Part II

Riemann Surfaces

Paper 1, Section II

24F Riemann Surfaces

Let D be a domain in $\mathbb{C}$. What is a *germ* on D? Define the *space of germs* $\mathcal{G}$ over D. Briefly describe the topology, the forgetful map $\pi : \mathcal{G} \to D$ and the complex structure on $\mathcal{G}$, all without proof. Define the evaluation map $\mathcal{E} : \mathcal{G} \to \mathbb{C}$, and prove that $\mathcal{E}$ is analytic.

Let D be the result of removing the eighth roots of unity from $\mathbb{C}$, and consider the function element $w = \sqrt{z^8 - 1}$ defined over D. Give an explicit gluing construction of the component R of the space of germs corresponding to w. You should construct the evaluation map and the forgetful map on R, and exhibit an analytic embedding $\Phi : R \hookrightarrow \mathcal{G}$. [You do not need to prove that the image of Φ is a component of $\mathcal{G}$.]

Assume that R can be embedded into a compact Riemann surface $\overline{R}$ by adding finitely many points. Assume, furthermore, that the forgetful map π extends to a meromorphic function $\bar{\pi} : \overline{R} \to \mathbb{C}_\infty$. How many points are in $\overline{R} \setminus R$? What is the genus of $\overline{R}$? [You may use standard theorems from the course, as long as you state them carefully.]

Paper 2, Section II

24F Riemann Surfaces

State the valency theorem, and define the *degree* $\deg f$ of an analytic map f of compact Riemann surfaces.

Consider a rational function f with derivative f'. Define the *degree* of f, and prove that $\deg f - 1 \leqslant \deg f' \leqslant 2 \deg f$. Give examples to show that these bounds can be achieved, for every possible value of $\deg f \geqslant 1$.

Consider a non-constant elliptic function g with derivative g'. Define the *degree* of g, and prove that $\deg g + 1 \leqslant \deg g' \leqslant 2 \deg g$. Give examples to show that these bounds can be achieved, for every odd value of $\deg g \geqslant 3$. [You may use properties of standard examples of elliptic functions without proof.]

Paper 3, Section II

23F Riemann Surfaces

State the uniformisation theorem.

Write down a list of all Riemann surfaces uniformised by $\mathbb{C}$ and $\mathbb{C}_\infty$, and prove that your list is complete. [You may assume that, if a Riemann surface R is uniformised by a Riemann surface X, then R is conformally equivalent to the quotient of X by a group of conformal equivalences of X acting freely and properly discontinuously. You may also assume standard facts about the groups of conformal equivalences of $\mathbb{C}$ and $\mathbb{C}_\infty$.]

Prove that any domain $D \subseteq \mathbb{C}$ with a complement containing more than one point is uniformised by the open unit disc $\mathbb{D}$.

Suppose there is a holomorphic embedding $\mathbb{C}_* \to R$, where R is a compact Riemann surface. Prove that R is conformally equivalent to the Riemann sphere.

Paper 1, Section II
24F Riemann Surfaces

(a) State the *Uniformisation theorem*, and deduce the Riemann mapping theorem.

(b) Let

$$E = \{x + iy \mid x, y \in \mathbb{R}\,,\ -\pi < x < \pi\}$$

be an infinite vertical strip in $\mathbb{C}$, and let $U \subseteq \mathbb{C}$ consist of $\mathbb{C}$ with the negative real axis (and zero) removed. A *Mercator projection* is a conformal equivalence $f : U \to E$ such that $\operatorname{Im} f(z) \to -\infty$ as $z \to 0$ and $\operatorname{Im} f(z) \to +\infty$ as $z \to \infty$. Exhibit an explicit Mercator projection.

(c) Consider a conformal equivalence $\phi : E \to E$ such that $\operatorname{Im} \phi(z) \to +\infty$ as $\operatorname{Im} z \to +\infty$ and $\operatorname{Im} \phi(z) \to -\infty$ as $\operatorname{Im} z \to -\infty$. Prove that ϕ is translation by an imaginary number, stating carefully any results that you use.

(d) Characterise all Mercator projections.

Paper 2, Section II
24F Riemann Surfaces

(a) Let $D = \{p_1, \ldots, p_n\}$ be a finite (possibly empty) subset of a Riemann surface R, and let $m_1, \ldots, m_n$ be strictly positive integers. Let V be the set of meromorphic functions f on R such that each pole of f is at some p_i, and the order of a pole at p_i is at most m_i. Prove that V is a vector space over $\mathbb{C}$.

(b) For any compact Riemann surface R, prove that

$$\dim_{\mathbb{C}} V \leqslant 1 + \sum_{i=1}^{n} m_i$$

by considering Laurent expansions about the p_i, or otherwise.

(c) Let $R = \mathbb{C}/\Lambda$ be a complex torus. For any meromorphic function f on R with poles $p_1, \ldots, p_n$, prove that

$$\sum_{i=1}^{n} \operatorname{res}_f(p_i) = 0\,.$$

Assuming that $n \geqslant 1$, deduce that $\dim_{\mathbb{C}} V = \sum_i m_i$.

Paper 3, Section II
23F Riemann Surfaces

(a) Consider a finite group H of conformal equivalences of the Riemann sphere $\mathbb{C}_\infty$ such that H fixes a point $p \in \mathbb{C}_\infty$. Prove that H is cyclic and that there is a neighbourhood U of p, invariant under H, so that the quotient $V = H\backslash U$ has the structure of a Riemann surface. Show furthermore that there are charts on U and V so that the quotient map takes the form $z \mapsto z^n$ for some $n \in \mathbb{N}$.

[*You may use without proof the fact that every Möbius transformation is conjugate to either a dilation* $z \mapsto \lambda z$ *or a translation* $z \mapsto z + c$.]

(b) Let G be a finite group of conformal automorphisms of $\mathbb{C}_\infty$. Prove that the quotient $R = G\backslash\mathbb{C}_\infty$ has a conformal structure such that the quotient map $\mathbb{C}_\infty \to R$ is holomorphic.

(c) For each positive integer $n \geqslant 2$, construct a faithful action of the dihedral group D_{2n} on $\mathbb{C}_\infty$. Furthermore, exhibit a rational function f such that z_1 and z_2 are in the same D_{2n}-orbit if and only if $f(z_1) = f(z_2)$.

Paper 1, Section II
24F Riemann Surfaces

(a) Consider an open disc $D \subseteq \mathbb{C}$. Prove that a real-valued function $u : D \to \mathbb{R}$ is harmonic if and only if

$$u = \mathrm{Re}(f)$$

for some analytic function f.

(b) Give an example of a domain D and a harmonic function $u : D \to \mathbb{R}$ that is not equal to the real part of an analytic function on D. Justify your answer carefully.

