Part II

Representation Theory

Paper 1, Section II

19H Representation Theory

What is a *representation* of a finite group G? What does it mean to say that a representation is *irreducible*? What is the *degree* of a representation? What does it mean to say that two representations are *isomorphic*?

Consider the dihedral group D_{2n} of order $2n$ for $n \geqslant 3$. Show directly that every irreducible complex representation of D_{2n} has degree at most 2.

For odd $n \geqslant 3$, explicitly construct $\frac{n+3}{2}$ pairwise non-isomorphic irreducible complex representations of D_{2n}. Justify your answer.

For even $n \geqslant 4$, explicitly construct $\frac{n+6}{2}$ pairwise non-isomorphic irreducible complex representations of D_{2n}. Justify your answer.

Paper 2, Section II

19H Representation Theory

Consider the subset H of $\mathrm{GL}_2(\mathbb{F}_{11})$ consisting of matrices of the form

$$\begin{pmatrix} a^2 & b \\ 0 & 1 \end{pmatrix} \text{ with } a, b \in \mathbb{F}_{11} \text{ and } a \neq 0.$$

Show that H is a non-abelian group of order 55 with 7 conjugacy classes and construct its character table. [*You may assume standard results from the course and that* 2 *is a generator of the cyclic group* $\mathbb{F}_{11}^{\times}$.]

Paper 3, Section II

19H Representation Theory

(a) State and prove Burnside's lemma. Deduce that if a finite group G acts 2-transitively on a set X then the corresponding permutation representation $\mathbb{C}X$ decomposes as a direct sum of two non-isomorphic irreducible representations.

(b) Let $G = S_n$ act naturally on the set $X = \{1, \ldots, n\}$. For each non-negative integer r, let X_r be the set of all r-element subsets of X equipped with the natural action of G, and π_r be the character of the corresponding permutation represention. If $0 \leqslant l \leqslant k \leqslant n/2$, show that

$$\langle \pi_k, \pi_l \rangle_G = l + 1.$$

Deduce that $\pi_r - \pi_{r-1}$ is a character of an irreducible representation for each $1 \leqslant r \leqslant n/2$.

What happens for $r > n/2$?

Paper 4, Section II

19H Representation Theory

State Schur's lemma.

What is a *complex representation* of a topological group G? What does it mean to say a complex representation (ρ, V) of G is *unitary*?

Explain why every complex representation of S^1 is unitary. Deduce that every complex representation of S^1 is a direct sum of 1-dimensional representations.

Let G be the group of 3×3 upper unitriangular real matrices

$$G := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

under matrix multiplication. Let Z be the centre of G and Z_0 the cyclic subgroup of Z given by

$$Z_0 = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{Z} \right\} \leqslant Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$$

By considering elements of the form $g^{-1}h^{-1}gh$ with $g, h \in G$, show that every 1-dimensional representation of G has kernel containing Z.

Show that any complex representation (ρ, V) of G/Z_0 decomposes as a direct sum of subrepresentations $(\rho_i, V_i)_{i=1,\ldots,d}$ with the property that

$$\mathrm{Res}^{G/Z_0}_{Z/Z_0} \rho_i = \theta_i \, \mathrm{id}_{V_i}$$

for some distinct 1-dimensional representations $\theta_1, \ldots, \theta_d$ of Z/Z_0. By considering $\det \rho_i$, or otherwise, deduce that $d = 1$ and that θ_1 is the trivial representation. Hence show that G/Z_0 does not have a faithful representation.

Paper 1, Section II
19H Representation Theory

Let G be a finite group.

State *Maschke's theorem* for complex representations of G. Deduce that every representation of G is isomorphic to a direct sum of irreducible representations.

Define the *character* χ_V of a complex representation V of G. Suppose that G acts on a finite set X. What is the *permutation representation* $\mathbb{C}X$? Describe its character $\chi_{\mathbb{C}X}$.

Show that if $V_1, \ldots, V_r$ are all the irreducible representations of G up to isomorphism then the regular representation decomposes as

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} (\dim V_i) V_i.$$

If V is a complex representation of G, let $\mathrm{Hom}_G(V,V)$ be the space of G-linear maps from V to V. If

$$V \cong \bigoplus_{i=1}^{r} n_i V_i,$$

what is the dimension of $\mathrm{Hom}_G(V,V)$? What is the dimension when $V = \mathbb{C}G$?

Now suppose V is a complex representation of G with character χ such that $\chi(g) = 0$ for all non-identity elements $g \in G$. Show that V is a direct sum of copies of the regular representation $\mathbb{C}G$.

Deduce that if W is any complex representation of G then

$$W \otimes \mathbb{C}G \cong \bigoplus_{i=1}^{\dim W} \mathbb{C}G.$$

[*You may assume that the irreducible complex characters of a finite group form an orthonormal basis of the space of class functions.*]

Paper 2, Section II
19H Representation Theory

Suppose that G is a group of order 16. Let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$ be the degrees of the irreducible characters of G. What are the possible values of r and $d_1, \ldots, d_r$? For each such collection $d_1, \ldots, d_r$ find a group of order 16 with these character degrees and construct the character table of the group. [You may assume any general results from the course provided that you state them clearly. You may restrict yourself to brief justifications of the values in each character table.]

Paper 3, Section II
19H Representation Theory

Let $G = SU(2)$ and let V_n be the complex vector space of homogeneous polynomials of degree n in two variables x, y. Construct a continuous homomorphism $\rho_n \colon G \to GL(V_n)$ so that (ρ_n, V_n) is an irreducible representation of G. Prove that (ρ_n, V_n) is indeed irreducible.

What is the character of V_n? Show that every irreducible representation of $SU(2)$ is isomorphic to (ρ_n, V_n) for some $n \geqslant 0$.

Suppose that χ is the character of a representation V of G. State a formula for the character of $\Lambda^2 V$ in terms of χ. Use it to decompose $\Lambda^2 V_4$ as a direct sum of irreducible representations up to isomorphism.

Express the character of $\Lambda^3 V$ in terms of χ. Justify your answer. Decompose $\Lambda^3 V_4$ as a direct sum of irreducible representations up to isomorphism.

Paper 4, Section II
19H Representation Theory

Suppose that H is a subgroup of a group G and χ is a complex character of H.

State *Mackey's restriction formula* and *Frobenius reciprocity* for characters. Use them to deduce Mackey's irreducibility criterion for an induced representation.

Suppose that k is a finite field of order $q \geqslant 4$, $G = SL_2(k)$ and

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in k, a \neq 0 \right\}.$$

Describe the degree 1 complex characters χ of B and explain, with justification, for which of them $\mathrm{Ind}_B^G \chi$ is irreducible.

Paper 1, Section II
19I Representation Theory

(a) What does it mean to say that a representation of a group is *completely reducible*? State Maschke's theorem for representations of finite groups over fields of characteristic 0. State and prove Schur's lemma. Deduce that if there exists a faithful irreducible complex representation of G, then $Z(G)$ is cyclic.

(b) If G is any finite group, show that the regular representation $\mathbb{C}G$ is faithful. Show further that for every finite simple group G, there exists a faithful irreducible complex representation of G.

(c) Which of the following groups have a faithful irreducible representation? Give brief justification of your answers.

(i) the cyclic groups C_n (n a positive integer);

(ii) the dihedral group D_8;

(iii) the direct product $C_2 \times D_8$.

Paper 2, Section II
19I Representation Theory

Let G be a finite group and work over $\mathbb{C}$.

(a) Let χ be a faithful character of G, and suppose that $\chi(g)$ takes precisely r different values as g varies over all the elements of G. Show that every irreducible character of G is a constituent of one of the powers $\chi^0, \chi^1, \ldots, \chi^{r-1}$. [Standard properties of the Vandermonde matrix may be assumed if stated correctly.]

