Part II

Probability and Measure

Paper 1, Section II

27K Probability and Measure

(a) State and prove Dynkin's lemma.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that if $\mathcal{A}_1, \mathcal{A}_2$ are π-systems contained in $\mathcal{F}$ such that

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) \quad \text{for all } A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2,$$

then the generated σ-algebras $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.

Paper 2, Section II

27K Probability and Measure

(a) Denote by $L^1(\mathbb{R}^d)$ the space of Lebesgue integrable functions on $\mathbb{R}^d$. For $f \in L^1(\mathbb{R}^d)$ with Fourier transform $\hat{f} \in L^1(\mathbb{R}^d)$, state (without proof) the Fourier inversion theorem and deduce Plancherel's identity for such f from it. Argue that if f is continuous, then the inversion formula holds everywhere.

(b) Show that the integral

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{4\sin^2(u/2)}{u^2} du, \; x \in \mathbb{R},$$

exists, and vanishes whenever $|x| > 1$. What is $\|g\|_2^2$? Justify your answers.

Paper 3, Section II

26K Probability and Measure

(a) State (without proof) Birkhoff's ergodic theorem. Show that convergence in that theorem holds in $L^1(\mu)$, whenever μ is a probability measure. [You may use convergence results for integrals without proof, provided they are clearly stated.]

(b) Now consider $(0, 1]$ equipped with its Borel σ-algebra $\mathcal{B}$ and Lebesgue measure μ. For $A \in \mathcal{B}$, $a \in (0, 1] \setminus \mathbb{Q}$, and

$$\theta(x) = x + a \bmod 1, \quad x \in (0, 1],$$

determine the μ-almost everywhere limit of $S_n(1_A)/n$ as $n \to \infty$, where

$$S_n(1_A) = 1_A + 1_A \circ \theta + \ldots 1_A \circ \theta^{n-1}.$$

[You may use without proof that θ is ergodic.]

(c) If $A = (a, b]$ for $0 < a < b < 1$, show that convergence in the last limit in fact occurs *everywhere* on $(0, 1]$. [*Hint: Use your result from (b) with* $A_k = (a + k^{-1}, b - k^{-1}]$ *for all* k *large enough.*]

Paper 4, Section II

26K Probability and Measure

(a) Let $(Y_n : n \in \mathbb{N})$ be an infinite sequence of i.i.d. random variables such that $\mathbb{E}|Y_1| = \infty$. Show that $\limsup_{n\to\infty} |Y_1 + \cdots + Y_n|/n = \infty$ almost surely.

(b) Show that one can find $(Y_n : n \in \mathbb{N})$ as in part (a) but such that $(Y_1 + \cdots + Y_n)/n$ converges weakly to some random variable Z.

[*You may use theorems from lectures provided you state them clearly.*]

Paper 1, Section II
27G Probability and Measure

(a) State and prove *Kolmogorov's zero-one law.*

(b) Consider the product space $E = \mathbb{R}^{\mathbb{N}}$ equipped with the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinder sets

$$\mathcal{C} = \left\{ A = \times_{n=1}^{\infty} A_n \,|\, A_n \subseteq \mathbb{R}, A_n \text{ Borel for } n \leqslant N, A_n = \mathbb{R} \text{ for } n > N, \text{ some } N \in \mathbb{N} \right\}.$$

For m a probability measure on $\mathbb{R}$, show that there exists a unique product measure μ on $(E, \sigma(\mathcal{C}))$ for which $\mu(A) = \prod_{n=1}^{\infty} m(A_n)$ for all $A \in \mathcal{C}$. Show further that the shift map θ defined on E by $\theta((x_1, x_2, \dots)) = (x_2, x_3, \dots)$ is measure-preserving and ergodic for μ.

[*You may use without proof the existence of an infinite sequence of i.i.d. real random variables defined on any probability space.*]

Paper 2, Section II
27G Probability and Measure

(a) State and prove the *monotone convergence theorem.*

(b) Let f_1 be a μ-integrable function and let f be a measurable function defined on some measure space $(E, \mathcal{E}, \mu)$. Suppose the sequence $(f_n : n \in \mathbb{N})$ of measurable functions on E is such that $f_n \uparrow f$ pointwise on E as $n \to \infty$. Show that $\mu(f_n) \uparrow \mu(f)$ as $n \to \infty$. Show that the conclusion may fail if f_1 is not integrable.

Paper 3, Section II
26G Probability and Measure

Suppose that as $n \to \infty$, a sequence of real random variables $X_n \to^d X$, i.e. X_n converges in distribution to some limiting random variable X. Suppose further that as $n \to \infty$ a sequence of real random variables $Y_n \to^P c$, i.e. Y_n converges in probability to some constant (non-random) limit $c > 0$. Show that $X_n Y_n \to^d cX$ as $n \to \infty$.

Now let $(Z_n : n \in \mathbb{N})$ be i.i.d. real random variables with $\mathbb{E} Z_i = 0$ and finite variance $\mathrm{Var}(Z_i) = 1$ for all i. Show that

$$\frac{\sqrt{n} \sum_{i=1}^{n} Z_i}{\sum_{i=1}^{n} Z_i^2} \to^d N(0,1)$$

as $n \to \infty$, where $N(0,1)$ denotes the standard normal distribution.

[*You may use the strong law of large numbers and the central limit theorem without proof, provided they are clearly stated. You may further use without proof the equivalence of weak convergence of laws of probability measures and convergence in distribution for real random variables.*]

Paper 4, Section II
26G Probability and Measure

Denote by L^1 the space of real-valued functions on $\mathbb{R}$ that are integrable with respect to Lebesgue measure. For $f \in L^1$ and g_t the probability density function of a normal $N(0,t)$ random variable with variance $t > 0$, show that their convolution

$$f * g_t(x) = \int_{\mathbb{R}} f(x-y) g_t(y) dy\,,\ x \in \mathbb{R},$$

defines another element of L^1. Show carefully that the Fourier inversion theorem holds for $f * g_t$.

Now suppose that the Fourier transform of f is also in L^1. Show that $f * g_t(x) \to f(x)$ for almost every $x \in \mathbb{R}$ as $t \to 0$.

[*You may use Fubini's theorem and the translation invariance of Lebesgue measure without proof.*]

Paper 1, Section II
27H Probability and Measure

(a) State and prove Fatou's lemma. [You may use the monotone convergence theorem without proof, provided it is clearly stated.]

(b) Show that the inequality in Fatou's lemma can be strict.

(c) Let $(X_n : n \in \mathbb{N})$ and X be non-negative random variables such that $X_n \to X$ almost surely as $n \to \infty$. Must we have $\mathbb{E}X \leqslant \sup_n \mathbb{E}X_n$?

Paper 2, Section II
27H Probability and Measure

Let $(E, \mathcal{E}, \mu)$ be a measure space. A function f is *simple* if it is of the form $f = \sum_{i=1}^{N} a_i 1_{A_i}$, where $a_i \in \mathbb{R}, N \in \mathbb{N}$ and $A_i \in \mathcal{E}$.

Now let $f : (E, \mathcal{E}, \mu) \to [0, \infty]$ be a Borel-measurable map. Show that there exists a sequence f_n of simple functions such that $f_n(x) \to f(x)$ for all $x \in E$ as $n \to \infty$.

Next suppose f is also μ-integrable. Construct a sequence f_n of simple μ-integrable functions such that $\int_E |f_n - f| d\mu \to 0$ as $n \to \infty$.

Finally, suppose f is also bounded. Show that there exists a sequence f_n of simple functions such that $f_n \to f$ uniformly on E as $n \to \infty$.

Paper 3, Section II
26H Probability and Measure

Show that random variables $X_1, \ldots, X_N$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are independent if and only if

$$\mathbb{E}\Big(\prod_{n=1}^{N} f_n(X_n)\Big) = \prod_{n=1}^{N} \mathbb{E}\big(f_n(X_n)\big)$$

for all bounded measurable functions $f_n : \mathbb{R} \to \mathbb{R}, n = 1, \ldots, N$.

Now let $(X_n : n \in \mathbb{N})$ be an infinite sequence of independent Gaussian random variables with zero means, $\mathbb{E}X_n = 0$, and finite variances, $\mathbb{E}X_n^2 = \sigma_n^2 > 0$. Show that the series $\sum_{n=1}^{\infty} X_n$ converges in $L^2(\mathbb{P})$ if and only if $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$.

[*You may use without proof that* $\mathbb{E}[e^{iuX_n}] = e^{-u^2\sigma_n^2/2}$ *for* $u \in \mathbb{R}$.]

Paper 4, Section II

26H Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that for any sequence $A_n \in \mathcal{F}$ satisfying $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ one necessarily has $\mathbb{P}(\limsup_n A_n) = 0$.

Let $(X_n : n \in \mathbb{N})$ and X be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $X_n \to X$ almost surely as $n \to \infty$ implies that $X_n \to X$ in probability as $n \to \infty$.

Show that $X_n \to X$ in probability as $n \to \infty$ if and only if for every subsequence $X_{n(k)}$ there exists a further subsequence $X_{n(k(r))}$ such that $X_{n(k(r))} \to X$ almost surely as $r \to \infty$.

