Part II

Principles of Quantum Mechanics

Paper 1, Section II

34B Principles of Quantum Mechanics

(a) Write down the Hamiltonian for a quantum harmonic oscillator of frequency ω in terms of the creation and annihilation operators $A^\dagger$ and A. You may work in units where $\hbar = 1$. Define the *number operator* N and state all commutation relations amongst A, $A^\dagger$ and N. Show that the eigenvalues of N are: (i) real, (ii) non-negative and (iii) integers.

(b) Consider a system of two independent harmonic oscillators of frequency $\omega = 1$ and $\omega = 2$. The $\omega = 1$ oscillator has creation and annihilation operators $A^\dagger$ and A, while the $\omega = 2$ oscillator has creation and annihilation operators $B^\dagger$ and B.

(i) Find the five lowest eigenvalues of the Hamiltonian H_0 of the combined system and determine the degeneracy of each of them.

(ii) The system is perturbed so that it is now described by the new Hamiltonian $H = H_0 + \lambda H'$, where $\lambda \in \mathbb{R}$ and $H' = A^\dagger A^\dagger B + AAB^\dagger$. Using degenerate perturbation theory, calculate to order λ the energies of the eigenstates associated with the level $E_0 = \frac{9}{2}$. Write down the perturbed eigenstates, to order λ, associated with these perturbed energies. By explicit evaluation show that they are in fact exact eigenstates of H with these energies as eigenvalues.

[In part (b) you may use without proof that for the harmonic oscillator

$$A\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad A^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.]$$

Paper 2, Section II

35B Principles of Quantum Mechanics

A two-state quantum system has Hamiltonian H_0 with eigenvectors $|-\rangle$ and $|+\rangle$, and corresponding eigenvalues E_- and $E_+ > E_-$. The system is perturbed by the Hermitian operator ΔH with matrix elements

$$\langle +|\,\Delta H\,|-\rangle = i\lambda\,, \qquad\qquad \langle +|\,\Delta H\,|+\rangle = \langle -|\,\Delta H\,|-\rangle = 0,$$

where λ is a real constant.

(i) Starting from the Schrödinger equation for $H_0 + \Delta H$ and explicitly deriving any necessary results, determine the corrections to the energy eigenstates and eigenvalues in perturbation theory up to linear order in λ.

(ii) Find the exact eigenstates and eigenvalues and show that they agree with the results of perturbation theory up to linear order in λ.

(iii) Determine the radius of convergence of perturbation theory in λ. [*Hint: the square root function has a branch point when its argument vanishes.*]

Paper 3, Section II

33B Principles of Quantum Mechanics

(a) Consider a composite system of several distinguishable particles. Describe how the multiparticle state is constructed from single-particle states. For the case of two identical particles, describe how considering the interchange symmetry leads to the definition of bosons and fermions.

(b) Consider two non-interacting, identical particles, each with spin 1. The single particle, spin-independent Hamiltonian $H(\mathbf{X}_i, \mathbf{P}_i)$ has non-degenerate eigenvalues E_n and wavefunctions $\psi_n(\mathbf{x}_i)$ where $i = 1, 2$ labels the particle and $n = 0, 1, 2, 3, \ldots$. In terms of these single-particle wavefunctions and single-particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the two-particle states and energies for (i) the ground states and (ii) the first excited states.

(c) For the system in part (b), assume now that E_n is a linear function of n. Find the degeneracy of the N^{th} energy level of the two-particle system for: (i) N even and (ii) N odd.

Paper 4, Section II

33B Principles of Quantum Mechanics

(a) A composite system is made of two sub-systems with total angular momenta j_1 and j_2, respectively. Let $\mathbf{J} = \{J_x, J_y, J_z\}$ be the angular momentum operator of the composite system and $|j, m\rangle$ a basis of eigenstates of $\mathbf{J}^2$ and J_z.

(i) Write $\mathbf{J}$ and the associated ladder operators $J_\pm$ in terms of the angular momentum operators $\mathbf{J}_{1,2}$ of each sub-system.

(ii) State the possible values of j in terms of j_1 and j_2 and specify under what conditions it is possible to have $j = 0$.

(iii) Write down all the states of definite j and m that have $m \geqslant j_1 + j_2 - 1$, in terms of the states of the sub-systems $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$.

(iv) Given a pure state, define what it means for the state to be a *product state* and what it means for the state to be an *entangled state*. Specify whether each of the states in (iii) is a product state or an entangled state.

(b) Let $j = j_1 + j_2$. For each of the two states of the system $|j, j\rangle$ and $|j, j-1\rangle$ compute the reduced density matrix of subsystem 1 and the associated entanglement entropy. Comment on the value of the entanglement entropy when $j_1 = j_2$.

(c) Explain why, if it exists, the state with $j = 0$ must be of the form

$$|0, 0\rangle = \sum_{m=-j_1}^{j_1} \alpha_m \, |j_1, m\rangle_1 \, |j_1, -m\rangle_2 \ .$$

By considering $J_+ |0, 0\rangle$, determine a relation between α_{m+1} and α_m, and hence find α_m.

[*Units in which $\hbar = 1$ have been used throughout. The states $|j, m\rangle$ obey*

$$J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \, |j, m \pm 1\rangle \ .]$$

Paper 1, Section II
34A Principles of Quantum Mechanics

Let A and $A^\dagger$ respectively be the lowering and raising operator for a one-dimensional quantum harmonic oscillator, with $[A, A^\dagger] = 1$. Also let $|n\rangle$ be the n^{th} excited state of the oscillator, obeying $N|n\rangle = n|n\rangle$ where $N = A^\dagger A$ is the number operator.

(a) Show that $A|n\rangle \propto |n-1\rangle$ and find the constant of proportionality.

(b) For any $z \in \mathbb{C}$, define the coherent state $|z\rangle$ by

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \ .$$

Show that $\langle z|z\rangle = 1$ and that $A|z\rangle = z|z\rangle$.

(c) Calculate the expectation value $\langle N\rangle$ and uncertainty ΔN of the number operator in the state $|z\rangle$. Show that the relative uncertainty $\Delta N/\langle N\rangle \to 0$ as $\langle N\rangle \to \infty$.

(d) A harmonic oscillator is prepared to be in state $|z\rangle$ at time $t = 0$. Using the properties of the Hamiltonian of the one-dimensional harmonic oscillator, show that the state evolved to time $t > 0$ is still an eigenstate of A and find its eigenvalue. Calculate the probability that the oscillator is found to be in the original state $|z\rangle$ at time t, and show that this probability is 1 whenever $t = kT$, where $k \in \mathbb{N}$ and T is the classical period of the oscillator.

Paper 2, Section II
35A Principles of Quantum Mechanics

(a) Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a basis of S_z eigenstates for a spin-$\frac{1}{2}$ particle. Find the eigenstates $|{\uparrow_\theta}\rangle$ and $|{\downarrow_\theta}\rangle$ of $\mathbf{n}\cdot\mathbf{S}$, where $\mathbf{n} = (\sin\theta, 0, \cos\theta)$, and give their corresponding eigenvalues.

(b) Two spin-$\frac{1}{2}$ particles are in the combined spin state

$$|\psi\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}\,.$$

Show that this state is unchanged under the substitution

$$(|\uparrow\rangle, |\downarrow\rangle) \mapsto (|{\uparrow_\theta}\rangle, |{\downarrow_\theta}\rangle).$$

Hence show that $|\psi\rangle$ is an eigenstate, with eigenvalue zero, of each Cartesian component of the combined spin operator $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$, where $\mathbf{S}^{(i)}$ is the spin operator of the i^{th} particle.

(c) Two spin-$\frac{1}{2}$ particles are in the spin state

$$|\chi\rangle = \frac{|\uparrow\rangle|{\downarrow_\theta}\rangle - |\downarrow\rangle|{\uparrow_\theta}\rangle}{\sqrt{2}}\,.$$

A measurement of S_z for the first particle is carried out, followed by a measurement of S_z for the second particle. List the possible outcomes for this pair of measurements and find the total probability, in terms of θ, for each pair of outcomes to occur. For which of these outcomes is the system left in an eigenstate of the combined total spin operator $\mathbf{S}\cdot\mathbf{S}$, and what are the corresponding eigenvalues?

[*Hint: The Pauli sigma matrices are given by*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad]$$

Paper 3, Section II
33A Principles of Quantum Mechanics

(a) Show that $[L_z, z] = 0$ and hence that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $m' = m$, where $|n, \ell, m\rangle$ is a simultaneous eigenstate of H, L^2 and L_z.

(b) Given that $[L^2, [L^2, z]] = 2\hbar^2(L^2 z + zL^2)$, show that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $|\ell' - \ell| = 1$ or $\ell' = \ell = 0$. By considering parity, show that this matrix element also vanishes if $\ell' = \ell$.

(c) A hydrogen atom in its ground state $|n, \ell, m\rangle = |1, 0, 0\rangle$ is placed in a constant, uniform electric field $\mathbf{E}$. With reference to the atom's charge distribution, but without detailed calculation, give a physical explanation of why there is no correction of first-order (in $\mathbf{E}$) to the ground state energy, but higher-order corrections are possible.

(d) Show that the second-order correction to the energy of the ground state caused by the electric field is

$$\frac{e^2|\mathbf{E}|^2}{\mathcal{R}} \sum_{n=2}^{\infty} \frac{n^2}{1-n^2} \left|\langle n, 1, 0|z|1, 0, 0\rangle\right|^2 ,$$

where $-\mathcal{R}$ is the unperturbed energy of $|1, 0, 0\rangle$.

[*You may assume that, when a Hamiltonian is perturbed by* ΔH, *the second-order correction to the ground state energy is*

$$\sum_{\alpha} \frac{|\langle\alpha|\Delta H|\phi\rangle|^2}{E_\phi - E_\alpha} ,$$

where $\{|\alpha\rangle\}$ *is a complete set of unperturbed eigenstates states orthogonal to the unperturbed ground state* $|\phi\rangle$, *and* E_α, E_ϕ *are their unperturbed energies.*]

Paper 4, Section II
33A Principles of Quantum Mechanics

A particle travels in one dimension subject to the Hamiltonian

$$H_0 = \frac{P^2}{2m} - U\,\delta(x)\,,$$

where U is a positive constant. Let $|0\rangle$ be the unique bound state of this potential and E_0 its energy. Further let $|k,\pm\rangle$ be unbound H_0 eigenstates of even/odd parity, each with energy E_k, chosen so that $\langle k',+|k,+\rangle = \langle k',-|k,-\rangle = \delta(k'-k)$.

(a) At times $t \leqslant 0$ the particle is trapped in the well. From $t=0$ it is disturbed by a time-dependent potential $v(x,t) = -Fx\,e^{-i\omega t}$ and subsequently its state may be expressed as

$$|\psi(t)\rangle = a(t)\,e^{-iE_0t/\hbar}|0\rangle + \int_0^\infty \Big(b_k(t)|k,+\rangle + c_k(t)|k,-\rangle\Big)\,e^{-iE_kt/\hbar}\,dk\,.$$

Show that

$$\dot{a}(t)\,e^{-iE_0t/\hbar}|0\rangle + \int_0^\infty e^{-iE_kt/\hbar}\Big(\dot{b}_k(t)|k,+\rangle + \dot{c}_k(t)|k,-\rangle\Big)\,dk = \frac{iF}{\hbar}e^{-i\omega t}\,x|\psi(t)\rangle$$

for all $t>0$.

(b) Working to first order in F, hence show that $b_k(t)=0$ and that

$$c_k(t) = \frac{iF}{\hbar}\langle k,-|x|0\rangle\,e^{i\Omega_kt/2}\,\frac{\sin(\Omega_kt/2)}{\Omega_k/2}\,,$$

where $\Omega_k = (E_k - E_0 - \hbar\omega)/\hbar$.

(c) The original bound state has position space wavefunction $\langle x|0\rangle = \sqrt{K}\,e^{-K|x|}$ where $K = mU/\hbar^2$, while the position space wavefunction of the odd parity unbound state is $\langle x|k,-\rangle = \sin(kx)\,/\sqrt{\pi}$ and its energy $E_k = \hbar^2k^2/2m$. Show that at late times the probability that the particle escapes from the original potential well is

$$P_{\text{free}}(t) = \frac{8\hbar F^2 t}{mE_0^2}\frac{\sqrt{E_f/|E_0|}}{(1+E_f/|E_0|)^4}$$

to lowest order in F, where $E_f>0$ is the final energy. [*You may assume that as* $t\to\infty$, *the function* $\sin^2(\lambda t)/(\lambda^2 t) \to \pi\,\delta(\lambda)$.]

Paper 1, Section II

34B Principles of Quantum Mechanics

(a) A group G of transformations acts on a quantum system. Briefly explain why the Born rule implies that these transformations may be represented by operators $U(g) : \mathcal{H} \to \mathcal{H}$ obeying

$$U(g)^\dagger\, U(g) = 1_{\mathcal{H}}$$
$$U(g_1)\, U(g_2) = e^{i\phi(g_1,g_2)}\, U(g_1 \cdot g_2)$$

for all $g_1, g_2 \in G$, where $\phi(g_1, g_2) \in \mathbb{R}$.

What additional property does $U(g)$ have when G is a group of symmetries of the Hamiltonian? Show that symmetries correspond to conserved quantities.

(b) The Coulomb Hamiltonian describing the gross structure of the hydrogen atom is invariant under time reversal, $t \mapsto -t$. Suppose we try to represent time reversal by a unitary operator T obeying $U(t)T = TU(-t)$, where $U(t)$ is the time-evolution operator. Show that this would imply that hydrogen has no stable ground state.

