Part II

Partial Differential Equations

Paper 4, Section II

29E Partial Differential Equations

(a) Show that the Cauchy problem for $u(x,t)$ satisfying

$$u_t + u = u_{xx}$$

with initial data $u(x,0) = u_0(x)$, which is a smooth 2π-periodic function of x, defines a *strongly continuous one parameter semi-group of contractions* on the Sobolev space H^s_{per} for any $s \in \{0,1,2,\dots\}$.

(b) Solve the Cauchy problem for the equation

$$u_{tt} + u_t + \frac{1}{4}u = u_{xx}$$

with $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$, where u_0, u_1 are smooth 2π-periodic functions of x, and show that the solution is smooth. Prove from first principles that the solution satisfies the property of *finite propagation speed.*

[In this question all functions are real-valued, and

$$H^s_{\text{per}} = \left\{ u = \sum_{m\in\mathbb{Z}} \hat{u}(m)e^{imx} \in L^2 \,:\, \|u\|^2_{H^s} = \sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{u}(m)|^2 < \infty \right\}$$

are the Sobolev spaces of functions which are 2π-periodic in x, for $s = 0,1,2,\dots$.]

Paper 3, Section II

30E Partial Differential Equations

(a) Show that if $f \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function and u is a tempered distribution which solves

$$-\Delta u + m^2 u = f$$

for some constant $m \neq 0$, then there exists a number $C > 0$ which depends only on m, such that $\|u\|_{H^{s+2}} \leqslant C\|f\|_{H^s}$ for any $s \geqslant 0$. Explain briefly why this inequality remains valid if f is only assumed to be in $H^s(\mathbb{R}^n)$.

Show that if $\epsilon > 0$ is given then $\|v\|_{H^1}^2 \leqslant \epsilon\|v\|_{H^2}^2 + \frac{1}{4\epsilon}\|v\|_{H^0}^2$ for any $v \in H^2(\mathbb{R}^n)$.

[*Hint: The inequality* $a \leqslant \epsilon a^2 + \frac{1}{4\epsilon}$ *holds for any positive* ϵ *and* $a \in \mathbb{R}$.]

Prove that if u is a smooth bounded function which solves

$$-\Delta u + m^2 u = u^3 + \alpha \cdot \nabla u$$

for some constant vector $\alpha \in \mathbb{R}^n$ and constant $m \neq 0$, then there exists a number $C' > 0$ such that $\|u\|_{H^2} \leqslant C'$ and C' depends only on $m, \alpha, \|u\|_{L^\infty}, \|u\|_{L^2}$.

[You may use the fact that, for non-negative s, the Sobolev space of functions

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty\}.]$$

(b) Let $u(x,t)$ be a smooth real-valued function, which is 2π-periodic in x and satisfies the equation

$$u_t = u^2 u_{xx} + u^3 .$$

Give a complete proof that if $u(x,0) > 0$ for all x then $u(x,t) > 0$ for all x and $t > 0$.

Paper 2, Section II
30E Partial Differential Equations

Prove that if $\phi \in C(\mathbb{R}^n)$ is absolutely integrable with $\int \phi(x)\,dx = 1$, and $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ for $\epsilon > 0$, then for every Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ the convolution

$$\phi_\epsilon * f(x) \to f(x)$$

uniformly in x as $\epsilon \downarrow 0$.

Show that the function $N_\epsilon \in C^\infty(\mathbb{R}^3)$ given by

$$N_\epsilon(x) = \frac{1}{4\pi\sqrt{|x|^2 + \epsilon^2}}$$

for $\epsilon > 0$ satisfies

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} -\Delta N_\epsilon(x)\, f(x)\, dx \;=\; f(0)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Hence prove that the tempered distribution determined by the function $N(x) = (4\pi|x|)^{-1}$ is a fundamental solution of the operator $-\Delta$.

[You may use the fact that $\int_0^\infty r^2/(1+r^2)^{5/2}\,dr = 1/3$.]