Paper 1, Section II

27K Probability and Measure

(a) Let $(X, \mathcal{F}, \nu)$ be a probability space. State the definition of the space $\mathbb{L}^2(X, \mathcal{F}, \nu)$. Show that it is a Hilbert space.

(b) Give an example of two real random variables Z_1, Z_2 that are not independent and yet have the same law.

(c) Let $Z_1, \ldots, Z_n$ be n random variables distributed uniformly on $[0,1]$. Let λ be the Lebesgue measure on the interval $[0,1]$, and let $\mathcal{B}$ be the Borel σ-algebra. Consider the expression

$$D(f) := \mathrm{Var}\Big[\frac{1}{n}(f(Z_1) + \ldots + f(Z_n)) - \int_{[0,1]} f d\lambda\Big]$$

where Var denotes the variance and $f \in \mathbb{L}^2([0,1], \mathcal{B}, \lambda)$.

Assume that $Z_1, \ldots, Z_n$ are pairwise independent. Compute $D(f)$ in terms of the variance $\mathrm{Var}(f) := \mathrm{Var}(f(Z_1))$.

(d) Now we no longer assume that $Z_1, \ldots, Z_n$ are pairwise independent. Show that

$$\sup D(f) \geqslant \frac{1}{n},$$

where the supremum ranges over functions $f \in \mathbb{L}^2([0,1], \mathcal{B}, \lambda)$ such that $\|f\|_2 = 1$ and $\int_{[0,1]} f d\lambda = 0$.

[Hint: you may wish to compute $D(f_{p,q})$ for the family of functions $f_{p,q} = \sqrt{\frac{k}{2}}(1_{I_p} - 1_{I_q})$ where $1 \leqslant p, q \leqslant k$, $I_j = [\frac{j}{k}, \frac{j+1}{k})$ and 1_A denotes the indicator function of the subset A.]

Paper 2, Section II

26K Probability and Measure

Let X be a set. Recall that a Boolean algebra $\mathcal{B}$ of subsets of X is a family of subsets containing the empty set, which is stable under finite union and under taking complements. As usual, let $\sigma(\mathcal{B})$ be the σ-algebra generated by $\mathcal{B}$.

(a) State the definitions of a *σ-algebra*, that of a *measure* on a measurable space, as well as the definition of a *probability measure*.

(b) State Carathéodory's extension theorem.

(c) Let $(X, \mathcal{F}, \mu)$ be a probability measure space. Let $\mathcal{B} \subset \mathcal{F}$ be a Boolean algebra of subsets of X. Let $\mathcal{C}$ be the family of all $A \in \mathcal{F}$ with the property that for every $\epsilon > 0$, there is $B \in \mathcal{B}$ such that

$$\mu(A \triangle B) < \epsilon,$$

where $A \triangle B$ denotes the symmetric difference of A and B, i.e., $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(i) Show that $\sigma(\mathcal{B})$ is contained in $\mathcal{C}$. Show by example that this may fail if $\mu(X) = +\infty$.

(ii) Now assume that $(X, \mathcal{F}, \mu) = ([0,1], \mathcal{L}_{[0,1]}, m)$, where $\mathcal{L}_{[0,1]}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and m is the Lebesgue measure. Let $\mathcal{B}$ be the family of all finite unions of sub-intervals. Is it true that $\mathcal{C}$ is equal to $\mathcal{L}_{[0,1]}$ in this case? Justify your answer.

Paper 3, Section II

26K Probability and Measure

Let $(X, \mathcal{A}, m, T)$ be a probability measure preserving system.

(a) State what it means for $(X, \mathcal{A}, m, T)$ to be *ergodic*.

(b) State Kolmogorov's 0-1 law for a sequence of independent random variables. What does it imply for the canonical model associated with an i.i.d. random process?

(c) Consider the special case when $X = [0,1]$, $\mathcal{A}$ is the σ-algebra of Borel subsets, and T is the map defined as

$$Tx = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}], \\ 2 - 2x, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

(i) Check that the Lebesgue measure m on $[0,1]$ is indeed an invariant probability measure for T.

(ii) Let $X_0 := 1_{(0,\frac{1}{2})}$ and $X_n := X_0 \circ T^n$ for $n \geqslant 1$. Show that $(X_n)_{n \geqslant 0}$ forms a sequence of i.i.d. random variables on $(X, \mathcal{A}, m)$, and that the σ-algebra $\sigma(X_0, X_1, \ldots)$ is all of $\mathcal{A}$. *[Hint: check first that for any integer $n \geqslant 0$, $T^{-n}(0, \frac{1}{2})$ is a disjoint union of 2^n intervals of length $1/2^{n+1}$.]*

(iii) Is $(X, \mathcal{A}, m, T)$ ergodic? Justify your answer.

Paper 4, Section II

26K Probability and Measure

(a) State and prove the strong law of large numbers for sequences of i.i.d. random variables with a finite moment of order 4.

(b) Let $(X_k)_{k\geqslant 1}$ be a sequence of independent random variables such that

$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}.$$

Let $(a_k)_{k\geqslant 1}$ be a sequence of real numbers such that

$$\sum_{k\geqslant 1} a_k^2 < \infty.$$

Set

$$S_n := \sum_{k=1}^{n} a_k X_k.$$

(i) Show that S_n converges in $\mathbb{L}^2$ to a random variable S as $n \to \infty$. Does it converge in $\mathbb{L}^1$? Does it converge in law?

(ii) Show that $\|S\|_4 \leqslant 3^{1/4}\|S\|_2$.

(iii) Let $(Y_k)_{k\geqslant 1}$ be a sequence of i.i.d. standard Gaussian random variables, i.e. each Y_k is distributed as $\mathcal{N}(0,1)$. Show that then $\sum_{k=1}^{n} a_k Y_k$ converges in law as $n \to \infty$ to a random variable and determine the law of the limit.

Paper 2, Section II

26K Probability and Measure

(a) Let $(X_i, \mathcal{A}_i)$ for $i = 1, 2$ be two measurable spaces. Define the product σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ on the Cartesian product $X_1 \times X_2$. Given a probability measure μ_i on $(X_i, \mathcal{A}_i)$ for each $i = 1, 2$, define the product measure $\mu_1 \otimes \mu_2$. Assuming the existence of a product measure, explain why it is unique. [You may use standard results from the course if clearly stated.]

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which the real random variables U and V are defined. Explain what is meant when one says that U has law μ. On what measurable space is the measure μ defined? Explain what it means for U and V to be independent random variables.

(c) Now let $X = [-\frac{1}{2}, \frac{1}{2}]$, let $\mathcal{A}$ be its Borel σ-algebra and let μ be Lebesgue measure. Give an example of a measure η on the product $(X \times X, \mathcal{A} \otimes \mathcal{A})$ such that $\eta(X \times A) = \mu(A) = \eta(A \times X)$ for every Borel set A, but such that η is *not* Lebesgue measure on $X \times X$.

(d) Let η be as in part (c) and let $I, J \subset X$ be intervals of length x and y respectively. Show that

$$x + y - 1 \leqslant \eta(I \times J) \leqslant \min\{x, y\}.$$

(e) Let X be as in part (c). Fix $d \geqslant 2$ and let Π_i denote the projection $\Pi_i(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$ from X^d to X^{d-1}. Construct a probability measure η on X^d, such that the image under each Π_i coincides with the $(d-1)$-dimensional Lebesgue measure, while η itself is *not* the d-dimensional Lebesgue measure. [*Hint: Consider the following collection of* $2d-1$ *independent random variables:* $U_1, \ldots, U_d$ *uniformly distributed on* $[0, \frac{1}{2}]$, *and* $\varepsilon_1, \ldots, \varepsilon_{d-1}$ *such that* $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$ *for each* i.]

Paper 3, Section II

26K Probability and Measure

(a) Let X and Y be real random variables such that $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$ for every compactly supported continuous function f. Show that X and Y have the same law.

(b) Given a real random variable Z, let $\varphi_Z(s) = \mathbb{E}(e^{isZ})$ be its characteristic function. Prove the identity

$$\iint g(\varepsilon s) f(x) e^{-isx} \varphi_Z(s) ds\, dx = \int \hat{g}(t)\ \mathbb{E}[f(Z - \varepsilon t)] dt$$

for real $\varepsilon > 0$, where is f is continuous and compactly supported, and where g is a Lebesgue integrable function such that $\hat{g}$ is also Lebesgue integrable, where

$$\hat{g}(t) = \int g(x) e^{itx} dx$$

is its Fourier transform. Use the above identity to derive a formula for $\mathbb{E}[f(Z)]$ in terms of φ_Z, and recover the fact that φ_Z determines the law of Z uniquely.

(c) Let X and Y be bounded random variables such that $\mathbb{E}(X^n) = \mathbb{E}(Y^n)$ for every positive integer n. Show that X and Y have the same law.

(d) The Laplace transform $\psi_Z(s)$ of a non-negative random variable Z is defined by the formula

$$\psi_Z(s) = \mathbb{E}(e^{-sZ})$$

for $s \geqslant 0$. Let X and Y be (possibly unbounded) non-negative random variables such that $\psi_X(s) = \psi_Y(s)$ for all $s \geqslant 0$. Show that X and Y have the same law.