(c) Let u be a harmonic function on $\mathbb{C}_*$ such that $u(2z) = u(z)$ for every $z \in \mathbb{C}_*$. Prove that u is constant, justifying your answer carefully. Exhibit a countable subset $S \subseteq \mathbb{C}_*$ and a non-constant harmonic function u on $\mathbb{C}_* \setminus S$ such that for all $z \in \mathbb{C}_* \setminus S$ we have $2z \in \mathbb{C}_* \setminus S$ and $u(2z) = u(z)$.

(d) Prove that every non-constant harmonic function $u : \mathbb{C} \to \mathbb{R}$ is surjective.

Paper 2, Section II
24F Riemann Surfaces

Let $D \subseteq \mathbb{C}$ be a domain, let (f, U) be a function element in D, and let $\alpha : [0,1] \to D$ be a path with $\alpha(0) \in U$. Define what it means for a function element (g, V) to be an *analytic continuation of* (f, U) *along* α.

Suppose that $\beta : [0,1] \to D$ is a path homotopic to α and that (h, V) is an analytic continuation of (f, U) along β. Suppose, furthermore, that (f, U) can be analytically continued along any path in D. Stating carefully any theorems that you use, prove that $g\big(\alpha(1)\big) = h\big(\beta(1)\big)$.

Give an example of a function element (f, U) that can be analytically continued to every point of $\mathbb{C}_*$ and a pair of homotopic paths α, β in $\mathbb{C}_*$ starting in U such that the analytic continuations of (f, U) along α and β take different values at $\alpha(1) = \beta(1)$.

Paper 3, Section II

23F Riemann Surfaces

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 0$, and let $m_1, \ldots, m_k$ be the multiplicities of the ramification points of f. Prove that

$$\sum_{i=1}^{k}(m_i - 1) = d - 1\,. \qquad (*)$$

Show that, for any list of integers $m_1, \ldots, m_k \geqslant 2$ satisfying $(*)$, there is a polynomial f of degree d such that the m_i are the multiplicities of the ramification points of f.

(b) Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be an analytic map, and let B be the set of branch points. Prove that the restriction $f : \mathbb{C}_\infty \setminus f^{-1}(B) \to \mathbb{C}_\infty \setminus B$ is a regular covering map. Given $z_0 \notin B$, explain how a closed loop γ in $\mathbb{C}_\infty \setminus B$ gives rise to a permutation σ_γ of $f^{-1}(z_0)$. Show that the group of all such permutations is transitive, and that the permutation σ_γ only depends on γ up to homotopy.

(c) Prove that there is no meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree 4 with branch points $B = \{0, 1, \infty\}$ such that every preimage of 0 and 1 has ramification index 2, while some preimage of ∞ has ramification index equal to 3. [*Hint: You may use the fact that every non-trivial product of* $(2,2)$*-cycles in the symmetric group* S_4 *is a* $(2,2)$*-cycle.*]

Paper 1, Section II

24F Riemann Surfaces

Assuming any facts about triangulations that you need, prove the Riemann–Hurwitz theorem.

Use the Riemann–Hurwitz theorem to prove that, for any cubic polynomial $f : \mathbb{C} \to \mathbb{C}$, there are affine transformations $g(z) = az + b$ and $h(z) = cz + d$ such that $k(z) = g \circ f \circ h(z)$ is of one of the following two forms:

$$k(z) = z^3 \;\text{ or }\; k(z) = z(z^2/3 - 1).$$

Paper 2, Section II

24F Riemann Surfaces

Assuming any facts about triangulations that you need, prove the Riemann–Hurwitz theorem.

Use the Riemann–Hurwitz theorem to prove that, for any cubic polynomial $f : \mathbb{C} \to \mathbb{C}$, there are affine transformations $g(z) = az + b$ and $h(z) = cz + d$ such that $k(z) = g \circ f \circ h(z)$ is of one of the following two forms:

$$k(z) = z^3 \;\text{ or }\; k(z) = z(z^2/3 - 1).$$

Paper 3, Section II

24F Riemann Surfaces

Assuming any facts about triangulations that you need, prove the Riemann–Hurwitz theorem.

Use the Riemann–Hurwitz theorem to prove that, for any cubic polynomial $f : \mathbb{C} \to \mathbb{C}$, there are affine transformations $g(z) = az + b$ and $h(z) = cz + d$ such that $k(z) = g \circ f \circ h(z)$ is of one of the following two forms:

$$k(z) = z^3 \;\text{ or }\; k(z) = z(z^2/3 - 1).$$

Paper 3, Section II

23F Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$, and $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ a holomorphic map of complex tori. Show that f lifts to a linear map $F : \mathbb{C} \to \mathbb{C}$.

Give the definition of $\wp(z) := \wp_\Lambda(z)$, the *Weierstrass $\wp$-function* for Λ. Show that there exist constants g_2, g_3 such that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Suppose $f \in \mathrm{Aut}(\mathbb{C}/\Lambda)$, that is, $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is a biholomorphic group homomorphism. Prove that there exists a lift $F(z) = \zeta z$ of f, where ζ is a root of unity for which there exist $m, n \in \mathbb{Z}$ such that $\zeta^2 + m\zeta + n = 0$.

Paper 2, Section II

23F Riemann Surfaces

(a) Prove that $z \mapsto z^4$ as a map from the upper half-plane $\mathbb{H}$ to $\mathbb{C} \setminus \{0\}$ is a covering map which is not regular.

(b) Determine the set of singular points on the unit circle for

$$h(z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^n.$$

(c) Suppose $f : \Delta \setminus \{0\} \to \Delta \setminus \{0\}$ is a holomorphic map where Δ is the unit disk. Prove that f extends to a holomorphic map $\tilde{f} : \Delta \to \Delta$. If additionally f is biholomorphic, prove that $\tilde{f}(0) = 0$.

(d) Suppose that $g : \mathbb{C} \hookrightarrow R$ is a holomorphic injection with R a compact Riemann surface. Prove that R has genus 0, stating carefully any theorems you use.

Paper 1, Section II

24F Riemann Surfaces

Define $X' := \{(x, y) \in \mathbb{C}^2 : x^3y + y^3 + x = 0\}$.

(a) Prove by defining an atlas that X' is a Riemann surface.

(b) Now assume that by adding finitely many points, it is possible to compactify X' to a Riemann surface X so that the coordinate projections extend to holomorphic maps π_x and π_y from X to $\mathbb{C}_\infty$. Compute the genus of X.

(c) Assume that any holomorphic automorphism of X' extends to a holomorphic automorphism of X. Prove that the group Aut(X) of holomorphic automorphisms of X contains an element ϕ of order 7. Prove further that there exists a holomorphic map $\pi : X \to \mathbb{C}_\infty$ which satisfies $\pi \circ \phi = \pi$.

Paper 2, Section II

23F Riemann Surfaces

State the uniformisation theorem. List without proof the Riemann surfaces which are uniformised by $\mathbb{C}_\infty$ and those uniformised by $\mathbb{C}$.