Paper 1, Section II
30E Partial Differential Equations

(a) State the Cauchy–Kovalevskaya theorem, and explain for which values of $a \in \mathbb{R}$ it implies the existence of solutions to the Cauchy problem

$$xu_x + yu_y + au_z = u\,, \qquad u(x,y,0) = f(x,y)\,,$$

where f is real analytic. Using the method of characteristics, solve this problem for these values of a, and comment on the behaviour of the characteristics as a approaches any value where the non-characteristic condition fails.

(b) Consider the Cauchy problem

$$u_y = v_x\,, \qquad v_y = -u_x$$

with initial data $u(x,0) = f(x)$ and $v(x,0) = 0$ which are 2π-periodic in x. Give an example of a sequence of smooth solutions (u_n, v_n) which are also 2π-periodic in x whose corresponding initial data $u_n(x,0) = f_n(x)$ and $v_n(x,0) = 0$ are such that $\displaystyle\int_0^{2\pi} |f_n(x)|^2 dx \to 0$ while $\displaystyle\int_0^{2\pi} |u_n(x,y)|^2 dx \to \infty$ for non-zero y as $n \to \infty$.

Comment on the significance of this in relation to the concept of *well-posedness.*

Paper 4, Section II

30D Partial Differential Equations

(a) Derive the solution of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0\,, \qquad u(0,x) = u_0(x)\,, \quad u_t(0,x) = u_1(x)\,, \tag{1}$$

with Cauchy data given by C^2 functions $u_j = u_j(x)\,, j = 0,1$, and where $x \in \mathbb{R}$ and $u_{tt} = \partial_t^2 u$ etc. Explain what is meant by the property of *finite propagation speed* for the wave equation. Verify that the solution to (1) satisfies this property.

(b) Consider the Cauchy problem

$$u_{tt} - u_{xx} + x^2 u = 0\,, \qquad u(0,x) = u_0(x)\,, \quad u_t(0,x) = u_1(x)\,. \tag{2}$$

By considering the quantities

$$e = \tfrac{1}{2}\big(u_t^2 + u_x^2 + x^2u^2\big) \qquad \text{and} \qquad p = -u_t u_x\,,$$

prove that solutions of (2) also satisfy the property of finite propagation speed.

(c) Define what is meant by a strongly continuous one-parameter group of unitary operators on a Hilbert space. Consider the Cauchy problem for the Schrödinger equation for $\psi(x,t) \in \mathbb{C}$:

$$i\psi_t \;=\; -\psi_{xx} + x^2\psi\,, \qquad \psi(x,0) = \psi_0(x)\,, \qquad -\infty < x < \infty\,. \tag{3}$$

[In the following you may use without proof the fact that there is an orthonormal set of (real-valued) Schwartz functions $\{f_j(x)\}_{j=1}^{\infty}$ which are eigenfunctions of the differential operator $P = -\partial_x^2 + x^2$ with eigenvalues $2j+1$, i.e.

$$Pf_j = (2j+1)f_j\,, \quad f_j \in \mathcal{S}(\mathbb{R})\,, \quad (f_j, f_k)_{L^2} = \int_{\mathbb{R}} f_j(x) f_k(x)dx = \delta_{jk}\,,$$

and which have the property that any function $u \in L^2$ can be written uniquely as a sum $u(x) = \sum_j (f_j, u)_{L^2} f_j(x)$ which converges in the metric defined by the L^2 norm.]

Write down the solution to (3) in the case that ψ_0 is given by a finite sum $\psi_0 = \sum_{j=1}^{N} (f_j, \psi_0)_{L^2} f_j$ and show that your formula extends to define a strongly continuous one-parameter group of unitary operators on the Hilbert space L^2 of square-integrable (complex-valued) functions, with inner product $(f,g)_{L^2} = \int_{\mathbb{R}} \overline{f(x)} g(x) dx$.

Paper 3, Section II

30D Partial Differential Equations

(a) Consider variable-coefficient operators of the form

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu$$

whose coefficients are defined on a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let a_{jk} satisfy the condition of uniform ellipticity, namely

$$m\|\xi\|^2 \leqslant \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leqslant M\|\xi\|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n$$

for suitably chosen positive numbers m, M.

State and prove the weak maximum principle for solutions of $Pu = 0$. [Any results from linear algebra and calculus needed in your proof should be stated clearly, but need not be proved.]