(e) Let

$$f(x;k) = 1_{\{x>0\}} \frac{1}{k!} x^k e^{-x}$$

where k is a non-negative integer and $1_{\{x>0\}}$ is the indicator function of the interval $(0, +\infty)$.

Given non-negative integers $k_1, \dots, k_n$, suppose that the random variables $X_1, \dots, X_n$ are independent with X_i having density function $f(\cdot; k_i)$. Find the density of the random variable $X_1 + \cdots + X_n$.

Paper 4, Section II
26K Probability and Measure

(a) Let $(X_n)_{n \geqslant 1}$ and X be real random variables with finite second moment on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that X_n converges to X almost surely. Show that the following assertions are equivalent:

(i) $X_n \to X$ in $\mathbf{L}^2$ as $n \to \infty$,

(ii) $\mathbb{E}(X_n^2) \to \mathbb{E}(X^2)$ as $n \to \infty$.

(b) Suppose now that $\Omega = (0,1)$, $\mathcal{F}$ is the Borel σ-algebra of $(0,1)$ and $\mathbb{P}$ is Lebesgue measure. Given a Borel probability measure μ on $\mathbb{R}$ we set

$$X_\mu(\omega) = \inf\{x \in \mathbb{R} | F_\mu(x) \geqslant \omega\},$$

where $F_\mu(x) := \mu((-\infty, x])$ is the distribution function of μ and $\omega \in \Omega$.

(i) Show that X_μ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with law μ.

(ii) Let $(\mu_n)_{n \geqslant 1}$ and ν be Borel probability measures on $\mathbb{R}$ with finite second moments. Show that

$$\mathbb{E}((X_{\mu_n} - X_\nu)^2) \to 0 \text{ as } n \to \infty$$

if and only if μ_n converges weakly to ν and $\int x^2 d\mu_n(x)$ converges to $\int x^2 d\nu(x)$ as $n \to \infty$.

[You may use any theorem proven in lectures as long as it is clearly stated. Furthermore, you may use without proof the fact that μ_n converges weakly to ν as $n \to \infty$ if and only if X_{μ_n} converges to X_ν almost surely.]

Paper 1, Section II

27K Probability and Measure

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued random variable. Given $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ we let

$$\phi_{\mathbf{X}}(u) = \mathbb{E}(e^{i\langle u, \mathbf{X}\rangle})$$

be its characteristic function, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^d$.

(a) Suppose $\mathbf{X}$ is a Gaussian vector with mean 0 and covariance matrix $\sigma^2 I_d$, where $\sigma > 0$ and I_d is the $d \times d$ identity matrix. What is the formula for the characteristic function $\phi_{\mathbf{X}}$ in the case $d = 1$? Derive from it a formula for $\phi_{\mathbf{X}}$ in the case $d \geqslant 2$.

(b) We now no longer assume that $\mathbf{X}$ is necessarily a Gaussian vector. Instead we assume that the X_i's are independent random variables and that the random vector $A\mathbf{X}$ has the same law as $\mathbf{X}$ for every orthogonal matrix A. Furthermore we assume that $d \geqslant 2$.

(i) Show that there exists a continuous function $f : [0, +\infty) \to \mathbb{R}$ such that

$$\phi_{\mathbf{X}}(u) = f(u_1^2 + \ldots + u_d^2).$$

[You may use the fact that for every two vectors $u, v \in \mathbb{R}^d$ such that $\langle u, u\rangle = \langle v, v\rangle$ there is an orthogonal matrix A such that $Au = v$.]

(ii) Show that for all $r_1, r_2 \geqslant 0$

$$f(r_1 + r_2) = f(r_1)f(r_2).$$

(iii) Deduce that f takes values in $(0, 1]$, and furthermore that there exists $\alpha \geqslant 0$ such that $f(r) = e^{-r\alpha}$, for all $r \geqslant 0$.

(iv) What must be the law of $\mathbf{X}$?

[Standard properties of characteristic functions from the course may be used without proof if clearly stated.]

Paper 4, Section II

26J Probability and Measure

Let $(X, \mathcal{A})$ be a measurable space. Let $T : X \to X$ be a measurable map, and μ a probability measure on $(X, \mathcal{A})$.

(a) State the definition of the following properties of the system $(X, \mathcal{A}, \mu, T)$:

(i) μ is T-*invariant*.

(ii) T is *ergodic* with respect to μ.

(b) State the pointwise ergodic theorem.

(c) Give an example of a probability measure preserving system $(X, \mathcal{A}, \mu, T)$ in which $\mathrm{Card}(T^{-1}\{x\}) > 1$ for μ-a.e. x.

(d) Assume X is finite and $\mathcal{A}$ is the boolean algebra of all subsets of X. Suppose that μ is a T-invariant probability measure on X such that $\mu(\{x\}) > 0$ for all $x \in X$. Show that T is a bijection.

(e) Let $X = \mathbb{N}$, the set of positive integers, and $\mathcal{A}$ be the σ-algebra of all subsets of X. Suppose that μ is a T-invariant ergodic probability measure on X. Show that there is a finite subset $Y \subseteq X$ with $\mu(Y) = 1$.

Paper 2, Section II

26J Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables with $\mathbb{E}(|X_n|^2) \leqslant 1$ for all $n \geqslant 1$.

(a) Suppose Z is another random variable such that $\mathbb{E}(|Z|^2) < \infty$. Why is ZX_n integrable for each n?

(b) Assume $\mathbb{E}(ZX_n) \xrightarrow[n\to\infty]{} 0$ for every random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(|Z|^2) < \infty$. Show that there is a subsequence $Y_k := X_{n_k}$, $k \geqslant 1$, such that

$$\frac{1}{N}\sum_{k=1}^{N} Y_k \xrightarrow[N\to\infty]{} 0 \text{ in } \mathbb{L}^2.$$

(c) Assume that $X_n \to X$ in probability. Show that $X \in \mathbb{L}^2$. Show that $X_n \to X$ in $\mathbb{L}^1$. Must it converge also in $\mathbb{L}^2$? Justify your answer.

(d) Assume that the $(X_n)_{n\geqslant 1}$ are independent. Give a necessary and sufficient condition on the sequence $(\mathbb{E}(X_n)_{n\geqslant 1})$ for the sequence

$$Y_N = \frac{1}{N}\sum_{k=1}^{N} X_k$$

to converge in $\mathbb{L}^2$.

Paper 3, Section II

26J Probability and Measure

Let m be the Lebesgue measure on the real line. Recall that if $E \subseteq \mathbb{R}$ is a Borel subset, then

$$m(E) = \inf \left\{ \sum_{n \geqslant 1} |I_n|,\ E \subseteq \bigcup_{n \geqslant 1} I_n \right\},$$

where the infimum is taken over all covers of E by countably many intervals, and $|I|$ denotes the length of an interval I.

(a) State the definition of a *Borel subset* of $\mathbb{R}$.

(b) State a definition of a *Lebesgue measurable subset* of $\mathbb{R}$.

(c) Explain why the following sets are Borel and compute their Lebesgue measure:

$$\mathbb{Q}, \quad \mathbb{R} \setminus \mathbb{Q}, \quad \bigcap_{n \geqslant 2} \left[\frac{1}{n}, n \right].$$

(d) State the definition of a *Borel measurable function* $f : \mathbb{R} \to \mathbb{R}$.

(e) Let f be a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$. Is it true that the subset of all $x \in \mathbb{R}$ where f is continuous at x is a Borel subset? Justify your answer.

(f) Let $E \subseteq [0, 1]$ be a Borel subset with $m(E) = 1/2 + \alpha$, $\alpha > 0$. Show that

$$E - E := \{x - y : x, y \in E\}$$

contains the interval $(-2\alpha, 2\alpha)$.

(g) Let $E \subseteq \mathbb{R}$ be a Borel subset such that $m(E) > 0$. Show that for every $\varepsilon > 0$, there exists $a < b$ in $\mathbb{R}$ such that

$$m(E \cap (a, b)) > (1 - \varepsilon) m((a, b)).$$

Deduce that $E - E$ contains an open interval around 0.

Paper 1, Section II

27J Probability and Measure

(a) Let X be a real random variable with $\mathbb{E}(X^2) < \infty$. Show that the variance of X is equal to $\inf_{a \in \mathbb{R}} (\mathbb{E}(X-a)^2)$.

(b) Let $f(x)$ be the indicator function of the interval $[-1,1]$ on the real line. Compute the Fourier transform of f.

(c) Show that

$$\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \frac{\pi}{2}.$$

(d) Let X be a real random variable and $\widehat{\mu_X}$ be its characteristic function.

(i) Assume that $|\widehat{\mu_X}(u)| = 1$ for some $u \in \mathbb{R}$. Show that there exists $\theta \in \mathbb{R}$ such that almost surely:

$$uX \in \theta + 2\pi\mathbb{Z}.$$

(ii) Assume that $|\widehat{\mu_X}(u)| = |\widehat{\mu_X}(v)| = 1$ for some real numbers u, v not equal to 0 and such that u/v is irrational. Prove that X is almost surely constant. [*Hint: You may wish to consider an independent copy of X.*]

Paper 2, Section II

24J Probability and Measure

(a) Give the definition of the *Fourier transform* $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^d)$.

(b) Explain what it means for Fourier inversion to hold.