An operator $A : \mathcal{H} \to \mathcal{H}$ is *anti*linear if

$$A\big(a|\alpha\rangle + b|\beta\rangle\big) = \bar{a}\, A|\alpha\rangle + \bar{b}\, A|\beta\rangle$$

for all $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$ and all $a, b \in \mathbb{C}$, and *anti*unitary if, in addition,

$$\langle\beta'|\alpha'\rangle = \overline{\langle\beta|\alpha\rangle}\,,$$

where $|\alpha'\rangle = A|\alpha\rangle$ and $|\beta'\rangle = A|\beta\rangle$. Show that if time reversal is instead represented by an antiunitary operator then the above instability of hydrogen is avoided.

Paper 2, Section II

35B Principles of Quantum Mechanics

(a) Let $\{|n\rangle\}$ be a basis of eigenstates of a non-degenerate Hamiltonian H, with corresponding eigenvalues $\{E_n\}$. Write down an expression for the energy levels of the perturbed Hamiltonian $H + \lambda\,\Delta H$, correct to second order in the dimensionless constant $\lambda \ll 1$.

(b) A particle travels in one dimension under the influence of the potential

$$V(X) = \frac{1}{2} m\omega^2 X^2 + \lambda\, \hbar\omega\, \frac{X^3}{L^3}\,,$$

where m is the mass, ω a frequency and $L = \sqrt{\hbar/2m\omega}$ a length scale. Show that, to first order in λ, all energy levels coincide with those of the harmonic oscillator. Calculate the energy of the ground state to second order in λ.

Does perturbation theory in λ converge for this potential? Briefly explain your answer.

Paper 3, Section II
33B Principles of Quantum Mechanics

(a) A quantum system with total angular momentum j_1 is combined with another of total angular momentum j_2. What are the possible values of the total angular momentum j of the combined system? For given j, what are the possible values of the angular momentum along any axis?

(b) Consider the case $j_1 = j_2$. Explain why all the states with $j = 2j_1 - 1$ are antisymmetric under exchange of the angular momenta of the two subsystems, while all the states with $j = 2j_1 - 2$ are symmetric.

(c) An exotic particle X of spin 0 and negative intrinsic parity decays into a pair of indistinguishable particles Y. Assume each Y particle has spin 1 and that the decay process conserves parity. Find the probability that the direction of travel of the Y particles is observed to lie at an angle $\theta \in (\pi/4, 3\pi/4)$ from some axis along which their total spin is observed to be $+\hbar$?

Paper 4, Section II
33B Principles of Quantum Mechanics

(a) A quantum system has Hamiltonian $H = H_0 + V(t)$. Let $\{|n\rangle\}_{n\in\mathbb{N}_0}$ be an orthonormal basis of H_0 eigenstates, with corresponding energies $E_n = \hbar\omega_n$. For $t < 0$, $V(t) = 0$ and the system is in state $|0\rangle$. Calculate the probability that it is found to be in state $|1\rangle$ at time $t > 0$, correct to lowest non-trivial order in V.

(b) Now suppose $\{|0\rangle, |1\rangle\}$ form a basis of the Hilbert space, with respect to which

$$\begin{pmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{pmatrix} = \begin{pmatrix} \hbar\omega_0 & \hbar v\,\Theta(t)e^{i\omega t} \\ \hbar v\,\Theta(t)e^{-i\omega t} & \hbar\omega_1 \end{pmatrix},$$

where $\Theta(t)$ is the Heaviside step function and v is a real constant. Calculate the exact probability that the system is in state $|1\rangle$ at time t. For which frequency ω is this probability maximized?

Paper 1, Section II

34A Principles of Quantum Mechanics

Let $A = (m\omega X + iP)/\sqrt{2m\hbar\omega}$ be the lowering operator of a one dimensional quantum harmonic oscillator of mass m and frequency ω, and let $|0\rangle$ be the ground state defined by $A|0\rangle = 0$.

a) Evaluate the commutator $[A, A^\dagger]$.

b) For $\gamma \in \mathbb{R}$, let $S(\gamma)$ be the unitary operator $S(\gamma) = \exp\left(-\frac{\gamma}{2}(A^\dagger A^\dagger - AA)\right)$ and define $A(\gamma) = S^\dagger(\gamma)AS(\gamma)$. By differentiating with respect to γ or otherwise, show that

$$A(\gamma) = A\cosh\gamma - A^\dagger \sinh\gamma \; .$$

c) The ground state of the harmonic oscillator saturates the uncertainty relation $\Delta X\, \Delta P \geqslant \hbar/2$. Compute $\Delta X\, \Delta P$ when the oscillator is in the state $|\gamma\rangle = S(\gamma)|0\rangle$.

Paper 2, Section II

34A Principles of Quantum Mechanics

(a) Consider the Hamiltonian $H(t) = H_0 + \delta H(t)$, where H_0 is time-independent and non-degenerate. The system is prepared to be in some state $|\psi\rangle = \sum_r a_r|r\rangle$ at time $t = 0$, where $\{|r\rangle\}$ is an orthonormal basis of eigenstates of H_0. Derive an expression for the state at time t, correct to first order in $\delta H(t)$, giving your answer in the interaction picture.

(b) An atom is modelled as a two–state system, where the excited state $|e\rangle$ has energy $\hbar\Omega$ above that of the ground state $|g\rangle$. The atom interacts with an electromagnetic field, modelled as a harmonic oscillator of frequency ω. The Hamiltonian is $H(t) = H_0 + \delta H(t)$, where

$$H_0 = \frac{\hbar\Omega}{2}\big(|e\rangle\langle e| - |g\rangle\langle g|\big) \otimes 1_{\text{field}} + 1_{\text{atom}} \otimes \hbar\omega \left(A^\dagger A + \frac{1}{2}\right)$$

is the Hamiltonian in the absence of interactions and

$$\delta H(t) = \begin{cases} 0\,, & t \leqslant 0\,, \\ \frac{1}{2}\hbar(\Omega-\omega)\Big(|e\rangle\langle g| \otimes A + \beta\,|g\rangle\langle e| \otimes A^\dagger\Big)\,, & t > 0\,, \end{cases}$$

describes the coupling between the atom and the field.

(i) Interpret each of the two terms in $\delta H(t)$. What value must the constant β take for time evolution to be unitary?

(ii) At $t = 0$ the atom is in state $(|e\rangle + |g\rangle)/\sqrt{2}$ while the field is described by the (normalized) state $e^{-1/2}\, e^{-A^\dagger}|0\rangle$ of the oscillator. Calculate the probability that at time t the atom will be in its excited state and the field will be described by the n^{th} excited state of the oscillator. Give your answer to first non-trivial order in perturbation theory. Show that this probability vanishes when $t = \pi/(\Omega - \omega)$.

Paper 3, Section II

33A Principles of Quantum Mechanics

Explain what is meant by the terms *boson* and *fermion*.

Three distinguishable spin-1 particles are governed by the Hamiltonian

$$H = \frac{2\lambda}{\hbar^2}\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1\right),$$

where $\mathbf{S}_i$ is the spin operator of particle i and λ is a positive constant. How many spin states are possible altogether? By considering the total spin operator, determine the eigenvalues and corresponding degeneracies of the Hamiltonian.

Now consider the case that all three particles are indistinguishable and all have the same spatial wavefunction. What are the degeneracies of the Hamiltonian in this case?

Paper 4, Section II

33 Principles of Quantum Mechanics

Briefly explain why the density operator ρ obeys $\rho \geqslant 0$ and $\mathrm{Tr}(\rho) = 1$. What is meant by a *pure* state and a *mixed* state?

A two-state system evolves under the Hamiltonian $H = \hbar\boldsymbol{\omega} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\omega}$ is a constant vector and $\boldsymbol{\sigma}$ are the Pauli matrices. At time t the system is described by a density operator

$$\rho(t) = \frac{1}{2}\left(1_{\mathcal{H}} + \mathbf{a}(t) \cdot \boldsymbol{\sigma}\right)$$

where $1_{\mathcal{H}}$ is the identity operator. Initially, the vector $\mathbf{a}(0) = \mathbf{a}$ obeys $|\mathbf{a}| < 1$ and $\mathbf{a}\cdot\boldsymbol{\omega} = 0$. Find $\rho(t)$ in terms of $\mathbf{a}$ and $\boldsymbol{\omega}$. At what time, if any, is the system definitely in the state $|\uparrow_x\rangle$ that obeys $\sigma_x|\uparrow_x\rangle = +|\uparrow_x\rangle$?

Paper 4, Section II

32B Principles of Quantum Mechanics

Define the *spin raising* and *spin lowering* operators S_+ and S_-. Show that

$$S_\pm|s,\sigma\rangle = \hbar\sqrt{s(s+1)-\sigma(\sigma\pm 1)}\,|s,\sigma\pm 1\rangle\,,$$

where $S_z|s,\sigma\rangle = \hbar\sigma|s,\sigma\rangle$ and $S^2|s,\sigma\rangle = s(s+1)\hbar^2|s,\sigma\rangle$.

Two spin-$\frac{1}{2}$ particles, with spin operators $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$, have a Hamiltonian

$$H = \alpha\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} + \mathbf{B}\cdot(\mathbf{S}^{(1)} - \mathbf{S}^{(2)})\,,$$

where α and $\mathbf{B} = (0,0,B)$ are constants. Express H in terms of the two particles' spin raising and spin lowering operators $S_\pm^{(1)}$, $S_\pm^{(2)}$ and the corresponding z-components $S_z^{(1)}$, $S_z^{(2)}$. Hence find the eigenvalues of H. Show that there is a unique groundstate in the limit $B \to 0$ and that the first excited state is triply degenerate in this limit. Explain this degeneracy by considering the action of the combined spin operator $\mathbf{S}^{(1)} + \mathbf{S}^{(2)}$ on the energy eigenstates.

Paper 3, Section II

33B Principles of Quantum Mechanics

Consider the Hamiltonian $H = H_0 + V$, where V is a small perturbation. If $H_0|n\rangle = E_n|n\rangle$, write down an expression for the eigenvalues of H, correct to second order in the perturbation, assuming the energy levels of H_0 are non-degenerate.

In a certain three-state system, H_0 and V take the form

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \qquad \text{and} \qquad V = V_0\begin{pmatrix} 0 & \epsilon & \epsilon^2 \\ \epsilon & 0 & 0 \\ \epsilon^2 & 0 & 0 \end{pmatrix}\,,$$

with V_0 and ϵ real, positive constants and $\epsilon \ll 1$.

(a) Consider first the case $E_1 = E_2 \neq E_3$ and $|\epsilon V_0/(E_3 - E_2)| \ll 1$. Use the results of degenerate perturbation theory to obtain the energy eigenvalues correct to order ϵ.

(b) Now consider the different case $E_3 = E_2 \neq E_1$ and $|\epsilon V_0/(E_2 - E_1)| \ll 1$. Use the results of non-degenerate perturbation theory to obtain the energy eigenvalues correct to order ϵ^2. Why is it not necessary to use degenerate perturbation theory in this case?

(c) Obtain the exact energy eigenvalues in case (b), and compare these to your perturbative results by expanding to second order in ϵ.

Paper 2, Section II

33B Principles of Quantum Mechanics

(a) Let $|i\rangle$ and $|j\rangle$ be two eigenstates of a time-independent Hamiltonian H_0, separated in energy by $\hbar\omega_{ij}$. At time $t = 0$ the system is perturbed by a small, time independent operator V. The perturbation is turned off at time $t = T$. Show that if the system is initially in state $|i\rangle$, the probability of a transition to state $|j\rangle$ is approximately

$$P_{ij} = 4|\langle i|V|j\rangle|^2 \frac{\sin^2(\omega_{ij}T/2)}{(\hbar\omega_{ij})^2} \,.$$

(b) An uncharged particle with spin one-half and magnetic moment μ travels at speed v through a region of uniform magnetic field $\mathbf{B} = (0, 0, B)$. Over a length L of its path, an additional perpendicular magnetic field b is applied. The spin-dependent part of the Hamiltonian is

$$H(t) = \begin{cases} -\mu(B\sigma_z + b\sigma_x) & \text{while } 0 < t < L/v \\ -\mu B\sigma_z & \text{otherwise,} \end{cases}$$

where σ_z and σ_x are Pauli matrices. The particle initially has its spin aligned along the direction of $\mathbf{B} = (0, 0, B)$. Find the probability that it makes a transition to the state with opposite spin

(i) by assuming $b \ll B$ and using your result from part (a),

(ii) by finding the exact evolution of the state.

[*Hint: for any 3-vector* $\mathbf{a}$, $e^{i\mathbf{a}\cdot\boldsymbol{\sigma}} = (\cos a)I + (i\sin a)\,\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}$, *where* I *is the* 2×2 *unit matrix,* $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, a=$|\mathbf{a}|$ *and* $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$.]

Paper 1, Section II

33B Principles of Quantum Mechanics

A $d = 3$ isotropic harmonic oscillator of mass μ and frequency ω has lowering operators

$$\mathbf{A} = \frac{1}{\sqrt{2\mu\hbar\omega}}\left(\mu\omega\mathbf{X} + \mathrm{i}\mathbf{P}\right) ,$$

where $\mathbf{X}$ and $\mathbf{P}$ are the position and momentum operators. Assuming the standard commutation relations for $\mathbf{X}$ and $\mathbf{P}$, evaluate the commutators $[A_i^\dagger, A_j^\dagger]$, $[A_i, A_j]$ and $[A_i, A_j^\dagger]$, for $i, j = 1, 2, 3$, among the components of the raising and lowering operators.

How is the ground state $|\mathbf{0}\rangle$ of the oscillator defined? How are normalised higher excited states obtained from $|\mathbf{0}\rangle$? [You should determine the appropriate normalisation constant for each energy eigenstate.]

By expressing the orbital angular momentum operator $\mathbf{L}$ in terms of the raising and lowering operators, show that each first excited state of the isotropic oscillator has total orbital angular momentum quantum number $\ell = 1$, and find a linear combination $|\psi\rangle$ of these first excited states obeying $L_z|\psi\rangle = +\hbar|\psi\rangle$ and $\||\psi\rangle\| = 1$.