Let U be a domain in $\mathbb{C}$ whose complement consists of more than one point. Deduce that U is uniformised by the open unit disk.

Let R be a compact Riemann surface of genus g and $P_1, \ldots, P_n$ be distinct points of R. Show that $R \setminus \{P_1, \ldots, P_n\}$ is uniformised by the open unit disk if and only if $2g - 2 + n > 0$, and by $\mathbb{C}$ if and only if $2g - 2 + n = 0$ or -1.

Let Λ be a lattice and $X = \mathbb{C}/\Lambda$ a complex torus. Show that an analytic map $f : \mathbb{C} \to X$ is either surjective or constant.

Give with proof an example of a pair of Riemann surfaces which are homeomorphic but not conformally equivalent.

Paper 3, Section II

23F Riemann Surfaces

Define the *degree* of an analytic map of compact Riemann surfaces, and state the Riemann–Hurwitz formula.

Let Λ be a lattice in $\mathbb{C}$ and $E = \mathbb{C}/\Lambda$ the associated complex torus. Show that the map

$$\psi : z + \Lambda \mapsto -z + \Lambda$$

is biholomorphic with four fixed points in E.

Let $S = E/\sim$ be the quotient surface (the topological surface obtained by identifying $z + \Lambda$ and $\psi(z + \Lambda)$), and let $p : E \to S$ be the associated projection map. Denote by E' the complement of the four fixed points of ψ, and let $S' = p(E')$. Describe briefly a family of charts making S' into a Riemann surface, so that $p : E' \to S'$ is a holomorphic map.

Now assume that, by adding finitely many points, it is possible to compactify S' to a Riemann surface S so that p extends to a regular map $E \to S$. Find the genus of S.

Paper 1, Section II

24F Riemann Surfaces

Given a complete analytic function $\mathcal{F}$ on a domain $G \subset \mathbb{C}$, define the *germ* of a function element (f, D) of $\mathcal{F}$ at $z \in D$. Let $\mathcal{G}$ be the set of all germs of function elements in G. Describe without proofs the topology and complex structure on $\mathcal{G}$ and the natural covering map $\pi : \mathcal{G} \to G$. Prove that the evaluation map $\mathcal{E} : \mathcal{G} \to \mathbb{C}$ defined by

$$\mathcal{E}([f]_z) = f(z)$$

is analytic on each component of $\mathcal{G}$.

Suppose $f : R \to S$ is an analytic map of compact Riemann surfaces with $B \subset S$ the set of branch points. Show that $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map.

Given $P \in S \setminus B$, explain how any closed curve in $S \setminus B$ with initial and final points P yields a permutation of the set $f^{-1}(P)$. Show that the group H obtained from all such closed curves is a transitive subgroup of the group of permutations of $f^{-1}(P)$.

Find the group H for the analytic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ where $f(z) = z^2 + z^{-2}$.

Paper 2, Section II

21F Riemann Surfaces

Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$. Let P be a fundamental parallelogram whose boundary contains no zeros or poles of f. Show that the number of zeros of f in P is the same as the number of poles of f in P, both counted with multiplicities.

Suppose additionally that f is even. Show that there exists a rational function $Q(z)$ such that $f = Q(\wp)$, where $\wp$ is the Weierstrass $\wp$-function.

Suppose f is a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$, and F is a meromorphic antiderivative of f, so that $F' = f$. Is it necessarily true that F is an elliptic function? Justify your answer.

[You may use standard properties of the Weierstrass $\wp$-function throughout.]

Paper 3, Section II

21F Riemann Surfaces

Let $n \geqslant 2$ be a positive even integer. Consider the subspace R of $\mathbb{C}^2$ given by the equation $w^2 = z^n - 1$, where (z, w) are coordinates in $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map to the first factor. Show that R has the structure of a Riemann surface in such a way that π becomes an analytic map. If τ denotes projection onto the second factor, show that τ is also analytic. [You may assume that R is connected.]

Find the ramification points and the branch points of both π and τ. Compute the ramification indices at the ramification points.

Assume that, by adding finitely many points, it is possible to compactify R to a Riemann surface $\overline{R}$ such that π extends to an analytic map $\overline{\pi} : \overline{R} \to \mathbb{C}_\infty$. Find the genus of $\overline{R}$ (as a function of n).

Paper 1, Section II

23F Riemann Surfaces

By considering the singularity at ∞, show that any injective analytic map $f : \mathbb{C} \to \mathbb{C}$ has the form $f(z) = az + b$ for some $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.

State the Riemann–Hurwitz formula for a non-constant analytic map $f : R \to S$ of compact Riemann surfaces R and S, explaining each term that appears.

Suppose $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is analytic of degree 2. Show that there exist Möbius transformations S and T such that

$$SfT : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is the map given by $z \mapsto z^2$.

Paper 3, Section II

20H Riemann Surfaces

Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$. Let $P \subset \mathbb{C}$ be a fundamental parallelogram and let the degree of f be n. Let $a_1, \ldots, a_n$ denote the zeros of f in P, and let $b_1, \ldots, b_n$ denote the poles (both with possible repeats). By considering the integral (if required, also slightly perturbing P)

$$\frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)} dz,$$

show that

$$\sum_{j=1}^{n} a_j \; - \; \sum_{j=1}^{n} b_j \; \in \Lambda.$$

Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to Λ. For $v, w \notin \Lambda$ with $\wp(v) \neq \wp(w)$ we set

$$f(z) = \det \begin{pmatrix} 1 & 1 & 1 \\ \wp(z) & \wp(v) & \wp(w) \\ \wp'(z) & \wp'(v) & \wp'(w) \end{pmatrix},$$

an elliptic function with periods Λ. Suppose $z \notin \Lambda$, $z - v \notin \Lambda$ and $z - w \notin \Lambda$. Prove that $f(z) = 0$ if and only if $z + v + w \in \Lambda$. [You may use standard properties of the Weierstrass $\wp$-function provided they are clearly stated.]

Paper 2, Section II

21H Riemann Surfaces

Suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is a holomorphic map of complex tori, and let π_j denote the projection map $\mathbb{C} \to \mathbb{C}/\Lambda_j$ for $j = 1, 2$. Show that there is a holomorphic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 F = f \pi_1$.

Prove that $F(z) = \lambda z + \mu$ for some $\lambda, \mu \in \mathbb{C}$. Hence deduce that two complex tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda \Lambda_1$ for some $\lambda \in \mathbb{C}^*$.

Paper 1, Section II

22H Riemann Surfaces

(a) Let $f : R \to S$ be a non-constant holomorphic map between Riemann surfaces. Prove that f takes open sets of R to open sets of S.

(b) Let U be a simply connected domain strictly contained in $\mathbb{C}$. Is there a conformal equivalence between U and $\mathbb{C}$? Justify your answer.

(c) Let R be a compact Riemann surface and $A \subset R$ a discrete subset. Given a non-constant holomorphic function $f : R \setminus A \to \mathbb{C}$, show that $f(R \setminus A)$ is dense in $\mathbb{C}$.