(c) Prove that Fourier inversion holds for $g_t(x) = (2\pi t)^{-d/2} e^{-\|x\|^2/(2t)}$. Show all of the steps in your computation. Deduce that Fourier inversion holds for Gaussian convolutions, i.e. any function of the form $f * g_t$ where $t > 0$ and $f \in L^1(\mathbb{R}^d)$.

(d) Prove that any function f for which Fourier inversion holds has a bounded, continuous version. In other words, there exists g bounded and continuous such that $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^d$.

(e) Does Fourier inversion hold for $f = \mathbf{1}_{[0,1]}$?

Paper 3, Section II

24J Probability and Measure

(a) Suppose that $\mathcal{X} = (X_n)$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Give the definition of what it means for $\mathcal{X}$ to be *uniformly integrable.*

(b) State and prove Hölder's inequality.

(c) Explain what it means for a family of random variables to be L^p *bounded.* Prove that an L^p bounded sequence is uniformly integrable provided $p > 1$.

(d) Prove or disprove: every sequence which is L^1 bounded is uniformly integrable.

Paper 4, Section II
25J Probability and Measure

(a) Suppose that $(E, \mathcal{E}, \mu)$ is a finite measure space and $\theta\colon E \to E$ is a measurable map. Prove that $\mu_\theta(A) = \mu(\theta^{-1}(A))$ defines a measure on $(E, \mathcal{E})$.

(b) Suppose that $\mathcal{A}$ is a π-system which generates $\mathcal{E}$. Using Dynkin's lemma, prove that θ is measure-preserving if and only if $\mu_\theta(A) = \mu(A)$ for all $A \in \mathcal{A}$.

(c) State Birkhoff's ergodic theorem and the maximal ergodic lemma.

(d) Consider the case $(E, \mathcal{E}, \mu) = ([0,1), \mathcal{B}([0,1)), \mu)$ where μ is Lebesgue measure on $[0,1)$. Let $\theta\colon [0,1) \to [0,1)$ be the following map. If $x = \sum_{n=1}^{\infty} 2^{-n}\omega_n$ is the binary expansion of x (where we disallow infinite sequences of 1s), then $\theta(x) = \sum_{n=1}^{\infty} 2^{-n}(\omega_{n-1}\mathbf{1}_{n\in E} + \omega_{n+1}\mathbf{1}_{n\in O})$ where E and O are respectively the even and odd elements of $\mathbb{N}$.

(i) Prove that θ is measure-preserving. [You may assume that θ is measurable.]

(ii) Prove or disprove: θ is ergodic.

Paper 1, Section II
26J Probability and Measure

(a) Give the definition of the *Borel σ-algebra* on $\mathbb{R}$ and a *Borel function* $f\colon E \to \mathbb{R}$ where $(E, \mathcal{E})$ is a measurable space.

(b) Suppose that (f_n) is a sequence of Borel functions which converges pointwise to a function f. Prove that f is a Borel function.

(c) Let $R_n\colon [0,1) \to \mathbb{R}$ be the function which gives the nth binary digit of a number in $[0,1)$ (where we do not allow for the possibility of an infinite sequence of 1s). Prove that R_n is a Borel function.

(d) Let $f\colon [0,1)^2 \to [0,\infty]$ be the function such that $f(x,y)$ for $x, y \in [0,1)^2$ is equal to the number of digits in the binary expansions of x, y which disagree. Prove that f is non-negative measurable.

(e) Compute the Lebesgue measure of $f^{-1}([0,\infty))$, i.e. the set of pairs of numbers in $[0,1)$ whose binary expansions disagree in a finite number of digits.

Paper 3, Section II

23J Probability and Measure

(a) Define the *Borel σ-algebra* $\mathcal{B}$ and the *Borel functions*.

(b) Give an example with proof of a set in $[0,1]$ which is not Lebesgue measurable.

(c) The Cantor set $\mathcal{C}$ is given by

$$\mathcal{C} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k} : (a_k) \text{ is a sequence with } a_k \in \{0,2\} \text{ for all } k \right\}.$$

(i) Explain why $\mathcal{C}$ is Lebesgue measurable.

(ii) Compute the Lebesgue measure of $\mathcal{C}$.

(iii) Is every subset of $\mathcal{C}$ Lebesgue measurable?

(iv) Let $f\colon [0,1] \to \mathcal{C}$ be the function given by

$$f(x) = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \quad \text{where} \quad a_k = \lfloor 2^k x \rfloor - 2\lfloor 2^{k-1} x \rfloor.$$

Explain why f is a Borel function.

(v) Using the previous parts, prove the existence of a Lebesgue measurable set which is not Borel.

Paper 4, Section II

24J Probability and Measure

Give the definitions of the *convolution* $f * g$ and of the *Fourier transform* $\widehat{f}$ of f, and show that $\widehat{f * g} = \widehat{f}\,\widehat{g}$. State what it means for Fourier inversion to hold for a function f.

State the Plancherel identity and compute the L^2 norm of the Fourier transform of the function $f(x) = e^{-x}\mathbf{1}_{[0,1]}$.

Suppose that (f_n), f are functions in L^1 such that $f_n \to f$ in L^1 as $n \to \infty$. Show that $\widehat{f_n} \to \widehat{f}$ uniformly.

Give the definition of weak convergence, and state and prove the Central Limit Theorem.

Paper 2, Section II

24J Probability and Measure

(a) State Jensen's inequality. Give the definition of $\|\cdot\|_{L^p}$ and the space L^p for $1 < p < \infty$. If $\|f - g\|_{L^p} = 0$, is it true that $f = g$? Justify your answer. State and prove Hölder's inequality using Jensen's inequality.

(b) Suppose that $(E, \mathcal{E}, \mu)$ is a finite measure space. Show that if $1 < q < p$ and $f \in L^p(E)$ then $f \in L^q(E)$. Give the definition of $\|\cdot\|_{L^\infty}$ and show that $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$.

(c) Suppose that $1 < q < p < \infty$. Show that if f belongs to both $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, then $f \in L^r(\mathbb{R})$ for any $r \in [q, p]$. If $f \in L^p(\mathbb{R})$, must we have $f \in L^q(\mathbb{R})$? Give a proof or a counterexample.

Paper 1, Section II

25J Probability and Measure

Throughout this question $(E, \mathcal{E}, \mu)$ is a measure space and (f_n), f are measurable functions.

(a) Give the definitions of *pointwise convergence*, *pointwise a.e. convergence*, and *convergence in measure*.

(b) If $f_n \to f$ pointwise a.e., does $f_n \to f$ in measure? Give a proof or a counterexample.

(c) If $f_n \to f$ in measure, does $f_n \to f$ pointwise a.e.? Give a proof or a counterexample.

(d) Now suppose that $(E, \mathcal{E}) = ([0,1], \mathcal{B}([0,1]))$ and that μ is Lebesgue measure on $[0,1]$. Suppose (f_n) is a sequence of Borel measurable functions on $[0,1]$ which converges pointwise a.e. to f.

(i) For each n, k let $E_{n,k} = \bigcup_{m \geqslant n} \{x : |f_m(x) - f(x)| > 1/k\}$. Show that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$ for each $k \in \mathbb{N}$.

(ii) Show that for every $\epsilon > 0$ there exists a set A with $\mu(A) < \epsilon$ so that $f_n \to f$ uniformly on $[0,1] \setminus A$.

(iii) Does (ii) hold with $[0,1]$ replaced by $\mathbb{R}$? Give a proof or a counterexample.

Paper 4, Section II
22J Probability and Measure

(a) State Fatou's lemma.

(b) Let X be a random variable on $\mathbb{R}^d$ and let $(X_k)_{k=1}^{\infty}$ be a sequence of random variables on $\mathbb{R}^d$. What does it mean to say that $X_k \to X$ *weakly*?

State and prove the Central Limit Theorem for i.i.d. real-valued random variables. [You may use auxiliary theorems proved in the course provided these are clearly stated.]

(c) Let X be a real-valued random variable with characteristic function φ. Let $(h_n)_{n=1}^{\infty}$ be a sequence of real numbers with $h_n \neq 0$ and $h_n \to 0$. Prove that if we have

$$\liminf_{n\to\infty} \frac{2\varphi(0) - \varphi(-h_n) - \varphi(h_n)}{h_n^2} < \infty,$$

then $\mathbb{E}[X^2] < \infty$.

Paper 3, Section II
22J Probability and Measure

(a) Let $(E, \mathcal{E}, \mu)$ be a measure space. What does it mean to say that $T\colon E \to E$ is a *measure-preserving transformation*? What does it mean to say that a set $A \in \mathcal{E}$ is *invariant under* T? Show that the class of invariant sets forms a σ-algebra.

(b) Take E to be $[0,1)$ with Lebesgue measure on its Borel σ-algebra. Show that the baker's map $T : [0,1) \to [0,1)$ defined by

$$T(x) = 2x - \lfloor 2x \rfloor$$

is measure-preserving.

(c) Describe in detail the construction of the canonical model for sequences of independent random variables having a given distribution m.

Define the Bernoulli shift map and prove it is a measure-preserving ergodic transformation.

[You may use without proof other results concerning sequences of independent random variables proved in the course, provided you state these clearly.]