Paper 4, Section II

33D Principles of Quantum Mechanics

The spin operators obey the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$. Let $|s,\sigma\rangle$ be an eigenstate of the spin operators S_z and $\mathbf{S}^2$, with $S_z|s,\sigma\rangle = \sigma\hbar\,|s,\sigma\rangle$ and $\mathbf{S}^2|s,\sigma\rangle = s(s+1)\hbar^2\,|s,\sigma\rangle$. Show that

$$S_\pm|s,\sigma\rangle = \sqrt{s(s+1)-\sigma(\sigma\pm1)}\,\hbar\,|s,\sigma\pm1\rangle\,,$$

where $S_\pm = S_x \pm iS_y$. When $s=1$, use this to derive the explicit matrix representation

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in a basis in which S_z is diagonal.

A beam of atoms, each with spin 1, is polarised to have spin $+\hbar$ along the direction $\mathbf{n} = (\sin\theta, 0, \cos\theta)$. This beam enters a Stern–Gerlach filter that splits the atoms according to their spin along the $\hat{\mathbf{z}}$-axis. Show that $N_+/N_- = \cot^4(\theta/2)$, where N_+ (respectively, N_-) is the number of atoms emerging from the filter with spins parallel (respectively, anti-parallel) to $\hat{\mathbf{z}}$.

Paper 1, Section II

33D Principles of Quantum Mechanics

A one-dimensional harmonic oscillator has Hamiltonian

$$H = \hbar\omega\left(A^\dagger A + \frac{1}{2}\right)$$

where $[A, A^\dagger] = 1$. Show that $A|n\rangle = \sqrt{n}|n-1\rangle$, where $H|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$ and $\langle n|n\rangle = 1$.

This oscillator is perturbed by adding a new term λX^4 to the Hamiltonian. Given that

$$A = \frac{m\omega X - iP}{\sqrt{2m\hbar\omega}}\,,$$

show that the ground state of the perturbed system is

$$|0_\lambda\rangle = |0\rangle - \frac{\hbar\lambda}{4m^2\omega^3}\left(3\sqrt{2}\,|2\rangle + \sqrt{\frac{3}{2}}\,|4\rangle\right)\,,$$

to first order in λ. [You may use the fact that, in non-degenerate perturbation theory, a perturbation Δ causes the first-order shift

$$|m^{(1)}\rangle = \sum_{n\neq m}\frac{\langle n|\Delta|m\rangle}{E_m - E_n}\,|n\rangle$$

in the m^{th} energy level.]

Paper 3, Section II

34D Principles of Quantum Mechanics

A quantum system is prepared in the ground state $|0\rangle$ at time $t = 0$. It is subjected to a time-varying Hamiltonian $H = H_0 + \Delta(t)$. Show that, to first order in $\Delta(t)$, the system evolves as

$$|\psi(t)\rangle = \sum_k c_k(t)\, \mathrm{e}^{-iE_k t/\hbar}|k\rangle\,,$$

where $H_0|k\rangle = E_k|k\rangle$ and

$$c_k(t) = \frac{1}{i\hbar}\int_0^t \langle k|\Delta(t')|0\rangle\, \mathrm{e}^{i(E_k-E_0)t'/\hbar}\,\mathrm{d}t'\,.$$

A large number of hydrogen atoms, each in the ground state, are subjected to an electric field

$$\mathbf{E}(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \hat{\mathbf{z}}\,\mathcal{E}_0 \exp(-t/\tau) & \text{for} \quad t > 0\,, \end{cases}$$

where $\mathcal{E}_0$ is a constant. Show that the fraction of atoms found in the state $|n,\ell,m\rangle = |2,1,0\rangle$ is, after a long time and to lowest non-trivial order in $\mathcal{E}_0$,

$$\frac{2^{15}}{3^{10}}\frac{a_0^2 e^2 \mathcal{E}_0^2}{\hbar^2(\omega^2 + 1/\tau^2)},$$

where $\hbar\omega$ is the energy difference between the $|2,1,0\rangle$ and $|1,0,0\rangle$ states, and e is the electron charge and a_0 the Bohr radius. What fraction of atoms lie in the $|2,0,0\rangle$ state?

[*Hint: You may assume the hydrogenic wavefunctions*

$$\langle \mathbf{r}|1,0,0\rangle = \frac{2}{\sqrt{4\pi}}\frac{1}{a_0^{3/2}}\exp\left(-\frac{r}{a_0}\right) \quad \textit{and} \quad \langle \mathbf{r}|2,1,0\rangle = \frac{1}{\sqrt{4\pi}}\frac{1}{(2a_0)^{3/2}}\frac{r}{a_0}\cos\theta\,\exp\left(-\frac{r}{2a_0}\right)$$

and the integral

$$\int_0^\infty r^m \mathrm{e}^{-\alpha r}\,\mathrm{d}r = \frac{m!}{\alpha^{m+1}}$$

for m a positive integer.]

Paper 2, Section II

34D Principles of Quantum Mechanics

Explain what is meant by the *intrinsic parity* of a particle.

In each of the decay processes below, parity is conserved.

A deuteron (d^+) has intrinsic parity $\eta_d = +1$ and spin $s = 1$. A negatively charged pion (π^-) has spin $s = 0$. The ground state of a hydrogenic 'atom' formed from a deuteron and a pion decays to two identical neutrons (n), each of spin $s = \frac{1}{2}$ and parity $\eta_n = +1$. Deduce the intrinsic parity of the pion.

The Δ^- particle has spin $s = \frac{3}{2}$ and decays as

$$\Delta^- \;\to\; \pi^- + n\,.$$

What are the allowed values of the orbital angular momentum? In the centre of mass frame, the vector $\mathbf{r}_\pi - \mathbf{r}_n$ joining the pion to the neutron makes an angle θ to the $\hat{\mathbf{z}}$-axis. The final state is an eigenstate of J_z and the spatial probability distribution is proportional to $\cos^2\theta$. Deduce the intrinsic parity of the Δ^-.

[*Hint: You may use the fact that the first three Legendre polynomials are given by*

$$P_0(x) = 1\ , \quad P_1(x) = x\ , \quad P_2(x) = \frac{1}{2}(3x^2 - 1)\ . \ ]$$

Paper 1, Section II

32C Principles of Quantum Mechanics

The position and momentum operators of the harmonic oscillator can be written as

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a + a^\dagger), \qquad \hat{p} = \left(\frac{\hbar m\omega}{2}\right)^{1/2} i(a^\dagger - a),$$

where m is the mass, ω is the frequency and the Hamiltonian is

$$H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2.$$

Assuming that

$$[\hat{x}, \hat{p}] = i\hbar$$

derive the commutation relations for a and $a^\dagger$. Construct the Hamiltonian in terms of a and $a^\dagger$. Assuming that there is a unique ground state, explain how all other energy eigenstates can be constructed from it. Determine the energy of each of these eigenstates.

Consider the modified Hamiltonian

$$H' = H + \lambda\hbar\omega\,(a^2 + a^{\dagger\,2}),$$

where λ is a dimensionless parameter. Use perturbation theory to calculate the modified energy levels to second order in λ, quoting any standard formulae that you require. Show that the modified Hamiltonian can be written as

$$H' = \frac{1}{2m}(1 - 2\lambda)\hat{p}^2 + \frac{1}{2}m\omega^2(1 + 2\lambda)\hat{x}^2\,.$$

Assuming $|\lambda| < \frac{1}{2}$, calculate the modified energies exactly. Show that the results are compatible with those obtained from perturbation theory.

Paper 2, Section II

32C Principles of Quantum Mechanics

Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be a set of Hermitian operators obeying

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \text{and} \quad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1, \qquad (*)$$

where $\mathbf{n}$ is any unit vector. Show that $(*)$ implies that

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma},$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$. Explain, with reference to the properties $(*)$, how $\boldsymbol{\sigma}$ can be related to the intrinsic angular momentum $\mathbf{S}$ for a particle of spin $\frac{1}{2}$.

Show that the operators $P_\pm = \frac{1}{2}(1 \pm \mathbf{n}\cdot\boldsymbol{\sigma})$ are Hermitian and obey

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0.$$

Show how $P_\pm$ can be used to write any state $|\chi\rangle$ as a linear combination of eigenstates of $\mathbf{n}\cdot\boldsymbol{\sigma}$. Use this to deduce that if the system is in a normalised state $|\chi\rangle$ when $\mathbf{n}\cdot\boldsymbol{\sigma}$ is measured, then the results ±1 will be obtained with probabilities

$$\|P_\pm|\chi\rangle\|^2 = \frac{1}{2}(1 \pm \langle\chi|\mathbf{n}\cdot\boldsymbol{\sigma}|\chi\rangle).$$

If $|\chi\rangle$ is a state corresponding to the system having spin up along a direction defined by a unit vector $\mathbf{m}$, show that a measurement will find the system to have spin up along $\mathbf{n}$ with probability $\frac{1}{2}(1 + \mathbf{n}\cdot\mathbf{m})$.

Paper 3, Section II

32C Principles of Quantum Mechanics

The angular momentum operators $\mathbf{J} = (J_1, J_2, J_3)$ obey the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \,,$$

$$[J_+, J_-] = 2J_3 \,,$$

where $J_\pm = J_1 \pm iJ_2$.

A quantum mechanical system involves the operators $a, a^\dagger, b$ and $b^\dagger$ such that

$$[a, a^\dagger] = [b, b^\dagger] = 1 \,,$$

$$[a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0.$$

Define $K_+ = a^\dagger b$, $K_- = ab^\dagger$ and $K_3 = \frac{1}{2}(a^\dagger a - b^\dagger b)$. Show that $K_\pm$ and K_3 obey the same commutation relations as $J_\pm$ and J_3.

Suppose that the system is in the state $|0\rangle$ such that $a|0\rangle = b|0\rangle = 0$. Show that $(a^\dagger)^2|0\rangle$ is an eigenstate of K_3. Let $K^2 = \frac{1}{2}(K_+K_- + K_-K_+) + K_3^2$. Show that $(a^\dagger)^2|0\rangle$ is an eigenstate of K^2 and find the eigenvalue. How many other states do you expect to find with same value of K^2? Find them.

Paper 4, Section II

32C Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + \lambda V(t)\,,$$

where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|n\rangle$ and $|m\rangle$ be eigenstates of H_0 with distinct eigenvalues E_n and E_m respectively. Show that if the system was in the state $|n\rangle$ in the remote past, then the probability of measuring it to be in a different state $|m\rangle$ at a time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_{-\infty}^{t} dt'\langle m|V(t')|n\rangle e^{i(E_m-E_n)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\;.$$

Let the system be a simple harmonic oscillator with $H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$, where $[a, a^\dagger] = 1$. Let $|0\rangle$ be the ground state which obeys $a|0\rangle = 0$. Suppose

$$V(t) = e^{-p|t|}(a + a^\dagger),$$

with $p > 0$. In the remote past the system was in the ground state. Find the probability, to lowest non-trivial order in λ, for the system to be in the first excited state in the far future.

Paper 1, Section II

31A Principles of Quantum Mechanics

A particle in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$ whose eigenstates obey

$$\langle x|x'\rangle = \delta(x-x')\,, \qquad \langle p|p'\rangle = \delta(p-p')\,, \qquad \langle x|p\rangle = (2\pi\hbar)^{-1/2} e^{ixp/\hbar}\,.$$

For a state $|\psi\rangle$, define the position-space and momentum-space wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ and show how each of these can be expressed in terms of the other.

Write down the translation operator $U(\alpha)$ and check that your expression is consistent with the property $U(\alpha)|x\rangle = |x+\alpha\rangle$. For a state $|\psi\rangle$, relate the position-space and momentum-space wavefunctions for $U(\alpha)|\psi\rangle$ to $\psi(x)$ and $\tilde{\psi}(p)$ respectively.

Now consider a harmonic oscillator with mass m, frequency ω, and annihilation and creation operators

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\,, \qquad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\,.$$

Let $\psi_n(x)$ and $\tilde{\psi}_n(p)$ be the wavefunctions corresponding to the normalised energy eigenstates $|n\rangle$, where $n = 0, 1, 2, \ldots$.

(i) Express $\psi_0(x-\alpha)$ explicitly in terms of the wavefunctions $\psi_n(x)$.

(ii) Given that $\tilde{\psi}_n(p) = f_n(u)\,\tilde{\psi}_0(p)$, where the f_n are polynomials and $u = (2/\hbar m\omega)^{1/2}p$, show that

$$e^{-i\gamma u} \;=\; e^{-\gamma^2/2}\sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}\,f_n(u) \quad \text{for any real } \gamma\,.$$

[You may quote standard results for a harmonic oscillator. You may also use, without proof, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators A and B which each commute with $[A,B]$.]

Paper 3, Section II

31A Principles of Quantum Mechanics

A three-dimensional oscillator has Hamiltonian

$$H = \frac{1}{2m}(\,\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2\,) \;+\; \frac{1}{2}m\omega^2(\,\alpha^2\hat{x}_1^2 + \beta^2\hat{x}_2^2 + \gamma^2\hat{x}_3^2\,)\,,$$

where the constants m, ω, α, β, γ are real and positive. Assuming a unique ground state, construct the general normalised eigenstate of H and give a formula for its energy eigenvalue. [You may quote without proof results for a one-dimensional harmonic oscillator of mass m and frequency ω that follow from writing $\hat{x} = (\hbar/2m\omega)^{1/2}(a + a^\dagger)$ and $\hat{p} = (\hbar m\omega/2)^{1/2}\, i\,(a^\dagger - a)\,.$]

List all states in the four lowest energy levels of H in the cases:

(i) $\alpha < \beta < \gamma < 2\alpha\,;$

(ii) $\alpha = \beta$ and $\gamma = \alpha + \epsilon\,,$ where $0 < \epsilon \ll \alpha\,.$

Now consider H with $\alpha = \beta = \gamma = 1$ subject to a perturbation

$$\lambda m\omega^2(\,\hat{x}_1\hat{x}_2 \,+\, \hat{x}_2\hat{x}_3 \,+\, \hat{x}_3\hat{x}_1\,)\,,$$

where λ is small. Compute the changes in energies for the ground state and the states at the first excited level of the original Hamiltonian, working to the leading order at which non-zero corrections occur. [You may quote without proof results from perturbation theory.]