(b) Assuming that the number of irreducible characters of G is equal to the number of conjugacy classes of G, show that the irreducible characters of G form a basis of the complex vector space of all class functions on G. Deduce that $g, h \in G$ are conjugate if and only if $\chi(g) = \chi(h)$ for all characters χ of G.

(c) Let χ be a character of G which is not faithful. Show that there is some irreducible character ψ of G such that $\langle \chi^n, \psi \rangle = 0$ for all integers $n \geqslant 0$.

Paper 3, Section II
19I Representation Theory
In this question we work over $\mathbb{C}$.

(a) (i) Let H be a subgroup of a finite group G. Given an H-space W, define the complex vector space $V = \mathrm{Ind}_H^G(W)$. Define, with justification, the G-action on V.

(ii) Write $\mathcal{C}(g)$ for the conjugacy class of $g \in G$. Suppose that $H \cap \mathcal{C}(g)$ breaks up into s conjugacy classes of H with representatives $x_1, \ldots, x_s$. If ψ is a character of H, write down, without proof, a formula for the induced character $\mathrm{Ind}_H^G(\psi)$ as a certain sum of character values $\psi(x_i)$.

(b) Define permutations $a, b \in S_7$ by $a = (1\ 2\ 3\ 4\ 5\ 6\ 7)$, $b = (2\ 3\ 5)(4\ 7\ 6)$ and let G be the subgroup $\langle a, b \rangle$ of S_7. It is given that the elements of G are all of the form $a^i b^j$ for $0 \leqslant i \leqslant 6$, $0 \leqslant j \leqslant 2$ and that G has order 21.

(i) Find the orders of the centralisers $C_G(a)$ and $C_G(b)$. Hence show that there are five conjugacy classes of G.

(ii) Find all characters of degree 1 of G by lifting from a suitable quotient group.

(iii) Let $H = \langle a \rangle$. By first inducing linear characters of H using the formula stated in part (a)(ii), find the remaining irreducible characters of G.

Paper 4, Section II
19I Representation Theory
(a) Define the group S^1. Sketch a proof of the classification of the irreducible continuous representations of S^1. Show directly that the characters obey an orthogonality relation.

(b) Define the group $SU(2)$.

(i) Show that there is a bijection between the conjugacy classes in $G = SU(2)$ and the subset $[-1, 1]$ of the real line. [If you use facts about a maximal torus T, you should prove them.]

(ii) Write $\mathcal{O}_x$ for the conjugacy class indexed by an element x, where $-1 < x < 1$. Show that $\mathcal{O}_x$ is homeomorphic to S^2. [*Hint: First show that $\mathcal{O}_x$ is in bijection with G/T.*]

(iii) Let $t\colon G \to [-1, 1]$ be the parametrisation of conjugacy classes from part (i). Determine the representation of G whose character is the function $g \mapsto 8t(g)^3$.

Paper 1, Section II

19F Representation Theory

State and prove Maschke's theorem.

Let G be the group of isometries of $\mathbb{Z}$. Recall that G is generated by the elements t, s where $t(n) = n + 1$ and $s(n) = -n$ for $n \in \mathbb{Z}$.

Show that every non-faithful finite-dimensional complex representation of G is a direct sum of subrepresentations of dimension at most two.

Write down a finite-dimensional complex representation of the group $(\mathbb{Z}, +)$ that is not a direct sum of one-dimensional subrepresentations. Hence, or otherwise, find a finite-dimensional complex representation of G that is not a direct sum of subrepresentations of dimension at most two. Briefly justify your answer.

[*Hint: You may assume that any non-trivial normal subgroup of G contains an element of the form t^m for some $m > 0$.*]

Paper 2, Section II

19F Representation Theory

Let G be the unique non-abelian group of order 21 up to isomorphism. Compute the character table of G.

[You may find it helpful to think of G as the group of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $a, b \in \mathbb{F}_7$ and $a^3 = 1$. You may use any standard results from the course provided you state them clearly.]

Paper 3, Section II

19F Representation Theory

State Mackey's restriction formula and Frobenius reciprocity for characters. Deduce Mackey's irreducibility criterion for an induced representation.

For $n \geqslant 2$ show that if S_{n-1} is the subgroup of S_n consisting of the elements that fix n, and W is a complex representation of S_{n-1}, then $\mathrm{Ind}_{S_{n-1}}^{S_n} W$ is not irreducible.

Paper 4, Section II

19F Representation Theory

(a) State and prove Burnside's lemma. Deduce that if a finite group G acts 2-transitively on a set X then the corresponding permutation character has precisely two (distinct) irreducible summands.

(b) Suppose that $\mathbb{F}_q$ is a field with q elements. Write down a list of conjugacy class representatives for $GL_2(\mathbb{F}_q)$. Consider the natural action of $GL_2(\mathbb{F}_q)$ on the set of lines through the origin in $\mathbb{F}_q^2$. What values does the corresponding permutation character take on each conjugacy class representative in your list? Decompose this permutation character into irreducible characters.

Paper 3, Section II

19I Representation Theory

In this question all representations are complex and G is a finite group.

(a) State and prove *Mackey's theorem.* State the *Frobenius reciprocity theorem.*

(b) Let X be a finite G-set and let $\mathbb{C}X$ be the corresponding permutation representation. Pick any orbit of G on X: it is isomorphic as a G-set to G/H for some subgroup H of G. Write down the character of $\mathbb{C}(G/H)$.

(i) Let $\mathbb{C}_G$ be the trivial representation of G. Show that $\mathbb{C}X$ may be written as a direct sum

$$\mathbb{C}X = \mathbb{C}_G \oplus V$$

for some representation V.

(ii) Using the results of (a) compute the character inner product $\langle 1_H \uparrow^G, 1_H \uparrow^G \rangle_G$ in terms of the number of (H, H) double cosets.

(iii) Now suppose that $|X| \geqslant 2$, so that $V \neq 0$. By writing $\mathbb{C}(G/H)$ as a direct sum of irreducible representations, deduce from (ii) that the representation V is irreducible if and only if G acts 2-transitively. In that case, show that V is not the trivial representation.

Paper 4, Section II

19I Representation Theory

(a) What is meant by a *compact topological group?* Explain why $\mathrm{SU}(n)$ is an example of such a group.

[In the following the existence of a Haar measure for any compact Hausdorff topological group may be assumed, if required.]

(b) Let G be any compact Hausdorff topological group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. [Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive-definite) symmetric bilinear form.]

(c) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the following vector space of matrices,

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation.

Show that

(i) this subspace is isomorphic to $\mathbb{R}^3$;

(ii) the trace map $(A, B) \mapsto -\mathrm{tr}(AB)$ induces an invariant positive definite symmetric bilinear form;

(iii) ρ is surjective with kernel $\{\pm I_2\}$. [You may assume, without proof, that $\mathrm{SU}(2)$ is connected.]

Paper 2, Section II

19I Representation Theory

(a) For any finite group G, let $\rho_1, \ldots, \rho_k$ be a complete set of non-isomorphic complex irreducible representations of G, with dimensions $n_1, \ldots n_k$, respectively. Show that

$$\sum_{j=1}^{k} n_j^2 = |G|.$$

(b) Let A, B, C, D be the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and let $G = \langle A, B, C, D \rangle$. Write $Z = -I_4$.

(i) Prove that the derived subgroup $G' = \langle Z \rangle$.

(ii) Show that for all $g \in G$, $g^2 \in \langle Z \rangle$, and deduce that G is a 2-group of order at most 32.

(iii) Prove that the given representation of G of degree 4 is irreducible.