Paper 2, Section II

23J Probability and Measure

(a) Let $(E, \mathcal{E}, \mu)$ be a measure space, and let $1 \leqslant p < \infty$. What does it mean to say that f belongs to $L^p(E, \mathcal{E}, \mu)$?

(b) State Hölder's inequality.

(c) Consider the measure space of the unit interval endowed with Lebesgue measure. Suppose $f \in L^2(0,1)$ and let $0 < \alpha < 1/2$.

(i) Show that for all $x \in \mathbb{R}$,

$$\int_0^1 |f(y)||x-y|^{-\alpha}\, dy < \infty\,.$$

(ii) For $x \in \mathbb{R}$, define

$$g(x) = \int_0^1 f(y)|x-y|^{-\alpha} dy\,.$$

Show that for $x \in \mathbb{R}$ fixed, the function g satisfies

$$|g(x+h) - g(x)| \leqslant \|f\|_2 \cdot (I(h))^{1/2},$$

where

$$I(h) = \int_0^1 \left(|x+h-y|^{-\alpha} - |x-y|^{-\alpha}\right)^2 dy.$$

(iii) Prove that g is a continuous function. [*Hint: You may find it helpful to split the integral defining $I(h)$ into several parts.*]

Paper 1, Section II

23J Probability and Measure

(a) Define the following concepts: a π-system, a d-system and a σ-algebra.

(b) State the Dominated Convergence Theorem.

(c) Does the set function

$$\mu(A) = \begin{cases} 0 & \text{for } A \text{ bounded,} \\ 1 & \text{for } A \text{ unbounded,} \end{cases}$$

furnish an example of a Borel measure?

(d) Suppose $g\colon [0,1] \to [0,1]$ is a measurable function. Let $f\colon [0,1] \to \mathbb{R}$ be continuous with $f(0) \leqslant f(1)$. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(\,g(x)^n)\,dx$$

exists and lies in the interval $[f(0), f(1)]$.

Paper 4, Section II
25K Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent identically distributed random variables. Set $S_n = X_1 + \cdots + X_n$.

(i) State the strong law of large numbers in terms of the random variables X_n.

(ii) Assume now that the X_n are non-negative and that their expectation is infinite. Let $R \in (0, \infty)$. What does the strong law of large numbers say about the limiting behaviour of S_n^R/n, where $S_n^R = (X_1 \wedge R) + \cdots + (X_n \wedge R)$?

Deduce that $S_n/n \to \infty$ almost surely.

Show that

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n \geqslant n) = \infty.$$

Show that $X_n \geqslant Rn$ infinitely often almost surely.

(iii) Now drop the assumption that the X_n are non-negative but continue to assume that $\mathbb{E}(|X_1|) = \infty$. Show that, almost surely,

$$\limsup_{n\to\infty} |S_n|/n = \infty.$$

Paper 3, Section II
25K Probability and Measure

(i) Let $(E, \mathcal{E}, \mu)$ be a measure space. What does it mean to say that a function $\theta : E \to E$ is a *measure-preserving transformation*?

What does it mean to say that θ is *ergodic*?

State Birkhoff's almost everywhere ergodic theorem.

(ii) Consider the set $E = (0, 1]^2$ equipped with its Borel σ-algebra and Lebesgue measure. Fix an irrational number $a \in (0, 1]$ and define $\theta : E \to E$ by

$$\theta(x_1, x_2) = (x_1 + a, x_2 + a),$$

where addition in each coordinate is understood to be modulo 1. Show that θ is a measure-preserving transformation. Is θ ergodic? Justify your answer.

Let f be an integrable function on E and let $\bar{f}$ be the invariant function associated with f by Birkhoff's theorem. Write down a formula for $\bar{f}$ in terms of f. [You are not expected to justify this answer.]

Paper 2, Section II

26K Probability and Measure

State and prove the monotone convergence theorem.

Let $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ be finite measure spaces. Define the *product* σ*-algebra* $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ on $E_1 \times E_2$.

Define the *product measure* $\mu = \mu_1 \otimes \mu_2$ on $\mathcal{E}$, and show carefully that μ is countably additive.

[You may use without proof any standard facts concerning measurability provided these are clearly stated.]

Paper 1, Section II

26K Probability and Measure

What is meant by the *Borel* σ*-algebra* on the real line $\mathbb{R}$?

Define the *Lebesgue measure* of a Borel subset of $\mathbb{R}$ using the concept of outer measure.

Let μ be the Lebesgue measure on $\mathbb{R}$. Show that, for any Borel set B which is contained in the interval $[0,1]$, and for any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and disjoint intervals $I_1, \ldots, I_n$ contained in $[0,1]$ such that, for $A = I_1 \cup \cdots \cup I_n$, we have

$$\mu(A \triangle B) \leqslant \varepsilon,$$

where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$.

Show that there does not exist a Borel set B contained in $[0,1]$ such that, for all intervals I contained in $[0,1]$,

$$\mu(B \cap I) = \mu(I)/2.$$

Paper 4, Section II
25K Probability and Measure

State Birkhoff's almost-everywhere ergodic theorem.

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = 1/2\,.$$

Define for $k \in \mathbb{N}$

$$Y_k = \sum_{n=1}^{\infty} X_{k+n-1}/2^n\,.$$

What is the distribution of Y_k? Show that the random variables Y_1 and Y_2 are not independent.

Set $S_n = Y_1 + \cdots + Y_n$. Show that S_n/n converges as $n \to \infty$ almost surely and determine the limit. [You may use without proof any standard theorem provided you state it clearly.]

Paper 3, Section II
25K Probability and Measure

Let X be an integrable random variable with $\mathbb{E}(X) = 0$. Show that the characteristic function ϕ_X is differentiable with $\phi_X'(0) = 0$. [You may use without proof standard convergence results for integrals provided you state them clearly.]

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, all having the same distribution as X. Set $S_n = X_1 + \cdots + X_n$. Show that $S_n/n \to 0$ in distribution. Deduce that $S_n/n \to 0$ in probability. [You may not use the Strong Law of Large Numbers.]

Paper 2, Section II
26K Probability and Measure

Let $(f_n : n \in \mathbb{N})$ be a sequence of non-negative measurable functions defined on a measure space $(E, \mathcal{E}, \mu)$. Show that $\liminf_n f_n$ is also a non-negative measurable function.

State the Monotone Convergence Theorem.

State and prove Fatou's Lemma.

Let $(f_n : n \in \mathbb{N})$ be as above. Suppose that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$. Show that

$$\mu(\min\{f_n, f\}) \to \mu(f)\,.$$

Deduce that, if f is integrable and $\mu(f_n) \to \mu(f)$, then f_n converges to f in L^1. [Still assume that f_n and f are as above.]

Paper 1, Section II

26K Probability and Measure

State Dynkin's π-system/d-system lemma.

Let μ and ν be probability measures on a measurable space $(E, \mathcal{E})$. Let $\mathcal{A}$ be a π-system on E generating $\mathcal{E}$. Suppose that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$. Show that $\mu = \nu$.

What does it mean to say that a sequence of random variables is independent?

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, all uniformly distributed on $[0, 1]$. Let Y be another random variable, independent of $(X_n : n \in \mathbb{N})$. Define random variables Z_n in $[0, 1]$ by $Z_n = (X_n + Y) \bmod 1$. What is the distribution of Z_1? Justify your answer.

Show that the sequence of random variables $(Z_n : n \in \mathbb{N})$ is independent.

Paper 4, Section II

25J Probability and Measure

State and prove Fatou's lemma. [You may use the monotone convergence theorem.]

For $(E, \mathcal{E}, \mu)$ a measure space, define $L^1 := L^1(E, \mathcal{E}, \mu)$ to be the vector space of μ-integrable functions on E, where functions equal almost everywhere are identified. Prove that L^1 is complete for the norm $\|\cdot\|_1$,

$$\|f\|_1 := \int_E |f| d\mu, \quad f \in L^1.$$

[You may assume that $\|\cdot\|_1$ indeed defines a norm on L^1.] Give an example of a measure space $(E, \mathcal{E}, \mu)$ and of a sequence $f_n \in L^1$ that converges to f almost everywhere such that $f \notin L^1$.

Now let

$$\mathcal{D} := \{f \in L^1 : f \geqslant 0 \text{ almost everywhere }, \int_E f d\mu = 1\}.$$

If a sequence $f_n \in \mathcal{D}$ converges to f in L^1, does it follow that $f \in \mathcal{D}$? If $f_n \in \mathcal{D}$ converges to f almost everywhere, does it follow that $f \in \mathcal{D}$? Justify your answers.

Paper 3, Section II

25J Probability and Measure

Carefully state and prove the first and second Borel–Cantelli lemmas.

Now let $(A_n : n \in \mathbb{N})$ be a sequence of events that are *pairwise independent*; that is, $\mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m)$ whenever $m \neq n$. For $N \geqslant 1$, let $S_N = \sum_{n=1}^{N} 1_{A_n}$. Show that $\mathrm{Var}(S_N) \leqslant \mathbb{E}(S_N)$.

Using Chebyshev's inequality or otherwise, deduce that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\lim_{N\to\infty} S_N = \infty$ almost surely. Conclude that $\mathbb{P}(A_n \text{ infinitely often}) = 1$.