Explain briefly why some energy levels of the perturbed Hamiltonian will be exactly degenerate. [*Hint: Compare with* (ii) *above.*]

Paper 4, Section II

31A Principles of Quantum Mechanics

(a) Consider a quantum system with Hamiltonian $H = H_0 + V$, where H_0 is independent of time. Define the interaction picture corresponding to this Hamiltonian and derive an expression for the time derivative of an operator in the interaction picture, assuming it is independent of time in the Schrödinger picture.

(b) The Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k \ .$$

Explain briefly how these properties allow $\boldsymbol{\sigma}$ to be used to describe a quantum system with spin $\frac{1}{2}$.

(c) A particle with spin $\frac{1}{2}$ has position and momentum operators $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$. The unitary operator corresponding to a rotation through an angle θ about an axis $\mathbf{n}$ is $U = \exp(-i\,\theta\,\mathbf{n}\cdot\mathbf{J}/\hbar)$ where $\mathbf{J}$ is the total angular momentum. Check this statement by considering the effect of an infinitesimal rotation on $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$ and $\boldsymbol{\sigma}$.

(d) Suppose that the particle in part (c) has Hamiltonian $H = H_0 + V$ with

$$H_0 = \frac{1}{2m}\hat{\mathbf{p}}^2 + \alpha\,\mathbf{L}\cdot\boldsymbol{\sigma} \qquad \text{and} \qquad V = B\,\sigma_3\,,$$

where $\mathbf{L}$ is the orbital angular momentum and α, B are constants. Show that all components of $\mathbf{J}$ are independent of time in the interaction picture. Is this true in the Heisenberg picture?

[You may quote commutation relations of $\mathbf{L}$ with $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$.]

Paper 2, Section II

32A Principles of Quantum Mechanics

(a) Let $|\, j\, m\,\rangle$ be standard, normalised angular momentum eigenstates with labels specifying eigenvalues for $\mathbf{J}^2$ and J_3. Taking units in which $\hbar = 1$,

$$J_{\pm}|\, j\, m\,\rangle \;=\; \big\{\, (j{\mp}m)(j{\pm}m{+}1)\,\big\}^{1/2}\,|\, j\, m{\pm}1\,\rangle\ .$$

Check the coefficients above by computing norms of states, quoting any angular momentum commutation relations that you require.

(b) Two particles, each of spin $s > 0$, have combined spin states $|\, J\, M\,\rangle$. Find expressions for all such states with $M = 2s{-}1$ in terms of product states.

(c) Suppose that the particles in part (b) move about their centre of mass with a spatial wavefunction that is a spherically symmetric function of relative position. If the particles are identical, what spin states $|\, J\ 2s{-}1\,\rangle$ are allowed? Justify your answer.

(d) Now consider two particles of spin 1 that are not identical and are both at rest. If the 3-component of the spin of each particle is zero, what is the probability that their total, combined spin is zero?

Paper 4, Section II

30A Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is $H_0 + \lambda V(t)$, where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Suppose that $|\chi\rangle$ and $|\phi\rangle$ are eigenstates of H_0 with distinct eigenvalues E and E', respectively. Show that if the system is in state $|\chi\rangle$ at time zero then the probability of measuring it to be in state $|\phi\rangle$ at time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt'\,\langle\phi|V(t')|\chi\rangle\, e^{i(E'-E)t'/\hbar}\right|^2 + O(\lambda^3)\,.$$

Let H_0 be the Hamiltonian for an isotropic three-dimensional harmonic oscillator of mass m and frequency ω, with $\chi(r)$ being the ground state wavefunction (where $r = |\mathbf{x}|$) and $\phi_i(\mathbf{x}) = (2m\omega/\hbar)^{1/2}\, x_i\, \chi(r)$ being wavefunctions for the states at the first excited energy level ($i = 1, 2, 3$). The oscillator is in its ground state at $t = 0$ when a perturbation

$$\lambda V(t) = \lambda\, \hat{x}_3\, e^{-\mu t}$$

is applied, with $\mu > 0$, and H_0 is then measured after a very large time has elapsed. Show that to first order in perturbation theory the oscillator will be found in one particular state at the first excited energy level with probability

$$\frac{\lambda^2}{2\hbar m\omega\,(\mu^2 + \omega^2)}\,,$$

but that the probability that it will be found in either of the other excited states is zero (to this order).

$$\left[\text{You may use the fact that } 4\pi\int_0^\infty r^4\,|\chi(r)|^2\,dr = \frac{3\hbar}{2m\omega}\,.\right]$$

Paper 3, Section II

31A Principles of Quantum Mechanics

Let $|\, j,\, m\,\rangle$ denote the normalised joint eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J}$ is the angular momentum operator for a quantum system. State clearly the possible values of the quantum numbers j and m and write down the corresponding eigenvalues in units with $\hbar = 1$.

Consider two quantum systems with angular momentum states $|\, \frac{1}{2},\, r\,\rangle$ and $|\, j,\, m\,\rangle$. The eigenstates corresponding to their combined angular momentum can be written as

$$|\, J,\, M\,\rangle \;=\; \sum_{r\,,\,m} C^{J\,M}_{r\,m}\, |\, \tfrac{1}{2},\, r\,\rangle\, |\, j,\, m\,\rangle\;,$$

where $C^{J\,M}_{r\,m}$ are Clebsch–Gordan coefficients for addition of angular momenta $\frac{1}{2}$ and j. What are the possible values of J and what is a necessary condition relating r, m and M in order that $C^{J\,M}_{r\,m} \neq 0$?

Calculate the values of $C^{J\,M}_{r\,m}$ for $j = 2$ and for all $M \geqslant \frac{3}{2}$. Use the sign convention that $C^{J\,J}_{r\,m} > 0$ when m takes its maximum value.

A particle X with spin $\frac{3}{2}$ and intrinsic parity η_X is at rest. It decays into two particles A and B with spin $\frac{1}{2}$ and spin 0, respectively. Both A and B have intrinsic parity -1. The relative orbital angular momentum quantum number for the two particle system is ℓ. What are the possible values of ℓ for the cases $\eta_X = +1$ and $\eta_X = -1$?

Suppose particle X is prepared in the state $|\, \frac{3}{2},\, \frac{3}{2}\,\rangle$ before it decays. Calculate the probability P for particle A to be found in the state $|\, \frac{1}{2},\, \frac{1}{2}\,\rangle$, given that $\eta_X = +1$.

What is the probability P if instead $\eta_X = -1$?

[Units with $\hbar = 1$ should be used throughout. You may also use without proof

$$J_{-}|\, j,\, m\,\rangle \;=\; \sqrt{(j{+}m)(j{-}m{+}1)}\;|\, j,\, m{-}1\,\rangle\,.]$$

Paper 2, Section II
31A Principles of Quantum Mechanics

Express the spin operator $\mathbf{S}$ for a particle of spin $\frac{1}{2}$ in terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Show that $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = \mathbb{I}$ for any unit vector $\mathbf{n}$ and deduce that

$$e^{-i\theta\, \mathbf{n}\cdot\mathbf{S}/\hbar} = \mathbb{I}\cos(\theta/2) - i(\mathbf{n} \cdot \boldsymbol{\sigma})\sin(\theta/2) .$$

The space of states V for a particle of spin $\frac{1}{2}$ has basis states $|\uparrow\rangle$, $|\downarrow\rangle$ which are eigenstates of S_3 with eigenvalues $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ respectively. If the Hamiltonian for the particle is $H = \frac{1}{2}\alpha\hbar\sigma_1$, find

$$e^{-itH/\hbar}|\uparrow\rangle \qquad \text{and} \qquad e^{-itH/\hbar}|\downarrow\rangle$$

as linear combinations of the basis states.

The space of states for a system of two spin $\frac{1}{2}$ particles is $V \otimes V$. Write down explicit expressions for the joint eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J}$ is the sum of the spin operators for the particles.

Suppose that the two-particle system has Hamiltonian $H = \frac{1}{2}\lambda\hbar(\sigma_1 \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_1)$ and that at time $t = 0$ the system is in the state with J_3 eigenvalue $\hbar$. Calculate the probability that at time $t > 0$ the system will be measured to be in the state with $\mathbf{J}^2$ eigenvalue zero.

Paper 1, Section II

31A Principles of Quantum Mechanics

If A and B are operators which each commute with their commutator $[A, B]$, show that

$$F(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)} \qquad \text{satisfies} \qquad F'(\lambda) = \lambda [A, B] F(\lambda) .$$

By solving this differential equation for $F(\lambda)$, deduce that

$$e^A e^B = e^{\frac{1}{2}[A,B]} e^{A+B} .$$

The annihilation and creation operators for a harmonic oscillator of mass m and frequency ω are defined by

$$a = \sqrt{\frac{m\omega}{2\hbar}} \Big(\hat{x} + \frac{i}{m\omega} \hat{p} \Big) , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \Big(\hat{x} - \frac{i}{m\omega} \hat{p} \Big) .$$

Write down an expression for the general normalised eigenstate $|n\rangle$ $(n = 0, 1, 2, \ldots)$ of the oscillator Hamiltonian H in terms of the ground state $|0\rangle$. What is the energy eigenvalue E_n of the state $|n\rangle$?

Suppose the oscillator is now subject to a small perturbation so that it is described by the modified Hamiltonian $H + \varepsilon V(\hat{x})$ with $V(\hat{x}) = \cos(\mu \hat{x})$. Show that

$$V(\hat{x}) = \tfrac{1}{2} e^{-\gamma^2/2} \left(e^{i\gamma a^\dagger} e^{i\gamma a} + e^{-i\gamma a^\dagger} e^{-i\gamma a} \right) ,$$

where γ is a constant, to be determined. Hence show that to $O(\varepsilon^2)$ the shift in the ground state energy as a result of the perturbation is

$$\varepsilon \, e^{-\mu^2\hbar/4m\omega} \; - \; \varepsilon^2 \, e^{-\mu^2\hbar/2m\omega} \frac{1}{\hbar\omega} \sum_{p=1}^{\infty} \frac{1}{(2p)! \, 2p} \Big(\frac{\mu^2\hbar}{2m\omega} \Big)^{2p} .$$

[Standard results of perturbation theory may be quoted without proof.]

Paper 4, Section II

32A Principles of Quantum Mechanics

Define the *interaction picture* for a quantum mechanical system with Schrödinger picture Hamiltonian $H_0 + V(t)$ and explain why the interaction and Schrödinger pictures give the same physical predictions for transition rates between eigenstates of H_0. Derive the equation of motion for the interaction picture states $|\overline{\psi(t)}\rangle$.

A system consists of just two states $|1\rangle$ and $|2\rangle$, with respect to which

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad V(t) = \hbar\lambda \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}.$$

Writing the interaction picture state as $|\overline{\psi(t)}\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle$, show that the interaction picture equation of motion can be written as

$$i\dot{a}_1(t) = \lambda e^{i\mu t} a_2(t), \qquad i\dot{a}_2(t) = \lambda e^{-i\mu t} a_1(t), \tag{$*$}$$

where $\mu = \omega - \omega_{21}$ and $\omega_{21} = (E_2 - E_1)/\hbar$. Hence show that $a_2(t)$ satisfies

$$\ddot{a}_2 + i\mu\,\dot{a}_2 + \lambda^2 a_2 = 0.$$

Given that $a_2(0) = 0$, show that the solution takes the form

$$a_2(t) = \alpha\, e^{-i\mu t/2} \sin \Omega t,$$

where Ω is a frequency to be determined and α is a complex constant of integration.

Substitute this solution for $a_2(t)$ into $(*)$ to determine $a_1(t)$ and, by imposing the normalization condition $\||\overline{\psi(t)}\rangle\|^2 = 1$ at $t = 0$, show that $|\alpha|^2 = \lambda^2/\Omega^2$.

At time $t = 0$ the system is in the state $|1\rangle$. Write down the probability of finding the system in the state $|2\rangle$ at time t.

Paper 3, Section II

33A Principles of Quantum Mechanics

Let $\mathbf{J} = (J_1, J_2, J_3)$ and $|j\,m\rangle$ denote the standard angular-momentum operators and states so that, in units where $\hbar = 1$,

$$\mathbf{J}^2|j\,m\rangle = j(j+1)|j\,m\rangle\,, \quad J_3|j\,m\rangle = m|j\,m\rangle\,.$$

Show that $U(\theta) = \exp(-i\theta J_2)$ is unitary. Define

$$J_i(\theta) \;=\; U(\theta)\,J_i\,U^{-1}(\theta) \quad \text{for } i = 1,2,3$$

and

$$|j\,m\rangle_\theta \;=\; U(\theta)|j\,m\rangle\,.$$

Find expressions for $J_1(\theta), J_2(\theta)$ and $J_3(\theta)$ as linear combinations of J_1, J_2 and J_3. Briefly explain why $U(\theta)$ represents a rotation of $\mathbf{J}$ through angle θ about the 2-axis.

Show that

$$J_3(\theta)|j\,m\rangle_\theta \;=\; m|j\,m\rangle_\theta\,. \qquad (*)$$

Express $|1\,0\rangle_\theta$ as a linear combination of the states $|1\,m\rangle$, $m = -1, 0, 1$. By expressing J_1 in terms of $J_\pm$, use $(*)$ to determine the coefficients in this expansion.