Paper 3, Section II

19F Riemann Surfaces

Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$ and let f be an even elliptic function with periods Λ. Prove that there exists a rational function Q such that $f(z) = Q(\wp(z))$. If we write $Q(w) = p(w)/q(w)$ where p and q are coprime polynomials, find the degree of f in terms of the degrees of the polynomials p and q. Describe all even elliptic functions of degree two. Justify your answers. [You may use standard properties of the Weierstrass $\wp$-function.]

Paper 2, Section II

20F Riemann Surfaces

Let G be a domain in $\mathbb{C}$. Define the germ of a function element (f, D) at $z \in D$. Let $\mathcal{G}$ be the set of all germs of function elements in G. Define the topology on $\mathcal{G}$. Show it is a topology, and that it is Hausdorff. Define the complex structure on $\mathcal{G}$, and show that there is a natural projection map $\pi : \mathcal{G} \to G$ which is an analytic covering map on each connected component of $\mathcal{G}$.

Given a complete analytic function $\mathcal{F}$ on G, describe how it determines a connected component $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{G}$. [You may assume that a function element (g, E) is an analytic continuation of a function element (f, D) along a path $\gamma : [0,1] \to G$ if and only if there is a lift of γ to $\mathcal{G}$ starting at the germ of (f, D) at $\gamma(0)$ and ending at the germ of (g, E) at $\gamma(1)$.]

In each of the following cases, give an example of a domain G in $\mathbb{C}$ and a complete analytic function $\mathcal{F}$ such that:

(i) $\pi : \mathcal{G}_{\mathcal{F}} \to G$ is regular but not bijective;

(ii) $\pi : \mathcal{G}_{\mathcal{F}} \to G$ is surjective but not regular.

Paper 1, Section II

20F Riemann Surfaces

Let $f : R \to S$ be a non-constant holomorphic map between compact connected Riemann surfaces and let $B \subset S$ denote the set of branch points. Show that the map $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map.

Given $w \in S \setminus B$ and a closed curve γ in $S \setminus B$ with initial and final point w, explain how this defines a permutation of the (finite) set $f^{-1}(w)$. Show that the group H obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(w)$.

Find the group H for $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ where $f(z) = z^3/(1-z^2)$.

Paper 3, Section II

22H Riemann Surfaces

State the Uniformization Theorem.

Show that any domain of $\mathbb{C}$ whose complement has more than one point is uniformized by the unit disc Δ. [You may use the fact that for $\mathbb{C}_\infty$ the group of automorphisms consists of Möbius transformations, and for $\mathbb{C}$ it consists of maps of the form $z \mapsto az + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.]

Let X be the torus $\mathbb{C}/\Lambda$, where Λ is a lattice. Given $p \in X$, show that $X \setminus \{p\}$ is uniformized by the unit disc Δ.

Is it true that a holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus two must be constant? Justify your answer.

Paper 2, Section II

23H Riemann Surfaces

State and prove the Valency Theorem and define the *degree* of a non-constant holomorphic map between compact Riemann surfaces.

Let X be a compact Riemann surface of genus g and $\pi : X \to \mathbb{C}_\infty$ a holomorphic map of degree two. Find the cardinality of the set R of ramification points of π. Find also the cardinality of the set of branch points of π. [You may use standard results from lectures provided they are clearly stated.]

Define $\sigma : X \to X$ as follows: if $p \in R$, then $\sigma(p) = p$; otherwise, $\sigma(p) = q$ where q is the unique point such that $\pi(q) = \pi(p)$ and $p \neq q$. Show that σ is a conformal equivalence with $\pi\sigma = \pi$ and $\sigma\sigma = \mathrm{id}$.

Paper 1, Section II

23H Riemann Surfaces

If X is a Riemann surface and $p : Y \to X$ is a covering map of topological spaces, show that there is a conformal structure on Y such that $p : Y \to X$ is analytic.

Let $f(z)$ be the complex polynomial $z^5 - 1$. Consider the subspace R of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ given by the equation $w^2 = f(z)$, where (z, w) denotes coordinates in $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map onto the first factor. Show that R has the structure of a Riemann surface which makes π an analytic map. If τ denotes projection onto the second factor, show that τ is also analytic. [You may assume that R is connected.]

Find the ramification points and the branch points of both π and τ. Compute also the ramification indices at the ramification points.

Assuming that it is possible to add a point P to R so that $X = R \cup \{P\}$ is a compact Riemann surface and τ extends to a holomorphic map $\tau : X \to \mathbb{C}_\infty$ such that $\tau^{-1}(\infty) = \{P\}$, compute the genus of X.

Paper 3, Section II
22I Riemann Surfaces
Let $\Lambda = \mathbb{Z} + \mathbb{Z}\lambda$ be a lattice in $\mathbb{C}$ where $\mathrm{Im}(\lambda) > 0$, and let X be the complex torus $\mathbb{C}/\Lambda$.

(i) Give the definition of an elliptic function with respect to Λ. Show that there is a bijection between the set of elliptic functions with respect to Λ and the set of holomorphic maps from X to the Riemann sphere. Next, show that if f is an elliptic function with respect to Λ and $f^{-1}\{\infty\} = \emptyset$, then f is constant.

(ii) Assume that

$$f(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

defines a meromorphic function on $\mathbb{C}$, where the sum converges uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$. Show that f is an elliptic function with respect to Λ. Calculate the order of f.

Let g be an elliptic function with respect to Λ on $\mathbb{C}$, which is holomorphic on $\mathbb{C} \setminus \Lambda$ and whose only zeroes in the closed parallelogram with vertices $\{0, 1, \lambda, \lambda + 1\}$ are simple zeroes at the points $\left\{\frac{1}{2}, \frac{\lambda}{2}, \frac{1}{2} + \frac{\lambda}{2}\right\}$. Show that g is a non-zero constant multiple of f'.

Paper 2, Section II
23I Riemann Surfaces
(i) Show that the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is biholomorphic to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

(ii) Define the degree of a non-constant holomorphic map between compact connected Riemann surfaces. State the Riemann–Hurwitz formula without proof. Now let X be a complex torus and $f \colon X \to Y$ a holomorphic map of degree 2, where Y is the Riemann sphere. Show that f has exactly four branch points.

(iii) List without proof those Riemann surfaces whose universal cover is the Riemann sphere or $\mathbb{C}$. Now let $f \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic map such that there are two distinct elements $a, b \in \mathbb{C}$ outside the image of f. Assuming the uniformization theorem and the monodromy theorem, show that f is constant.

Paper 1, Section II

23I Riemann Surfaces

(i) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence r in $(0, \infty)$. Show that there is at least one point a on the circle $C = \{z \in \mathbb{C} : |z| = r\}$ which is a singular point of f, that is, there is no direct analytic continuation of f in any neighbourhood of a.

