad \theta_m = V^{-1}(\mu - (1 + r)S_0)$$

where λ is to be determined.

- (b) Show that the same portfolio is optimal for a utility-maximizing investor with CARA utility function

$$U(x) = -\exp\{-\gamma x\}$$

for a unique choice of γ , also to be determined.

Paper 2, Section II**29K Stochastic Financial Models**

Let $(S_n^0, S_n)_{0 \leq n \leq T}$ be a discrete-time asset price model in \mathbb{R}^{d+1} with numéraire.

- (i) What is meant by an *arbitrage* for such a model?
- (ii) What does it mean to say that the model is *complete*?

Consider now the case where $d = 1$ and where

$$S_n^0 = (1 + r)^n, \quad S_n = S_0 \prod_{k=1}^n Z_k$$

for some $r > 0$ and some independent positive random variables Z_1, \dots, Z_T with $\log Z_n \sim N(\mu, \sigma^2)$ for all n .

(iii) Find an equivalent probability measure \mathbb{P}^* such that the discounted asset price $(S_n/S_n^0)_{0 \leq n \leq T}$ is a martingale.

(iv) Does this model have an arbitrage? Justify your answer.

(v) By considering the contingent claim $(S_1)^2$ or otherwise, show that this model is not complete.

Paper 3, Section II**29K Stochastic Financial Models**

(a) Let $(B_t)_{t \geq 0}$ be a real-valued random process.

(i) What does it mean to say that $(B_t)_{t \geq 0}$ is a *Brownian motion*?

(ii) State the reflection principle for Brownian motion.

(b) Suppose that $(B_t)_{t \geq 0}$ is a Brownian motion and set $M_t = \sup_{s \leq t} B_s$ and $Z_t = M_t - B_t$.

(i) Find the joint distribution function of B_t and M_t .

(ii) Show that (M_t, Z_t) has a joint density function on $[0, \infty)^2$ given by

$$\mathbb{P}(M_t \in dy \text{ and } Z_t \in dz) = \frac{2}{\sqrt{2\pi t}} \frac{(y+z)}{t} e^{-(y+z)^2/(2t)} dy dz.$$

(iii) You are given that two of the three processes $(|B_t|)_{t \geq 0}$, $(M_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ have the same distribution. Identify which two, justifying your answer.

Paper 4, Section II**29K Stochastic Financial Models**

(i) What does it mean to say that $(S_t^0, S_t)_{0 \leq t \leq T}$ is a *Black–Scholes model* with interest rate r , drift μ and volatility σ ?

(ii) Write down the Black–Scholes pricing formula for the time-0 value V_0 of a time- T contingent claim C .

(iii) Show that if C is a European call of strike K and maturity T then

$$V_0 \geq S_0 - e^{-rT}K.$$

(iv) For the European call, derive the Black–Scholes pricing formula

$$V_0 = S_0 \Phi(d^+) - e^{-rT}K \Phi(d^-),$$

where Φ is the standard normal distribution function and d^+ and d^- are to be determined.

(v) Fix $t \in (0, T)$ and consider a modified contract which gives the investor the right but not the obligation to buy one unit of the risky asset at price K , either at time t or time T but not both, where the choice of exercise time is to be made by the investor at time t . Determine whether the investor should exercise the contract at time t .

Paper 4, Section II**29K Stochastic Financial Models**

(a) Describe the (*Cox-Ross-Rubinstein*) *binomial model*. What are the necessary and sufficient conditions on the model parameters for it to be arbitrage-free? How is the equivalent martingale measure \mathbb{Q} characterised in this case?

(b) Consider a discounted claim H of the form $H = h(S_0^1, S_1^1, \dots, S_T^1)$ for some function h . Show that the value process of H is of the form

$$V_t(\omega) = v_t(S_0^1, S_1^1(\omega), \dots, S_t^1(\omega)),$$

for $t \in \{0, \dots, T\}$, where the function v_t is given by

$$v_t(x_0, \dots, x_t) = \mathbb{E}_{\mathbb{Q}} \left[h \left(x_0, \dots, x_t, x_t \cdot \frac{S_1^1}{S_0^1}, \dots, x_t \cdot \frac{S_{T-t}^1}{S_0^1} \right) \right].$$

You may use any property of conditional expectations without proof.

(c) Suppose that $H = h(S_T^1)$ only depends on the terminal value S_T^1 of the stock price. Derive an explicit formula for the value of H at time $t \in \{0, \dots, T\}$.

(d) Suppose that H is of the form $H = h(S_T^1, M_T)$, where $M_t := \max_{s \in \{0, \dots, t\}} S_s^1$. Show that the value process of H is of the form

$$V_t(\omega) = v_t(S_t^1(\omega), M_t(\omega)),$$

for $t \in \{0, \dots, T\}$, where the function v_t is given by

$$v_t(x, m) = \mathbb{E}_{\mathbb{Q}} [g(x, m, S_0^1, S_{T-t}^1, M_{T-t})]$$

for a function g to be determined.

Paper 3, Section II**29K Stochastic Financial Models**

In the Black–Scholes model the price $\pi(C)$ at time 0 for a European option of the form $C = f(S_T)$ with maturity $T > 0$ is given by

$$\pi(C) = e^{-rT} \int_{-\infty}^{\infty} f\left(S_0 \exp\left(\sigma\sqrt{T}y + \left(r - \frac{1}{2}\sigma^2\right)T\right)\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

(a) Find the price at time 0 of a European call option with maturity $T > 0$ and strike price $K > 0$ in terms of the standard normal distribution function. Derive the put-call parity to find the price of the corresponding European put option.

(b) The digital call option with maturity $T > 0$ and strike price $K > 0$ has payoff given by

$$C_{\text{digCall}} = \begin{cases} 1 & \text{if } S_T \geq K, \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of the option at any time $t \in [0, T]$? Determine the number of units of the risky asset that are held in the hedging strategy at time t .

(c) The digital put option with maturity $T > 0$ and strike price $K > 0$ has payoff

$$C_{\text{digPut}} = \begin{cases} 1 & \text{if } S_T < K, \\ 0 & \text{otherwise.} \end{cases}$$

Find the put-call parity for digital options and deduce the Black–Scholes price at time 0 for a digital put.

Paper 2, Section II**29K Stochastic Financial Models**

(a) In the context of a multi-period model in discrete time, what does it mean to say that a probability measure is an *equivalent martingale measure*?

(b) State the *fundamental theorem of asset pricing*.

(c) Consider a single-period model with one risky asset S^1 having initial price $S_0^1 = 1$. At time 1 its value S_1^1 is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$S_1^1 = \exp(\sigma Z + m), \quad m \in \mathbb{R}, \sigma > 0,$$

where $Z \sim \mathcal{N}(0, 1)$. Assume that there is a riskless numéraire S^0 with $S_0^0 = S_1^0 = 1$. Show that there is no arbitrage in this model.

[Hint: You may find it useful to consider a density of the form $\exp(\tilde{\sigma}Z + \tilde{m})$ and find suitable \tilde{m} and $\tilde{\sigma}$. You may use without proof that if X is a normal random variable then $\mathbb{E}(e^X) = \exp(\mathbb{E}(X) + \frac{1}{2}\text{Var}(X))$.]

(d) Now consider a multi-period model with one risky asset S^1 having a non-random initial price $S_0^1 = 1$ and a price process $(S_t^1)_{t \in \{0, \dots, T\}}$ of the form

$$S_t^1 = \prod_{i=1}^t \exp(\sigma_i Z_i + m_i), \quad m_i \in \mathbb{R}, \sigma_i > 0,$$

where Z_i are i.i.d. $\mathcal{N}(0, 1)$ -distributed random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that there is a constant riskless numéraire S^0 with $S_t^0 = 1$ for all $t \in \{0, \dots, T\}$. Show that there exists no arbitrage in this model.

Paper 1, Section II**30K Stochastic Financial Models**

(a) What does it mean to say that $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a *martingale*?

(b) Let $(X_n)_{n \geq 0}$ be a Markov chain defined by $X_0 = 0$ and

$$\begin{aligned}\mathbb{P}[X_n = 1 \mid X_{n-1} = 0] &= \mathbb{P}[X_n = -1 \mid X_{n-1} = 0] = \frac{1}{2n}, \\ \mathbb{P}[X_n = 0 \mid X_{n-1} = 0] &= 1 - \frac{1}{n}\end{aligned}$$

and

$$\mathbb{P}[X_n = nX_{n-1} \mid X_{n-1} \neq 0] = \frac{1}{n}, \quad \mathbb{P}[X_n = 0 \mid X_{n-1} \neq 0] = 1 - \frac{1}{n}$$

for $n \geq 1$. Show that $(X_n)_{n \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ where \mathcal{F}_0 is trivial and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ for $n \geq 1$.

(c) Let $M = (M_n)_{n \geq 0}$ be adapted with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$ with $\mathbb{E}[|M_n|] < \infty$ for all n . Show that the following are equivalent:

- (i) M is a martingale.
- (ii) For every stopping time τ , the stopped process M^τ defined by $M_n^\tau := M_{n \wedge \tau}$, $n \geq 0$, is a martingale.
- (iii) $\mathbb{E}[M_{n \wedge \tau}] = \mathbb{E}[M_0]$ for all $n \geq 0$ and every stopping time τ .

[Hint: To show that (iii) implies (i) you might find it useful to consider the stopping time

$$T(\omega) := \begin{cases} n & \text{if } \omega \in A, \\ n+1 & \text{if } \omega \notin A, \end{cases}$$

for any $A \in \mathcal{F}_n$.]

Paper 4, Section II**29K Stochastic Financial Models**

Consider a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$, which is assumed to be concave, strictly increasing and twice differentiable. Further, U satisfies

$$|U'(x)| \leq c|x|^\alpha, \quad \forall x \in \mathbb{R},$$

for some positive constants c and α . Let X be an $\mathcal{N}(\mu, \sigma^2)$ -distributed random variable and set $f(\mu, \sigma) := \mathbb{E}[U(X)]$.