(b) Consider the nonlinear elliptic equation

$$-\Delta u + e^u = f \tag{1}$$

for $u : \mathbb{R}^n \to \mathbb{R}$ satisfying the additional condition

$$\lim_{|x|\to\infty} u(x) = 0\,. \tag{2}$$

Assume that $f \in \mathcal{S}(\mathbb{R}^n)$. Prove that any two C^2 solutions of (1) which also satisfy (2) are equal.

Now let $u \in C^2(\mathbb{R}^n)$ be a solution of (1) and (2). Prove that if $f(x) < 1$ for all x then $u(x) < 0$ for all x. Prove that if $\max_x f(x) = L \geqslant 1$ then $u(x) \leqslant \ln L$ for all x.

Paper 1, Section II

30D Partial Differential Equations

State the Cauchy–Kovalevskaya theorem, including a definition of the term *non-characteristic.*

For which values of the real number a, and for which functions f, does the Cauchy–Kovalevskaya theorem ensure that the Cauchy problem

$$u_{tt} = u_{xx} + a u_{xxxx}\,, \quad u(x,0) = 0\,,\; u_t(x,0) = f(x) \tag{1}$$

has a local solution?

Now consider the Cauchy problem (1) in the case that $f(x) = \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{imx}$ is a smooth 2π-periodic function.

(i) Show that if $a \leqslant 0$ there exists a unique smooth solution u for all times, and show that for all $T \geqslant 0$ there exists a number $C = C(T) > 0$, independent of f, such that

$$\int_{-\pi}^{+\pi} |u(x,t)|^2 dx \;\leqslant\; C \int_{-\pi}^{+\pi} |f(x)|^2 dx \tag{2}$$

for all $t : |t| \leqslant T$.

(ii) If $a = 1$ does there exist a choice of $C = C(T)$ for which (2) holds? Give a full justification for your answer.

Paper 2, Section II

31D Partial Differential Equations

In this question, functions are all real-valued, and

$$H^s_{per} = \{u = \sum_{m\in\mathbb{Z}} \hat{u}(m)e^{imx} \in L^2 : \|u\|^2_{H^s} = \sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{u}(m)|^2 < \infty\}$$

are the Sobolev spaces of functions 2π-periodic in x, for $s = 0, 1, 2, \ldots$.

State Parseval's theorem. For $s = 0, 1$ prove that the norm $\|u\|_{H^s}$ is equivalent to the norm $\| \, \|_s$ defined by

$$\|u\|^2_s = \sum_{r=0}^{s} \int_{-\pi}^{+\pi} (\partial^r_x u)^2 \, dx \, .$$

Consider the Cauchy problem

$$u_t - u_{xx} = f \, , \qquad u(x,0) = u_0(x) \, , \qquad t \geqslant 0 \, , \tag{1}$$

where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial value u_0 is also smooth and 2π-periodic. Prove that if u is a smooth solution which is 2π-periodic in x, then it satisfies

$$\int_0^T (\, u_t^2 + u_{xx}^2 \,) \, dt \leqslant C \left(\|u_0\|^2_{H^1} + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dx \, dt \right)$$

for some number $C > 0$ which does not depend on u or f.

State the Lax–Milgram lemma. Prove, using the Lax–Milgram lemma, that if

$$f(x,t) = e^{\lambda t} g(x)$$

with $g \in H^0_{per}$ and $\lambda > 0$, then there exists a weak solution to (1) of the form $u(x,t) = e^{\lambda t}\phi(x)$ with $\phi \in H^1_{per}$. Does the same hold for all $\lambda \in \mathbb{R}$? Briefly explain your answer.

Paper 4, Section II

30C Partial Differential Equations

(i) Show that an arbitrary C^2 solution of the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ can be written in the form $u = F(x-t) + G(x+t)$.

Hence, deduce the formula for the solution at arbitrary $t > 0$ of the Cauchy problem

$$u_{tt} - u_{xx} = 0\,, \qquad u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,, \tag{$*$}$$

where u_0, u_1 are arbitrary Schwartz functions.

Deduce from this formula a theorem on finite propagation speed for the one-dimensional wave equation.