A particle of spin 1 is in the state $|1\,0\rangle$ at time $t = 0$. It is subject to the Hamiltonian

$$H = -\mu\mathbf{B}\cdot\mathbf{J}\,,$$

where $\mathbf{B} = (0, \mathbf{B}, 0)$. At time t the value of J_3 is measured and found to be $J_3 = 0$. At time $2t$ the value of J_3 is measured again and found to be $J_3 = 1$. Show that the joint probability for these two values to be measured is

$$\frac{1}{8}\sin^2(2\mu Bt)\,.$$

[*The following result may be quoted:* $J_\pm\,|j\,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\;|j\;m \pm 1\rangle$.]

Paper 2, Section II
33A Principles of Quantum Mechanics

(i) Let a and $a^\dagger$ be the annihilation and creation operators, respectively, for a simple harmonic oscillator whose Hamiltonian is

$$H_0 \;=\; \omega\left(a^\dagger a + \tfrac{1}{2}\right),$$

with $[a, a^\dagger] = 1$. Explain how the set of eigenstates $\{\, |n\rangle \;:\; n = 0, 1, 2, \ldots\}$ of H_0 is obtained and deduce the corresponding eigenvalues. Show that

$$a|0\rangle = 0\,,$$

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \qquad n \geqslant 1\,,$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \qquad n \geqslant 0\,.$$

(ii) Consider a system whose unperturbed Hamiltonian is

$$H_0 = \left(a^\dagger a + \tfrac{1}{2}\right) \;+\; 2\left(b^\dagger b + \tfrac{1}{2}\right),$$

where $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$ and all other commutators are zero. Find the degeneracies of the eigenvalues of H_0 with energies $E_0 = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$ and $\frac{11}{2}$.

The system is perturbed so that it is now described by the Hamiltonian

$$H = H_0 + \lambda H',$$

where $H' = (a^\dagger)^2 b + a^2 b^\dagger$. Using degenerate perturbation theory, calculate to $O(\lambda)$ the energies of the eigenstates associated with the level $E_0 = \frac{9}{2}$.

Write down the eigenstates, to $O(\lambda)$, associated with these perturbed energies. By explicit evaluation show that they are in fact exact eigenstates of H with these energies as eigenvalues.

Paper 1, Section II

33A Principles of Quantum Mechanics

Let $\hat{x}, \hat{p}$ and $H(\hat{x}, \hat{p}) = \hat{p}^2/2m + V(\hat{x})$ be the position operator, momentum operator and Hamiltonian for a particle moving in one dimension. Let $|\psi\rangle$ be the state vector for the particle. The position and momentum eigenstates have inner products

$$\langle x|p\rangle \;=\; \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar)\,, \qquad \langle x|x'\rangle = \delta(x-x') \qquad \text{and} \qquad \langle p|p'\rangle = \delta(p-p')\,.$$

Show that

$$\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x) \quad \text{and} \quad \langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\tilde{\psi}(p)\,,$$

where $\psi(x)$ and $\tilde{\psi}(p)$ are the wavefunctions in the position representation and momentum representation, respectively. Show how $\psi(x)$ and $\tilde{\psi}(p)$ may be expressed in terms of each other.

For general $V(\hat{x})$, express $\langle p|V(\hat{x})|\psi\rangle$ in terms of $\tilde{\psi}(p)$, and hence write down the time-independent Schrödinger equation in the momentum representation satisfied by $\tilde{\psi}(p)$.

Consider now the case $V(x) = -(\hbar^2\lambda/m)\delta(x),\ \lambda > 0$. Show that there is a bound state with energy $E = -\varepsilon,\ \varepsilon > 0$, with wavefunction $\tilde{\psi}(p)$ satisfying

$$\tilde{\psi}(p) = \frac{\hbar\lambda}{\pi}\frac{1}{2m\varepsilon + p^2}\int_{-\infty}^{\infty}\tilde{\psi}(p')\,dp'\,.$$

Hence show that there is a unique value for ε and determine this value.

Paper 4, Section II
32E Principles of Quantum Mechanics

(i) The creation and annihilation operators for a harmonic oscillator of angular frequency ω satisfy the commutation relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian H and number operator N in terms of a and $a^\dagger$. Explain how the space of eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$, of H is formed, and deduce the eigenenergies for these states. Show that

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,.$$

(ii) The operator K_r is defined to be

$$K_r = \frac{(a^\dagger)^r a^r}{r!}\,,$$

for $r = 0, 1, 2, \ldots$. Show that K_r commutes with N. Show that if $r \leqslant n$, then

$$K_r|n\rangle = \frac{n!}{(n-r)!r!}|n\rangle\,,$$

and $K_r|n\rangle = 0$ otherwise. By considering the action of K_r on the state $|n\rangle$, deduce that

$$\sum_{r=0}^{\infty}(-1)^r K_r = |0\rangle\langle 0|\,.$$

Paper 3, Section II
33E Principles of Quantum Mechanics

A particle moves in one dimension in an infinite square-well potential $V(x) = 0$ for $|x| < a$ and ∞ for $|x| > a$. Find the energy eigenstates. Show that the energy eigenvalues are given by $E_n = E_1 n^2$ for integer n, where E_1 is a constant which you should find.

The system is perturbed by the potential $\delta V = \epsilon x/a$. Show that the energy of the n^{th} level E_n remains unchanged to first order in ϵ. Show that the ground-state wavefunction is

$$\psi_1(x) = \frac{1}{\sqrt{a}}\left[\cos\frac{\pi x}{2a} + \frac{D\epsilon}{\pi^2 E_1}\sum_{n=2,4,\ldots}(-1)^{An}\frac{n^B}{(n^2-1)^C}\sin\frac{n\pi x}{2a} + \mathcal{O}(\epsilon^2)\right]\,,$$

where A, B, C and D are numerical constants which you should find. Briefly comment on the conservation of parity in the unperturbed and perturbed systems.

Paper 2, Section II

33E Principles of Quantum Mechanics

(i) In units where $\hbar = 1$, angular momentum states $|j\ m\rangle$ obey

$$J^2|j\ m\rangle = j(j+1)|j\ m\rangle, \qquad J_3|j\ m\rangle = m|j\ m\rangle.$$

Use the algebra of angular momentum $[J_i,\ J_j] = i\epsilon_{ijk}J_k$ to derive the following in terms of J^2, $J_\pm = J_1 \pm iJ_2$ and J_3:

(a) $[J^2,\ J_i]$;

(b) $[J_3,\ J_\pm]$;

(c) $[J^2,\ J_\pm]$.

(ii) Find J_+J_- in terms of J^2 and J_3. Thus calculate the quantum numbers of the state $J_\pm|j\ m\rangle$ in terms of j and m. Derive the normalisation of the state $J_-|j\ m\rangle$. Therefore, show that

$$\langle j\ j-1|J_+^{j-1}J_-^{j}|j\ j\rangle = \sqrt{A}\,(2j-1)!,$$

finding A in terms of j.

(iii) Consider the combination of a spinless particle with an electron of spin 1/2 and orbital angular momentum 1. Calculate the probability that the electron has a spin of $+1/2$ in the $z-$direction if the combined system has an angular momentum of $+1/2$ in the $z-$direction and a total angular momentum of $+3/2$. Repeat the calculation for a total angular momentum of $+1/2$.

Paper 1, Section II

33E Principles of Quantum Mechanics

Consider a composite system of several identical particles. Describe how the multi-particle state is constructed from single-particle states. For the case of two identical particles, describe how considering the interchange symmetry leads to the definition of bosons and fermions.

Consider two non-interacting, identical particles, each with spin 1. The single-particle, spin-independent Hamiltonian $H(\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_i)$ has non-degenerate eigenvalues E_n and wavefunctions $\psi_n(\mathbf{x}_i)$ where $i = 1, 2$ labels the particle and $n = 0, 1, 2, 3, \ldots$. In terms of these single-particle wavefunctions and single-particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the two-particle states and energies for:

(i) the ground state;

(ii) the first excited state.

Assume now that E_n is a linear function of n. Find the degeneracy of the N^{th} energy level of the two-particle system for:

(iii) N even;

(iv) N odd.

Paper 4, Section II
32A Principles of Quantum Mechanics

Setting $\hbar = 1$, the raising and lowering operators $J_\pm = J_1 \pm iJ_2$ for angular momentum satisfy

$$[J_3, J_\pm\,] = \pm J_\pm\,, \qquad J_\pm|j\ m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\ |j\ m\pm 1\rangle\,,$$

where $J_3|j\ m\rangle = m|j\ m\rangle$. Find the matrix representation $S_\pm$ for $J_\pm$ in the basis $\{|1\ 1\rangle, |1\ 0\rangle, |1\ -1\rangle\}$ of $j=1$ states. Hence, calculate the matrix representation $\mathbf{S}$ of $\mathbf{J}$.

Suppose that the angular momentum of the state $\mathbf{v} = |1\ m\rangle$ is measured in the direction $\mathbf{n} = (0, \sin\theta, \cos\theta)$ to be $+1$. Find the components of $\mathbf{v}$, expressing each component by a single term consisting of products of powers of $\sin(\theta/2)$ and $\cos(\theta/2)$ multiplied by constants.

Suppose that two measurements of a total angular momentum 1 system are made. The first is made in the third direction with value $+1$, and the second measurement is subsequently immediately made in direction $\mathbf{n}$. What is the probability that the second measurement is also $+1$?

Paper 3, Section II
33A Principles of Quantum Mechanics

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$\frac{-2e^2}{4\pi\epsilon_0 r}$$

has normalised, spherically-symmetric real wavefunctions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. The helium atom can be modelled by considering two non-interacting spin 1/2 particles in the above potential. What are the consequences of the Pauli exclusion principle for the ground state? Write down the two-electron state for this model in the form of a spatial wavefunction times a spin state. Assuming that wavefunctions are spherically-symmetric, find the states of the first excited energy level of the helium atom. What combined angular momentum quantum numbers J, M does each state have?

Assuming standard perturbation theory results, arrive at a multi-dimensional integral in terms of the one-particle wavefunctions for the first-order correction to the helium ground state energy, arising from the electron-electron interaction.

Paper 2, Section II

33A Principles of Quantum Mechanics

(a) Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture. Explain how the two pictures provide equivalent descriptions of physical results.

(b) Derive the equation of motion for an operator in the Heisenberg picture.

For a particle of mass m moving in one dimension, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, and the state vector is $|\Psi\rangle$. The eigenstates of $\hat{x}$ and $\hat{p}$ satisfy

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}, \qquad \langle x|x'\rangle = \delta(x-x'), \qquad \langle p|p'\rangle = \delta(p-p').$$

Use standard methods in the Dirac formalism to show that

$$\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x'),$$

$$\langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p-p').$$

Calculate $\langle x|\hat{H}|x'\rangle$ and express $\langle x|\hat{p}|\Psi\rangle$, $\langle x|\hat{H}|\Psi\rangle$ in terms of the position space wavefunction $\Psi(x)$.

Write down the momentum space Hamiltonian for the potential

$$V(\hat{x}) = m\omega^2\hat{x}^4/2.$$

Paper 1, Section II

33A Principles of Quantum Mechanics

Let a and $a^\dagger$ be the simple harmonic oscillator annihilation and creation operators, respectively. Write down the commutator $[a,\ a^\dagger]$.

Consider a new operator $b = ca + sa^\dagger$, where $c \equiv \cosh\theta$, $s \equiv \sinh\theta$ with θ a real constant. Show that

$$[b, b^\dagger] = 1.$$

Consider the Hamiltonian

$$H = \epsilon a^\dagger a + \frac{1}{2}\lambda({a^\dagger}^2 + a^2)\,,$$

where ϵ and λ are real and such that $\epsilon > \lambda > 0$. Assuming that $\epsilon c - \lambda s = Ec$ and $\lambda c - \epsilon s = Es$, with E a real constant, show that

$$[b, H] = Eb\,.$$

Thus, calculate the energy of $b|E_a\rangle$ in terms of E and E_a, where E_a is an eigenvalue of H.

Assuming that $b|E_{\text{min}}\rangle = 0$, calculate E_{min} in terms of λ, s and c. Find the possible values of $x = s/c$. Finally, show that the energy eigenvalues of the system are

$$E_n = -\frac{\epsilon}{2} + (n + \frac{1}{2})\sqrt{\epsilon^2 - \lambda^2}\,.$$

Paper 1, Section II

33D Principles of Quantum Mechanics

Two individual angular momentum states $|j_1,\ m_1\rangle$, $|j_2,\ m_2\rangle$, acted on by $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ respectively, can be combined to form a combined state $|J,\ M\rangle$. What is the combined angular momentum operator $\mathbf{J}$ in terms of $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$? [Units in which $\hbar = 1$ are to be used throughout.]

Defining raising and lowering operators $J_\pm^{(i)}$, where $i \in \{1,2\}$, find an expression for $\mathbf{J}^2$ in terms of $\mathbf{J}^{(i)2}$, $J_\pm^{(i)}$ and $J_3^{(i)}$. Show that this implies

$$\left[\mathbf{J}^2,\ J_3\right] = 0\,.$$

Write down the state with $J = j_1 + j_2$ and with J_3 eigenvalue $M = -j_1 - j_2$ in terms of the individual angular momentum states. From this starting point, calculate the combined state with eigenvalues $J = j_1 + j_2 - 1$ and $M = -j_1 - j_2 + 1$ in terms of the individual angular momentum states.

If $j_1 = 3$ and $j_2 = 1$ and the combined system is in the state $|3,\ -3\rangle$, what is the probability of measuring the $J_3^{(i)}$ eigenvalues of individual angular momentum states to be -3 and 0, respectively?

[You may assume without proof that standard angular momentum states $|j,\ m\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_\pm|j,\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\ |j,\ m+1\rangle,$$

and that

$$[J_\pm,\ J_3] = \pm J_\pm\,.\ ]$$

Paper 2, Section II

33D Principles of Quantum Mechanics

A quantum system has energy eigenstates $|n\rangle$ with eigenvalues $E_n = n\hbar$, $n \in \{1, 2, 3, \ldots\}$. An observable Q is such that $Q|n\rangle = q_n|n\rangle$.