(a) Show that

$$\mathbb{E}[U'(X)(X - \mu)] = \sigma^2 \mathbb{E}[U''(X)].$$

(b) Show that $\frac{\partial f}{\partial \mu} > 0$ and $\frac{\partial f}{\partial \sigma} \leq 0$. Discuss this result in the context of mean-variance analysis.

(c) Show that f is concave in μ and σ , i.e. check that the matrix of second derivatives is negative semi-definite. [You may use without proof the fact that if a 2×2 matrix has non-positive diagonal entries and a non-negative determinant, then it is negative semi-definite.]

Paper 3, Section II**29K Stochastic Financial Models**

Consider a multi-period model with asset prices $\bar{S}_t = (S_t^0, \dots, S_t^d)$, $t \in \{0, \dots, T\}$, modelled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a filtration $(\mathcal{F}_t)_{t \in \{0, \dots, T\}}$. Assume that \mathcal{F}_0 is \mathbb{P} -trivial, i.e. $\mathbb{P}[A] \in \{0, 1\}$ for all $A \in \mathcal{F}_0$, and assume that S^0 is a \mathbb{P} -a.s. strictly positive numéraire, i.e. $S_t^0 > 0$ \mathbb{P} -a.s. for all $t \in \{0, \dots, T\}$. Further, let $X_t = (X_t^1, \dots, X_t^d)$ denote the discounted price process defined by $X_t^i := S_t^i / S_t^0$, $t \in \{0, \dots, T\}$, $i \in \{1, \dots, d\}$.

- (a) What does it mean to say that a self-financing strategy $\bar{\theta}$ is an *arbitrage*?
- (b) State the fundamental theorem of asset pricing.
- (c) Let \mathbb{Q} be a probability measure on (Ω, \mathcal{F}) which is equivalent to \mathbb{P} and for which $\mathbb{E}_{\mathbb{Q}}[|X_t|] < \infty$ for all t . Show that the following are equivalent:

- (i) \mathbb{Q} is a martingale measure.
- (ii) If $\bar{\theta} = (\theta^0, \theta)$ is self-financing and θ is bounded, i.e. $\max_{t=1, \dots, T} |\theta_t| \leq c < \infty$ for a suitable $c > 0$, then the value process V of $\bar{\theta}$ is a \mathbb{Q} -martingale.
- (iii) If $\bar{\theta} = (\theta^0, \theta)$ is self-financing and θ is bounded, then the value process V of $\bar{\theta}$ satisfies

$$\mathbb{E}_{\mathbb{Q}}[V_T] = V_0.$$

[Hint: To show that (iii) implies (i) you might find it useful to consider self-financing strategies $\bar{\theta} = (\theta^0, \theta)$ with θ of the form

$$\theta_s := \begin{cases} \mathbf{1}_A & \text{if } s = t, \\ 0 & \text{otherwise,} \end{cases}$$

for any $A \in \mathcal{F}_{t-1}$ and any $t \in \{1, \dots, T\}$.]

- (d) Prove that if there exists a martingale measure \mathbb{Q} satisfying the conditions in (c) then there is no arbitrage.

Paper 2, Section II**29K Stochastic Financial Models**

Consider the Black–Scholes model, i.e. a market model with one risky asset with price S_t at time t given by

$$S_t = S_0 \exp(\sigma B_t + \mu t),$$

where $(B_t)_{t \geq 0}$ denotes a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mu > 0$ the constant growth rate, $\sigma > 0$ the constant volatility and $S_0 > 0$ the initial price of the asset. Assume that the riskless rate of interest is $r \geq 0$.

(a) Consider a European option $C = f(S_T)$ with expiry $T > 0$ for any bounded, continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. Use the Cameron–Martin theorem to characterize the equivalent martingale measure and deduce the following formula for the price π_C of C at time 0:

$$\pi_C = e^{-rT} \int_{-\infty}^{\infty} f\left(S_0 \exp\left(\sigma\sqrt{T}y + \left(r - \frac{1}{2}\sigma^2\right)T\right)\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

(b) Find the price at time 0 of a European option with maturity $T > 0$ and payoff $C = (S_T)^\gamma$ for some $\gamma > 1$. What is the value of the option at any time $t \in [0, T]$? Determine a hedging strategy (you only need to specify how many units of the risky asset are held at any time t).

Paper 1, Section II**30K Stochastic Financial Models**

(a) What does it mean to say that $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a *martingale*?

(b) Let Y_1, Y_2, \dots be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $Y_i > 0$ \mathbb{P} -a.s. and $\mathbb{E}[Y_i] = 1$, $i \geq 1$. Further, let

$$M_0 = 1 \quad \text{and} \quad M_n = \prod_{i=1}^n Y_i, \quad n \geq 1.$$

Show that $(M_n)_{n \geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

(c) Let $X = (X_n)_{n \geq 0}$ be an adapted process with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$ such that $\mathbb{E}[|X_n|] < \infty$ for every n . Show that X admits a unique decomposition

$$X_n = M_n + A_n, \quad n \geq 0,$$

where $M = (M_n)_{n \geq 0}$ is a martingale and $A = (A_n)_{n \geq 0}$ is a previsible process with $A_0 = 0$, which can recursively be constructed from X as follows,

$$A_0 := 0, \quad A_{n+1} - A_n := \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n].$$

(d) Let $(X_n)_{n \geq 0}$ be a super-martingale. Show that the following are equivalent:

- (i) $(X_n)_{n \geq 0}$ is a martingale.
- (ii) $\mathbb{E}[X_n] = \mathbb{E}[X_0]$ for all $n \in \mathbb{N}$.

Paper 2, Section II**27J Stochastic Financial Models**

- (a) What is a *Brownian motion*?
- (b) Let $(B_t, t \geq 0)$ be a Brownian motion. Show that the process $\tilde{B}_t := \frac{1}{c}B_{c^2t}$, $c \in \mathbb{R} \setminus \{0\}$, is also a Brownian motion.
- (c) Let $Z := \sup_{t \geq 0} B_t$. Show that $cZ \stackrel{(d)}{=} Z$ for all $c > 0$ (i.e. cZ and Z have the same laws). Conclude that $Z \in \{0, +\infty\}$ a.s.
- (d) Show that $\mathbb{P}[Z = +\infty] = 1$.

Paper 3, Section II**27J Stochastic Financial Models**

- (a) State the fundamental theorem of asset pricing for a multi-period model.

Consider a market model in which there is no arbitrage, the prices for all European put and call options are already known and there is a riskless asset $S^0 = (S_t^0)_{t \in \{0, \dots, T\}}$ with $S_t^0 = (1+r)^t$ for some $r \geq 0$. The holder of a so-called ‘chooser option’ $C(K, t_0, T)$ has the right to choose at a preassigned time $t_0 \in \{0, 1, \dots, T\}$ between a European call and a European put option on the same asset S^1 , both with the same strike price K and the same maturity T . [We assume that at time t_0 the holder will take the option having the higher price at that time.]

- (b) Show that the payoff function of the chooser option is given by

$$C(K, t_0, T) = \begin{cases} (S_T^1 - K)^+ & \text{if } S_{t_0}^1 > K(1+r)^{t_0-T}, \\ (K - S_T^1)^+ & \text{otherwise.} \end{cases}$$

- (c) Show that the price $\pi(C(K, t_0, T))$ of the chooser option $C(K, t_0, T)$ is given by

$$\pi(C(K, t_0, T)) = \pi(EC(K, T)) + \pi(EP(K(1+r)^{t_0-T}, t_0)),$$

where $\pi(EC(K, T))$ and $\pi(EP(K, T))$ denote the price of a European call and put option, respectively, with strike K and maturity T .

Paper 4, Section II**28J Stochastic Financial Models**

- (a) Describe the (Cox–Ross–Rubinstein) *binomial model*. When is the model arbitrage-free? How is the equivalent martingale measure characterised in this case?
- (b) What is the price and the hedging strategy for any given contingent claim C in the binomial model?
- (c) For any fixed $0 < t < T$ and $K > 0$, the payoff function of a forward-start-option is given by

$$\left(\frac{S_T^1}{S_t^1} - K \right)^+.$$

Find a formula for the price of the forward-start-option in the binomial model.

Paper 1, Section II**29J Stochastic Financial Models**

- (a) What does it mean to say that $(X_n, \mathcal{F}_n)_{n \geq 0}$ is a *martingale*?
- (b) Let $\Delta_0, \Delta_1, \dots$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[|\Delta_i|] < \infty$ and $\mathbb{E}[\Delta_i] = 0$, $i \geq 0$. Further, let

$$X_0 = \Delta_0 \quad \text{and} \quad X_{n+1} = X_n + \Delta_{n+1} f_n(X_0, \dots, X_n), \quad n \geq 0,$$

where

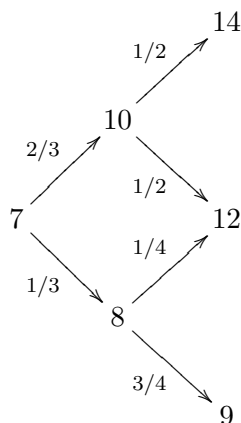
$$f_n(x_0, \dots, x_n) = \frac{1}{n+1} \sum_{i=0}^n x_i.$$

Show that $(X_n)_{n \geq 0}$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$.

- (c) State and prove the optional stopping theorem for a bounded stopping time τ .

Paper 3, Section II**26K Stochastic Financial Models**

Consider the following two-period market model. There is a risk-free asset which pays interest at rate $r = 1/4$. There is also a risky stock with prices $(S_t)_{t \in \{0,1,2\}}$ given by



The above diagram should be read as

$$\mathbb{P}(S_1 = 10 \mid S_0 = 7) = 2/3, \quad \mathbb{P}(S_2 = 14 \mid S_1 = 10) = 1/2$$

and so forth.

(a) Find the risk-neutral probabilities.

(b) Consider a European put option with strike $K = 10$ expiring at time $T = 2$. What is the initial no-arbitrage price of the option? How many shares should be held in each period to replicate the payout?