(ii) Define the Fourier transform of a tempered distribution. Compute the Fourier transform of the tempered distribution $T_t \in \mathcal{S}'(\mathbb{R})$ defined for all $t > 0$ by the function

$$T_t(y) = \begin{cases} \frac{1}{2} & \text{if } |y| \leqslant t, \\ 0 & \text{if } |y| > t, \end{cases}$$

that is, $\langle T_t\,,\, f\,\rangle = \frac{1}{2}\int_{-t}^{+t} f(y)\,dy$ for all $f \in \mathcal{S}(\mathbb{R})$. By considering the Fourier transform in x, deduce from this the formula for the solution of $(*)$ that you obtained in part (i) in the case $u_0 = 0$.

Paper 3, Section II

30C Partial Differential Equations

Define the parabolic boundary $\partial_{par}\Omega_T$ of the domain $\Omega_T = [0,1] \times (0,T]$ for $T > 0$.

Let $u = u(x,t)$ be a smooth real-valued function on Ω_T which satisfies the inequality

$$u_t - au_{xx} + bu_x + cu \leqslant 0\,.$$

Assume that the coefficients a, b and c are smooth functions and that there exist positive constants m, M such that $m \leqslant a \leqslant M$ everywhere, and $c \geqslant 0$. Prove that

$$\max_{(x,t)\in\overline{\Omega}_T} u(x,t) \leqslant \max_{(x,t)\in\partial_{par}\Omega_T} u^+(x,t)\,. \qquad (*)$$

[Here $u^+ = \max\{u, 0\}$ is the positive part of the function u.]

Consider a smooth real-valued function ϕ on Ω_T such that

$$\phi_t - \phi_{xx} - (1-\phi^2)\phi = 0\,, \qquad \phi(x,0) = f(x)$$

everywhere, and $\phi(0,t) = 1 = \phi(1,t)$ for all $t \geqslant 0$. Deduce from $(*)$ that if $f(x) \leqslant 1$ for all $x \in [0,1]$ then $\phi(x,t) \leqslant 1$ for all $(x,t) \in \Omega_T$. [*Hint: Consider* $u = \phi^2 - 1$ *and compute* $u_t - u_{xx}$.]

Paper 1, Section II

30C Partial Differential Equations

(i) Discuss briefly the concept of *well-posedness* of a Cauchy problem for a partial differential equation.

Solve the Cauchy problem

$$\partial_2 u + x_1 \partial_1 u = a u^2 \,, \qquad u(x_1, 0) = \phi(x_1) \,,$$

where $a \in \mathbb{R}$, $\phi \in C^1(\mathbb{R})$ and ∂_i denotes the partial derivative with respect to x_i for $i = 1, 2$.

For the case $a = 0$ show that the solution satisfies $\max_{x_1 \in \mathbb{R}} |u(x_1, x_2)| = \|\phi\|_{C^0}$, where the C^r norm on functions $\phi = \phi(x_1)$ of one variable is defined by

$$\|\phi\|_{C^r} = \sum_{i=0}^{r} \max_{x \in \mathbb{R}} |\partial_1^i \phi(x_1)|.$$

Deduce that the Cauchy problem is then well-posed in the uniform metric (i.e. the metric determined by the C^0 norm).

(ii) State the Cauchy–Kovalevskaya theorem and deduce that the following Cauchy problem for the Laplace equation,

$$\partial_1^2 u + \partial_2^2 u = 0 \,, \qquad u(x_1, 0) = 0 \,,\ \partial_2 u(x_1, 0) = \phi(x_1) \,, \qquad (*)$$

has a unique analytic solution in some neighbourhood of $x_2 = 0$ for any analytic function $\phi = \phi(x_1)$. Write down the solution for the case $\phi(x_1) = \sin(n x_1)$, and hence give a sequence of initial data $\{\phi_n(x_1)\}_{n=1}^{\infty}$ with the property that

$$\|\phi_n\|_{C^r} \ \to \ 0 \,, \quad \text{as } n \to \infty, \text{ for each } \ r \in \mathbb{N} \,,$$

whereas u_n, the corresponding solution of $(*)$, satisfies

$$\max_{x_1 \in \mathbb{R}} |u_n(x_1, x_2)| \ \to \ +\infty \,, \quad \text{as } n \to \infty,$$

for any $x_2 \neq 0$.