(a) What is the commutator of Q with the Hamiltonian H?

(b) Given $q_n = \frac{1}{n}$, consider the state

$$|\psi\rangle \propto \sum_{n=1}^{N} \sqrt{n}|n\rangle\,.$$

Determine:

(i) The probability of measuring Q to be $1/N$.

(ii) The probability of measuring energy $\hbar$ followed by another immediate measurement of energy $2\hbar$.

(iii) The average of many separate measurements of Q, each measurement being on a state $|\psi\rangle$, as $N \to \infty$.

(c) Given $q_1 = 1$ and $q_n = -1$ for $n > 1$, consider the state

$$|\psi\rangle \propto \sum_{n=1}^{\infty} \alpha^{n/2}|n\rangle\,,$$

where $0 < \alpha < 1$.

(i) Show that the probability of measuring an eigenvalue $q = -1$ of $|\psi\rangle$ is

$$A + B\alpha\,,$$

where A and B are integers that you should find.

(ii) Show that $\langle Q\rangle_\psi$ is $C + D\alpha$, where C and D are integers that you should find.

(iii) Given that Q is measured to be -1 at time $t = 0$, write down the state after a time t has passed. What is then the subsequent probability at time t of measuring the energy to be $2\hbar$?

Paper 3, Section II

33D Principles of Quantum Mechanics

The Pauli matrices $\boldsymbol{\sigma} = (\sigma_x,\ \sigma_y,\ \sigma_z) = (\sigma_1,\ \sigma_2,\ \sigma_3)$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are used to represent angular momentum operators with respect to basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ corresponding to spin up and spin down along the z-axis. They satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \,.$$

(i) How are $|\uparrow\rangle$ and $|\downarrow\rangle$ represented? How is the spin operator $\mathbf{s}$ related to $\boldsymbol{\sigma}$ and $\hbar$? Check that the commutation relations between the spin operators are as desired. Check that $\mathbf{s}^2$ acting on a spin one-half state has the correct eigenvalue.

What are the states obtained by applying s_x, s_y to the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of s_z?

(ii) Let V be the space of states for a spin one-half system. Consider a combination of three such systems with states belonging to $V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ and spin operators acting on each subsystem denoted by $s_x^{(i)}$, $s_y^{(i)}$ with $i = 1, 2, 3$. Find the eigenvalues of the operators

$$s_x^{(1)} s_y^{(2)} s_y^{(3)}\,, \quad s_y^{(1)} s_x^{(2)} s_y^{(3)}\,, \quad s_y^{(1)} s_y^{(2)} s_x^{(3)} \quad \text{and} \quad s_x^{(1)} s_x^{(2)} s_x^{(3)}$$

of the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big[\, |\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 \; - |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3 \,\big]\,.$$

(iii) Consider now whether these outcomes for measurements of particular combinations of the operators $s_x^{(i)}$, $s_y^{(i)}$ in the state $|\Psi\rangle$ could be reproduced by replacing the spin operators with classical variables $\tilde{s}_x^{(i)}$, $\tilde{s}_y^{(i)}$ which take values $\pm\hbar/2$ according to some probabilities. Assume that these variables are identical to the quantum measurements of $s_x^{(1)} s_y^{(2)} s_y^{(3)}$, $s_y^{(1)} s_x^{(2)} s_y^{(3)}$, $s_y^{(1)} s_y^{(2)} s_x^{(3)}$ on $|\Psi\rangle$. Show that classically this implies a unique possibility for

$$\tilde{s}_x^{(1)} \tilde{s}_x^{(2)} \tilde{s}_x^{(3)}\,,$$

and find its value.

State briefly how this result could be used to experimentally test quantum mechanics.

Paper 4, Section II

32D Principles of Quantum Mechanics

The quantum-mechanical observable Q has just two orthonormal eigenstates $|1\rangle$ and $|2\rangle$ with eigenvalues -1 and 1, respectively. The operator Q' is defined by $Q' = Q + \epsilon T$, where

$$T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} .$$

Defining orthonormal eigenstates of Q' to be $|1'\rangle$ and $|2'\rangle$ with eigenvalues q_1', q_2', respectively, consider a perturbation to first order in $\epsilon \in \mathbb{R}$ for the states

$$|1'\rangle = a_1|1\rangle + a_2\epsilon|2\rangle \,, \qquad |2'\rangle = b_1|2\rangle + b_2\epsilon|1\rangle \,,$$

where a_1, a_2, b_1, b_2 are complex coefficients. The real eigenvalues are also expanded to first order in ϵ:

$$q_1' = -1 + c_1\epsilon \,, \qquad q_2' = 1 + c_2\epsilon \,.$$

From first principles, find a_1, a_2, b_1, b_2, c_1, c_2.

Working exactly to all orders, find the real eigenvalues q_1', q_2' directly. Show that the exact eigenvectors of Q' may be taken to be of the form

$$A_j(\epsilon) \begin{pmatrix} 1 \\ -i(1 + Bq_j')/\epsilon \end{pmatrix} ,$$

finding $A_j(\epsilon)$ and the real numerical coefficient B in the process.

By expanding the exact expressions, again find a_1, a_2, b_1, b_2, c_1, c_2, verifying the perturbation theory results above.

Paper 1, Section II

33C Principles of Quantum Mechanics

Two states $|j_1\ m_1\rangle_1$, $|j_2\ m_2\rangle_2$, with angular momenta j_1, j_2, are combined to form states $|J\ M\rangle$ with total angular momentum

$$J = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 .$$

Write down the state with $J = M = j_1 + j_2$ in terms of the original angular momentum states. Briefly describe how the other combined angular momentum states may be found in terms of the original angular momentum states.

If $j_1 = j_2 = j$, explain why the state with $J = 0$ must be of the form

$$|0\ 0\rangle = \sum_{m=-j}^{j} \alpha_m |j\ m\rangle_1 |j\ {-m}\rangle_2 .$$

By considering $J_+|0\ 0\rangle$, determine a relation between α_{m+1} and α_m, hence find α_m .

If the system is in the state $|j\ j\rangle_1|j\ {-j}\rangle_2$ what is the probability, written in terms of j, of measuring the combined total angular momentum to be zero?

[Standard angular momentum states $|j\ m\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_\pm|j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m{\pm}1\rangle .$$

Units in which $\hbar = 1$ have been used throughout.]

Paper 2, Section II

33C Principles of Quantum Mechanics

Consider a joint eigenstate of $\mathbf{J}^2$ and J_3, $|j\ m\rangle$. Write down a unitary operator $U(\mathbf{n},\theta)$ for rotation of the state by an angle θ about an axis with direction $\mathbf{n}$, where $\mathbf{n}$ is a unit vector. How would a state with zero orbital angular momentum transform under such a rotation?

What is the relation between the angular momentum operator $\mathbf{J}$ and the Pauli matrices $\boldsymbol{\sigma}$ when $j=\frac{1}{2}$? Explicitly calculate $(\mathbf{J}\cdot\mathbf{a})^2$, for an arbitrary real vector $\mathbf{a}$, in this case. What are the eigenvalues of the operator $\mathbf{J}\cdot\mathbf{a}$? Show that the unitary rotation operator for $j=\frac{1}{2}$ can be expressed as

$$U(\mathbf{n},\theta) = \cos\frac{\theta}{2} - i\,\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}\,. \qquad (*)$$

Starting with a state $|\frac{1}{2}\ m\rangle$ the component of angular momentum along a direction $\mathbf{n}'$, making and angle θ with the z-axis, is susequently measured to be m'. Immediately after this measurement the state is $|\frac{1}{2}\ m'\rangle_\theta$. Write down an eigenvalue equation for $|\frac{1}{2}\ m'\rangle_\theta$ in terms of $\mathbf{n}'\cdot\mathbf{J}$. Show that the probability for measuring an angular momentum of $m'\hbar$ along the direction $\mathbf{n}'$ is, assuming $\mathbf{n}'$ is in the x-z plane,

$$\left|\langle\tfrac{1}{2}\ m|\tfrac{1}{2}\ m'\rangle_\theta\right|^2 = \left|\langle\tfrac{1}{2}\ m|U(\mathbf{y},\theta)|\tfrac{1}{2}\ m'\rangle\right|^2,$$

where $\mathbf{y}$ is a unit vector in the y-direction. Using $(*)$ show that the probability that $m=+\frac{1}{2}$, $m'=-\frac{1}{2}$ is of the form

$$A + B\cos^2\frac{\theta}{2}\,,$$

determining the integers A and B in the process.

[Assume $\hbar=1$. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
]

Paper 3, Section II

33C Principles of Quantum Mechanics

What are the commutation relations between the position operator $\hat{x}$ and momentum operator $\hat{p}$? Show that this is consistent with $\hat{x}$, $\hat{p}$ being hermitian.

The annihilation operator for a harmonic oscillator is

$$a = \sqrt{\frac{1}{2\hbar}}\,(\hat{x} + i\hat{p})$$

in units where the mass and frequency of the oscillator are 1. Derive the relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian

$$H = \tfrac{1}{2}\,\hat{p}^2 + \tfrac{1}{2}\,\hat{x}^2$$

in terms of the operator $N = a^\dagger a$.

Assume there exists a unique ground state $|0\rangle$ of H such that $a|0\rangle = 0$. Explain how the space of eigenstates $|n\rangle$, is formed, and deduce the energy eigenvalues for these states. Show that

$$a|n\rangle = A|n{-}1\rangle\,, \qquad a^\dagger|n\rangle = B|n{+}1\rangle\,,$$

finding A and B in terms of n.

Calculate the energy eigenvalues of the Hamiltonian for two harmonic oscillators

$$H = H_1 + H_2\,, \qquad H_i = \tfrac{1}{2}\,\hat{p}_i^2 + \tfrac{1}{2}\,\hat{x}_i^2\,, \quad i = 1, 2\,.$$

What is the degeneracy of the n^{th} energy level? Suppose that the two oscillators are then coupled by adding the extra term

$$\Delta H = \lambda \hat{x}_1\,\hat{x}_2$$

to H, where $\lambda \ll 1$. Calculate the energies for the states of the unperturbed system with the three lowest energy eigenvalues to first order in λ using perturbation theory.

[You may assume standard perturbation theory results.]

Paper 4, Section II

32C Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + V(t)\,,$$

where H_0 is independent of time. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|a\rangle$ and $|b\rangle$ be orthonormal eigenstates of H_0 with eigenvalues E_a and E_b respectively. Assume $V(t) = 0$ for $t \leqslant 0$. Show that if the system is initially, at $t = 0$, in the state $|a\rangle$ then the probability of measuring it to be the state $|b\rangle$ after a time t is

$$\frac{1}{\hbar^2}\left|\int_0^t dt' \langle b|V(t')|a\rangle e^{\,i(E_b-E_a)t'/\hbar}\right|^2 \qquad (*)$$

to order $V(t)^2$.

Suppose a system has a basis of just two orthonormal states $|1\rangle$ and $|2\rangle$, with respect to which

$$H_0 \,=\, E\,I\,, \qquad V(t) = vt\,\sigma_1\,, \quad t \geqslant 0\,,$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 \,=\, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Use $(*)$ to calculate the probability of a transition from state $|1\rangle$ to state $|2\rangle$ after a time t to order v^2.

Show that the time dependent Schrödinger equation has a solution

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\Big(Et\,I + \tfrac{1}{2}vt^2\,\sigma_1\Big)\right)|\psi(0)\rangle\,.$$

Calculate the transition probability exactly. Hence find the condition for the order v^2 approximation to be valid.

Paper 1, Section II

33C Principles of Quantum Mechanics

The position and momentum for a harmonic oscillator can be written

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\, a + a^{\dagger}\,), \qquad \hat{p} = \left(\frac{\hbar m\omega}{2}\right)^{1/2} i(\, a^{\dagger} - a\,),$$

where m is the mass, ω is the frequency, and the Hamiltonian is

$$H \;=\; \frac{1}{2m}\hat{p}^2 \;+\; \frac{1}{2}m\omega^2\hat{x}^2 \;=\; \hbar\omega\Big(\, a^{\dagger}a + \frac{1}{2}\,\Big).$$

Starting from the commutation relations for a and $a^{\dagger}$, determine the energy levels of the oscillator. Assuming a unique ground state, explain how all other energy eigenstates can be constructed from it.

Consider a modified Hamiltonian

$$H' \;=\; H \;+\; \lambda\hbar\omega\,(\, a^2 + a^{\dagger\, 2}\,),$$

where λ is a dimensionless parameter. Calculate the modified energy levels to second order in λ, quoting any standard formulas which you require. Show that the modified Hamiltonian can be written as

$$H' \;=\; \frac{1}{2m}\alpha\hat{p}^2 \;+\; \frac{1}{2}m\omega^2\beta\hat{x}^2,$$

where α and β depend on λ. Hence find the modified energies exactly, assuming $|\lambda| < \frac{1}{2}$, and show that the results are compatible with those obtained from perturbation theory.

Paper 2, Section II

33C Principles of Quantum Mechanics

Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be a set of Hermitian operators obeying

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad \text{and} \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1\,, \tag{$*$}$$

where $\mathbf{n}$ is any unit vector. Show that $(*)$ implies

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) \;=\; \mathbf{a}\cdot\mathbf{b} \,+\, i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}\,,$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$. Explain, with reference to the properties $(*)$, how $\boldsymbol{\sigma}$ can be related to the intrinsic angular momentum $\mathbf{S}$ for a particle of spin $\frac{1}{2}$.