(iv) Prove that G has order 32, and find all the irreducible representations of G.

Paper 1, Section II

19I Representation Theory

(a) State and prove *Schur's lemma* over $\mathbb{C}$.

In the remainder of this question we work over $\mathbb{R}$.

(b) Let G be the cyclic group of order 3.

(i) Write the regular $\mathbb{R}G$-module as a direct sum of irreducible submodules.

(ii) Find all the intertwining homomorphisms between the irreducible $\mathbb{R}G$-modules. Deduce that the conclusion of Schur's lemma is false if we replace $\mathbb{C}$ by $\mathbb{R}$.

(c) Henceforth let G be a cyclic group of order n. Show that

(i) if n is even, the regular $\mathbb{R}G$-module is a direct sum of two (non-isomorphic) 1-dimensional irreducible submodules and $(n-2)/2$ (non-isomorphic) 2-dimensional irreducible submodules;

(ii) if n is odd, the regular $\mathbb{R}G$-module is a direct sum of one 1-dimensional irreducible submodule and $(n-1)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

Paper 1, Section II

19I Representation Theory

(a) Define the *derived subgroup*, G', of a finite group G. Show that if χ is a linear character of G, then $G' \leqslant \ker \chi$. Prove that the linear characters of G are precisely the lifts to G of the irreducible characters of G/G'. [You should state clearly any additional results that you require.]

(b) For $n \geqslant 1$, you may take as given that the group

$$G_{6n} := \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$$

has order $6n$.

(i) Let $\omega = e^{2\pi i/3}$. Show that if ε is any $(2n)$-th root of unity in $\mathbb{C}$, then there is a representation of G_{6n} over $\mathbb{C}$ which sends

$$a \mapsto \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

(ii) Find all the irreducible representations of G_{6n}.

(iii) Find the character table of G_{6n}.

Paper 2, Section II

19I Representation Theory

(a) Suppose H is a subgroup of a finite group G, χ is an irreducible character of G and $\varphi_1, \ldots, \varphi_r$ are the irreducible characters of H. Show that in the restriction $\chi\downarrow_H = a_1\varphi_1 + \cdots + a_r\varphi_r$, the multiplicities $a_1, \ldots, a_r$ satisfy

$$\sum_{i=1}^{r} a_i^2 \leqslant |G:H|. \qquad (\dagger)$$

Determine necessary and sufficient conditions under which the inequality in (†) is actually an equality.

(b) Henceforth suppose that H is a (normal) subgroup of index 2 in G, and that χ is an irreducible character of G.

Lift the non-trivial linear character of G/H to obtain a linear character of G which satisfies

$$\lambda(g) = \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H \end{cases}.$$

(i) Show that the following are equivalent:

(1) $\chi\downarrow_H$ is irreducible;

(2) $\chi(g) \neq 0$ for some $g \in G$ with $g \notin H$;

(3) the characters χ and $\chi\lambda$ of G are not equal.

(ii) Suppose now that $\chi\downarrow_H$ is irreducible. Show that if ψ is an irreducible character of G which satisfies

$$\psi\downarrow_H = \chi\downarrow_H,$$

then either $\psi = \chi$ or $\psi = \chi\lambda$.

(iii) Suppose that $\chi\downarrow_H$ is the sum of two irreducible characters of H, say $\chi\downarrow_H = \psi_1 + \psi_2$. If ϕ is an irreducible character of G such that $\phi\downarrow_H$ has ψ_1 or ψ_2 as a constituent, show that $\phi = \chi$.

(c) Suppose that G is a finite group with a subgroup K of index 3, and let χ be an irreducible character of G. Prove that

$$\langle \chi\downarrow_K, \chi\downarrow_K \rangle_K = 1, 2 \text{ or } 3.$$

Give examples to show that each possibility can occur, giving brief justification in each case.

Paper 3, Section II

19I Representation Theory

State the row orthogonality relations. Prove that if χ is an irreducible character of the finite group G, then $\chi(1)$ divides the order of G.

Stating clearly any additional results you use, deduce the following statements:

(i) Groups of order p^2, where p is prime, are abelian.

(ii) If G is a group of order $2p$, where p is prime, then either the degrees of the irreducible characters of G are all 1, or they are

$$1, 1, 2, \ldots, 2 \text{ (with } (p-1)/2 \text{ of degree 2).}$$

(iii) No simple group has an irreducible character of degree 2.

(iv) Let p and q be prime numbers with $p > q$, and let G be a non-abelian group of order pq. Then q divides $p-1$ and G has $q + ((p-1)/q)$ conjugacy classes.

Paper 4, Section II

19I Representation Theory

Define $G = \mathrm{SU}(2)$ and write down a complete list

$$\{V_n : n = 0, 1, 2, \ldots\}$$

of its continuous finite-dimensional irreducible representations. You should define all the terms you use but proofs are not required. Find the character χ_{V_n} of V_n. State the Clebsch–Gordan formula.

(a) Stating clearly any properties of symmetric powers that you need, decompose the following spaces into irreducible representations of G:

(i) $V_4 \otimes V_3$, $V_3 \otimes V_3$, S^2V_3;

(ii) $V_1 \otimes \cdots \otimes V_1$ (with n multiplicands);

(iii) S^3V_2.

(b) Let G act on the space $M_3(\mathbb{C})$ of 3×3 complex matrices by

$$A : X \mapsto A_1 X A_1^{-1},$$

where A_1 is the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Show that this gives a representation of G and decompose it into irreducible summands.

Paper 2, Section II

17G Representation Theory

In this question you may assume the following result. Let χ be a character of a finite group G and let $g \in G$. If $\chi(g)$ is a rational number, then $\chi(g)$ is an integer.

(a) If a and b are positive integers, we denote their highest common factor by (a, b). Let g be an element of order n in the finite group G. Suppose that g is conjugate to g^i for all i with $1 \leqslant i \leqslant n$ and $(i, n) = 1$. Prove that $\chi(g)$ is an integer for all characters χ of G.

[You may use the following result without proof. Let ω be an nth root of unity. Then

$$\sum_{\substack{1 \leqslant i \leqslant n, \\ (i,n)=1}} \omega^i$$

is an integer.]

Deduce that all the character values of symmetric groups are integers.

(b) Let G be a group of odd order.

Let χ be an irreducible character of G with $\chi = \bar{\chi}$. Prove that

$$\langle \chi, 1_G \rangle = \frac{1}{|G|}(\chi(1) + 2\alpha),$$

where α is an algebraic integer. Deduce that $\chi = 1_G$.

Paper 3, Section II

17G Representation Theory

(a) State Burnside's $p^a q^b$ theorem.

(b) Let P be a non-trivial group of prime power order. Show that if H is a non-trivial normal subgroup of P, then $H \cap Z(P) \neq \{1\}$.

Deduce that a non-abelian simple group cannot have an abelian subgroup of prime power index.

(c) Let ρ be a representation of the finite group G over $\mathbb{C}$. Show that $\delta : g \mapsto \det(\rho(g))$ is a linear character of G. Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

Now let E be a group of order $2k$, where k is an odd integer. By considering the regular representation of E, or otherwise, show that E has a normal subgroup of index 2.

Deduce that if H is a non-abelian simple group of order less than 80, then H has order 60.

Paper 1, Section II
18G Representation Theory

(a) Prove that if there exists a faithful irreducible complex representation of a finite group G, then the centre $Z(G)$ is cyclic.

(b) Define the permutations $a, b, c \in S_6$ by

$$a = (1\ 2\ 3),\ b = (4\ 5\ 6),\ c = (2\ 3)(4\ 5),$$

and let $E = \langle a, b, c\rangle$.