(c) Now consider an American put option with the same strike and expiration date. Find the optimal exercise policy, assuming immediate exercise is not allowed. Would your answer change if you were allowed to exercise the option at time 0?

Paper 4, Section II**27K Stochastic Financial Models**

Let U be concave and strictly increasing, and let \mathcal{A} be a vector space of random variables. For every random variable Z let

$$F(Z) = \sup_{X \in \mathcal{A}} \mathbb{E}[U(X + Z)]$$

and suppose there exists a random variable $X_Z \in \mathcal{A}$ such that

$$F(Z) = \mathbb{E}[U(X_Z + Z)].$$

For a random variable Y , let $\pi(Y)$ be such that $F(Y - \pi(Y)) = F(0)$.

(a) Show that for every constant a we have $\pi(Y + a) = \pi(Y) + a$, and that if $\mathbb{P}(Y_1 \leq Y_2) = 1$, then $\pi(Y_1) \leq \pi(Y_2)$. Hence show that if $\mathbb{P}(a \leq Y \leq b) = 1$ for constants $a \leq b$, then $a \leq \pi(Y) \leq b$.

(b) Show that $Y \mapsto \pi(Y)$ is concave, and hence show $t \mapsto \pi(tY)/t$ is decreasing for $t > 0$.

(c) Assuming U is continuously differentiable, show that $\pi(tY)/t$ converges as $t \rightarrow 0$, and that there exists a random variable X_0 such that

$$\lim_{t \rightarrow 0} \frac{\pi(tY)}{t} = \frac{\mathbb{E}[U'(X_0)Y]}{\mathbb{E}[U'(X_0)]}.$$

Paper 2, Section II**27K Stochastic Financial Models**

In the context of the Black–Scholes model, let S_0 be the initial price of the stock, and let σ be its volatility. Assume that the risk-free interest rate is zero and the stock pays no dividends. Let $\text{EC}(S_0, K, \sigma, T)$ denote the initial price of a European call option with strike K and maturity date T .

- (a) Show that the Black–Scholes formula can be written in the form

$$\text{EC}(S_0, K, \sigma, T) = S_0 \Phi(d_1) - K \Phi(d_2),$$

where d_1 and d_2 depend on S_0 , K , σ and T , and Φ is the standard normal distribution function.

- (b) Let $\text{EP}(S_0, K, \sigma, T)$ be the initial price of a put option with strike K and maturity T . Show that

$$\text{EP}(S_0, K, \sigma, T) = \text{EC}(S_0, K, \sigma, T) + K - S_0.$$

- (c) Show that

$$\text{EP}(S_0, K, \sigma, T) = \text{EC}(K, S_0, \sigma, T).$$

- (d) Consider a European contingent claim with maturity T and payout

$$S_T I_{\{S_T \leq K\}} - K I_{\{S_T > K\}}.$$

Assuming $K > S_0$, show that its initial price can be written as $\text{EC}(S_0, K, \hat{\sigma}, T)$ for a volatility parameter $\hat{\sigma}$ which you should express in terms of S_0 , K , σ and T .

Paper 1, Section II**28K Stochastic Financial Models**

- (a) What is a Brownian motion?

(b) State the Brownian reflection principle. State the Cameron–Martin theorem for Brownian motion with constant drift.

- (c) Let $(W_t)_{t \geq 0}$ be a Brownian motion. Show that

$$\mathbb{P} \left(\max_{0 \leq s \leq t} (W_s + as) \leq b \right) = \Phi \left(\frac{b - at}{\sqrt{t}} \right) - e^{2ab} \Phi \left(\frac{-b - at}{\sqrt{t}} \right),$$

where Φ is the standard normal distribution function.

- (d) Find

$$\mathbb{P} \left(\max_{u \geq t} (W_u + au) \leq b \right).$$

Paper 4, Section II**26K Stochastic Financial Models**

(i) An investor in a single-period market with time-0 wealth w_0 may generate any time-1 wealth w_1 of the form $w_1 = w_0 + X$, where X is any element of a vector space V of random variables. The investor's objective is to maximize $E[U(w_1)]$, where U is strictly increasing, concave and C^2 . Define the *utility indifference price* $\pi(Y)$ of a random variable Y .

Prove that the map $Y \mapsto \pi(Y)$ is concave. [You may assume that any supremum is attained.]

(ii) Agent j has utility $U_j(x) = -\exp(-\gamma_j x)$, $j = 1, \dots, J$. The agents may buy for time-0 price p a risky asset which will be worth X at time 1, where X is random and has density

$$f(x) = \frac{1}{2}\alpha e^{-\alpha|x|}, \quad -\infty < x < \infty.$$

Assuming zero interest, prove that agent j will optimally choose to buy

$$\theta_j = -\frac{\sqrt{1 + p^2 \alpha^2} - 1}{\gamma_j p}$$

units of the risky asset at time 0.

If the asset is in unit net supply, if $\Gamma^{-1} \equiv \sum_j \gamma_j^{-1}$, and if $\alpha > \Gamma$, prove that the market for the risky asset will clear at price

$$p = -\frac{2\Gamma}{\alpha^2 - \Gamma^2}.$$

What happens if $\alpha \leq \Gamma$?

Paper 3, Section II**26K Stochastic Financial Models**

A single-period market consists of n assets whose prices at time t are denoted by $S_t = (S_t^1, \dots, S_t^n)^T$, $t = 0, 1$, and a riskless bank account bearing interest rate r . The value of S_0 is given, and $S_1 \sim N(\mu, V)$. An investor with utility $U(x) = -\exp(-\gamma x)$ wishes to choose a portfolio of the available assets so as to maximize the expected utility of her wealth at time 1. Find her optimal investment.

What is the *market portfolio* for this problem? What is the *beta* of asset i ? Derive the Capital Asset Pricing Model, that

$$\text{Excess return of asset } i = \text{Excess return of market portfolio} \times \beta_i.$$

The Sharpe ratio of a portfolio θ is defined to be the excess return of the portfolio θ divided by the standard deviation of the portfolio θ . If ρ_i is the correlation of the return on asset i with the return on the market portfolio, prove that

$$\text{Sharpe ratio of asset } i = \text{Sharpe ratio of market portfolio} \times \rho_i.$$

Paper 1, Section II**26K Stochastic Financial Models**

(i) What does it mean to say that $(X_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale?

(ii) If Y is an integrable random variable and $Y_n = E[Y \mid \mathcal{F}_n]$, prove that (Y_n, \mathcal{F}_n) is a martingale. [Standard facts about conditional expectation may be used without proof provided they are clearly stated.] When is it the case that the limit $\lim_{n \rightarrow \infty} Y_n$ exists almost surely?

(iii) An urn contains initially one red ball and one blue ball. A ball is drawn at random and then returned to the urn with a new ball of the *other* colour. This process is repeated, adding one ball at each stage to the urn. If the number of red balls after n draws and replacements is X_n , and the number of blue balls is Y_n , show that $M_n = h(X_n, Y_n)$ is a martingale, where

$$h(x, y) = (x - y)(x + y - 1).$$

Does this martingale converge almost surely?

Paper 2, Section II**27K Stochastic Financial Models**

(i) What is Brownian motion?

(ii) Suppose that $(B_t)_{t \geq 0}$ is Brownian motion, and the price S_t at time t of a risky asset is given by

$$S_t = S_0 \exp\left\{ \sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right)t \right\}$$

where $\mu > 0$ is the constant growth rate, and $\sigma > 0$ is the constant volatility of the asset. Assuming that the riskless rate of interest is $r > 0$, derive an expression for the price at time 0 of a European call option with strike K and expiry T , explaining briefly the basis for your calculation.

(iii) With the same notation, derive the time-0 price of a European option with expiry T which at expiry pays

$$\{(S_T - K)^+\}^2 / S_T.$$

Paper 4, Section II**29K Stochastic Financial Models**

Write down the Black–Scholes partial differential equation (PDE), and explain briefly its relevance to option pricing.

Show how a change of variables reduces the Black–Scholes PDE to the heat equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0 \text{ for all } (t, x) \in [0, T) \times \mathbb{R},$$

$$f(T, x) = \varphi(x) \text{ for all } x \in \mathbb{R},$$

where φ is a given boundary function.

Consider the following numerical scheme for solving the heat equation on the equally spaced grid $(t_n, x_k) \in [0, T] \times \mathbb{R}$ where $t_n = n\Delta t$ and $x_k = k\Delta x$, $n = 0, 1, \dots, N$ and $k \in \mathbb{Z}$, and $\Delta t = T/N$. We approximate $f(t_n, x_k)$ by f_k^n where

$$0 = \frac{f_k^{n+1} - f_k^n}{\Delta t} + \theta L f_k^{n+1} + (1 - \theta) L f_k^n, \quad f_k^N = \varphi(x_k), \quad (*)$$

and $\theta \in [0, 1]$ is a constant and the operator L is the matrix with non-zero entries $L_{kk} = -\frac{1}{(\Delta x)^2}$ and $L_{k,k+1} = L_{k,k-1} = \frac{1}{2(\Delta x)^2}$. By considering what happens when $\varphi(x) = \exp(i\omega x)$, show that the finite-difference scheme $(*)$ is stable if and only if

$$1 \geq \lambda(2\theta - 1),$$

where $\lambda \equiv \Delta t/(\Delta x)^2$. For what values of θ is the scheme $(*)$ unconditionally stable?

Paper 3, Section II**29K Stochastic Financial Models**

Derive the Black–Scholes formula $C(S_0, K, r, T, \sigma)$ for the time-0 price of a European call option with expiry T and strike K written on an asset with volatility σ and time-0 price S_0 , and where r is the riskless rate of interest. Explain what is meant by the *delta hedge* for this option, and determine it explicitly.

In terms of the Black–Scholes call option price formula C , find the time-0 price of a forward-starting option, which pays $(S_T - \lambda S_t)^+$ at time T , where $0 < t < T$ and $\lambda > 0$ are given. Find the price of an option which pays $\max\{S_T, \lambda S_t\}$ at time T . How would this option be hedged?