(ii) Let X and Y be connected Riemann surfaces. Define the space $\mathcal{G}$ of germs of function elements of X into Y. Define the natural topology on $\mathcal{G}$ and the natural map $\pi \colon \mathcal{G} \to X$. [You may assume without proof that the topology on $\mathcal{G}$ is Hausdorff.] Show that π is continuous. Define the natural complex structure on $\mathcal{G}$ which makes it into a Riemann surface. Finally, show that there is a bijection between the connected components of $\mathcal{G}$ and the complete holomorphic functions of X into Y.

Paper 3, Section II

22I Riemann Surfaces

Let Λ be the lattice $\mathbb{Z}+\mathbb{Z}i$, X the torus $\mathbb{C}/\Lambda$, and $\wp$ the Weierstrass elliptic function with respect to Λ.

(i) Let $x \in X$ be the point given by $0 \in \Lambda$. Determine the group

$$G = \{f \in \mathrm{Aut}(X) \mid f(x) = x\}\,.$$

(ii) Show that $\wp^2$ defines a degree 4 holomorphic map $h\colon X \to \mathbb{C} \cup \{\infty\}$, which is invariant under the action of G, that is, $h(f(y)) = h(y)$ for any $y \in X$ and any $f \in G$. Identify a ramification point of h distinct from x which is fixed by every element of G.

[If you use the Monodromy theorem, then you should state it correctly. You may use the fact that $\mathrm{Aut}(\mathbb{C}) = \{az + b \mid a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}\}$, and may assume without proof standard facts about $\wp$.]

Paper 2, Section II

23I Riemann Surfaces

Let X be the algebraic curve in $\mathbb{C}^2$ defined by the polynomial $p(z,w) = z^d + w^d + 1$ where d is a natural number. Using the implicit function theorem, or otherwise, show that there is a natural complex structure on X. Let $f\colon X \to \mathbb{C}$ be the function defined by $f(a,b) = b$. Show that f is holomorphic. Find the ramification points and the corresponding branching orders of f.

Assume that f extends to a holomorphic map $g\colon Y \to \mathbb{C} \cup \{\infty\}$ from a compact Riemann surface Y to the Riemann sphere so that $g^{-1}(\infty) = Y \setminus X$ and that g has no ramification points in $g^{-1}(\infty)$. State the Riemann–Hurwitz formula and apply it to g to calculate the Euler characteristic and the genus of Y.

Paper 1, Section II

23I Riemann Surfaces

(i) Let $f(z) = \sum_{n=1}^{\infty} z^{2^n}$. Show that the unit circle is the natural boundary of the function element $(D(0,1), f)$.

(ii) Let $U = \{z \in \mathbf{C} \ : \ \mathrm{Re}(z) > 0\} \subset \mathbf{C}$; explain carefully how a holomorphic function f may be defined on U satisfying the equation

$$(f(z)^2 - 1)^2 = z\,.$$

Let $\mathcal{F}$ denote the connected component of the space of germs $\mathcal{G}$ (of holomorphic functions on $\mathbf{C} \setminus \{0\}$) corresponding to the function element (U, f), with associated holomorphic map $\pi : \mathcal{F} \to \mathbf{C} \setminus \{0\}$. Determine the number of points of $\mathcal{F}$ in $\pi^{-1}(w)$ when (a) $w = \frac{1}{2}$, and (b) $w = 1$.

[You may assume any standard facts about analytic continuations that you may need.]

Paper 1, Section II
23G Riemann Surfaces

Suppose that R_1 and R_2 are Riemann surfaces, and A is a discrete subset of R_1. For any continuous map $\alpha : R_1 \to R_2$ which restricts to an analytic map of Riemann surfaces $R_1 \setminus A \to R_2$, show that α is an analytic map.

Suppose that f is a non-constant analytic function on a Riemann surface R. Show that there is a discrete subset $A \subset R$ such that, for $P \in R \setminus A$, f defines a local chart on some neighbourhood of P.

Deduce that, if $\alpha : R_1 \to R_2$ is a homeomorphism of Riemann surfaces and f is a non-constant analytic function on R_2 for which the composite $f \circ \alpha$ is analytic on R_1, then α is a conformal equivalence. Give an example of a pair of Riemann surfaces which are homeomorphic but not conformally equivalent.

[You may assume standard results for analytic functions on domains in the complex plane.]

Paper 2, Section II
23G Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where τ is a fixed complex number with non-zero imaginary part. Suppose that f is a meromorphic function on $\mathbb{C}$ for which the poles of f are precisely the points in Λ, and for which $f(z) - 1/z^2 \to 0$ as $z \to 0$. Assume moreover that $f'(z)$ determines a doubly periodic function with respect to Λ with $f'(-z) = -f'(z)$ for all $z \in \mathbb{C} \setminus \Lambda$. Prove that:

(i) $f(-z) = f(z)$ for all $z \in \mathbb{C} \setminus \Lambda$.

(ii) f is doubly periodic with respect to Λ.

(iii) If it exists, f is uniquely determined by the above properties.

(iv) For some complex number A, f satisfies the differential equation $f''(z) = 6f(z)^2 + A$.

Paper 3, Section II

22G Riemann Surfaces

State the Classical Monodromy Theorem for analytic continuations in subdomains of the plane.

Let n, r be positive integers with $r > 1$ and set $h(z) = z^n - 1$. By removing n semi-infinite rays from $\mathbb{C}$, find a subdomain $U \subset \mathbb{C}$ on which an analytic function $h^{1/r}$ may be defined, justifying this assertion. Describe *briefly* a gluing procedure which will produce the Riemann surface R for the complete analytic function $h^{1/r}$.

Let Z denote the set of nth roots of unity and assume that the natural analytic covering map $\pi : R \to \mathbb{C} \setminus Z$ extends to an analytic map of Riemann surfaces $\tilde{\pi} : \tilde{R} \to \mathbb{C}_\infty$, where $\tilde{R}$ is a compactification of R and $\mathbb{C}_\infty$ denotes the extended complex plane. Show that $\tilde{\pi}$ has precisely n branch points if and only if r divides n.

Paper 1, Section II

23G Riemann Surfaces

Given a lattice $\Lambda \subset \mathbb{C}$, we may define the corresponding Weierstrass $\wp$-function to be the unique even Λ-periodic elliptic function $\wp$ with poles only on Λ and for which $\wp(z) - 1/z^2 \to 0$ as $z \to 0$. For $w \notin \Lambda$, we set

$$f(z) = \det \begin{pmatrix} 1 & 1 & 1 \\ \wp(z) & \wp(w) & \wp(-z-w) \\ \wp'(z) & \wp'(w) & \wp'(-z-w) \end{pmatrix},$$

an elliptic function with periods Λ. By considering the poles of f, show that f has valency at most 4 (i.e. is at most 4 to 1 on a period parallelogram).