Paper 2, Section II

26J Probability and Measure

The Fourier transform of a Lebesgue integrable function $f \in L^1(\mathbb{R})$ is given by

$$\hat{f}(u) = \int_{\mathbb{R}} f(x) e^{ixu} d\mu(x),$$

where μ is Lebesgue measure on the real line. For $f(x) = e^{-ax^2}, x \in \mathbb{R}, a > 0$, prove that

$$\hat{f}(u) = \sqrt{\frac{\pi}{a}} e^{-\frac{u^2}{4a}}.$$

[You may use properties of derivatives of Fourier transforms without proof provided they are clearly stated, as well as the fact that $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ is a probability density function.]

State and prove the almost everywhere Fourier inversion theorem for Lebesgue integrable functions on the real line. [You may use standard results from the course, such as the dominated convergence and Fubini's theorem. You may also use that $g_t * f(x) := \int_{\mathbb{R}} g_t(x-y) f(y) dy$ where $g_t(z) = t^{-1}\phi(z/t), t > 0$, converges to f in $L^1(\mathbb{R})$ as $t \to 0$ whenever $f \in L^1(\mathbb{R})$.]

The probability density function of a Gamma distribution with scalar parameters $\lambda > 0, \alpha > 0$ is given by

$$f_{\alpha,\lambda}(x) = \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} 1_{[0,\infty)}(x).$$

Let $0 < \alpha < 1, \lambda > 0$. Is $\widehat{f_{\alpha,\lambda}}$ integrable?

Paper 1, Section II

26J Probability and Measure

Carefully state and prove Jensen's inequality for a convex function $c : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval. Assuming that c is strictly convex, give necessary and sufficient conditions for the inequality to be strict.

Let μ be a Borel probability measure on $\mathbb{R}$, and suppose μ has a strictly positive probability density function f_0 with respect to Lebesgue measure. Let $\mathcal{P}$ be the family of all strictly positive probability density functions f on $\mathbb{R}$ with respect to Lebesgue measure such that $\log(f/f_0) \in L^1(\mu)$. Let X be a random variable with distribution μ. Prove that the mapping

$$f \mapsto \mathbb{E}\left[\log \frac{f}{f_0}(X)\right]$$

has a unique maximiser over $\mathcal{P}$, attained when $f = f_0$ almost everywhere.

Paper 1, Section II
26K Probability and Measure

(i) Let $(E, \mathcal{E}, \mu)$ be a measure space and let $1 \leqslant p < \infty$. For a measurable function f, let $\|f\|_p = (\int |f|^p d\mu)^{1/p}$. Give the definition of the space L^p. Prove that $(L^p, \|\cdot\|_p)$ forms a Banach space.

[You may assume that L^p is a normed vector space. You may also use in your proof any other result from the course provided that it is clearly stated.]

(ii) Show that convergence in probability implies convergence in distribution.

[*Hint: Show the pointwise convergence of the characteristic function, using without proof the inequality* $|e^{iy} - e^{ix}| \leqslant |x - y|$ *for* $x, y \in \mathbb{R}$.]

(iii) Let $(\alpha_j)_{j \geqslant 1}$ be a given real-valued sequence such that $\sum_{j=1}^{\infty} \alpha_j^2 = \sigma^2 < \infty$. Let $(X_j)_{j \geqslant 1}$ be a sequence of independent standard Gaussian random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let

$$Y_n = \sum_{j=1}^{n} \alpha_j X_j \, .$$

Prove that there exists a random variable Y such that $Y_n \to Y$ in L^2.

(iv) Specify the distribution of the random variable Y defined in part (iii), justifying carefully your answer.

Paper 2, Section II
26K Probability and Measure

(i) Define the notions of a π-system and a d-system. State and prove Dynkin's lemma.

(ii) Let $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ denote two finite measure spaces. Define the σ-algebra $\mathcal{E}_1 \otimes \mathcal{E}_2$ and the product measure $\mu_1 \otimes \mu_2$. [You do not need to verify that such a measure exists.] State (without proof) Fubini's Theorem.

(iii) Let $(E, \mathcal{E}, \mu)$ be a measure space, and let f be a non-negative Borel-measurable function. Let G be the subset of $E \times \mathbb{R}$ defined by

$$G = \{(x, y) \in E \times \mathbb{R} : 0 \leqslant y \leqslant f(x)\}.$$

Show that $G \in \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$. Show further that

$$\int f \, d\mu = (\mu \otimes \lambda)(G),$$

where λ is Lebesgue measure.

Paper 3, Section II
25K Probability and Measure

(i) State and prove Kolmogorov's zero-one law.

(ii) Let $(E, \mathcal{E}, \mu)$ be a finite measure space and suppose that $(B_n)_{n \geqslant 1}$ is a sequence of events such that $B_{n+1} \subset B_n$ for all $n \geqslant 1$. Show carefully that $\mu(B_n) \to \mu(B)$, where $B = \cap_{n=1}^{\infty} B_n$.

(iii) Let $(X_i)_{i \geqslant 1}$ be a sequence of independent and identically distributed random variables such that $\mathbb{E}(X_1^2) = \sigma^2 < \infty$ and $\mathbb{E}(X_1) = 0$. Let $K > 0$ and consider the event A_n defined by

$$A_n = \left\{ \frac{S_n}{\sqrt{n}} \geqslant K \right\}, \qquad \text{where} \quad S_n = \sum_{i=1}^{n} X_i \,.$$

Prove that there exists $c > 0$ such that for all n large enough, $\mathbb{P}(A_n) \geqslant c$. Any result used in the proof must be stated clearly.

(iv) Prove using the results above that A_n occurs infinitely often, almost surely. Deduce that

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = \infty \,,$$

almost surely.

Paper 4, Section II
25K Probability and Measure

(i) State and prove Fatou's lemma. State and prove Lebesgue's dominated convergence theorem. [You may assume the monotone convergence theorem.]

In the rest of the question, let f_n be a sequence of integrable functions on some measure space $(E, \mathcal{E}, \mu)$, and assume that $f_n \to f$ almost everywhere, where f is a given integrable function. We also assume that $\int |f_n| d\mu \to \int |f| d\mu$ as $n \to \infty$.

(ii) Show that $\int f_n^+ d\mu \to \int f^+ d\mu$ and that $\int f_n^- d\mu \to \int f^- d\mu$, where $\phi^+ = \max(\phi, 0)$ and $\phi^- = \max(-\phi, 0)$ denote the positive and negative parts of a function ϕ.

(iii) Here we assume also that $f_n \geqslant 0$. Deduce that $\int |f - f_n| d\mu \to 0$.

Paper 1, Section II

26I Probability and Measure

State Carathéodory's extension theorem. Define all terms used in the statement.

Let $\mathcal{A}$ be the ring of finite unions of disjoint bounded intervals of the form

$$A = \bigcup_{i=1}^{m} (a_i, b_i]$$

where $m \in \mathbb{Z}^+$ and $a_1 < b_1 < \ldots < a_m < b_m$. Consider the set function μ defined on $\mathcal{A}$ by

$$\mu(A) = \sum_{i=1}^{m} (b_i - a_i)\,.$$

You may assume that μ is additive. Show that for any decreasing sequence $(B_n : n \in \mathbb{N})$ in $\mathcal{A}$ with empty intersection we have $\mu(B_n) \to 0$ as $n \to \infty$.

Explain how this fact can be used in conjunction with Carathéodory's extension theorem to prove the existence of Lebesgue measure.

Paper 2, Section II

26I Probability and Measure

Show that any two probability measures which agree on a π-system also agree on the σ-algebra generated by that π-system.

State Fubini's theorem for non-negative measurable functions.

Let μ denote Lebesgue measure on $\mathbb{R}^2$. Fix $s \in [0,1)$. Set $c = \sqrt{1-s^2}$ and $\lambda = \sqrt{c}$. Consider the linear maps $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (\lambda^{-1}x,\, \lambda y)\,, \qquad g(x,y) = (x,\, sx+y)\,, \qquad h(x,y) = (x - sy,\, y)\,.$$

Show that $\mu = \mu \circ f^{-1}$ and that $\mu = \mu \circ g^{-1}$. You must justify any assertion you make concerning the values taken by μ.

Compute $r = f \circ h \circ g \circ f$. Deduce that μ is invariant under rotations.

Paper 3, Section II

25I Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables with common density function

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Fix $\alpha \in [0,1]$ and set

$$Y_n = \operatorname{sgn}(X_n)|X_n|^\alpha, \quad S_n = Y_1 + \ldots + Y_n.$$

Show that for all $\alpha \in [0,1]$ the sequence of random variables S_n/n converges in distribution and determine the limit.

[*Hint: In the case* $\alpha = 1$ *it may be useful to prove that* $\mathbb{E}(e^{iuX_1}) = e^{-|u|}$, *for all* $u \in \mathbb{R}$.]

Show further that for all $\alpha \in [0,1/2)$ the sequence of random variables $S_n/\sqrt{n}$ converges in distribution and determine the limit.