(i) Using the relations $a^3 = b^3 = c^2 = 1$, $ab = ba$, $c^{-1}ac = a^{-1}$ and $c^{-1}bc = b^{-1}$, prove that E has order 18.

(ii) Suppose that ε and η are complex cube roots of unity. Prove that there is a (matrix) representation ρ of E over $\mathbb{C}$ such that

$$a \mapsto \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},\quad b \mapsto \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix},\quad c \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) For which values of ε, η is ρ faithful? For which values of ε, η is ρ irreducible?

(c) Note that $\langle a, b\rangle$ is a normal subgroup of E which is isomorphic to $C_3 \times C_3$. By inducing linear characters of this subgroup, or otherwise, obtain the character table of E.

Deduce that E has the property that $Z(E)$ is cyclic but E has no faithful irreducible representation over $\mathbb{C}$.

Paper 4, Section II
18G Representation Theory

Let G =SU(2) and let V_n be the vector space of complex homogeneous polynomials of degree n in two variables.

(a) Prove that V_n has the structure of an irreducible representation for G.

(b) State and prove the Clebsch–Gordan theorem.

(c) Quoting without proof any properties of symmetric and exterior powers which you need, decompose S^2V_n and Λ^2V_n $(n \geqslant 1)$ into irreducible G-spaces.

Paper 3, Section II

17I Representation Theory

(a) Let the finite group G act on a finite set X and let π be the permutation character. If G is 2-transitive on X, show that $\pi = 1_G + \chi$, where χ is an irreducible character of G.

(b) Let $n \geqslant 4$, and let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$. For any integer $r \leqslant n/2$, write X_r for the set of all r-element subsets of X, and let π_r be the permutation character of the action of G on X_r. Compute the degree of π_r. If $0 \leqslant \ell \leqslant k \leqslant n/2$, compute the character inner product $\langle \pi_k, \pi_\ell \rangle$.

Let $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd. Deduce that S_n has distinct irreducible characters $\chi^{(n)} = 1_G, \chi^{(n-1,1)}, \chi^{(n-2,2)}, \ldots, \chi^{(n-m,m)}$ such that for all $r \leqslant m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \cdots + \chi^{(n-r,r)}.$$

(c) Let Ω be the set of all ordered pairs (i, j) with $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Let S_n act on Ω in the obvious way. Write $\pi^{(n-2,1,1)}$ for the permutation character of S_n in this action. By considering inner products, or otherwise, prove that

$$\pi^{(n-2,1,1)} = 1 + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \psi,$$

where ψ is an irreducible character. Calculate the degree of ψ, and calculate its value on the elements $(1\ 2)$ and $(1\ 2\ 3)$ of S_n.

Paper 2, Section II

17I Representation Theory

Show that the 1-dimensional (complex) characters of a finite group G form a group under pointwise multiplication. Denote this group by $\widehat{G}$. Show that if $g \in G$, the map $\chi \mapsto \chi(g)$ from $\widehat{G}$ to $\mathbb{C}$ is a character of $\widehat{G}$, hence an element of $\widehat{\widehat{G}}$. What is the kernel of the map $G \to \widehat{\widehat{G}}$?

Show that if G is abelian the map $G \to \widehat{\widehat{G}}$ is an isomorphism. Deduce, from the structure theorem for finite abelian groups, that the groups G and $\widehat{G}$ are isomorphic as abstract groups.

Paper 4, Section II

18I Representation Theory

Let N be a proper normal subgroup of a finite group G and let U be an irreducible complex representation of G. Show that either U restricted to N is a sum of copies of a single irreducible representation of N, or else U is induced from an irreducible representation of some proper subgroup of G.

Recall that a p-group is a group whose order is a power of the prime number p. Deduce, by induction on the order of the group, or otherwise, that every irreducible complex representation of a p-group is induced from a 1-dimensional representation of some subgroup.

[You may assume that a non-abelian p-group G has an abelian normal subgroup which is not contained in the centre of G.]

Paper 1, Section II

18I Representation Theory

Let N be a normal subgroup of the finite group G. Explain how a (complex) representation of G/N gives rise to an associated representation of G, and briefly describe which representations of G arise this way.

Let G be the group of order 54 which is given by

$$G = \langle a, b : a^9 = b^6 = 1, b^{-1}ab = a^2 \rangle.$$

Find the conjugacy classes of G. By observing that $N_1 = \langle a \rangle$ and $N_2 = \langle a^3, b^2 \rangle$ are normal in G, or otherwise, construct the character table of G.

Paper 4, Section II
15F Representation Theory

(a) Let S^1 be the circle group. Assuming any required facts about continuous functions from real analysis, show that every 1-dimensional continuous representation of S^1 is of the form

$$z \mapsto z^n$$

for some $n \in \mathbb{Z}$.

(b) Let $G = SU(2)$, and let ρ_V be a continuous representation of G on a finite-dimensional vector space V.

(i) Define the character χ_V of ρ_V, and show that $\chi_V \in \mathbb{N}[z, z^{-1}]$.

(ii) Show that $\chi_V(z) = \chi_V(z^{-1})$.

(iii) Let V be the irreducible 4-dimensional representation of G. Decompose $V \otimes V$ into irreducible representations. Hence decompose the exterior square $\Lambda^2 V$ into irreducible representations.

Paper 3, Section II
15F Representation Theory

(a) State Mackey's theorem, defining carefully all the terms used in the statement.

(b) Let G be a finite group and suppose that G acts on the set Ω.

If $n \in \mathbb{N}$, we say that the action of G on Ω is *n-transitive* if Ω has at least n elements and for every pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ such that the a_i are distinct elements of Ω and the b_i are distinct elements of Ω, there exists $g \in G$ with $ga_i = b_i$ for every i.

(i) Let Ω have at least n elements, where $n \geqslant 1$ and let $\omega \in \Omega$. Show that G acts n-transitively on Ω if and only if G acts transitively on Ω and the stabiliser G_ω acts $(n-1)$-transitively on $\Omega \setminus \{\omega\}$.

(ii) Show that the permutation module $\mathbb{C}\Omega$ can be decomposed as

$$\mathbb{C}\Omega = \mathbb{C}_G \oplus V,$$

where $\mathbb{C}_G$ is the trivial module and V is some $\mathbb{C}G$-module.

(iii) Assume that $|\Omega| \geqslant 2$, so that $V \neq 0$. Prove that V is irreducible if and only if G acts 2-transitively on Ω. In that case show also that V is not the trivial representation. [*Hint: Pick any orbit of G on Ω; it is isomorphic as a G-set to G/H for some subgroup $H \leqslant G$. Consider the induced character* $\mathrm{Ind}_H^G 1_H$.]

Paper 2, Section II
15F Representation Theory

Let G be a finite group. Suppose that $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional complex representation of dimension d. Let $n \in \mathbb{N}$ be arbitrary.

(i) Define the nth *symmetric power* $S^n V$ and the nth *exterior power* $\Lambda^n V$ and write down their respective dimensions.

Let $g \in G$ and let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of g on V. What are the eigenvalues of g on $S^n V$ and on $\Lambda^n V$?

(ii) Let X be an indeterminate. For any $g \in G$, define the *characteristic polynomial* $Q = Q(g, X)$ of g on V by $Q(g, X) := \det(g - XI)$. What is the relationship between the coefficients of Q and the character $\chi_{\Lambda^n V}$ of the exterior power?

Find a relation between the character $\chi_{S^n V}$ of the symmetric power and the polynomial Q.

Paper 1, Section II
15F Representation Theory

(a) Let G be a finite group and let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Use Maschke's theorem to prove that ρ is irreducible.

(b) Let n be a positive integer. You are given that the *dicyclic* group

$$G_{4n} = \langle a, b : a^{2n} = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle$$

has order $4n$.