Paper 1, Section II**29K Stochastic Financial Models**

Suppose that $\bar{S}_t \equiv (S_t^0, \dots, S_t^d)^T$ denotes the vector of prices of $d+1$ assets at times $t = 0, 1, \dots$, and that $\bar{\theta}_t \equiv (\theta_t^0, \dots, \theta_t^d)^T$ denotes the vector of the numbers of the $d+1$ different assets held by an investor from time $t-1$ to time t . Assuming that asset 0 is a bank account paying zero interest, that is, $S_t^0 = 1$ for all $t \geq 0$, explain what is meant by the statement that the portfolio process $(\bar{\theta}_t)_{t \geq 0}$ is *self-financing*. If the portfolio process is self-financing, prove that for any $t > 0$

$$\bar{\theta}_t \cdot \bar{S}_t - \bar{\theta}_0 \cdot \bar{S}_0 = \sum_{j=1}^t \theta_j \cdot \Delta S_j,$$

where $S_j \equiv (S_j^1, \dots, S_j^d)^T$, $\Delta S_j = S_j - S_{j-1}$, and $\theta_j \equiv (\theta_j^1, \dots, \theta_j^d)^T$.

Suppose now that the ΔS_t are independent with common $N(0, V)$ distribution. Let

$$F(z) = \inf E \left[\sum_{t \geq 1} (1 - \beta) \beta^t \left\{ (\bar{\theta}_t \cdot \bar{S}_t - \bar{\theta}_0 \cdot \bar{S}_0)^2 + \sum_{j=1}^t |\Delta \theta_j|^2 \right\} \mid \theta_0 = z \right],$$

where $\beta \in (0, 1)$ and the infimum is taken over all self-financing portfolio processes $(\bar{\theta}_t)_{t \geq 0}$ with $\theta_0^0 = 0$. Explain why F should satisfy the equation

$$F(z) = \beta \inf_y [y \cdot Vy + |y - z|^2 + F(y)]. \quad (*)$$

If Q is a positive-definite symmetric matrix satisfying the equation

$$Q = \beta(V + I + Q)^{-1}(V + Q),$$

show that $(*)$ has a solution of the form $F(z) = z \cdot Qz$.

Paper 2, Section II**30K Stochastic Financial Models**

An agent has expected-utility preferences over his possible wealth at time 1; that is, the wealth Z is preferred to wealth Z' if and only if $E U(Z) \geq E U(Z')$, where the function $U : \mathbb{R} \rightarrow \mathbb{R}$ is strictly concave and twice continuously differentiable. The agent can trade in a market, with the time-1 value of his portfolio lying in an affine space \mathcal{A} of random variables. Assuming cash can be held between time 0 and time 1, define the agent's time-0 *utility indifference price* $\pi(Y)$ for a contingent claim with time-1 value Y . Assuming any regularity conditions you may require, prove that the map $Y \mapsto \pi(Y)$ is concave.

Comment briefly on the limit $\lim_{\lambda \rightarrow 0} \pi(\lambda Y)/\lambda$.

Consider a market with two claims with time-1 values X and Y . Their joint distribution is

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} V_{XX} & V_{XY} \\ V_{YX} & V_{YY} \end{pmatrix}\right).$$

At time 0, arbitrary quantities of the claim X can be bought at price p , but Y is not marketed. Derive an explicit expression for $\pi(Y)$ in the case where

$$U(x) = -\exp(-\gamma x),$$

where $\gamma > 0$ is a given constant.

Paper 4, Section II**29J Stochastic Financial Models**

Let $S_t := (S_t^1, S_t^2, \dots, S_t^n)^T$ denote the time- t prices of n risky assets in which an agent may invest, $t = 0, 1$. He may also invest his money in a bank account, which will return interest at rate $r > 0$. At time 0, he knows S_0 and r , and he knows that $S_1 \sim N(\mu, V)$. If he chooses at time 0 to invest cash value θ_i in risky asset i , express his wealth w_1 at time 1 in terms of his initial wealth $w_0 > 0$, the choices $\theta := (\theta_1, \dots, \theta_n)^T$, the value of S_1 , and r .

Suppose that his goal is to minimize the variance of w_1 subject to the requirement that the mean $E(w_1)$ should be at least m , where $m \geq (1+r)w_0$ is given. What portfolio θ should he choose to achieve this?

Suppose instead that his goal is to minimize $E(w_1^2)$ subject to the same constraint. Show that his optimal portfolio is unchanged.

Paper 3, Section II**29J Stochastic Financial Models**

Suppose that $(\varepsilon_t)_{t=0,1,\dots,T}$ is a sequence of independent and identically distributed random variables such that $E \exp(z\varepsilon_1) < \infty$ for all $z \in \mathbb{R}$. Each day, an agent receives an income, the income on day t being ε_t . After receiving this income, his wealth is w_t . From this wealth, he chooses to consume c_t , and invests the remainder $w_t - c_t$ in a bank account which pays a daily interest rate of $r > 0$. Write down the equation for the evolution of w_t .

Suppose we are given constants $\beta \in (0, 1)$, A_T , $\gamma > 0$, and define the functions

$$U(x) = -\exp(-\gamma x), \quad U_T(x) = -A_T \exp(-\nu x),$$

where $\nu := \gamma r / (1 + r)$. The agent's objective is to attain

$$V_0(w) := \sup E \left\{ \sum_{t=0}^{T-1} \beta^t U(c_t) + \beta^T U_T(w_T) \mid w_0 = w \right\},$$

where the supremum is taken over all adapted sequences (c_t) . If the value function is defined for $0 \leq n < T$ by

$$V_n(w) = \sup E \left\{ \sum_{t=n}^{T-1} \beta^{t-n} U(c_t) + \beta^{T-n} U_T(w_T) \mid w_n = w \right\},$$

with $V_T = U_T$, explain briefly why you expect the V_n to satisfy

$$V_n(w) = \sup_c \left[U(c) + \beta E \{ V_{n+1}((1+r)(w-c) + \varepsilon_{n+1}) \} \right]. \quad (*)$$

Show that the solution to $(*)$ has the form

$$V_n(w) = -A_n \exp(-\nu w),$$

for constants A_n to be identified. What is the form of the consumption choices that achieve the supremum in $(*)$?

Paper 1, Section II**29J Stochastic Financial Models**

(i) Suppose that the price S_t of an asset at time t is given by

$$S_t = S_0 \exp\left\{\sigma B_t + \left(r - \frac{1}{2}\sigma^2\right)t\right\},$$

where B is a Brownian motion, S_0 and σ are positive constants, and r is the riskless rate of interest, assumed constant. In this model, explain briefly why the time-0 price of a derivative which delivers a bounded random variable Y at time T should be given by $E(e^{-rT}Y)$. What feature of this model ensures that the price is unique?

Derive an expression $C(S_0, K, T, r, \sigma)$ for the time-0 price of a *European call option* with *strike* K and *expiry* T . Explain the italicized terms.

(ii) Suppose now that the price X_t of an asset at time t is given by

$$X_t = \sum_{j=1}^n w_j \exp\left\{\sigma_j B_t + \left(r - \frac{1}{2}\sigma_j^2\right)t\right\},$$

where the w_j and σ_j are positive constants, and the other notation is as in part (i) above. Show that the time-0 price of a European call option with strike K and expiry T written on this asset can be expressed as

$$\sum_{j=1}^n C(w_j, k_j, T, r, \sigma_j),$$

where the k_j are constants. Explain how the k_j are characterized.

Paper 2, Section II**30J Stochastic Financial Models**

What does it mean to say that $(Y_n, \mathcal{F}_n)_{n \geq 0}$ is a *supermartingale*?

State and prove Doob's Upcrossing Inequality for a supermartingale.

Let $(M_n, \mathcal{F}_n)_{n \leq 0}$ be a martingale indexed by negative time, that is, for each $n \leq 0$, $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$, $M_n \in L^1(\mathcal{F}_n)$ and $E[M_n | \mathcal{F}_{n-1}] = M_{n-1}$. Using Doob's Upcrossing Inequality, prove that the limit $\lim_{n \rightarrow -\infty} M_n$ exists almost surely.

Paper 1, Section II**29J Stochastic Financial Models**

Consider a multi-period binomial model with a risky asset (S_0, \dots, S_T) and a riskless asset (B_0, \dots, B_T) . In each period, the value of the risky asset S is multiplied by u if the period was good, and by d otherwise. The riskless asset is worth $B_t = (1 + r)^t$ at time $0 \leq t \leq T$, where $r \geq 0$.

(i) Assuming that $T = 1$ and that

$$d < 1 + r < u, \quad (1)$$

show how any contingent claim to be paid at time 1 can be priced and exactly replicated. Briefly explain the significance of the condition (1), and indicate how the analysis of the single-period model extends to many periods.

(ii) Now suppose that $T = 2$. We assume that $u = 2$, $d = 1/3$, $r = 1/2$, and that the risky asset is worth $S_0 = 27$ at time zero. Find the time-0 value of an American put option with strike price $K = 28$ and expiry at time $T = 2$, and find the optimal exercise policy. (Assume that the option cannot be exercised immediately at time zero.)

Paper 4, Section II**29J Stochastic Financial Models**

In a one-period market, there are n risky assets whose returns at time 1 are given by a column vector $R = (R^1, \dots, R^n)'$. The return vector R has a multivariate Gaussian distribution with expectation μ and non-singular covariance matrix V . In addition, there is a bank account giving interest $r > 0$, so that one unit of cash invested at time 0 in the bank account will be worth $R_f = 1 + r$ units of cash at time 1.

An agent with the initial wealth w invests $x = (x_1, \dots, x_n)'$ in risky assets and keeps the remainder $x_0 = w - x \cdot \mathbf{1}$ in the bank account. The return on the agent's portfolio is

$$Z := x \cdot R + (w - x \cdot \mathbf{1})R_f.$$

The agent's utility function is $u(Z) = -\exp(-\gamma Z)$, where $\gamma > 0$ is a parameter. His objective is to maximize $\mathbb{E}(u(Z))$.