If $w \notin \frac{1}{3}\Lambda$, show that f has at least six distinct zeros. If $w \in \frac{1}{3}\Lambda$, show that f has at least four distinct zeros, at least one of which is a multiple zero. Deduce that the meromorphic function f is identically zero.

If z_1, z_2, z_3 are distinct non-lattice points in a period parallelogram such that $z_1 + z_2 + z_3 \in \Lambda$, what can be said about the points $(\wp(z_i), \wp'(z_i)) \in \mathbb{C}^2$ $(i = 1, 2, 3)$?

Paper 2, Section II

23G Riemann Surfaces

Given a complete analytic function $\mathcal{F}$ on a domain $U \subset \mathbb{C}$, describe *briefly* how the space of germs construction yields a Riemann surface R associated to $\mathcal{F}$ together with a covering map $\pi : R \to U$ (*proofs not required*).

In the case when π is regular, explain briefly how, given a point $P \in U$, any closed curve in U with initial and final points P yields a permutation of the set $\pi^{-1}(P)$.

Now consider the Riemann surface R associated with the complete analytic function

$$(z^2-1)^{1/2} + (z^2-4)^{1/2}$$

on $U = \mathbb{C} \setminus \{\pm 1, \pm 2\}$, with regular covering map $\pi : R \to U$. Which subgroup of the full symmetric group of $\pi^{-1}(P)$ is obtained in this way from all such closed curves (with initial and final points P)?

Paper 3, Section II

22G Riemann Surfaces

Show that the analytic isomorphisms (i.e. conformal equivalences) of the Riemann sphere $\mathbb{C}_\infty$ to itself are given by the non-constant Möbius transformations.

State the Riemann–Hurwitz formula for a non-constant analytic map between compact Riemann surfaces, carefully explaining the terms which occur.

Suppose now that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an analytic map of degree 2; show that there exist Möbius transformations S and T such that

$$SfT : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

is the map given by $z \mapsto z^2$.

Paper 1, Section II

23G Riemann Surfaces

(a) Let $X = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Define the notion of a *rational function* r and describe the function $f\colon X \to X$ determined by r. Assuming that f is holomorphic and non-constant, define the *degree* of r as a rational function and the *degree* of f as a holomorphic map, and prove that the two degrees coincide. [You are not required to prove that the degree of f is well-defined.]

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be two subsets of X each containing three distinct elements. Prove that $X \setminus A$ is biholomorphic to $X \setminus B$.

(b) Let $Z \subset \mathbb{C}^2$ be the algebraic curve defined by the vanishing of the polynomial $p(z, w) = w^2 - z^3 + z^2 + z$. Prove that Z is smooth at every point. State the implicit function theorem and define a complex structure on Z, so that the maps $g, h : Z \to \mathbb{C}$ given by $g(z, w) = w$, $h(z, w) = z$ are holomorphic.

Define what is meant by a *ramification point* of a holomorphic map between Riemann surfaces. Give an example of a ramification point of g and calculate the branching order of g at that point.

Paper 2, Section II

23G Riemann Surfaces

(a) Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ be a lattice in $\mathbb{C}$, where the imaginary part of τ is positive. Define the terms *elliptic function* with respect to Λ and *order* of an elliptic function.

Suppose that f is an elliptic function with respect to Λ of order $m > 0$. Show that the derivative f' is also an elliptic function with respect to Λ and that its order n satisfies $m + 1 \leqslant n \leqslant 2m$. Give an example of an elliptic function f with $m = 5$ and $n = 6$, and an example of an elliptic function f with $m = 5$ and $n = 9$.
[Basic results about holomorphic maps may be used without proof, provided these are accurately stated.]

(b) State the monodromy theorem. Using the monodromy theorem, or otherwise, prove that if two tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent then the lattices satisfy $\Lambda_2 = a\Lambda_1$, for some $a \in \mathbb{C} \setminus \{0\}$.
[You may assume that $\mathbb{C}$ is simply connected and every biholomorphic map of $\mathbb{C}$ onto itself is of the form $z \mapsto cz+d$, for some $c, d \in \mathbb{C}$, $c \neq 0$.]

Paper 3, Section II

22G Riemann Surfaces

(i) Let $f(z) = \sum_{n=1}^{\infty} z^{2^n}$. Show that the unit circle is the natural boundary of the function element $(D(0,1), f)$, where $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

(ii) Let X be a connected Riemann surface and (D, h) a function element on X into $\mathbb{C}$. Define a *germ* of (D, h) at a point $p \in D$. Let $\mathcal{G}$ be the set of all the germs of function elements on X into $\mathbb{C}$. Describe the topology and the complex structure on $\mathcal{G}$, and show that $\mathcal{G}$ is a covering of X (in the sense of complex analysis). Show that there is a one-to-one correspondence between complete holomorphic functions on X into $\mathbb{C}$ and the connected components of $\mathcal{G}$. [You are not required to prove that the topology on $\mathcal{G}$ is second-countable.]

1/II/23H Riemann Surfaces

Define the terms *Riemann surface*, *holomorphic map* between Riemann surfaces and *biholomorphic map*.

Show, without using the notion of degree, that a non-constant holomorphic map between compact connected Riemann surfaces must be surjective.

Let ϕ be a biholomorphic map of the punctured unit disc $\Delta^* = \{0 < |z| < 1\} \subset \mathbb{C}$ onto itself. Show that ϕ extends to a biholomorphic map of the open unit disc Δ to itself such that $\phi(0) = 0$.

Suppose that $f : R \to S$ is a continuous holomorphic map between Riemann surfaces and f is holomorphic on $R \setminus \{p\}$, where p is a point in R. Show that f is then holomorphic on all of R.

[The Open Mapping Theorem may be used without proof if clearly stated.]

2/II/23H Riemann Surfaces

Explain what is meant by a divisor D on a compact connected Riemann surface S. Explain briefly what is meant by a canonical divisor. Define the degree of D and the notion of linear equivalence between divisors. If two divisors on S have the same degree must they be linearly equivalent? Give a proof or a counterexample as appropriate, stating accurately any auxiliary results that you require.

Define $\ell(D)$ for a divisor D, and state the Riemann–Roch theorem. Deduce that the dimension of the space of holomorphic differentials is determined by the genus g of S and that the same is true for the degree of a canonical divisor. Show further that if $g = 2$ then S admits a non-constant meromorphic function with at most two poles (counting with multiplicities).

[General properties of meromorphic functions and meromorphic differentials on S may be used without proof if clearly stated.]

3/II/22H Riemann Surfaces

Define the degree of a non-constant holomorphic map between compact connected Riemann surfaces and state the Riemann–Hurwitz formula.

Show that there exists a compact connected Riemann surface of any genus $g \geqslant 0$.

[You may use without proof any foundational results about holomorphic maps and complex algebraic curves from the course, provided that these are accurately stated. You may also assume that if $h(s)$ is a non-constant complex polynomial without repeated roots then the algebraic curve $C = \{(s,t) \in \mathbb{C}^2 : t^2 - h(s) = 0\}$ is path connected.]