[You should state clearly any result about random variables from the course to which you appeal. You are not expected to evaluate explicitly the integral

$$m(\alpha) = \int_0^\infty \frac{x^\alpha}{\pi(1+x^2)} dx. \quad]$$

Paper 4, Section II

25I Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent normal random variables having mean 0 and variance 1. Set $S_n = X_1 + \ldots + X_n$ and $U_n = S_n - \lfloor S_n \rfloor$. Thus U_n is the fractional part of S_n. Show that U_n converges to U in distribution, as $n \to \infty$ where U is uniformly distributed on $[0,1]$.

Paper 1, Section II

26J Probability and Measure

Let $(E, \mathcal{E}, \mu)$ be a measure space. Explain what is meant by a *simple function* on $(E, \mathcal{E}, \mu)$ and state the definition of the *integral* of a simple function with respect to μ.

Explain what is meant by an *integrable function* on $(E, \mathcal{E}, \mu)$ and explain how the integral of such a function is defined.

State the monotone convergence theorem.

Show that the following map is linear

$$f \mapsto \mu(f) : L^1(E, \mathcal{E}, \mu) \to \mathbb{R},$$

where $\mu(f)$ denotes the integral of f with respect to μ.

[You may assume without proof any fact concerning simple functions and their integrals. You are not expected to prove the monotone convergence theorem.]

Paper 2, Section II

26J Probability and Measure

State Kolmogorov's zero-one law.

State Birkhoff's almost everywhere ergodic theorem and von Neumann's L^p-ergodic theorem.

State the strong law of large numbers for independent and identically distributed integrable random variables, and use the results above to prove it.

Paper 3, Section II

25J Probability and Measure

State and prove the first and second Borel–Cantelli lemmas.

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent Cauchy random variables. Thus, each X_n is real-valued, with density function

$$f(x) = \frac{1}{\pi(1 + x^2)}.$$

Show that

$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} = c, \quad \text{almost surely,}$$

for some constant c, to be determined.

Paper 4, Section II

25J Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Show that, for any random variable $X \in L^2(\mathbb{P})$, there exists a $\mathcal{G}$-measurable random variable $Y \in L^2(\mathbb{P})$ such that $\mathbb{E}((X-Y)Z) = 0$ for all $\mathcal{G}$-measurable random variables $Z \in L^2(\mathbb{P})$.

[You may assume without proof the completeness of $L^2(\mathbb{P})$.]

Let (G, X) be a Gaussian random variable in $\mathbb{R}^2$, with mean (μ, ν) and covariance matrix $\begin{pmatrix} u & v \\ v & w \end{pmatrix}$. Assume that $\mathcal{F} = \sigma(G, X)$ and $\mathcal{G} = \sigma(G)$. Find the random variable Y explicitly in this case.

1/II/25J Probability and Measure

State the Dominated Convergence Theorem.

Hence or otherwise prove Kronecker's Lemma: if (a_j) is a sequence of non-negative reals such that

$$\sum_{j=1}^{\infty} \frac{a_j}{j} < \infty,$$

then

$$n^{-1} \sum_{j=1}^{n} a_j \to 0 \quad (n \to \infty).$$

Let $\xi_1, \xi_2, \ldots$ be independent $N(0,1)$ random variables and set $S_n = \xi_1 + \ldots + \xi_n$. Let $\mathcal{F}_0$ be the collection of all finite unions of intervals of the form (a, b), where a and b are rational, together with the whole line $\mathbb{R}$. Prove that with probability 1 the limit

$$m(B) \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} I_B(S_j)$$

exists for all $B \in \mathcal{F}_0$, and identify it. Is it possible to extend m defined on $\mathcal{F}_0$ to a measure on the Borel σ-algebra of $\mathbb{R}$? Justify your answer.

2/II/25J Probability and Measure

Explain what is meant by a *simple function* on a measurable space $(S, \mathcal{S})$.

Let $(S, \mathcal{S}, \mu)$ be a finite measure space and let $f : S \to \mathbb{R}$ be a non-negative Borel measurable function. State the definition of the integral of f with respect to μ.

Prove that, for any sequence of simple functions (g_n) such that $0 \leqslant g_n(x) \uparrow f(x)$ for all $x \in S$, we have

$$\int g_n d\mu \uparrow \int f d\mu.$$

State and prove the Monotone Convergence Theorem for finite measure spaces.

3/II/24J **Probability and Measure**

(i) What does it mean to say that a sequence of random variables (X_n) converges *in probability* to X? What does it mean to say that the sequence (X_n) converges *in distribution* to X? Prove that if $X_n \to X$ in probability, then $X_n \to X$ in distribution.

(ii) What does it mean to say that a sequence of random variables (X_n) is *uniformly integrable*? Show that, if (X_n) is uniformly integrable and $X_n \to X$ in distribution, then $\mathbb{E}(X_n) \to \mathbb{E}(X)$.

[Standard results from the course may be used without proof if clearly stated.]

4/II/25J **Probability and Measure**

(i) A *stepfunction* is any function s on $\mathbb{R}$ which can be written in the form

$$s(x) = \sum_{k=1}^{n} c_k I_{(a_k,b_k]}(x), \quad x \in \mathbb{R},$$

where a_k, b_k, c_k are real numbers, with $a_k < b_k$ for all k. Show that the set of all stepfunctions is dense in $L^1(\mathbb{R}, \mathcal{B}, \mu)$. Here, $\mathcal{B}$ denotes the Borel σ-algebra, and μ denotes Lebesgue measure.

[You may use without proof the fact that, for any Borel set B of finite measure, and any $\varepsilon > 0$, there exists a finite union of intervals A such that $\mu(A \triangle B) < \varepsilon$.]

(ii) Show that the Fourier transform

$$\hat{s}(t) = \int_{\mathbb{R}} s(x) e^{itx}\, dx$$

of a stepfunction has the property that $\hat{s}(t) \to 0$ as $|t| \to \infty$.

(iii) Deduce that the Fourier transform of any integrable function has the same property.

1/II/25J Probability and Measure

Let E be a set and $\mathcal{E} \subseteq \mathcal{P}(E)$ be a set system.

(a) Explain what is meant by a π-system, a d-system and a σ-algebra.

(b) Show that $\mathcal{E}$ is a σ-algebra if and only if $\mathcal{E}$ is a π-system and a d-system.

(c) Which of the following set systems $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ are π-systems, d-systems or σ-algebras? Justify your answers. ($\#(A)$ denotes the number of elements in A.)

$E_1 = \{1, 2, \ldots, 10\}$ and $\mathcal{E}_1 = \{A \subseteq E_1 : \#(A) \text{ is even}\}$,
$E_2 = \mathbb{N} = \{1, 2, \ldots\}$ and $\mathcal{E}_2 = \{A \subseteq E_2 : \#(A) \text{ is even or } \#(A) = \infty\}$,
$E_3 = \mathbb{R}$ and $\mathcal{E}_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{\emptyset\}$.

(d) State and prove the theorem on the uniqueness of extension of a measure.

[*You may use standard results from the lectures without proof, provided they are clearly stated.*]

2/II/25J Probability and Measure

(a) State and prove the first Borel–Cantelli lemma. State the second Borel–Cantelli lemma.

(b) Let $X_1, X_2, \ldots$ be a sequence of independent random variables that converges in probability to the limit X. Show that X is almost surely constant.

A sequence $X_1, X_2, \ldots$ of random variables is said to be *completely convergent* to X if

$$\sum_{n \in \mathbb{N}} \mathbb{P}\big(A_n(\epsilon)\big) < \infty \quad \text{for all } \epsilon > 0 \ , \quad \text{where } A_n(\epsilon) = \big\{|X_n - X| > \epsilon\big\} \ .$$

(c) Show that complete convergence implies almost sure convergence.

(d) Show that, for sequences of independent random variables, almost sure convergence also implies complete convergence.

(e) Find a sequence of (dependent) random variables that converges almost surely but does not converge completely.

3/II/24J Probability and Measure

Let $(E, \mathcal{E}, \mu)$ be a finite measure space, i.e. $\mu(E) < \infty$, and let $1 \leqslant p \leqslant \infty$.

(a) Define the L^p-norm $\|f\|_p$ of a measurable function $f : E \to \overline{\mathbb{R}}$, define the space $L^p(E, \mathcal{E}, \mu)$ and define convergence in L^p.

In the following you may use inequalities from the lectures without proof, provided they are clearly stated.

(b) Let $f, f_1, f_2, \ldots \in L^p(E, \mathcal{E}, \mu)$. Show that $f_n \to f$ in L^p implies $\|f_n\|_p \to \|f\|_p$.

(c) Let $f : E \to \mathbb{R}$ be a bounded measurable function with $\|f\|_\infty > 0$. Let

$$M_n = \int_E |f|^n d\mu \, .$$

Show that $M_n \in (0, \infty)$ and $M_{n+1} M_{n-1} \geqslant M_n^2$.

By using Jensen's inequality, or otherwise, show that

$$\mu(E)^{-1/n} \|f\|_n \leqslant M_{n+1}/M_n \leqslant \|f\|_\infty .$$

Prove that $\lim_{n \to \infty} M_{n+1}/M_n = \|f\|_\infty$.

$\Big[$*Observe that* $|f| \geqslant \mathbf{1}_{\{|f| > \|f\|_\infty - \epsilon\}} \big(\|f\|_\infty - \epsilon\big)$.$\Big]$

4/II/25J Probability and Measure

Let $(E, \mathcal{E}, \mu)$ be a measure space with $\mu(E) < \infty$ and let $\theta : E \to E$ be measurable.