(i) Find the agent's optimal portfolio and its expected return.

[Hint. Relate $\mathbb{E}(u(Z))$ to $\mathbb{E}(Z)$ and $\text{Var}(Z)$.]

(ii) Under which conditions does the optimal portfolio that you found in (i) require borrowing from the bank account?

(iii) Find the optimal portfolio if it is required that all of the agent's wealth be invested in risky assets.

Paper 3, Section II**29J Stochastic Financial Models**

(i) Let $\mathcal{F} = \{\mathcal{F}_n\}_{n=0}^{\infty}$ be a filtration. Give the definition of a martingale and a stopping time with respect to the filtration \mathcal{F} .

(ii) State Doob's optional stopping theorem. Give an example of a martingale M and a stopping time T such that $\mathbb{E}(M_T) \neq \mathbb{E}(M_0)$.

(iii) Let S_n be a standard random walk on \mathbb{Z} , that is, $S_0 = 0$, $S_n = X_1 + \dots + X_n$, where X_i are i.i.d. and $X_i = 1$ or -1 with probability $1/2$.

Let $T_a = \inf \{n \geq 0 : S_n = a\}$ where a is a positive integer. Show that for all $\theta > 0$,

$$\mathbb{E} \left(e^{-\theta T_a} \right) = \left(e^{\theta} - \sqrt{e^{2\theta} - 1} \right)^a.$$

Carefully justify all steps in your derivation.

[Hint. For all $\lambda > 0$ find θ such that $M_n = \exp(-\theta n + \lambda S_n)$ is a martingale. You may assume that T_a is almost surely finite.]

Let $T = T_a \wedge T_{-a} = \inf \{n \geq 0 : |S_n| = a\}$. By introducing a suitable martingale, compute $\mathbb{E}(e^{-\theta T})$.

Paper 2, Section II**30J Stochastic Financial Models**

(i) Give the definition of Brownian motion.

(ii) The price S_t of an asset evolving in continuous time is represented as

$$S_t = S_0 \exp(\sigma W_t + \mu t) ,$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion and σ and μ are constants. If riskless investment in a bank account returns a continuously compounded rate of interest r , derive the Black-Scholes formula for the time-0 price of a European call option on asset S with strike price K and expiry T . [Standard results from the course may be used without proof but must be stated clearly.]

(iii) In the same financial market, a certain contingent claim C pays $(S_T)^n$ at time T , where $n \geq 1$. Find the closed-form expression for the time-0 value of this contingent claim.

Show that for every $s > 0$ and $n \geq 1$,

$$s^n = n(n-1) \int_0^s k^{n-2}(s-k)dk.$$

Using this identity, how would you replicate (at least approximately) the contingent claim C with a portfolio consisting only of European calls?

Paper 1, Section II**29J Stochastic Financial Models**

In a one-period market, there are n assets whose prices at time t are given by $S_t = (S_t^1, \dots, S_t^n)^T$, $t = 0, 1$. The prices S_1 of the assets at time 1 have a $N(\mu, V)$ distribution, with non-singular covariance V , and the prices S_0 at time 0 are known constants. In addition, there is a bank account giving interest r , so that one unit of cash invested at time 0 will be worth $(1 + r)$ units of cash at time 1.

An agent with initial wealth w_0 chooses a portfolio $\theta = (\theta^1, \dots, \theta^n)$ of the assets to hold, leaving him with $x = w_0 - \theta \cdot S_0$ in the bank account. His objective is to maximize his expected utility

$$E(-\exp[-\gamma\{x(1+r) + \theta \cdot S_1\}]) \quad (\gamma > 0).$$

Find his optimal portfolio in each of the following three situations:

- (i) θ is unrestricted;
- (ii) no investment in the bank account is allowed: $x = 0$;
- (iii) the initial holdings x of cash must be non-negative.

For the third problem, show that the optimal initial holdings of cash will be zero if and only if

$$\frac{S_0 \cdot (\gamma V)^{-1} \mu - w_0}{S_0 \cdot (\gamma V)^{-1} S_0} \geq 1 + r.$$

Paper 2, Section II**30J Stochastic Financial Models**

Consider a symmetric simple random walk $(Z_n)_{n \in \mathbb{Z}^+}$ taking values in statespace $I = h\mathbb{Z}^2 \equiv \{(ih, jh) : i, j \in \mathbb{Z}\}$, where $h \equiv N^{-1}$ (N an integer). Writing $Z_n \equiv (X_n, Y_n)$, the transition probabilities are given by

$$P(\Delta Z_n = (h, 0)) = P(\Delta Z_n = (0, h)) = P(\Delta Z_n = (-h, 0)) = P(\Delta Z_n = (0, -h)) = \frac{1}{4},$$

where $\Delta Z_n \equiv Z_n - Z_{n-1}$.

What does it mean to say that $(M_n, \mathcal{F}_n)_{n \in \mathbb{Z}^+}$ is a *martingale*? Find a condition on θ and λ such that

$$M_n = \exp(\theta X_n - \lambda Y_n)$$

is a martingale. If $\theta = i\alpha$ for some real α , show that M is a martingale if

$$e^{-\lambda h} = 2 - \cos(\alpha h) - \sqrt{(2 - \cos(\alpha h))^2 - 1}. \quad (*)$$

Suppose that the random walk Z starts at position $(0, 1) \equiv (0, Nh)$ at time 0, and suppose that

$$\tau = \inf\{n : Y_n = 0\}.$$

Stating fully any results to which you appeal, prove that

$$E \exp(i\alpha X_\tau) = e^{-\lambda},$$

where λ is as given at $(*)$. Deduce that as $N \rightarrow \infty$

$$E \exp(i\alpha X_\tau) \rightarrow e^{-|\alpha|}$$

and comment briefly on this result.

Paper 3, Section II**29J Stochastic Financial Models**

First, what is a *Brownian motion*?

- (i) The price S_t of an asset evolving in continuous time is represented as

$$S_t = S_0 \exp(\sigma W_t + \mu t),$$

where W is a standard Brownian motion, and σ and μ are constants. If riskless investment in a bank account returns a continuously-compounded rate of interest r , derive a formula for the time-0 price of a European call option on the asset S with strike K and expiry T . You may use any general results, but should state them clearly.

- (ii) In the same financial market, consider now a derivative which pays

$$Y = \left\{ \exp\left(T^{-1} \int_0^T \log(S_u) du\right) - K \right\}^+$$

at time T . Find the time-0 price for this derivative. Show that it is less than the price of the European call option which you derived in (i).

Paper 4, Section II**29J Stochastic Financial Models**

In a two-period model, two agents enter a negotiation at time 0. Agent j knows that he will receive a random payment X_j at time 1 ($j = 1, 2$), where the joint distribution of (X_1, X_2) is known to both agents, and $X_1 + X_2 > 0$. At the outcome of the negotiation, there will be an agreed *risk transfer* random variable Y which agent 1 will pay to agent 2 at time 1. The objective of agent 1 is to maximize $EU_1(X_1 - Y)$, and the objective of agent 2 is to maximize $EU_2(X_2 + Y)$, where the functions U_j are strictly increasing, strictly concave, C^2 , and have the properties that

$$\lim_{x \downarrow 0} U'_j(x) = +\infty, \quad \lim_{x \uparrow \infty} U'_j(x) = 0.$$

Show that, unless there exists some $\lambda \in (0, \infty)$ such that

$$\frac{U'_1(X_1 - Y)}{U'_2(X_2 + Y)} = \lambda \quad \text{almost surely}, \quad (*)$$

the risk transfer Y could be altered to the benefit of both agents, and so would not be the conclusion of the negotiation.

Show that, for given $\lambda > 0$, the relation $(*)$ determines a unique risk transfer $Y = Y_\lambda$, and that $X_2 + Y_\lambda$ is a function of $X_1 + X_2$.

Paper 1, Section II**29I Stochastic Financial Models**

What is a Brownian motion? State the reflection principle for Brownian motion.

Let $W = (W_t)_{t \geq 0}$ be a Brownian motion. Let $M = \max_{0 \leq t \leq 1} W_t$. Prove

$$\mathbb{P}(M \geq x, W_1 \leq x - y) = \mathbb{P}(M \geq x, W_1 \geq x + y)$$

for all $x, y \geq 0$. Hence, show that the random variables M and $|W_1|$ have the same distribution.

Find the density function of the random variable $R = W_1/M$.

Paper 2, Section II**30I Stochastic Financial Models**

What is a martingale? What is a supermartingale? What is a stopping time?

Let $M = (M_n)_{n \geq 0}$ be a martingale and $\hat{M} = (\hat{M}_n)_{n \geq 0}$ a supermartingale with respect to a common filtration. If $M_0 = \hat{M}_0$, show that $\mathbb{E}M_T \geq \mathbb{E}\hat{M}_T$ for any bounded stopping time T .

[If you use a general result about supermartingales, you must prove it.]

Consider a market with one stock with prices $S = (S_n)_{n \geq 0}$ and constant interest rate r . Explain why an investor's wealth X satisfies

$$X_n = (1 + r) X_{n-1} + \pi_n [S_n - (1 + r) S_{n-1}]$$

where π_n is the number of shares of the stock held during the n th period.

Given an initial wealth X_0 , an investor seeks to maximize $\mathbb{E}U(X_N)$ where U is a given utility function. Suppose the stock price is such that $S_n = S_{n-1} \xi_n$ where $(\xi_n)_{n \geq 1}$ is a sequence of independent and identically distributed random variables. Let V be defined inductively by

$$V(n, x, s) = \sup_{p \in \mathbb{R}} \mathbb{E}V[n + 1, (1 + r)x - ps(1 + r - \xi_1), s\xi_1]$$

with terminal condition $V(N, x, s) = U(x)$ for all $x, s \in \mathbb{R}$.

Show that the process $(V(n, X_n, S_n))_{0 \leq n \leq N}$ is a supermartingale for any trading strategy π .