(i) Show that if ϵ is any $(2n)$th root of unity in $\mathbb{C}$, then there is a representation of G_{4n} over $\mathbb{C}$ which sends

$$a \mapsto \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ \epsilon^n & 0 \end{pmatrix}.$$

(ii) Find all the irreducible representations of G_{4n}.

(iii) Find the character table of G_{4n}.

[*Hint: You may find it helpful to consider the cases n odd and n even separately.*]

Paper 4, Section II
19H Representation Theory

Let G =SU(2).

(i) Sketch a proof that there is an isomorphism of topological groups $G/\{\pm I\} \cong$ SO(3).

(ii) Let V_2 be the irreducible complex representation of G of dimension 3. Compute the character of the (symmetric power) representation $S^n(V_2)$ of G for any $n \geqslant 0$. Show that the dimension of the space of invariants $(S^n(V_2))^G$, meaning the subspace of $S^n(V_2)$ where G acts trivially, is 1 for n even and 0 for n odd. [*Hint: You may find it helpful to restrict to the unit circle subgroup $S^1 \leqslant G$. The irreducible characters of G may be quoted without proof.*]

Using the fact that V_2 yields the standard 3-dimensional representation of SO(3), show that $\bigoplus_{n \geqslant 0} S^n V_2 \cong \mathbb{C}[x, y, z]$. Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of SO(3) is a polynomial ring in one generator. Find a generator for this polynomial ring.

Paper 3, Section II
19H Representation Theory

(i) State Frobenius' theorem for transitive permutation groups acting on a finite set. Define *Frobenius group* and show that any finite Frobenius group (with an appropriate action) satisfies the hypotheses of Frobenius' theorem.

(ii) Consider the group

$$F_{p,q} := \langle a, b : a^p = b^q = 1,\ b^{-1}ab = a^u \rangle,$$

where p is prime, q divides $p-1$ (q not necessarily prime), and u has multiplicative order q modulo p (such elements u exist since q divides $p-1$). Let S be the subgroup of $\mathbb{Z}_p^\times$ consisting of the powers of u, so that $|S| = q$. Write $r = (p-1)/q$, and let $v_1, \ldots, v_r$ be coset representatives for S in $\mathbb{Z}_p^\times$.

(a) Show that $F_{p,q}$ has $q+r$ conjugacy classes and that a complete list of the classes comprises $\{1\}$, $\{a^{v_j s} : s \in S\}$ $(1 \leqslant j \leqslant r)$ and $\{a^m b^n : 0 \leqslant m \leqslant p-1\}$ $(1 \leqslant n \leqslant q-1)$.

(b) By observing that the derived subgroup $F'_{p,q} = \langle a \rangle$, find q 1-dimensional characters of $F_{p,q}$. [Appropriate results may be quoted without proof.]

(c) Let $\varepsilon = e^{2\pi i/p}$. For $v \in \mathbb{Z}_p^\times$ denote by ψ_v the character of $\langle a \rangle$ defined by $\psi_v(a^x) = \varepsilon^{vx}$ $(0 \leqslant x \leqslant p-1)$. By inducing these characters to $F_{p,q}$, or otherwise, find r distinct irreducible characters of degree q.

Paper 2, Section II

19H Representation Theory

In this question work over $\mathbb{C}$. Let H be a subgroup of G. State Mackey's restriction formula, defining all the terms you use. Deduce Mackey's irreducibility criterion.

Let $G = \langle g, r : g^m = r^2 = 1,\ rgr^{-1} = g^{-1} \rangle$ (the dihedral group of order $2m$) and let $H = \langle g \rangle$ (the cyclic subgroup of G of order m). Write down the m inequivalent irreducible characters χ_k $(1 \leqslant k \leqslant m)$ of H. Determine the values of k for which the induced character $\mathrm{Ind}_H^G \chi_k$ is irreducible.

Paper 1, Section II

19H Representation Theory

(i) Let K be any field and let $\lambda \in K$. Let $J_{\lambda,n}$ be the $n \times n$ Jordan block

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

Compute $J_{\lambda,n}^r$ for each $r \geqslant 0$.

(ii) Let G be a cyclic group of order N, and let K be an algebraically closed field of characteristic $p \geqslant 0$. Determine all the representations of G on vector spaces over K, up to equivalence. Which are irreducible? Which do not split as a direct sum $W \oplus W'$, with $W \neq 0$ and $W' \neq 0$?

Paper 3, Section II

19G Representation Theory

Suppose that (ρ_1, V_1) and (ρ_2, V_2) are complex representations of the finite groups G_1 and G_2 respectively. Use ρ_1 and ρ_2 to construct a representation $\rho_1 \otimes \rho_2$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ and show that its character satisfies

$$\chi_{\rho_1 \otimes \rho_2}(g_1, g_2) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)$$

for each $g_1 \in G_1$, $g_2 \in G_2$.

Prove that if ρ_1 and ρ_2 are irreducible then $\rho_1 \otimes \rho_2$ is irreducible as a representation of $G_1 \times G_2$. Moreover, show that every irreducible complex representation of $G_1 \times G_2$ arises in this way.

Is it true that every complex representation of $G_1 \times G_2$ is of the form $\rho_1 \otimes \rho_2$ with ρ_i a complex representation of G_i for $i = 1, 2$? Justify your answer.

Paper 2, Section II

19G Representation Theory

Recall that a regular icosahedron has 20 faces, 30 edges and 12 vertices. Let G be the group of rotational symmetries of a regular icosahedron.

Compute the conjugacy classes of G. Hence, or otherwise, construct the character table of G. Using the character table explain why G must be a simple group.

[You may use any general theorems provided that you state them clearly.]

Paper 4, Section II

19G Representation Theory

State and prove Burnside's $p^a q^b$-theorem.

Paper 1, Section II

19G Representation Theory

State and prove Maschke's Theorem for complex representations of finite groups.

Without using character theory, show that every irreducible complex representation of the dihedral group of order 10, D_{10}, has dimension at most two. List the irreducible complex representations of D_{10} up to isomorphism.

Let V be the set of vertices of a regular pentagon with the usual action of D_{10}. Explicitly decompose the permutation representation $\mathbb{C}V$ into a direct sum of irreducible subrepresentations.

Paper 4, Section II

19H Representation Theory

Write an essay on the finite-dimensional representations of SU(2), including a proof of their complete reducibility, and a description of the irreducible representations and the decomposition of their tensor products.

Paper 3, Section II

19H Representation Theory

Show that every complex representation of a finite group G is equivalent to a unitary representation. Let χ be a character of some finite group G and let $g \in G$. Explain why there are roots of unity $\omega_1, \ldots, \omega_d$ such that

$$\chi(g^i) = \omega_1^i + \cdots + \omega_d^i$$

for all integers i.

For the rest of the question let G be the symmetric group on some finite set. Explain why $\chi(g) = \chi(g^i)$ whenever i is coprime to the order of g.

Prove that $\chi(g) \in \mathbb{Z}$.

State without proof a formula for $\sum_{g \in G} \chi(g)^2$ when χ is irreducible. Is there an irreducible character χ of degree at least 2 with $\chi(g) \neq 0$ for all $g \in G$? Explain your answer.

[You may assume basic facts about the symmetric group, and about algebraic integers, without proof. You may also use without proof the fact that $\sum_{\substack{1 \leqslant i \leqslant n \\ \gcd(i,n)=1}} \omega^i \in \mathbb{Z}$ for any nth root of unity ω.]

Paper 2, Section II

19H Representation Theory

Suppose that G is a finite group. Define the inner product of two complex-valued class functions on G. Prove that the characters of the irreducible representations of G form an orthonormal basis for the space of complex-valued class functions.