(a) Define an invariant set $A \in \mathcal{E}$ and an invariant function $f : E \to \mathbb{R}$.
What is meant by saying that θ is measure-preserving?
What is meant by saying that θ is ergodic?

(b) Which of the following functions θ_1 to θ_4 is ergodic? Justify your answer.

On the measure space $\big([0,1], \mathcal{B}([0,1]), \mu\big)$ with Lebesgue measure μ consider

$$\theta_1(x) = 1 + x \ , \qquad \theta_2(x) = x^2 \ , \qquad \theta_3(x) = 1 - x \ .$$

On the discrete measure space $\big(\{-1, 1\}, \mathcal{P}(\{-1, 1\}), \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\big)$ consider

$$\theta_4(x) = -x \ .$$

(c) State Birkhoff's almost everywhere ergodic theorem.

(d) Let θ be measure-preserving and let $f : E \to \mathbb{R}$ be bounded.
Prove that $\frac{1}{n}\big(f + f \circ \theta + \ldots + f \circ \theta^{n-1}\big)$ converges in L^p for all $p \in [1, \infty)$.

1/II/25J **Probability and Measure**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of (real-valued, Borel-measurable) random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

(a) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events in $\mathcal{A}$.
What does it mean for the events $(A_n)_{n\in\mathbb{N}}$ to be independent?
What does it mean for the random variables $(X_n)_{n\in\mathbb{N}}$ to be independent?

(b) Define the tail σ-algebra $\mathcal{T}$ for a sequence $(X_n)_{n\in\mathbb{N}}$ and state Kolmogorov's 0-1 law.

(c) Consider the following events in $\mathcal{A}$,

$$\{X_n \leqslant 0 \ \text{ eventually}\}\ ,$$
$$\{\lim_{n\to\infty} X_1 + \ldots + X_n \text{ exists}\}\ ,$$
$$\{X_1 + \ldots + X_n \leqslant 0 \ \text{ infinitely often}\}\ .$$

Which of them are tail events for $(X_n)_{n\in\mathbb{N}}$? Justify your answers.

(d) Let $(X_n)_{n\in\mathbb{N}}$ be independent random variables with

$$\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \tfrac{1}{2} \quad \text{for all } n \in \mathbb{N}\ ,$$

and define $U_n = X_1X_2 + X_2X_3 + \ldots + X_{2n}X_{2n+1}$.
Show that $U_n/n \to c$ a.s. for some $c \in \mathbb{R}$, and determine c.
[*Standard results may be used without proof, but should be clearly stated.*]

2/II/25J Probability and Measure

(a) What is meant by saying that $(\Omega, \mathcal{A}, \mu)$ is a measure space? Your answer should include clear definitions of any terms used.

(b) Consider the following sequence of Borel-measurable functions on the measure space $(\mathbb{R}, \mathcal{L}, \lambda)$, with the Lebesgue σ-algebra $\mathcal{L}$ and Lebesgue measure λ:

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 \leqslant x \leqslant e^n; \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \ n \in \mathbb{N} \ .$$

For each $p \in [1, \infty]$, decide whether the sequence $(f_n)_{n \in \mathbb{N}}$ converges in L^p as $n \to \infty$.
Does $(f_n)_{n \in \mathbb{N}}$ converge almost everywhere?
Does $(f_n)_{n \in \mathbb{N}}$ converge in measure?
Justify your answers.

For parts (c) and (d), let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued, Borel-measurable functions on a probability space $(\Omega, \mathcal{A}, \mu)$.

(c) Prove that $\{x \in \Omega : f_n(x) \text{ converges to a finite limit}\} \in \mathcal{A}$.

(d) Show that $f_n \to 0$ almost surely if and only if $\sup_{m \geqslant n} |f_m| \to 0$ in probability.

3/II/24J Probability and Measure

Let X be a real-valued random variable. Define the characteristic function ϕ_X. Show that $\phi_X(u) \in \mathbb{R}$ for all $u \in \mathbb{R}$ if and only if X and $-X$ have the same distribution.

For parts (a) and (b) below, let X and Y be independent and identically distributed random variables.

(a) Show that $X = Y$ almost surely implies that X is almost surely constant.

(b) Suppose that there exists $\varepsilon > 0$ such that $|\phi_X(u)| = 1$ for all $|u| < \varepsilon$. Calculate ϕ_{X-Y} to show that $\mathbb{E}\big(1 - \cos(u(X - Y))\big) = 0$ for all $|u| < \varepsilon$, and conclude that X is almost surely constant.

(c) Let X, Y, and Z be independent $\mathrm{N}(0,1)$ random variables. Calculate the characteristic function of $\eta = XY - Z$, given that $\phi_X(u) = e^{-u^2/2}$.

4/II/25J **Probability and Measure**

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ a measurable function.

(a) Explain what is meant by saying that f is *integrable*, and how the integral $\int_\Omega f \, d\mu$ is defined, starting with integrals of $\mathcal{A}$-simple functions.

[*Your answer should consist of clear definitions, including the ones for $\mathcal{A}$-simple functions and their integrals.*]

(b) For $f : \Omega \to [0, \infty)$ give a specific sequence $(g_n)_{n \in \mathbb{N}}$ of $\mathcal{A}$-simple functions such that $0 \leqslant g_n \leqslant f$ and $g_n(x) \to f(x)$ for all $x \in \Omega$. Justify your answer.

(c) Suppose that that $\mu(\Omega) < \infty$ and let $f_1, f_2, \ldots : \Omega \to \mathbb{R}$ be measurable functions such that $f_n(x) \to 0$ for all $x \in \Omega$. Prove that, if

$$\lim_{c \to \infty} \sup_{n \in \mathbb{N}} \int_{|f_n| > c} |f_n| \, d\mu = 0,$$

then $\int_\Omega f_n \, d\mu \to 0$.

Give an example with $\mu(\Omega) < \infty$ such that $f_n(x) \to 0$ for all $x \in \Omega$, but $\int_\Omega f_n \, d\mu \not\to 0$, and justify your answer.

(d) State and prove Fatou's Lemma for a sequence of non-negative measurable functions.

[*Standard results on measurability and integration may be used without proof.*]

1/II/25J **Probability and Measure**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $\mathcal{G} \subseteq \mathcal{F}$, what is meant by saying that $\mathcal{G}$ is a π-system? State the 'uniqueness of extension' theorem for measures on $\sigma(\mathcal{G})$ having given values on $\mathcal{G}$.

For $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$, we call $\mathcal{G}$, $\mathcal{H}$ independent if

$$\mathbb{P}(G \cap H) = \mathbb{P}(G)\mathbb{P}(H) \quad \text{for all} \quad G \in \mathcal{G},\ H \in \mathcal{H}.$$

If $\mathcal{G}$ and $\mathcal{H}$ are independent π-systems, show that $\sigma(\mathcal{G})$ and $\sigma(\mathcal{H})$ are independent.

Let $Y_1, Y_2, \ldots, Y_m,\ Z_1, Z_2, \ldots, Z_n$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the σ-fields $\sigma(Y) = \sigma(Y_1, Y_2, \ldots, Y_m)$ and $\sigma(Z) = \sigma(Z_1, Z_2, \ldots, Z_n)$ are independent.

2/II/25J **Probability and Measure**

Let $\mathcal{R}$ be a family of random variables on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. What is meant by saying that $\mathcal{R}$ is uniformly integrable? Explain the use of uniform integrability in the study of convergence in probability and in L^1. [*Clear definitions should be given of any terms used, but proofs may be omitted.*]

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly integrable families of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family $\mathcal{R}$ given by

$$\mathcal{R} = \{X + Y : X \in \mathcal{R}_1,\ Y \in \mathcal{R}_2\}$$

is uniformly integrable.

3/II/24J **Probability and Measure**

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For a measurable function $f : \Omega \to \mathbb{R}$, and $p \in [1, \infty)$, let $||f||_p = [\mu(|f|^p)]^{1/p}$. Let L^p be the space of all such f with $||f||_p < \infty$. Explain what is meant by each of the following statements:

(a) A sequence of functions $(f_n : n \geqslant 1)$ is Cauchy in L^p.

(b) L^p is complete.

Show that L^p is complete for $p \in [1, \infty)$.

Take $\Omega = (1, \infty)$, $\mathcal{F}$ the Borel σ-field of Ω, and μ the Lebesgue measure on $(\Omega, \mathcal{F})$. For $p = 1, 2$, determine which if any of the following sequences of functions are Cauchy in L^p:

(i) $f_n(x) = x^{-1} 1_{(1,n)}(x)$,

(ii) $g_n(x) = x^{-2} 1_{(1,n)}(x)$,

where 1_A denotes the indicator function of the set A.

4/II/25J **Probability and Measure**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel-measurable. State Fubini's theorem for the double integral

$$\int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} f(x, y) \, dx \, dy \, .$$

Let $0 < a < b$. Show that the function

$$f(x, y) = \begin{cases} e^{-xy} & \text{if } \ x \in (0, \infty), \ y \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

is measurable and integrable on $\mathbb{R}^2$.

Evaluate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx$$

by Fubini's theorem or otherwise.