Suppose π^* is a trading strategy such that the corresponding wealth process X^* makes $(V(n, X_n^*, S_n))_{0 \leq n \leq N}$ a martingale. Show that π^* is optimal.

Paper 3, Section II**29I Stochastic Financial Models**

Consider a market with two assets, a riskless bond and a risky stock, both of whose initial (time-0) prices are $B_0 = 1 = S_0$. At time 1, the price of the bond is a constant $B_1 = R > 0$ and the price of the stock S_1 is uniformly distributed on the interval $[0, C]$ where $C > R$ is a constant.

Describe the set of state price densities.

Consider a contingent claim whose payout at time 1 is given by S_1^2 . Use the fundamental theorem of asset pricing to show that, if there is no arbitrage, the initial price of the claim is larger than R and smaller than C .

Now consider an investor with initial wealth $X_0 = 1$, and assume $C = 3R$. The investor's goal is to maximize his expected utility of time-1 wealth $\mathbb{E}U[R + \pi(S_1 - R)]$, where $U(x) = \sqrt{x}$. Show that the optimal number of shares of stock to hold is $\pi^* = 1$.

What would be the investor's marginal utility price of the contingent claim described above?

Paper 4, Section II**29I Stochastic Financial Models**

Consider a market with no riskless asset and d risky stocks where the price of stock $i \in \{1, \dots, d\}$ at time $t \in \{0, 1\}$ is denoted S_t^i . We assume the vector $S_0 \in \mathbb{R}^d$ is not random, and we let $\mu = \mathbb{E}S_1$ and $V = \mathbb{E}[(S_1 - \mu)(S_1 - \mu)^T]$. Assume V is not singular.

Suppose an investor has initial wealth $X_0 = x$, which he invests in the d stocks so that his wealth at time 1 is $X_1 = \pi^T S_1$ for some $\pi \in \mathbb{R}^d$. He seeks to minimize the $\text{var}(X_1)$ subject to his budget constraint and the condition that $\mathbb{E}X_1 = m$ for a given constant $m \in \mathbb{R}$.

Illustrate this investor's problem by drawing a diagram of the mean-variance efficient frontier. Write down the Lagrangian for the problem. Show that there are two vectors π_A and π_B (which do not depend on the constants x and m) such that the investor's optimal portfolio is a linear combination of π_A and π_B .

Another investor with initial wealth $Y_0 = y$ seeks to maximize $\mathbb{E}U(Y_1)$ his expected utility of time 1 wealth, subject to his budget constraint. Assuming that S_1 is Gaussian and $U(w) = -e^{-\gamma w}$ for a constant $\gamma > 0$, show that the optimal portfolio in this case is also a linear combination of π_A and π_B .

[You may use the moment generating function of the Gaussian distribution without derivation.]

Continue to assume S_1 is Gaussian, but now assume that U is increasing, concave, and twice differentiable, and that U, U' and U'' are of exponential growth but not necessarily of the form $U(w) = -e^{-\gamma w}$. (Recall that a function f is of exponential growth if $|f(w)| \leq ae^{b|w|}$ for some constants positive constants a, b .) Prove that the utility maximizing investor still holds a linear combination of π_A and π_B .

[You may use the Gaussian integration by parts formula

$$\mathbb{E}[\nabla f(Z)] = \mathbb{E}[Zf(Z)]$$

where $Z = (Z_1, \dots, Z_d)^T$ is a vector of independent standard normal random variables, and f is differentiable and of exponential growth. You may also interchange integration and differentiation without justification.]

Paper 1, Section II**29J Stochastic Financial Models**

An investor must decide how to invest his initial wealth w_0 in n assets for the coming year. At the end of the year, one unit of asset i will be worth X_i , $i = 1, \dots, n$, where $X = (X_1, \dots, X_n)^T$ has a multivariate normal distribution with mean μ and non-singular covariance matrix V . At the beginning of the year, one unit of asset i costs p_i . In addition, he may invest in a riskless bank account; an initial investment of 1 in the bank account will have grown to $1 + r > 1$ at the end of the year.

(a) The investor chooses to hold θ_i units of asset i , with the remaining $\varphi = w_0 - \theta \cdot p$ in the bank account. His objective is to minimise the variance of his wealth $w_1 = \varphi(1 + r) + \theta \cdot X$ at the end of the year, subject to a required mean value m for w_1 . Derive the optimal portfolio θ^* , and the minimised variance.

(b) Describe the set $A \subseteq \mathbb{R}^2$ of achievable pairs $(\mathbb{E}[w_1], \text{var}(w_1))$ of mean and variance of the terminal wealth. Explain what is meant by the *mean-variance efficient frontier* as you do so.

(c) Suppose that the investor requires expected mean wealth at time 1 to be m . He wishes to minimise the expected shortfall $\mathbb{E}[(w_1 - (1 + r)w_0)^-]$ subject to this requirement. Show that he will choose a portfolio corresponding to a point on the mean-variance efficient frontier.

Paper 2, Section II**30J Stochastic Financial Models**

What is a *martingale*? What is a *stopping time*? State and prove the optional sampling theorem.

Suppose that ξ_i are independent random variables with values in $\{-1, 1\}$ and common distribution $\mathbb{P}(\xi = 1) = p = 1 - q$. Assume that $p > q$. Let S_n be the random walk such that $S_0 = 0$, $S_n = S_{n-1} + \xi_n$ for $n \geq 1$. For $z \in (0, 1)$, determine the set of values of θ for which the process $M_n = \theta^{S_n} z^n$ is a martingale. Hence derive the probability generating function of the random time

$$\tau_k = \inf\{t : S_t = k\},$$

where k is a positive integer. Hence find the mean of τ_k .

Let $\tau'_k = \inf\{t > \tau_k : S_t = k\}$. Clearly the mean of τ'_k is greater than the mean of τ_k ; identify the point in your derivation of the mean of τ_k where the argument fails if τ_k is replaced by τ'_k .

Paper 3, Section II**29J Stochastic Financial Models**

What is a *Brownian motion*? State the assumptions of the Black–Scholes model of an asset price, and derive the time-0 price of a European call option struck at K , and expiring at T .

Find the time-0 price of a European call option expiring at T , but struck at S_t , where $t \in (0, T)$, and S_t is the price of the underlying asset at time t .

Paper 4, Section II**29J Stochastic Financial Models**

An agent with utility $U(x) = -\exp(-\gamma x)$, where $\gamma > 0$ is a constant, may select at time 0 a portfolio of n assets, which he then holds to time 1. The values $X = (X_1, \dots, X_n)^T$ of the assets at time 1 have a multivariate normal distribution with mean μ and nonsingular covariance matrix V . Prove that the agent will prefer portfolio $\psi \in \mathbb{R}^n$ to portfolio $\theta \in \mathbb{R}^n$ if and only if $q(\psi) > q(\theta)$, where

$$q(x) = x \cdot \mu - \frac{\gamma}{2} x \cdot Vx.$$

Determine his optimal portfolio.

The agent initially holds portfolio θ , which he may change to portfolio $\theta + z$ at cost $\varepsilon \sum_{i=1}^n |z_i|$, where ε is some positive transaction cost. By considering the function $t \mapsto q(\theta + tz)$ for $0 \leq t \leq 1$, or otherwise, prove that the agent will have no reason to change his initial portfolio θ if and only if, for every $i = 1, \dots, n$,

$$|\mu_i - \gamma(V\theta)_i| \leq \varepsilon.$$

1/II/28J **Stochastic Financial Models**

(a) In the context of the Black–Scholes formula, let S_0 be the time-0 spot price, K be the strike price, T be the time to maturity, and let σ be the volatility. Assume that the interest rate r is constant and assume absence of dividends. Write $\text{EC}(S_0, K, \sigma, r, T)$ for the time-0 price of a standard European call. The Black–Scholes formula can be written in the following form

$$\text{EC}(S_0, K, \sigma, r, T) = S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2).$$

State how the quantities d_1 and d_2 depend on S_0, K, σ, r and T .

Assume that you sell this option at time 0. What is your replicating portfolio at time 0?

[No proofs are required.]

(b) Compute the limit of $\text{EC}(S_0, K, \sigma, r, T)$ as $\sigma \rightarrow \infty$. Construct an explicit arbitrage under the assumption that European calls are traded above this limiting price.

(c) Compute the limit of $\text{EC}(S_0, K, \sigma, r, T)$ as $\sigma \rightarrow 0$. Construct an explicit arbitrage under the assumption that European calls are traded below this limiting price.

(d) Express in terms of S_0, d_1 and T the derivative

$$\frac{\partial}{\partial \sigma} \text{EC}(S_0, K, \sigma, r, T).$$

[Hint: you may find it useful to express $\frac{\partial}{\partial \sigma} d_1$ in terms of $\frac{\partial}{\partial \sigma} d_2$.]

[You may use without proof the formula $S_0 \Phi'(d_1) - e^{-rT} K \Phi'(d_2) = 0$.]

(e) Say what is meant by implied volatility and explain why the previous results make it well-defined.

2/II/28J **Stochastic Financial Models**

(a) Let $(B_t : t \geq 0)$ be a Brownian motion and consider the process

$$Y_t = Y_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t}$$

for $Y_0 > 0$ deterministic. For which values of μ is $(Y_t : t \geq 0)$ a supermartingale? For which values of μ is $(Y_t : t \geq 0)$ a martingale? For which values of μ is $(1/Y_t : t \geq 0)$ a martingale? Justify your answers.

(b) Assume that the riskless rates of return for Dollar investors and Euro investors are r_D and r_E respectively. Thus, 1 Dollar at time 0 in the bank account of a Dollar investor will grow to $e^{r_D t}$ Dollars at time t . For a Euro investor, the Dollar is a risky, tradable asset. Let \mathbb{Q}_E be his equivalent martingale measure and assume that the EUR/USD exchange rate at time t , that is, the number of Euros that one Dollar will buy at time t , is given by

$$Y_t = Y_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t},$$

where (B_t) is a Brownian motion under \mathbb{Q}_E . Determine μ as function of r_D and r_E . Verify that Y is a martingale if $r_D = r_E$.