Suppose that p is a prime and $\mathbb{F}_p$ is the field of p elements. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$. List the conjugacy classes of G.

Let G act naturally on the set of lines in the space $\mathbb{F}_p^2$. Compute the corresponding permutation character and show that it is reducible. Decompose this character as a sum of two irreducible characters.

Paper 1, Section II

19H Representation Theory

Write down the character table of D_{10}.

Suppose that G is a group of order 60 containing 24 elements of order 5, 20 elements of order 3 and 15 elements of order 2. Calculate the character table of G, justifying your answer.

[You may assume the formula for induction of characters, provided you state it clearly.]

Paper 1, Section II

19I Representation Theory

Let G be a finite group and Z its centre. Suppose that G has order n and Z has order m. Suppose that $\rho : G \to \mathrm{GL}(V)$ is a complex irreducible representation of degree d.

(i) For $g \in Z$, show that $\rho(g)$ is a scalar multiple of the identity.

(ii) Deduce that $d^2 \leqslant n/m$.

(iii) Show that, if ρ is faithful, then Z is cyclic.

[Standard results may be quoted without proof, provided they are stated clearly.]

Now let G be a group of order 18 containing an elementary abelian subgroup P of order 9 and an element t of order 2 with $txt^{-1} = x^{-1}$ for each $x \in P$. By considering the action of P on an irreducible $\mathbb{C}G$-module prove that G has no faithful irreducible complex representation.

Paper 2, Section II

19I Representation Theory

State Maschke's Theorem for finite-dimensional complex representations of the finite group G. Show by means of an example that the requirement that G be finite is indispensable.

Now let G be a (possibly infinite) group and let H be a normal subgroup of finite index r in G. Let $g_1, \ldots, g_r$ be representatives of the cosets of H in G. Suppose that V is a finite-dimensional completely reducible $\mathbb{C}G$-module. Show that

(i) if U is a $\mathbb{C}H$-submodule of V and $g \in G$, then the set $gU = \{gu : u \in U\}$ is a $\mathbb{C}H$-submodule of V;

(ii) if U is a $\mathbb{C}H$-submodule of V, then $\sum_{i=1}^{r} g_i U$ is a $\mathbb{C}G$-submodule of V;

(iii) V is completely reducible regarded as a $\mathbb{C}H$-module.

Hence deduce that if χ is an irreducible character of the finite group G then all the constituents of χ_H have the same degree.

Paper 3, Section II

19I Representation Theory

Define the character $\mathrm{Ind}_H^G\psi$ of a finite group G which is induced by a character ψ of a subgroup H of G.

State and prove the Frobenius reciprocity formula for the characters ψ of H and χ of G.

Now suppose that H has index 2 in G. An irreducible character ψ of H and an irreducible character χ of G are said to be 'related' if

$$\langle \mathrm{Ind}_H^G\psi, \chi\rangle_G = \langle \psi, \mathrm{Res}_H^G\chi\rangle_H > 0.$$

Show that each ψ of degree d is either 'monogamous' in the sense that it is related to one χ (of degree $2d$), or 'bigamous' in the sense that it is related to precisely two distinct characters χ_1, χ_2 (of degree d). Show that each χ is related to one bigamous ψ, or to two monogamous characters ψ_1, ψ_2 (of the same degree).

Write down the degrees of the complex irreducible characters of the alternating group A_5. Find the degrees of the irreducible characters of a group G containing A_5 as a subgroup of index 2, distinguishing two possible cases.

Paper 4, Section II

19I Representation Theory

Define the groups SU(2) and SO(3).

Show that $G = \mathrm{SU}(2)$ acts on the vector space of 2×2 complex matrices of the form

$$V = \left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Denote the corresponding representation of SU(2) on V by ρ.

Prove the following assertions about this action:

(i) The subspace V is isomorphic to $\mathbb{R}^3$.

(ii) The pairing $(A, B) \mapsto -\mathrm{tr}(AB)$ defines a positive definite non-degenerate SU(2)-invariant bilinear form.

(iii) The representation ρ maps G into SO(3). [You may assume that for any compact group H, and any $n \in \mathbb{N}$, there is a continuous group homomorphism $H \to \mathrm{O}(n)$ if and only if H has an n-dimensional representation over $\mathbb{R}$.]

Write down an orthonormal basis for V and use it to show that ρ is surjective with kernel $\{\pm I\}$.

Use the isomorphism $\mathrm{SO}(3) \cong G/\{\pm I\}$ to write down a list of irreducible representations of SO(3) in terms of irreducibles for SU(2). [Detailed explanations are not required.]

Paper 1, Section II

19F Representation Theory

(i) Let N be a normal subgroup of the finite group G. Without giving detailed proofs, define the process of lifting characters from G/N to G. State also the orthogonality relations for G.

(ii) Let a, b be the following two permutations in S_{12},

$$a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)\,,$$

$$b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)\,,$$

and let $G = \langle a, b \rangle$, a subgroup of S_{12} . Prove that G is a group of order 12 and list the conjugacy classes of G. By identifying a normal subgroup of G of index 4 and lifting irreducible characters, calculate all the linear characters of G. Calculate the complete character table of G. By considering 6th roots of unity, find explicit matrix representations affording the non-linear characters of G.

Paper 2, Section II

19F Representation Theory

Define the concepts of induction and restriction of characters. State and prove the Frobenius Reciprocity Theorem.

Let H be a subgroup of G and let $g \in G$. We write $\mathcal{C}(g)$ for the conjugacy class of g in G, and write $C_G(g)$ for the centraliser of g in G. Suppose that $H \cap \mathcal{C}(g)$ breaks up into m conjugacy classes of H, with representatives $x_1, x_2, \ldots, x_m$.

Let ψ be a character of H. Writing $\mathrm{Ind}_H^G(\psi)$ for the induced character, prove that

(i) if no element of $\mathcal{C}(g)$ lies in H, then $\mathrm{Ind}_H^G(\psi)(g) = 0$,

(ii) if some element of $\mathcal{C}(g)$ lies in H, then

$$\mathrm{Ind}_H^G(\psi)(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|}\,.$$

Let $G = S_4$ and let $H = \langle a, b \rangle$, where $a = (1\ 2\ 3\ 4)$ and $b = (1\ 3)$. Identify H as a dihedral group and write down its character table. Restrict each G-conjugacy class to H and calculate the H-conjugacy classes contained in each restriction. Given a character ψ of H, express $\mathrm{Ind}_H^G(\psi)(g)$ in terms of ψ, where g runs through a set of conjugacy classes of G. Use your calculation to find the values of all the irreducible characters of H induced to G.

Paper 3, Section II

19F Representation Theory

Show that the degree of a complex irreducible character of a finite group is a factor of the order of the group.

State and prove Burnside's $p^a q^b$ theorem. You should quote clearly any results you use.

Prove that for any group of odd order n having precisely k conjugacy classes, the integer $n - k$ is divisible by 16.

Paper 4, Section II

19F Representation Theory

Define the circle group $U(1)$. Give a complete list of the irreducible representations of $U(1)$.

Define the spin group $G = SU(2)$, and explain briefly why it is homeomorphic to the unit 3-sphere in $\mathbb{R}^4$. Identify the conjugacy classes of G and describe the classification of the irreducible representations of G. Identify the characters afforded by the irreducible representations. You need not give detailed proofs but you should define all the terms you use.

Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices by conjugation, where $A \in SU(2)$ acts by

$$A : M \mapsto A_1 M {A_1}^{-1} ,$$

in which A_1 denotes the 3×3 block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Show that this gives a representation of G and decompose it into irreducibles.