Show that the operators $P_\pm = \frac{1}{2}(1 \pm \mathbf{n}\cdot\boldsymbol{\sigma})$ are Hermitian and obey

$$P_\pm^2 = P_\pm\ , \qquad P_+P_- = P_-P_+ = 0\,.$$

Show also how $P_\pm$ can be used to write any state $|\chi\rangle$ as a linear combination of eigenstates of $\mathbf{n}\cdot\boldsymbol{\sigma}$. Use this to deduce that if the system is in a normalised state $|\chi\rangle$ when $\mathbf{n}\cdot\boldsymbol{\sigma}$ is measured, then the results ±1 will be obtained with probabilities

$$||\,P_\pm|\chi\rangle\,||^2 \;=\; \tfrac{1}{2}(\,1\,\pm\,\langle\chi|\mathbf{n}\cdot\boldsymbol{\sigma}|\chi\rangle\,)\,.$$

If $|\chi\rangle$ is a state corresponding to the system having spin up along a direction defined by a unit vector $\mathbf{m}$, show that a measurement will find the system to have spin up along $\mathbf{n}$ with probability $\frac{1}{2}(1+\mathbf{n}\cdot\mathbf{m})$.

Paper 3, Section II

33C Principles of Quantum Mechanics

(i) Consider two quantum systems with angular momentum states $|\, j\, m\,\rangle$ and $|\, 1\, q\,\rangle$. The eigenstates corresponding to their combined angular momentum can be written as

$$|\, J\, M\,\rangle \;=\; \sum_{q\,m} C^{J\,M}_{q\,m}\,|\, 1\, q\,\rangle |\, j\, m\,\rangle\,,$$

where $C^{J\,M}_{q\,m}$ are Clebsch–Gordan coefficients for addition of angular momenta one and j. What are the possible values of J and how must q, m and M be related for $C^{J\,M}_{q\,m} \neq 0$?

Construct all states $|\, J\, M\,\rangle$ in terms of product states in the case $j = \frac{1}{2}$.

(ii) A general stationary state for an electron in a hydrogen atom $|n\,\ell\,m\rangle$ is specified by the principal quantum number n in addition to the labels ℓ and m corresponding to the total orbital angular momentum and its component in the 3-direction (electron spin is ignored). An oscillating electromagnetic field can induce a transition to a new state $|n'\,\ell'\,m'\rangle$ and, in a suitable approximation, the amplitude for this to occur is proportional to

$$\langle n'\,\ell'\,m'|\,\hat{x}_i\,|n\,\ell\,m\rangle\,,$$

where $\hat{x}_i$ $(i = 1, 2, 3)$ are components of the electron's position. Give clear but concise arguments based on angular momentum which lead to conditions on ℓ, m, ℓ', m' and i for the amplitude to be non-zero.

Explain briefly how parity can be used to obtain an additional selection rule.

[Standard angular momentum states $|\, j\, m\,\rangle$ are joint eigenstates of $\mathbf{J}^2$ and J_3, obeying

$$J_\pm|\, j\, m\,\rangle \;=\; \sqrt{(j{\mp}m)(j{\pm}m{+}1)}\,|\, j\, m{\pm}1\,\rangle, \quad J_3|\, j\, m\,\rangle = m|\, j\, m\,\rangle.$$

You may also assume that $X_{\pm1} = \frac{1}{\sqrt{2}}(\mp\hat{x}_1 - i\hat{x}_2)$ and $X_0 = \hat{x}_3$ have commutation relations with orbital angular momentum $\mathbf{L}$ given by

$$[L_3, X_q] \;=\; qX_q\;, \qquad [L_\pm, X_q] \;=\; \sqrt{(1{\mp}q)(2{\pm}q)}\,X_{q\pm1}\,.$$

Units in which $\hbar = 1$ are to be used throughout.]

Paper 4, Section II

32C Principles of Quantum Mechanics

For any given operators A and B, show that $F(\lambda) = e^{\lambda A} B e^{-\lambda A}$ has derivative $F'(\lambda) = e^{\lambda A}[A, B]e^{-\lambda A}$ and deduce an analogous formula for the nth derivative. Hence, by considering $F(\lambda)$ as a power series in λ, show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \ldots + \frac{1}{n!}[A,[A,\ldots[A,B]\ldots]] + \ldots \qquad (*)$$

A particle of unit mass in one dimension has position $\hat{x}$ and momentum $\hat{p}$ in the Schrödinger picture, and Hamiltonian

$$H = \tfrac{1}{2}\hat{p}^2 - \alpha\hat{x},$$

where α is a constant. Apply $(*)$ to find the Heisenberg picture operators $\hat{x}(t)$ and $\hat{p}(t)$ in terms of $\hat{x}$ and $\hat{p}$, and check explicitly that $H(\hat{x}(t), \hat{p}(t)) = H(\hat{x}, \hat{p})$.

A particle of unit mass in two dimensions has position $\hat{x}_i$ and momentum $\hat{p}_i$ in the Schrödinger picture, and Hamiltonian

$$H = \tfrac{1}{2}\left(\hat{p}_1^2 + \hat{p}_2^2\right) - \beta\left(\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1\right),$$

where β is a constant. Calculate the Heisenberg picture momentum components $\hat{p}_i(t)$ in terms of $\hat{p}_i$ and verify that $\hat{p}_1(t)^2 + \hat{p}_2(t)^2$ is independent of time. Now consider the interaction picture corresponding to $H = H_0 + V$: show that if $H_0 = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2)$ then the interaction picture position operators are $\hat{x}_i + t\hat{p}_i$, and use this to find the Heisenberg picture position operators $\hat{x}_i(t)$ in terms of $\hat{x}_i$ and $\hat{p}_i$.

[*Hint: If* $[H_0, V] = 0$ *and* $\bar{Q}(t)$ *is an operator in the interaction picture, then the corresponding operator in the Heisenberg picture is* $Q(t) = e^{itV/\hbar}\bar{Q}(t)e^{-itV/\hbar}$.]

1/II/32D **Principles of Quantum Mechanics**

(a) If A and B are operators which each commute with their commutator $[A,B]$, show that $[A, e^B] = [A,B]e^B$. By considering

$$F(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$$

and differentiating with respect to the parameter λ, show also that

$$e^A e^B \;=\; C e^{A+B} \;=\; e^{A+B} C$$

where $C = e^{\frac{1}{2}[A,B]}$.

(b) Consider a one-dimensional quantum system with position $\hat{x}$ and momentum $\hat{p}$. Write down a formula for the operator $U(\alpha)$ corresponding to translation through α, calculate $[\hat{x}, U(\alpha)]$, and show that your answer is consistent with the assumption that position eigenstates obey $|x+\alpha\rangle \;=\; U(\alpha)|x\rangle$. Given this assumption, express the wavefunction for $U(\alpha)|\psi\rangle$ in terms of the wavefunction $\psi(x)$ for $|\psi\rangle$.

Now suppose the one-dimensional system is a harmonic oscillator of mass m and frequency ω. Show that

$$\psi_0(x-\alpha) \;=\; e^{-m\omega\alpha^2/4\hbar} \sum_{n=0}^{\infty} \left(\frac{m\omega}{2\hbar}\right)^{n/2} \frac{\alpha^n}{\sqrt{n!}}\, \psi_n(x),$$

where $\psi_n(x)$ are normalised wavefunctions with energies $E_n = \hbar\omega(n + \frac{1}{2})$.

[Standard results for constructing normalised energy eigenstates in terms of annihilation and creation operators

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\!\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \qquad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\!\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

may be quoted without proof.]

2/II/32D **Principles of Quantum Mechanics**

Derive approximate expressions for the eigenvalues of a Hamiltonian $H + \lambda V$, working to second order in the parameter λ and assuming the eigenstates and eigenvalues of H are known and non-degenerate.

Let $\mathbf{J} = (J_1, J_2, J_3)$ be angular momentum operators with $|j\, m\rangle$ joint eigenstates of $\mathbf{J}^2$ and J_3. What are the possible values of the labels j and m and what are the corresponding eigenvalues of the operators?

A particle with spin j is trapped in space (its position and momentum can be ignored) but is subject to a magnetic field of the form $\mathbf{B} = (B_1, 0, B_3)$, resulting in a Hamiltonian $-\gamma(B_1 J_1 + B_3 J_3)$. Starting from the eigenstates and eigenvalues of this Hamiltonian when $B_1 = 0$, use perturbation theory to compute the leading order corrections to the energies when B_1 is non-zero but much smaller than B_3. Compare with the exact result.

[You may set $\hbar = 1$ and use $J_\pm |j\, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\, m{\pm}1\rangle$.]

3/II/32D **Principles of Quantum Mechanics**

Explain, in a few lines, how the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are used to represent angular momentum operators with respect to basis states $|\!\uparrow\rangle$ and $|\!\downarrow\rangle$ corresponding to spin up and spin down along the 3-axis. You should state clearly which properties of the matrices correspond to general features of angular momentum and which are specific to spin half.

Consider two spin-half particles labelled A and B, each with its spin operators and spin eigenstates. Find the matrix representation of

$$\boldsymbol{\sigma}^{(\mathrm{A})} \cdot \boldsymbol{\sigma}^{(\mathrm{B})} = \sigma_1^{(\mathrm{A})}\sigma_1^{(\mathrm{B})} + \sigma_2^{(\mathrm{A})}\sigma_2^{(\mathrm{B})} + \sigma_3^{(\mathrm{A})}\sigma_3^{(\mathrm{B})}$$

with respect to a basis of two-particle states $|\!\uparrow\rangle_{\mathrm{A}}|\!\uparrow\rangle_{\mathrm{B}}\,,\ |\!\downarrow\rangle_{\mathrm{A}}|\!\uparrow\rangle_{\mathrm{B}}\,,\ |\!\uparrow\rangle_{\mathrm{A}}|\!\downarrow\rangle_{\mathrm{B}}\,,\ |\!\downarrow\rangle_{\mathrm{A}}|\!\downarrow\rangle_{\mathrm{B}}$. Show that the eigenvalues of the matrix are $1, 1, 1, -3$ and find the eigenvectors.

What is the behaviour of each eigenvector under interchange of A and B? If the particles are identical, and there are no other relevant degrees of freedom, which of the two-particle states are allowed?

By relating $(\boldsymbol{\sigma}^{(\mathrm{A})} + \boldsymbol{\sigma}^{(\mathrm{B})})^2$ to the operator discussed above, show that your findings are consistent with standard results for addition of angular momentum.

4/II/32D **Principles of Quantum Mechanics**

Define the interaction picture for a quantum mechanical system with Schrödinger picture Hamiltonian $H_0 + V(t)$ and explain why either picture gives the same physical predictions. Derive an equation of motion for interaction picture states and use this to show that the probability of a transition from a state $|n\rangle$ at time zero to a state $|m\rangle$ at time t is

$$P(t) = \frac{1}{\hbar^2} \left| \int_0^t e^{i(E_m - E_n)t'/\hbar} \langle m|V(t')|n\rangle \, dt' \right|^2$$

correct to second order in V, where the initial and final states are orthogonal eigenstates of H_0 with eigenvalues E_n and E_m.

Consider a perturbed harmonic oscillator:

$$H_0 = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \, , \qquad V(t) = \hbar\lambda(a e^{i\nu t} + a^\dagger e^{-i\nu t})$$

with a and $a^\dagger$ annihilation and creation operators (all usual properties may be assumed). Working to order λ^2, find the probability for a transition from an initial state with $E_n = \hbar\omega(n + \frac{1}{2})$ to a final state with $E_m = \hbar\omega(m + \frac{1}{2})$ after time t.

Suppose t becomes large and perturbation theory still applies. Explain why the rate $P(t)/t$ for each allowed transition is sharply peaked, as a function of ν, around $\nu = \omega$.

1/II/32D Principles of Quantum Mechanics

A particle in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$ whose eigenstates obey

$$\langle x|x'\rangle = \delta(x-x')\ , \qquad \langle p|p'\rangle = \delta(p-p')\ , \qquad \langle x|p\rangle = (2\pi\hbar)^{-1/2}e^{ixp/\hbar}\ .$$

Given a state $|\psi\rangle$, define the corresponding position-space and momentum-space wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ and show how each of these can be expressed in terms of the other. Derive the form taken in momentum space by the time-independent Schrödinger equation

$$\Big(\frac{\hat{p}^2}{2m} + V(\hat{x})\Big)|\psi\rangle = E|\psi\rangle$$

for a general potential V.

Now let $V(x) = -(\hbar^2\lambda/m)\delta(x)$ with λ a positive constant. Show that the Schrödinger equation can be written

$$\Big(\frac{p^2}{2m} - E\Big)\tilde{\psi}(p) = \frac{\hbar\lambda}{2\pi m}\int_{-\infty}^{\infty} dp'\,\tilde{\psi}(p')$$

and verify that it has a solution $\tilde{\psi}(p) = N/(p^2+\alpha^2)$ for unique choices of α and E, to be determined (you need not find the normalisation constant, N). Check that this momentum space wavefunction can also be obtained from the position space solution $\psi(x) = \sqrt{\lambda}e^{-\lambda|x|}$.

2/II/32D Principles of Quantum Mechanics

Let $|s\,m\rangle$ denote the combined spin eigenstates for a system of two particles, each with spin 1. Derive expressions for all states with $m = s$ in terms of product states.

Given that the particles are identical, and that the spatial wavefunction describing their relative position has definite orbital angular momentum ℓ, show that $\ell + s$ must be even. Suppose that this two-particle state is known to arise from the decay of a single particle, X, also of spin 1. Assuming that total angular momentum and parity are conserved in this process, find the values of ℓ and s that are allowed, depending on whether the intrinsic parity of X is even or odd.

[*You may set* $\hbar = 1$ *and use* $J_\pm|\,j\,m\,\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\ |\,j\ m{\pm}1\,\rangle$.]