(c) Let r_D, r_E be as in part (b). Let now \mathbb{Q}_D be an equivalent martingale measure for a Dollar investor and assume that the EUR/USD exchange rate at time t is given by

$$Y_t = Y_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t},$$

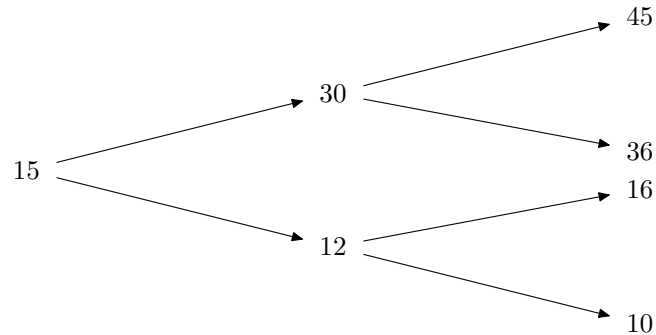
where now (B_t) is a Brownian motion under \mathbb{Q}_D . Determine μ as function of r_D and r_E . Given $r_D = r_E$, check, under \mathbb{Q}_D , that Y is not a martingale but that $1/Y$ is a martingale.

(d) Assuming still that $r_D = r_E$, rederive the final conclusion of part (c), namely the martingale property of $1/Y$, directly from part (b).

3/II/27J **Stochastic Financial Models**

Consider a vector of asset prices evolving over time $\bar{S} = (S_t^0, S_t^1, \dots, S_t^d)_{t \in \{0, 1, \dots, T\}}$. The asset price S^0 is assumed constant over time. In this context, explain what is an arbitrage and prove that the existence of an equivalent martingale measure implies no-arbitrage.

Suppose that over two periods a stock price moves on a binomial tree



Assume riskless rate $r = 1/4$. Determine the equivalent martingale measure. [No proof is required.]

Sell an American put with strike 15 and expiry 2 at its no-arbitrage price, which you should determine.

Verify that the buyer of the option should use his early exercise right if the first period is bad.

Assume that the first period is bad, and that the buyer forgets to exercise. How much risk-free profit can you lock in?

4/II/28J **Stochastic Financial Models**

(a) Consider a family $(X_n : n \geq 0)$ of independent, identically distributed, positive random variables and fix $z_0 > 0$. Define inductively

$$z_{n+1} = z_n X_n, \quad n \geq 0.$$

Compute, for $n \in \{1, \dots, N\}$, the conditional expectation $\mathbb{E}(z_N | z_n)$.

(b) Fix $R \in [0, 1)$. In the setting of part (a), compute also $\mathbb{E}(U(z_N) | z_n)$, where

$$U(x) = x^{1-R}/(1-R), \quad x \geq 0.$$

(c) Let U be as in part (b). An investor with wealth $w_0 > 0$ at time 0 wishes to invest it in such a way as to maximise $\mathbb{E}(U(w_N))$ where w_N is the wealth at the start of day N . Let $\alpha \in [0, 1]$ be fixed. On day n , he chooses the proportion $\theta \in [\alpha, 1]$ of his wealth to invest in a single risky asset, so that his wealth at the start of day $n+1$ will be

$$w_{n+1} = w_n \{\theta X_n + (1-\theta)(1+r)\}.$$

Here, $(X_n : n \geq 0)$ is as in part (a) and r is the per-period riskless rate of interest. If $V_n(w) = \sup \mathbb{E}(U(w_N) | w_n = w)$ denotes the value function of this optimization problem, show that $V_n(w_n) = a_n U(w_n)$ and give a formula for a_n . Verify that, in the case $\alpha = 1$, your answer is in agreement with part (b).

1/II/28J **Stochastic Financial Models**

- (i) What does it mean to say that a process $(M_t)_{t \geq 0}$ is a *martingale*? What does the *martingale convergence theorem* tell us when applied to *positive martingales*?
- (ii) What does it mean to say that a process $(B_t)_{t \geq 0}$ is a *Brownian motion*? Show that $\sup_{t \geq 0} B_t = \infty$ with probability one.
- (iii) Suppose that $(B_t)_{t \geq 0}$ is a Brownian motion. Find μ such that

$$S_t = \exp(x_0 + \sigma B_t + \mu t)$$

is a martingale. Discuss the limiting behaviour of S_t and $\mathbb{E}(S_t)$ for this μ as $t \rightarrow \infty$.

2/II/28J **Stochastic Financial Models**

In the context of a single-period financial market with n traded assets, what is an arbitrage? What is an equivalent martingale measure?

Fix $\epsilon \in (0, 1)$ and consider the following single-period market with 3 assets:

Asset 1 is a riskless bond and pays no interest.

Asset 2 is a stock with initial price £1 per share; its possible final prices are $u = 1 + \epsilon$ with probability $3/5$ and $d = 1 - \epsilon$ with probability $2/5$.

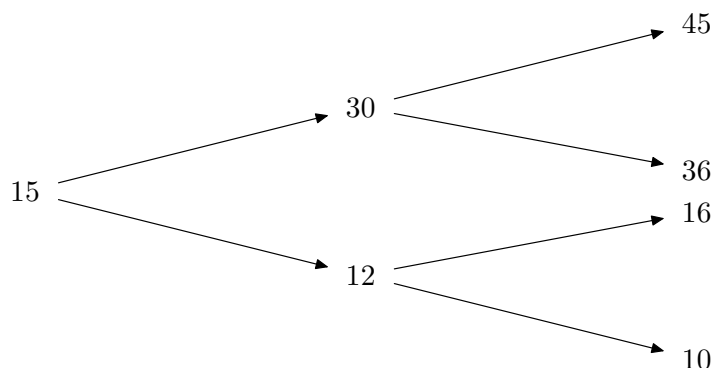
Asset 3 is another stock that behaves like an independent copy of asset 2.

Find all equivalent martingale measures for the problem and characterise all contingent claims that can be replicated.

Consider a contingent claim Y that pays 1 if both risky assets move in the same direction and zero otherwise. Show that the lower arbitrage bound, simply obtained by calculating all possible prices as the pricing measure ranges over all equivalent martingale measures, is zero. Why might someone pay for such a contract?

3/II/27J **Stochastic Financial Models**

Suppose that over two periods a stock price moves on a binomial tree



- (i) Determine for what values of the riskless rate r there is no arbitrage. From here on, fix $r = 1/4$ and determine the equivalent martingale measure.
- (ii) Find the time-zero price and replicating portfolio for a European put option with strike 15 and expiry 2.
- (iii) Find the time-zero price and optimal exercise policy for an American put option with the same strike and expiry.
- (iv) Deduce the corresponding (European and American) call option prices for the same strike and expiry.

4/II/28J **Stochastic Financial Models**

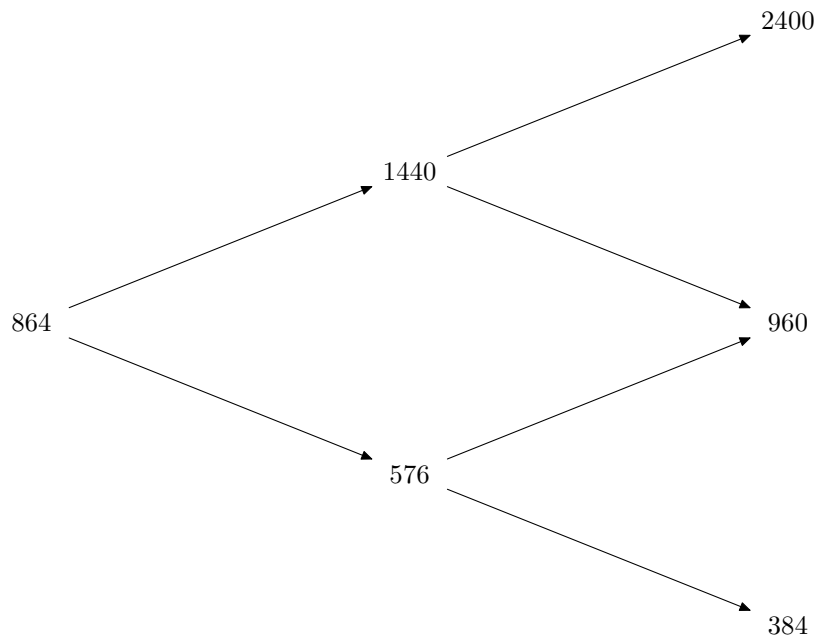
Briefly describe the Black–Scholes model. Consider a “cash-or-nothing” option with strike price K , i.e. an option whose payoff at maturity is

$$f(S_T) = \begin{cases} 1 & \text{if } S_T > K, \\ 0 & \text{if } S_T \leq K. \end{cases}$$

It can be interpreted as a bet that the stock will be worth at least K at time T . Find a formula for its value at time t , in terms of the spot price S_t . Find a formula for its Delta (i.e. its hedge ratio). How does the Delta behave as $t \rightarrow T$? Why is it difficult, in practice, to hedge such an instrument?

1/II/28I **Stochastic Financial Models**

Over two periods a stock price $\{S_t : t = 0, 1, 2\}$ moves on a binomial tree.



Assuming that the riskless rate is constant at $r = 1/3$, verify that all risk-neutral up-probabilities are given by one value $p \in (0, 1)$. Find the time-0 value of the following three put options all struck at $K = S_0 = 864 = 2^5 \times 3^3$, with expiry 2:

- (a) a European put;
- (b) an American put;
- (c) a European put modified by raising the strike to $K = 992$ at time 1 if the stock went down in the first period.

2/II/28I **Stochastic Financial Models**

- (a) In the context of a single-period financial market with n traded assets and a single riskless asset earning interest at rate r , what is an arbitrage? What is an equivalent martingale measure? Explain marginal utility pricing, and how it leads to an equivalent martingale measure.
- (b) Consider the following single-period market with two assets. The first is a riskless bond, worth 1 at time 0, and 1 at time 1. The second is a share, worth 1 at time 0 and worth S_1 at time 1, where S_1 is uniformly distributed on the interval $[0, a]$, where $a > 0$. Under what condition on a is this model arbitrage free? When it is, characterise the set \mathcal{E} of equivalent martingale measures.