4/II/23H Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where Im $\tau > 0$. The Weierstrass function $\wp$ is the unique meromorphic Λ-periodic function on $\mathbb{C}$, such that the only poles of $\wp$ are at points of Λ and $\wp(z) - 1/z^2 \to 0$ as $z \to 0$.

Show that $\wp$ is an even function. Find all the zeroes of $\wp'$.

Suppose that a is a complex number such that $2a \notin \Lambda$. Show that the function

$$h(z) = (\wp(z-a) - \wp(z+a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$

has no poles in $\mathbb{C} \setminus \Lambda$. By considering the Laurent expansion of h at $z = 0$, or otherwise, deduce that h is constant.

[General properties of meromorphic doubly-periodic functions may be used without proof if accurately stated.]

1/II/23F Riemann Surfaces

Define a complex structure on the unit sphere $S^2 \subset \mathbb{R}^3$ using stereographic projection charts φ, ψ. Let $U \subset \mathbb{C}$ be an open set. Show that a continuous non-constant map $F : U \to S^2$ is holomorphic if and only if $\varphi \circ F$ is a meromorphic function. Deduce that a non-constant rational function determines a holomorphic map $S^2 \to S^2$. Define what is meant by a rational function taking the value $a \in \mathbb{C} \cup \{\infty\}$ with multiplicity m at infinity.

Define the degree of a rational function. Show that any rational function f satisfies $(\deg f) - 1 \leqslant \deg f' \leqslant 2 \deg f$ and give examples to show that the bounds are attained. Is it true that the product $f.g$ satisfies $\deg(f.g) = \deg f + \deg g$, for any non-constant rational functions f and g? Justify your answer.

2/II/23F Riemann Surfaces

A function ψ is defined for $z \in \mathbb{C}$ by

$$\psi(z) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(n + \tfrac{1}{2}\right)^2 \tau + 2\pi i \left(n + \tfrac{1}{2}\right)\left(z + \tfrac{1}{2}\right)\right)$$

where τ is a complex parameter with $\mathrm{Im}(\tau) > 0$. Prove that this series converges uniformly on the subsets $\{|\mathrm{Im}(z)| \leqslant R\}$ for $R > 0$ and deduce that ψ is holomorphic on $\mathbb{C}$.

You may assume without proof that

$$\psi(z+1) = -\psi(z) \qquad \text{and} \qquad \psi(z+\tau) = -\exp(-\pi i \tau - 2\pi i z)\,\psi(z)$$

for all $z \in \mathbb{C}$. Let $\ell(z)$ be the logarithmic derivative $\ell(z) = \dfrac{\psi'(z)}{\psi(z)}$. Show that

$$\ell(z+1) = \ell(z) \qquad \text{and} \qquad \ell(z+\tau) = -2\pi i + \ell(z)$$

for all $z \in \mathbb{C}$. Deduce that ψ has only one zero in the parallelogram P with vertices $\frac{1}{2}(\pm 1 \pm \tau)$. Find all of the zeros of ψ.

Let Λ be the lattice in $\mathbb{C}$ generated by 1 and τ. Show that, for $\lambda_j, a_j \in \mathbb{C}$ $(j = 1, \ldots, n)$, the formula

$$f(z) = \lambda_1 \frac{\psi'(z-a_1)}{\psi(z-a_1)} + \ldots + \lambda_n \frac{\psi'(z-a_n)}{\psi(z-a_n)}$$

gives a Λ-periodic meromorphic function f if and only if $\lambda_1 + \ldots + \lambda_n = 0$. Deduce that $\dfrac{d}{dz}\left(\dfrac{\psi'(z-a)}{\psi(z-a)}\right)$ is Λ-periodic.

3/II/22F Riemann Surfaces

(i) Let R and S be compact connected Riemann surfaces and $f : R \to S$ a non-constant holomorphic map. Define the branching order $v_f(p)$ at $p \in R$ showing that it is well defined. Prove that the set of ramification points $\{p \in R : v_f(p) > 1\}$ is finite. State the Riemann–Hurwitz formula.

Now suppose that R and S have the same genus g. Prove that, if $g > 1$, then f is biholomorphic. In the case when $g = 1$, write down an example where f is not biholomorphic.

[*The inverse mapping theorem for holomorphic functions on domains in* $\mathbb{C}$ *may be assumed without proof if accurately stated.*]

(ii) Let Y be a non-singular algebraic curve in $\mathbb{C}^2$. Describe, without detailed proofs, a family of charts for Y, so that the restrictions to Y of the first and second projections $\mathbb{C}^2 \to \mathbb{C}$ are holomorphic maps. Show that the algebraic curve

$$Y = \{(s,t) \in \mathbb{C}^2 : t^4 = (s^2-1)(s-4)\}$$

is non-singular. Find all the ramification points of the map $f : Y \to \mathbb{C}; (s,t) \mapsto s$.

4/II/23F Riemann Surfaces

Let R be a Riemann surface, $\widetilde{R}$ a topological surface, and $p : \widetilde{R} \to R$ a continuous map. Suppose that every point $x \in \widetilde{R}$ admits a neighbourhood $\widetilde{U}$ such that p maps $\widetilde{U}$ homeomorphically onto its image. Prove that $\widetilde{R}$ has a complex structure such that p is a holomorphic map.

A holomorphic map $\pi : Y \to X$ between Riemann surfaces is called a *covering map* if every $x \in X$ has a neighbourhood V with $\pi^{-1}(V)$ a disjoint union of open sets W_k in Y, so that $\pi : W_k \to V$ is biholomorphic for each W_k. Suppose that a Riemann surface Y admits a holomorphic covering map from the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Prove that any holomorphic map $\mathbb{C} \to Y$ is constant.

[*You may assume any form of the monodromy theorem and basic results about the lifts of paths, provided that these are accurately stated.*]

1/II/23F Riemann Surfaces

Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ be a lattice in $\mathbb{C}$, where τ is a fixed complex number with positive imaginary part. The Weierstrass $\wp$-function is the unique meromorphic Λ-periodic function on $\mathbb{C}$ such that $\wp$ is holomorphic on $\mathbb{C} \setminus \Lambda$, and $\wp(z) - 1/z^2 \to 0$ as $z \to 0$.

Show that $\wp(-z) = \wp(z)$ and find all the zeros of $\wp'$ in $\mathbb{C}$.

Show that $\wp$ satisfies a differential equation

$$\wp'(z)^2 = Q(\wp(z)),$$

for some cubic polynomial $Q(w)$. Further show that

$$Q(w) = 4\left(w - \wp\left(\tfrac{1}{2}\right)\right)\left(w - \wp\left(\tfrac{1}{2}\tau\right)\right)\left(w - \wp\left(\tfrac{1}{2}(1+\tau)\right)\right)$$

and that the three roots of Q are distinct.