Paper 1, Section II

19F Representation Theory

Let G be a finite group, and suppose G acts on the finite sets X_1, X_2. Define the permutation representation ρ_{X_1} corresponding to the action of G on X_1, and compute its character π_{X_1}. State and prove "Burnside's Lemma".

Let G act on $X_1 \times X_2$ via the usual diagonal action. Prove that the character inner product $\langle \pi_{X_1}, \pi_{X_2} \rangle$ is equal to the number of G-orbits on $X_1 \times X_2$.

Hence, or otherwise, show that the general linear group $\mathrm{GL}_2(q)$ of invertible 2×2 matrices over the finite field of q elements has an irreducible complex representation of dimension equal to q.

Let S_n be the symmetric group acting on the set $X = \{1, 2, \ldots, n\}$. Denote by Z the set of all 2-element subsets $\{i, j\}$ $(i \neq j)$ of elements of X, with the natural action of S_n. If $n \geqslant 4$, decompose π_Z into irreducible complex representations, and determine the dimension of each irreducible constituent. What can you say when $n = 3$?

Paper 2, Section II

19F Representation Theory

(i) Let G be a finite group. Show that

(1) If χ is an irreducible character of G then so is its conjugate $\bar{\chi}$.

(2) The product of any two characters of G is again a character of G.

(3) If χ and ψ are irreducible characters of G then

$$\langle \chi\psi, 1_G \rangle = \begin{cases} 1, & \text{if } \chi = \bar{\psi}, \\ 0, & \text{if } \chi \neq \bar{\psi}. \end{cases}$$

(ii) If χ is a character of the finite group G, define χ_S and χ_A. For $g \in G$ prove that

$$\chi_S(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right) \quad \text{and} \quad \chi_A(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right).$$

(iii) A certain group of order 24 has precisely seven conjugacy classes with representatives $g_1, \ldots, g_7$; further, G has a character χ with values as follows:

g_i	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\lvert C_G(g_i)\rvert$	24	24	4	6	6	6	6
χ	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2

where $\omega = e^{2\pi i/3}$.

It is given that $g_1^2, g_2^2, g_3^2, g_4^2, g_5^2, g_6^2, g_7^2$ are conjugate to $g_1, g_1, g_2, g_5, g_4, g_4, g_5$ respectively.

Determine χ_S and χ_A, and show that both are irreducible.

Paper 3, Section II

19F Representation Theory

Let $G = \mathrm{SU}(2)$. Let V_n be the complex vector space of homogeneous polynomials of degree n in two variables z_1, z_2. Define the usual left action of G on V_n and denote by $\rho_n : G \to \mathrm{GL}(V_n)$ the representation induced by this action. Describe the character χ_n afforded by ρ_n.

Quoting carefully any results you need, show that

(i) The representation ρ_n has dimension $n+1$ and is irreducible for $n \in \mathbb{Z}_{\geqslant 0}$;

(ii) Every finite-dimensional continuous irreducible representation of G is one of the ρ_n;

(iii) V_n is isomorphic to its dual V_n^*.

Paper 4, Section II

19F Representation Theory

Let $H \leqslant G$ be finite groups.

(a) Let ρ be a representation of G affording the character χ. Define the restriction, $\mathrm{Res}_H^G \rho$ of ρ to H.

Suppose χ is irreducible and suppose $\mathrm{Res}_H^G \rho$ affords the character χ_H. Let $\psi_1, \ldots, \psi_r$ be the irreducible characters of H. Prove that $\chi_H = d_1\psi_1 + \cdots + d_r\psi_r$, where the non-negative integers $d_1, \ldots, d_r$ satisfy the inequality

$$\sum_{i=1}^{r} d_i^2 \leqslant |G : H|. \tag{1}$$

Prove that there is equality in (1) if and only if $\chi(g) = 0$ for all elements g of G which lie outside H.

(b) Let ψ be a class function of H. Define the induced class function, $\mathrm{Ind}_H^G \psi$.

State the Frobenius reciprocity theorem for class functions and deduce that if ψ is a character of H then $\mathrm{Ind}_H^G \psi$ is a character of G.

Assuming ψ is a character, identify a G-space affording the character $\mathrm{Ind}_H^G \psi$. Briefly justify your answer.

(c) Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G and let ψ be an irreducible character of H. Show that the integers $e_1, \ldots, e_k$, which are given by $\mathrm{Ind}_H^G(\psi) = e_1\chi_1 + \cdots + e_k\chi_k$, satisfy

$$\sum_{i=1}^{k} e_i^2 \leqslant |G : H|.$$

1/II/19G Representation Theory

For a complex representation V of a finite group G, define the action of G on the dual representation V^*. If α denotes the character of V, compute the character β of V^*.

[Your formula should express $\beta(g)$ just in terms of the character α.]

Using your formula, how can you tell from the character whether a given representation is self-dual, that is, isomorphic to the dual representation?

Let V be an irreducible representation of G. Show that the trivial representation occurs as a summand of $V \otimes V$ with multiplicity either 0 or 1. Show that it occurs once if and only if V is self-dual.

For a self-dual irreducible representation V, show that V either has a nondegenerate G-invariant symmetric bilinear form or a nondegenerate G-invariant alternating bilinear form, but not both.

If V is an irreducible self-dual representation of odd dimension n, show that the corresponding homomorphism $G \to GL(n, \mathbf{C})$ is conjugate to a homomorphism into the orthogonal group $O(n, \mathbf{C})$. Here $O(n, \mathbf{C})$ means the subgroup of $GL(n, \mathbf{C})$ that preserves a nondegenerate symmetric bilinear form on $\mathbf{C}^n$.

2/II/19G Representation Theory

A finite group G of order 360 has conjugacy classes $C_1 = \{1\}, C_2, \ldots, C_7$ of sizes 1, 45, 40, 40, 90, 72, 72. The values of four of its irreducible characters are given in the following table.

C_1	C_2	C_3	C_4	C_5	C_6	C_7
5	1	2	-1	-1	0	0
8	0	-1	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$
8	0	-1	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
10	-2	1	1	0	0	0

Complete the character table.

[*Hint: it will not suffice just to use orthogonality of characters.*]

Deduce that the group G is simple.

3/II/19G Representation Theory

Let V_2 denote the irreducible representation $\mathrm{Sym}^2(\mathbb{C}^2)$ of $SU(2)$; thus V_2 has dimension 3. Compute the character of the representation $\mathrm{Sym}^n(V_2)$ of $SU(2)$ for any $n \geqslant 0$. Compute the dimension of the invariants $\mathrm{Sym}^n(V_2)^{SU(2)}$, meaning the subspace of $\mathrm{Sym}^n(V_2)$ where $SU(2)$ acts trivially.

Hence, or otherwise, show that the ring of complex polynomials in three variables x, y, z which are invariant under the action of $SO(3)$ is a polynomial ring. Find a generator for this polynomial ring.

4/II/19G Representation Theory

(a) Let A be a normal subgroup of a finite group G, and let V be an irreducible representation of G. Show that either V restricted to A is isotypic (a sum of copies of one irreducible representation of A), or else V is induced from an irreducible representation of some proper subgroup of G.

(b) Using (a), show that every (complex) irreducible representation of a p-group is induced from a 1-dimensional representation of some subgroup.

[You may assume that a nonabelian p-group G has an abelian normal subgroup A which is not contained in the centre of G.]

1/II/19H Representation Theory

A finite group G has seven conjugacy classes $C_1 = \{e\}, C_2, \ldots, C_7$ and the values of five of its irreducible characters are given in the following table.

C_1	C_2	C_3	C_4	C_5	C_6	C_7
1	1	1	1	1	1	1
1	1	1	1	-1	-1	-1
4	0	1	-1	2	-1	0
4	0	1	-1	-2	1	0
5	1	-1	0	1	1	-1

Calculate the number of elements in the various conjugacy classes and complete the character table.