An agent with C^2 utility U and with wealth w at time 0 aims to pick the number θ of shares to hold so as to maximise his expected utility of wealth at time 1. Show that he will choose θ to be positive if and only if $a > 2$.

An option pays $(S_1 - 1)^+$ at time 1. Assuming that $a = 2$, deduce that the agent's price for this option will be $1/4$, and show that the range of possible prices for this option as the pricing measure varies in \mathcal{E} is the interval $(0, \frac{1}{2})$.

3/II/27I **Stochastic Financial Models**

Let r denote the riskless rate and let $\sigma > 0$ be a fixed volatility parameter.

- (a) Let $(S_t)_{t \geq 0}$ be a Black–Scholes asset with zero dividends:

$$S_t = S_0 \exp(\sigma B_t + (r - \sigma^2/2)t) ,$$

where B is standard Brownian motion. Derive the Black–Scholes partial differential equation for the price of a European option on S with bounded payoff $\varphi(S_T)$ at expiry T :

$$\partial_t V + \frac{1}{2} \sigma^2 S^2 \partial_{SS} V + r S \partial_S V - rV = 0, \quad V(T, \cdot) = \varphi(\cdot) .$$

[You may use the fact that for C^2 functions $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying exponential growth conditions, and standard Brownian motion B , the process

$$C_t^f = f(t, B_t) - \int_0^t (\partial_t f + \frac{1}{2} \partial_{BB} f)(s, B_s) ds$$

is a martingale.]

- (b) Indicate the changes in your argument when the asset pays dividends continuously at rate $D > 0$. Find the corresponding Black–Scholes partial differential equation.
- (c) Assume $D = 0$. Find a closed form solution for the time-0 price of a European power option with payoff S_T^n .

4/II/28I **Stochastic Financial Models**

State the definitions of a *martingale* and a *stopping time*.

State and prove the optional sampling theorem.

If $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale, under what conditions is it true that M_n converges with probability 1 as $n \rightarrow \infty$? Show by an example that some condition is necessary.

A market consists of $K > 1$ agents, each of whom is either optimistic or pessimistic. At each time $n = 0, 1, \dots$, one of the agents is selected at random, and chooses to talk to one of the other agents (again selected at random), whose type he then adopts. If choices in different periods are independent, show that the proportion of pessimists is a martingale. What can you say about the limiting behaviour of the proportion of pessimists as time n tends to infinity?

1/II/28J **Stochastic Financial Models**

Let $X \equiv (X_0, X_1, \dots, X_J)^T$ be a zero-mean Gaussian vector, with covariance matrix $V = (v_{jk})$. In a simple single-period economy with J agents, agent i will receive X_i at time 1 ($i = 1, \dots, J$). If Y is a contingent claim to be paid at time 1, define agent i 's *reservation bid price* for Y , assuming his preferences are given by $E[U_i(X_i + Z)]$ for any contingent claim Z .

Assuming that $U_i(x) \equiv -\exp(-\gamma_i x)$ for each i , where $\gamma_i > 0$, show that agent i 's reservation bid price for λ units of X_0 is

$$p_i(\lambda) = -\frac{1}{2}\gamma_i(\lambda^2 v_{00} + 2\lambda v_{0i}).$$

As $\lambda \rightarrow 0$, find the limit of agent i 's per-unit reservation bid price for X_0 , and comment on the expression you obtain.

The agents bargain, and reach an equilibrium. Assuming that the contingent claim X_0 is in zero net supply, show that the equilibrium price of X_0 will be

$$p = -\Gamma v_{0\bullet},$$

where $\Gamma^{-1} = \sum_{i=1}^J \gamma_i^{-1}$ and $v_{0\bullet} = \sum_{i=1}^J v_{0i}$. Show that at that price agent i will choose to buy

$$\theta_i = (\Gamma v_{0\bullet} - \gamma_i v_{0i}) / (\gamma_i v_{00})$$

units of X_0 .

By computing the improvement in agent i 's expected utility, show that the value to agent i of access to this market is equal to a fixed payment of

$$\frac{(\gamma_i v_{0i} - \Gamma v_{0\bullet})^2}{2\gamma_i v_{00}}.$$

2/II/28J **Stochastic Financial Models**

(i) At the beginning of year n , an investor makes decisions about his investment and consumption for the coming year. He first takes out an amount c_n from his current wealth w_n , and sets this aside for consumption. He splits his remaining wealth between a bank account (unit wealth invested at the start of the year will have grown to a sure amount $r > 1$ by the end of the year), and the stock market. Unit wealth invested in the stock market will have become the random amount $X_{n+1} > 0$ by the end of the year.

The investor's objective is to invest and consume so as to maximise the expected value of $\sum_{n=1}^N U(c_n)$, where U is strictly increasing and strictly convex. Consider the dynamic programming equation (Bellman equation) for his problem,

$$V_n(w) = \sup_{c, \theta} \left\{ U(c) + E_n [V_{n+1}(\theta(w - c)X_{n+1} + (1 - \theta)(w - c)r)] \right\} \quad (0 \leq n < N),$$

$$V_N(w) = U(w).$$

Explain all undefined notation, and explain briefly why the equation holds.

(ii) Supposing that the X_i are independent and identically distributed, and that $U(x) = x^{1-R}/(1-R)$, where $R > 0$ is different from 1, find as explicitly as you can the form of the agent's optimal policy.

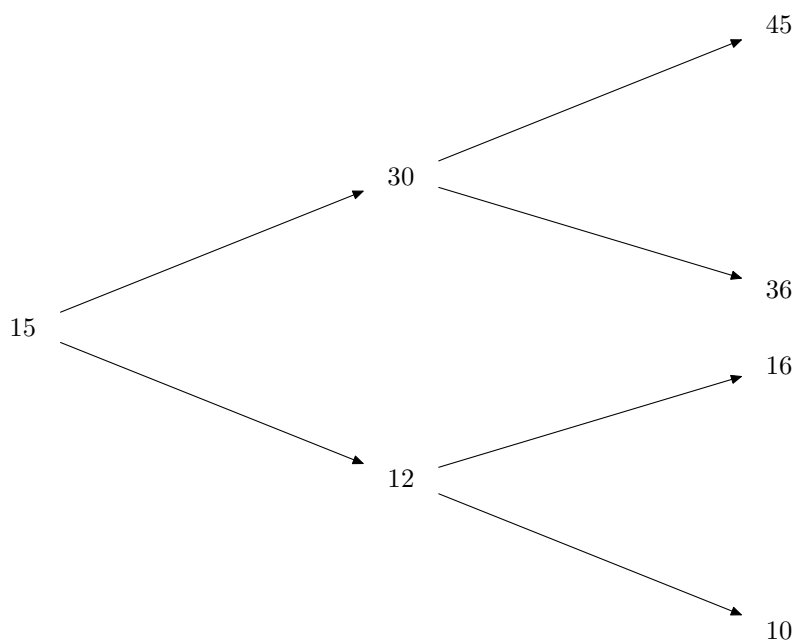
(iii) Return to the general problem of (i). Assuming that the sample space Ω is finite, and that all suprema are attained, show that

$$\begin{aligned} E_n [V'_{n+1}(w_{n+1}^*)(X_{n+1} - r)] &= 0, \\ rE_n [V'_{n+1}(w_{n+1}^*)] &= U'(c_n^*), \\ rE_n [V'_{n+1}(w_{n+1}^*)] &= V'_n(w_n^*), \end{aligned}$$

where $(c_n^*, w_n^*)_{0 \leq n \leq N}$ denotes the optimal consumption and wealth process for the problem. Explain the significance of each of these equalities.

3/II/27J Stochastic Financial Models

Suppose that over two periods a stock price moves on a binomial tree:



- Find an arbitrage opportunity when the riskless rate equals $1/10$. Give precise details of when and how much you buy, borrow and sell.
- From here on, assume instead that the riskless rate equals $1/4$. Determine the equivalent martingale measure. [*No proof is required.*]
- Determine the time-zero price of an American put with strike 15 and expiry 2. Assume you sell it at this price. Which hedge do you put on at time zero? Consider the scenario of two bad periods. How does your hedge work?
- The buyer of the American put turns out to be an unsophisticated investor who fails to use his early exercise right when he should. Assume the first period was bad. How much profit can you make out of this? You should detail your exact strategy.

4/II/28J **Stochastic Financial Models**

- (a) In the context of the Black–Scholes formula, let S_0 be spot price, K be strike price, T be time to maturity, and assume constant interest rate r , volatility σ and absence of dividends. Write down explicitly the prices of a European call and put,

$$EC(S_0, K, \sigma, r, T) \text{ and } EP(S_0, K, \sigma, r, T).$$

Use Φ for the normal distribution function. [No proof is required.]

- (b) From here on assume $r = 0$. Keeping T, σ fixed, we shorten the notation to $EC(S_0, K)$ and similarly for EP . Show that *put-call symmetry* holds:

$$EC(S_0, K) = EP(K, S_0).$$

Check homogeneity: for every real $\alpha > 0$

$$EC(\alpha S_0, \alpha K) = \alpha EC(S_0, K).$$

- (c) Show that the price of a down-and-out European call with strike $K < S_0$ and barrier $B \leq K$ is given by

$$EC(S_0, K) - \frac{S_0}{B} EC\left(\frac{B^2}{S_0}, K\right).$$

(d)

- (i) Specialize the last expression to $B = K$ and simplify.
- (ii) Answer a popular interview question in investment banks: What is the fair value of a down-and-out call given that $S_0 = 100$, $B = K = 80$, $\sigma = 20\%$, $r = 0$, $T = 1$? Identify the corresponding hedge. [It may be helpful to compute Delta first.]
- (iii) Does this hedge work beyond the Black–Scholes model? When does it fail?