[*Standard properties of meromorphic doubly-periodic functions may be used without proof provided these are accurately stated, but any properties of the $\wp$-function that you use must be deduced from first principles.*]

2/II/23F Riemann Surfaces

Define the terms *Riemann surface*, *holomorphic map* between Riemann surfaces, and *biholomorphic map*.

(a) Prove that if two holomorphic maps f, g coincide on a non-empty open subset of a connected Riemann surface R then $f = g$ everywhere on R.

(b) Prove that if $f : R \to S$ is a non-constant holomorphic map between Riemann surfaces and $p \in R$ then there is a choice of co-ordinate charts ϕ near p and ψ near $f(p)$, such that $(\psi \circ f \circ \phi^{-1})(z) = z^n$, for some non-negative integer n. Deduce that a holomorphic bijective map between Riemann surfaces is biholomorphic.

[*The inverse function theorem for holomorphic functions on open domains in $\mathbb{C}$ may be used without proof if accurately stated.*]

3/II/22F Riemann Surfaces

Define the *branching order* $v_f(p)$ at a point p and the *degree* of a non-constant holomorphic map f between compact Riemann surfaces. State the Riemann–Hurwitz formula.

Let $W_m \subset \mathbb{C}^2$ be an affine curve defined by the equation $s^m = t^m + 1$, where $m \geqslant 2$ is an integer. Show that the projective curve $\overline{W}_m \subset \mathbb{P}^2$ corresponding to W_m is non-singular and identify the points of $\overline{W}_m \setminus W_m$. Let F be a continuous map from $\overline{W}_m$ to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, such that the restriction of F to W_m is given by $F(s,t) = s$. Show that F is holomorphic on $\overline{W}_m$. Find the degree and the ramification points of F on $\overline{W}_m$ and their branching orders. Determine the genus of $\overline{W}_m$.

[*Basic properties of the complex structure on an algebraic curve may be used without proof if accurately stated.*]

4/II/23F Riemann Surfaces

Define what is meant by a *divisor* on a compact Riemann surface, the *degree* of a divisor, and a *linear equivalence* between divisors. For a divisor D, define $\ell(D)$ and show that if a divisor D' is linearly equivalent to D then $\ell(D) = \ell(D')$. Determine, without using the Riemann–Roch theorem, the value $\ell(P)$ in the case when P is a point on the Riemann sphere S^2.

[*You may use without proof any results about holomorphic maps on S^2 provided that these are accurately stated.*]

State the Riemann–Roch theorem for a compact connected Riemann surface C. (You are *not* required to give a definition of a canonical divisor.) Show, by considering an appropriate divisor, that if C has genus g then C admits a non-constant meromorphic function (that is a holomorphic map $C \to S^2$) of degree at most $g + 1$.

1/II/23H Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where τ is a fixed complex number with $\mathrm{Im}\tau > 0$. The Weierstrass $\wp$-function is defined as a Λ-periodic meromorphic function such that

(1) the only poles of $\wp$ are at points of Λ, and

(2) there exist positive constants ε and M such that for all $|z| < \varepsilon$, we have

$$|\wp(z) - 1/z^2| < M|z|.$$

Show that $\wp$ is uniquely determined by the above properties and that $\wp(-z) = \wp(z)$. By considering the valency of $\wp$ at $z = 1/2$, show that $\wp''(1/2) \neq 0$.

Show that $\wp$ satisfies the differential equation

$$\wp''(z) = 6\wp^2(z) + A,$$

for some complex constant A.

[*Standard theorems about doubly-periodic meromorphic functions may be used without proof provided they are accurately stated, but any properties of the $\wp$-function that you use must be deduced from first principles.*]

2/II/23H Riemann Surfaces

Define the terms *function element* and *complete analytic function.*

Let (f, D) be a function element such that $f(z)^n = p(z)$, for some integer $n \geqslant 2$, where $p(z)$ is a complex polynomial with no multiple roots. Let F be the complete analytic function containing (f, D). Show that every function element $(\tilde{f}, \tilde{D})$ in F satisfies $\tilde{f}(z)^n = p(z)$.

Describe how the non-singular complex algebraic curve

$$C = \{(z, w) \in \mathbb{C}^2 \mid w^n - p(z) = 0\}$$

can be made into a Riemann surface such that the first and second projections $\mathbb{C}^2 \to \mathbb{C}$ define, by restriction, holomorphic maps $f_1, f_2 : C \to \mathbb{C}$.

Explain briefly the relation between C and the Riemann surface $S(F)$ for the complete analytic function F given earlier.

[*You do not need to prove the Inverse Function Theorem, provided that you state it accurately.*]

3/II/22H Riemann Surfaces

Explain what is meant by a *meromorphic differential* on a compact connected Riemann surface S. Show that if f is a meromorphic function on S then df defines a meromorphic differential on S. Show also that if η and ω are two meromorphic differentials on S which are not identically zero then $\eta = h\omega$ for some meromorphic function h. Show that zeros and poles of a meromorphic differential are well-defined and explain, without proof, how to obtain the genus of S by counting zeros and poles of ω.

Let $V_0 \subset \mathbb{C}^2$ be the affine curve with equation $u^2 = v^2 + 1$ and let $V \subset \mathbb{P}^2$ be the corresponding projective curve. Show that V is non-singular with two points at infinity, and that dv extends to a meromorphic differential on V.

[*You may assume without proof that that the map*

$$(u,v) = \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right), \qquad t \in \mathbb{C} \setminus \{-1,1\},$$

is onto $V_0 \setminus \{(1,0)\}$ *and extends to a biholomorphic map from* $\mathbb{P}^1$ *onto* V.]

4/II/23H Riemann Surfaces

Define what is meant by the *degree* of a non-constant holomorphic map between compact connected Riemann surfaces, and state the Riemann–Hurwitz formula.

Let $E_\Lambda = \mathbb{C}/\Lambda$ be an elliptic curve defined by some lattice Λ. Show that the map

$$\psi : z + \Lambda \in E_\Lambda \to -z + \Lambda \in E_\Lambda$$

is biholomorphic, and that there are four points in E_Λ fixed by ψ.

Let $S = E_\Lambda/\sim$ be the quotient surface (the topological surface obtained by identifying $z + \Lambda$ and $\psi(z+\Lambda)$, for each z) and let $\pi : E_\Lambda \to S$ be the corresponding projection map. Denote by $E_\Lambda^0 \subset E_\Lambda$ the complement of the four points fixed by ψ, and let $S^0 = \pi(E_\Lambda^0)$. Describe briefly a family of charts making S^0 into a Riemann surface, so that $\pi : E_\Lambda^0 \to S^0$ is a holomorphic map.

Now assume that the complex structure of S^0 extends to S, so that S is a Riemann surface, and that the map π is in fact holomorphic on all of E_Λ. Calculate the genus of S.