[*You may not identify G with any known group, unless you justify doing so.*]

2/II/19H Representation Theory

Let G be a finite group and let Z be its centre. Show that if ρ is a complex irreducible representation of G, assumed to be faithful (that is, the kernel of ρ is trivial), then Z is cyclic.

Now assume that G is a p-group (that is, the order of G is a power of the prime p), and assume that Z is cyclic. If ρ is a faithful representation of G, show that some irreducible component of ρ is faithful.

[*You may use without proof the fact that, since G is a p-group, Z is non-trivial and any non-trivial normal subgroup of G intersects Z non-trivially.*]

Deduce that a finite p-group has a faithful irreducible representation if and only if its centre is cyclic.

3/II/19H **Representation Theory**

Let G be a finite group with a permutation action on the set X. Describe the corresponding permutation character π_X. Show that the multiplicity in π_X of the principal character 1_G equals the number of orbits of G on X.

Assume that G is transitive on X, with $|X| > 1$. Show that G contains an element g which is fixed-point-free on X, that is, $g\alpha \neq \alpha$ for all α in X.

Assume that $\pi_X = 1_G + m\chi$, with χ an irreducible character of G, for some natural number m. Show that $m = 1$.

[*You may use without proof any facts about algebraic integers, provided you state them correctly.*]

Explain how the action of G on X induces an action of G on X^2. Assume that G has r orbits on X^2. If now

$$\pi_X = 1_G + m_2\chi_2 + \ldots + m_k\chi_k,$$

with $1_G, \chi_2, \ldots, \chi_k$ distinct irreducible characters of G, and $m_2, \ldots, m_k$ natural numbers, show that $r = 1 + m_2^2 + \ldots + m_k^2$. Deduce that, if $r \leqslant 5$, then $k = r$ and $m_2 = \ldots = m_k = 1$.

4/II/19H **Representation Theory**

Write an essay on the representation theory of SU_2.

Your answer should include a description of each irreducible representation and an explanation of how to decompose arbitrary representations into a direct sum of these.

1/II/19F **Representation Theory**

(a) Let G be a finite group and X a finite set on which G acts. Define the permutation representation $\mathbb{C}[X]$ and compute its character.

(b) Let G and U be the following subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$, where p is a prime,

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\} \ , \quad U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{F}_p \right\} \ .$$

(i) Decompose $\mathbb{C}[G/U]$ into irreducible representations.

(ii) Let $\psi : U \to \mathbb{C}^\times$ be a non-trivial, one-dimensional representation. Determine the character of the induced representation $\mathrm{Ind}_U^G \psi$, and decompose $\mathrm{Ind}_U^G \psi$ into irreducible representations.

(iii) List all of the irreducible representations of G and show that your list is complete.

2/II/19F **Representation Theory**

(a) Let G be S_4, the symmetric group on four letters. Determine the character table of G.
[*Begin by listing the conjugacy classes and their orders.*]

(b) For each irreducible representation V of $G = S_4$, decompose $\mathrm{Res}_{A_4}^{S_4}(V)$ into irreducible representations. You must justify your answer.

3/II/19F **Representation Theory**

(a) Let $G = \mathrm{SU}_2$, and let V_n be the space of homogeneous polynomials of degree n in the variables x and y. Thus $\dim V_n = n + 1$. Define the action of G on V_n and show that V_n is an irreducible representation of G.

(b) Decompose $V_3 \otimes V_3$ into irreducible representations. Decompose $\wedge^2 V_3$ and $\mathrm{S}^2 V_3$ into irreducible representations.

(c) Given any representation V of a group G, define the dual representation V^*. Show that V_n^* is isomorphic to V_n as a representation of SU_2.

[*You may use any results from the lectures provided that you state them clearly.*]

4/II/19F **Representation Theory**

In this question, all vector spaces will be complex.

(a) Let A be a finite abelian group.

(i) Show directly from the definitions that any irreducible representation must be one-dimensional.

(ii) Show that A has a faithful one-dimensional representation if and only if A is cyclic.

(b) Now let G be an arbitrary finite group and suppose that the centre of G is non-trivial. Write $Z = \{z \in G \mid zg = gz \quad \forall g \in G\}$ for this centre.

(i) Let W be an irreducible representation of G. Show that $\mathrm{Res}_Z^G W = \dim W . \chi$, where χ is an irreducible representation of Z.

(ii) Show that every irreducible representation of Z occurs in this way.

(iii) Suppose that Z is not a cyclic group. Show that there does not exist an irreducible representation W of G such that every irreducible representation V occurs as a summand of $W^{\otimes n}$ for some n.

1/II/19G Representation Theory

Let the finite group G act on finite sets X and Y, and denote by $\mathbb{C}[X]$, $\mathbb{C}[Y]$ the associated permutation representations on the spaces of complex functions on X and Y. Call their characters χ_X and χ_Y.

(i) Show that the inner product $\langle \chi_X | \chi_Y \rangle$ is the number of orbits for the diagonal action of G on $X \times Y$.

(ii) Assume that $|X| > 1$, and let $S \subset \mathbb{C}[X]$ be the subspace of those functions whose values sum to zero. By considering $\|\chi_X\|^2$, show that S is irreducible if and only if the G-action on X is *doubly transitive*: this means that for any two pairs (x_1, x_2) and (x_1', x_2') of points in X with $x_1 \neq x_2$ and $x_1' \neq x_2'$, there exists some $g \in G$ with $gx_1 = x_1'$ and $gx_2 = x_2'$.

(iii) Let now $G = S_n$ acting on the set $X = \{1, 2, \ldots, n\}$. Call Y the set of 2-element subsets of X, with the natural action of S_n. If $n \geqslant 4$, show that $\mathbb{C}[Y]$ decomposes under S_n into three irreducible representations, one of which is the trivial representation and another of which is S. What happens when $n = 3$?

[*Hint: Consider* $\langle 1 | \chi_Y \rangle$, $\langle \chi_X | \chi_Y \rangle$ *and* $\|\chi_Y\|^2$.]

2/II/19G Representation Theory

Let G be a finite group and $\{\chi_i\}$ the set of its irreducible characters. Also choose representatives g_j for the conjugacy classes, and denote by $Z(g_j)$ their centralisers.

(i) State the orthogonality and completeness relations for the χ_k.

(ii) Using Part (i), or otherwise, show that

$$\sum\nolimits_i \overline{\chi_i(g_j)} \cdot \chi_i(g_k) = \delta_{jk} \cdot |Z(g_j)| \,.$$

(iii) Let A be the matrix with $A_{ij} = \chi_i(g_j)$. Prove that

$$|\det A|^2 = \prod\nolimits_j |Z(g_j)|.$$

(iv) Show that $\det A$ is either real or purely imaginary, explaining when each situation occurs.

[*Hint for (iv): Consider the effect of complex conjugation on the rows of the matrix A.*]

3/II/19G **Representation Theory**

Let G be the group with 21 elements generated by a and b, subject to the relations $a^7 = b^3 = 1$ and $ba = a^2b$.

(i) Find the conjugacy classes of G.

(ii) Find three non-isomorphic one-dimensional representations of G.

(iii) For a subgroup H of a finite group K, write down (without proof) the formula for the character of the K-representation induced from a representation of H.

(iv) By applying Part (iii) to the case when H is the subgroup $\langle a \rangle$ of $K = G$, find the remaining irreducible characters of G.

4/II/19G **Representation Theory**

(i) State and prove the Weyl integration formula for $SU(2)$.

(ii) Determine the characters of the symmetric powers of the standard 2-dimensional representation of $SU(2)$ and prove that they are irreducible.

[*Any general theorems from the course may be used.*]