3/II/32D **Principles of Quantum Mechanics**

Let
$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^\dagger) \ , \qquad \hat{p} = \left(\frac{\hbar m\omega}{2}\right)^{1/2} i(a^\dagger - a)$$
be the position and momentum operators for a one-dimensional harmonic oscillator of mass m and frequency ω. Write down the commutation relations obeyed by a and $a^\dagger$ and give an expression for the oscillator Hamiltonian $H(\hat{x}, \hat{p})$ in terms of them. Prove that the only energies allowed are $E_n = \hbar\omega(n + \frac{1}{2})$ with $n = 0, 1, 2, \ldots$ and give, without proof, a formula for a general normalised eigenstate $|n\rangle$ in terms of $|0\rangle$.

A three-dimensional oscillator with charge is subjected to a weak electric field so that its total Hamiltonian is
$$H_1 \ + \ H_2 \ + \ H_3 \ + \ \lambda m\omega^2(\,\hat{x}_1\hat{x}_2 + \hat{x}_2\hat{x}_3 + \hat{x}_3\hat{x}_1\,)$$
where $H_i = H(\hat{x}_i, \hat{p}_i)$ for $i = 1, 2, 3$ and λ is a small, dimensionless parameter. Express the general eigenstate for the Hamiltonian with $\lambda = 0$ in terms of one-dimensional oscillator states, and give the corresponding energy eigenvalue. Use perturbation theory to compute the changes in energies of states in the lowest two levels when $\lambda \neq 0$, working to the leading order at which non-vanishing corrections occur.

4/II/32D **Principles of Quantum Mechanics**

The Hamiltonian for a particle of spin $\frac{1}{2}$ in a magnetic field $\mathbf{B}$ is

$$H = -\frac{1}{2}\hbar\gamma\mathbf{B}\cdot\boldsymbol{\sigma} \qquad \text{where} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and γ is a constant (the motion of the particle in space can be ignored). Consider a magnetic field which is independent of time. Writing $\mathbf{B} = B\mathbf{n}$, where $\mathbf{n}$ is a unit vector, calculate the time evolution operator and show that if the particle is initially in a state $|\chi\rangle$ the probability of measuring it to be in an orthogonal state $|\chi'\rangle$ after a time t is

$$|\langle\chi'|\mathbf{n}\cdot\boldsymbol{\sigma}|\chi\rangle|^2 \sin^2\frac{\gamma Bt}{2}\ .$$

Evaluate this to find the probability for a transition from a state of spin up along the z direction to one of spin down along the z direction when $\mathbf{B} = (B_x, 0, B_z)$.

Now consider a magnetic field whose x and y components are time-dependent but small:

$$\mathbf{B} = (\,A\cos\alpha t,\ A\sin\alpha t,\ B_z\,)\,.$$

Show that the probability for a transition from a spin-up state at time zero to a spin-down state at time t (with spin measured along the z direction, as before) is approximately

$$\left(\frac{\gamma A}{\gamma B_z+\alpha}\right)^2 \sin^2\frac{(\gamma B_z+\alpha)t}{2}\ ,$$

where you may assume $|A| \ll |B_z + \alpha\gamma^{-1}|$. Comment on how this compares, when $\alpha = 0$, with the result for a time-independent field.

[*The first-order transition amplitude due to a perturbation $V(t)$ is*

$$-\frac{i}{\hbar}\int_0^t dt' e^{i(E'-E)t'/\hbar}\langle\chi'|V(t')|\chi\rangle$$

where $|\chi\rangle$ and $|\chi'\rangle$ are orthogonal eigenstates of the unperturbed Hamiltonian with eigenvalues E and E' respectively.]

1/II/32D **Principles of Quantum Mechanics**

A particle in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$. Explain how to introduce the position-space wavefunction $\psi(x)$ for a quantum state $|\psi\rangle$ and use this to derive a formula for $\| \, |\psi\rangle \, \|^2$. Find the wavefunctions for $\hat{x}|\psi\rangle$ and $\hat{p}|\psi\rangle$ in terms of $\psi(x)$, stating clearly any standard properties of position and momentum eigenstates which you require.

Define annihilation and creation operators a and $a^\dagger$ for a harmonic oscillator of unit mass and frequency and write the Hamiltonian

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2$$

in terms of them. Let $|\psi_\alpha\rangle$ be a normalized eigenstate of a with eigenvalue α, a complex number. Show that $|\psi_\alpha\rangle$ cannot be an eigenstate of H unless $\alpha = 0$, and that $|\psi_0\rangle$ is an eigenstate of H with the lowest possible energy. Find a normalized wavefunction for $|\psi_\alpha\rangle$ for any α. Do there exist normalizable eigenstates of $a^\dagger$? Justify your answer.

2/II/32A Principles of Quantum Mechanics

Let $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ denote the eigenstates of S_z for a particle of spin $\frac{1}{2}$. Show that

$$|{\uparrow}\theta\rangle = \cos\frac{\theta}{2}\,|{\uparrow}\rangle + \sin\frac{\theta}{2}\,|{\downarrow}\rangle\ , \qquad |{\downarrow}\theta\rangle = -\sin\frac{\theta}{2}\,|{\uparrow}\rangle + \cos\frac{\theta}{2}\,|{\downarrow}\rangle$$

are eigenstates of $S_z \cos\theta + S_x \sin\theta$ for any θ. Show also that the composite state

$$|\chi\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle\,|{\downarrow}\rangle\ -\ |{\downarrow}\rangle\,|{\uparrow}\rangle\right)\ ,$$

for two spin-$\frac{1}{2}$ particles, is unchanged under a transformation

$$|{\uparrow}\rangle\ \mapsto\ |{\uparrow}\theta\rangle\ , \qquad |{\downarrow}\rangle\ \mapsto\ |{\downarrow}\theta\rangle \qquad (*)$$

applied to all one-particle states. Hence, by considering the action of certain components of the spin operator for the composite system, show that $|\chi\rangle$ is a state of total spin zero.

Two spin-$\frac{1}{2}$ particles A and B have combined spin zero (as in the state $|\chi\rangle$ above) but are widely separated in space. A magnetic field is applied to particle B in such a way that its spin states are transformed according to $(*)$, for a certain value of θ, while the spin states of particle A are unaffected. Once this has been done, a measurement is made of S_z for particle A, followed by a measurement of S_z for particle B. List the possible results for this pair of measurements and find the total probability, in terms of θ, for each pair of outcomes to occur. For which outcomes is the two-particle system left in an eigenstate of the combined total spin operator, S^2, and what is the eigenvalue for each such outcome?

$$\left[\qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \qquad \right]$$

3/II/32D **Principles of Quantum Mechanics**

Consider a Hamiltonian H with known eigenstates and eigenvalues (possibly degenerate). Derive a general method for calculating the energies of a new Hamiltonian $H + \lambda V$ to first order in the parameter λ. Apply this method to find approximate expressions for the new energies close to an eigenvalue E of H, given that there are just two orthonormal eigenstates $|1\rangle$ and $|2\rangle$ corresponding to E and that

$$\langle 1|V|1\rangle = \langle 2|V|2\rangle = \alpha \ , \qquad \langle 1|V|2\rangle = \langle 2|V|1\rangle = \beta \ .$$

A charged particle of mass m moves in two-dimensional space but is confined to a square box $0 \leqslant x, y \leqslant a$. In the absence of any potential within this region the allowed wavefunctions are

$$\psi_{pq}(x,y) = \frac{2}{a}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{a} \ , \qquad p,\, q = 1, 2, \ldots ,$$

inside the box, and zero outside. A weak electric field is now applied, modifying the Hamiltonian by a term $\lambda xy/a^2$, where $\lambda m a^2/\hbar^2$ is small. Show that the three lowest new energy levels for the particle are approximately

$$\frac{\hbar^2\pi^2}{ma^2} + \frac{\lambda}{4} \ , \qquad \frac{5\hbar^2\pi^2}{2ma^2} + \lambda\Big(\frac{1}{4} \pm \Big(\frac{4}{3\pi}\Big)^4\Big) \ .$$

[*It may help to recall that* $2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$.]

4/II/32A **Principles of Quantum Mechanics**

Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture and explain how these formulations give rise to identical physical predictions. Derive an equation of motion for an operator in the Heisenberg picture, assuming the operator is independent of time in the Schrödinger picture.

State clearly the form of the unitary operator corresponding to a rotation through an angle θ about an axis $\mathbf{n}$ (a unit vector) for a general quantum system. Verify your statement for the case in which the system is a single particle by considering the effect of an infinitesimal rotation on the particle's position $\hat{\mathbf{x}}$ and on its spin $\mathbf{S}$.

Show that if the Hamiltonian for a particle is of the form

$$H \, = \, \frac{1}{2m}\hat{\mathbf{p}}^2 \, + \, U(\hat{\mathbf{x}}^2)\hat{\mathbf{x}}\cdot\mathbf{S}$$

then all components of the total angular momentum are independent of time in the Heisenberg picture. Is the same true for either orbital or spin angular momentum?

[*You may quote commutation relations involving components of* $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$, $\mathbf{L}$ *and* $\mathbf{S}$.]

1/II/32D **Principles of Quantum Mechanics**

A one-dimensional harmonic oscillator has Hamiltonian

$$H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\Big(a^\dagger a + \frac{1}{2}\Big)\,,$$

where

$$a = \Big(\frac{m\omega}{2\hbar}\Big)^{1/2}\Big(\hat{x} + \frac{i}{m\omega}\hat{p}\Big)\ ,\quad a^\dagger = \Big(\frac{m\omega}{2\hbar}\Big)^{1/2}\Big(\hat{x} - \frac{i}{m\omega}\hat{p}\Big)\ \text{ obey }\ [a, a^\dagger] = 1.$$

Assuming the existence of a normalised state $|0\rangle$ with $a|0\rangle = 0$, verify that

$$|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger\,n}|0\rangle\ ,\qquad n = 0, 1, 2, \ldots$$

are eigenstates of H with energies E_n, to be determined, and that these states all have unit norm.

The Hamiltonian is now modified by a term

$$\lambda V = \lambda\hbar\omega(\,a^r + a^{\dagger\,r}\,)$$

where r is a positive integer. Use perturbation theory to find the change in the lowest energy level to order λ^2 for any r. [*You may quote any standard formula you need.*]

Compute by perturbation theory, again to order λ^2, the change in the first excited energy level when $r = 1$. Show that in this special case, $r = 1$, the *exact* change in *all* energy levels as a result of the perturbation is $-\lambda^2\hbar\omega$.

2/II/32D Principles of Quantum Mechanics

The components of $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are 2×2 hermitian matrices obeying

$$[\,\sigma_i,\,\sigma_j\,] \;=\; 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1 \qquad\qquad (*)$$

for any unit vector $\mathbf{n}$. Show that these properties imply

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) \;=\; \mathbf{a}\cdot\mathbf{b} \,+\, i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$$

for any constant vectors $\mathbf{a}$ and $\mathbf{b}$. Assuming that θ is real, explain why the matrix $U = \exp(-i\mathbf{n}\cdot\boldsymbol{\sigma}\,\theta/2)$ is unitary, and show that

$$U \;=\; \cos(\theta/2) \,-\, i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)\;.$$

Hence deduce that

$$U\mathbf{m}\cdot\boldsymbol{\sigma}U^{-1} \;=\; \mathbf{m}\cdot\boldsymbol{\sigma}\cos\theta \,+\, (\mathbf{n}\times\mathbf{m})\cdot\boldsymbol{\sigma}\sin\theta$$

where $\mathbf{m}$ is any unit vector orthogonal to $\mathbf{n}$.

Write down an equation relating the matrices $\boldsymbol{\sigma}$ and the angular momentum operator $\mathbf{S}$ for a particle of spin one half, and explain *briefly* the significance of the conditions $(*)$. Show that if $|\chi\rangle$ is a state with spin 'up' measured along the direction $(0,0,1)$ then, for a certain choice of $\mathbf{n}$, $U|\chi\rangle$ is a state with spin 'up' measured along the direction $(\sin\theta, 0, \cos\theta)$.

3/II/32D Principles of Quantum Mechanics

The angular momentum operators $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ refer to independent systems, each with total angular momentum one. The combination of these systems has a basis of states which are of product form $|m_1; m_2\rangle = |1\,m_1\rangle|1\,m_2\rangle$ where m_1 and m_2 are the eigenvalues of $J_3^{(1)}$ and $J_3^{(2)}$ respectively. Let $|J\,M\rangle$ denote the alternative basis states which are simultaneous eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$ is the combined angular momentum. What are the possible values of J and M? Find expressions for all states with $J = 1$ in terms of product states. How do these states behave when the constituent systems are interchanged?

Two spin-one particles A and B have no mutual interaction but they each move in a potential $V(\mathbf{r})$ which is independent of spin. The single-particle energy levels E_i and the corresponding wavefunctions $\psi_i(\mathbf{r})$ $(i = 1, 2, \ldots)$ are the same for either A or B. Given that $E_1 < E_2 < \ldots$, explain how to construct the two-particle states of lowest energy and combined total spin $J = 1$ for the cases that (i) A and B are identical, and (ii) A and B are not identical.

[*You may assume* $\hbar = 1$ *and use the result* $J_\pm|j\,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\;m\pm 1\rangle$.]

4/II/32D **Principles of Quantum Mechanics**

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + \lambda V(t)\,,$$

where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|a\rangle$ and $|b\rangle$ be eigenstates of H_0 with distinct eigenvalues E_a and E_b respectively. Show that if the system is initially in state $|a\rangle$ then the probability of measuring it to be in state $|b\rangle$ after a time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt' \langle b|V(t')|a\rangle e^{i(E_b-E_a)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\;.$$

Deduce that if $V(t) = e^{-\mu t/\hbar}W$, where W is a time-independent operator and μ is a positive constant, then the probability for such a transition to have occurred after a very long time is approximately

$$\frac{\lambda^2}{\mu^2 + (E_b - E_a)^2}|\langle b|W|a\rangle|^2\;.$$