Paper 2, Section II

31C Partial Differential Equations

State the Lax–Milgram lemma.

Let $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3)$ be a smooth vector field which is 2π-periodic in each coordinate x_j for $j = 1, 2, 3$. Write down the definition of a weak H^1_{per} solution for the equation

$$-\Delta u + \sum_j V_j \partial_j u + u = f \qquad (*)$$

to be solved for $u = u(x_1, x_2, x_3)$ given $f = f(x_1, x_2, x_3)$ in H^0, with both u and f also 2π-periodic in each co-ordinate. [In this question use the definition

$$H^s_{per} = \left\{ u = \sum_{m \in \mathbb{Z}^3} \hat{u}(m) e^{im \cdot x} \in L^2 : \|u\|^2_{H^s} = \sum_{m \in \mathbb{Z}^3} (1 + \|m\|^2)^s |\hat{u}(m)|^2 < \infty \right\}$$

for the Sobolev spaces of functions 2π-periodic in each coordinate x_j and for $s = 0, 1, 2, \ldots$.]

If the vector field is divergence-free, prove that there exists a unique weak H^1_{per} solution for all such f.

Supposing that $\mathbf{V}$ is the constant vector field with components $(1, 0, 0)$, write down the solution of $(*)$ in terms of Fourier series and show that there exists $C > 0$ such that

$$\|u\|_{H^2} \leqslant C \|f\|_{H^0} .$$

Paper 4, Section II

30B Partial Differential Equations

i) State the Lax–Milgram lemma.

ii) Consider the boundary value problem

$$\begin{aligned} \Delta^2 u - \Delta u + u = f \quad &\text{in } \Omega, \\ u = \nabla u \cdot \gamma = 0 \quad &\text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, γ is the exterior unit normal vector to $\partial\Omega$, and $f \in L^2(\Omega)$. Show (using the Lax–Milgram lemma) that the boundary value problem has a unique weak solution in the space

$$H_0^2(\Omega) := \big\{ u : \Omega \to \mathbb{R}; u = \nabla u \cdot \gamma = 0 \text{ on } \partial\Omega \big\}.$$

[*Hint. Show that*

$$\|\Delta u\|^2_{L^2(\Omega)} = \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(\Omega)} \quad \textit{for all } u \in C_0^\infty(\Omega),$$

and then use the fact that $C_0^\infty(\Omega)$ *is dense in* $H_0^2(\Omega)$.]

Paper 3, Section II

30B Partial Differential Equations

Consider the nonlinear partial differential equation for a function $u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$,

$$u_t = \Delta u - \alpha |\nabla u|^2, \tag{1}$$

$$\text{subject to} \quad u(x,0) = u_0(x), \tag{2}$$

where $u_0 \in L^\infty(\mathbb{R}^n)$.

(i) Find a transformation $w := F(u)$ such that w satisfies the heat equation

$$w_t = \Delta w, \quad x \in \mathbb{R}^n,$$

if (1) holds for u.

(ii) Use the transformation obtained in (i) (and its inverse) to find a solution to the initial value problem (1), (2).
[*Hint. Use the fundamental solution of the heat equation.*]

(iii) The equation (1) is posed on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary, subject to the initial condition (2) on Ω and inhomogeneous Dirichlet boundary conditions

$$u = u_D \quad \text{on } \partial\Omega,$$

where u_D is a bounded function. Use the maximum-minimum principle to prove that there exists at most one classical solution of this boundary value problem.

Paper 1, Section II

30B Partial Differential Equations

Let $u_0 : \mathbb{R} \to \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, $u_0(x) \geqslant 0$ for all $x \in \mathbb{R}$. Consider the partial differential equation for $u = u(x,y)$,

$$4yu_x + 3u_y = u^2, \qquad (x,y) \in \mathbb{R}^2$$

subject to the Cauchy condition $u(x,0) = u_0(x)$.

i) Compute the solution of the Cauchy problem by the method of characteristics.

ii) Prove that the domain of definition of the solution contains

$$(x,y) \in \mathbb{R} \times \left(-\infty, \frac{3}{\sup_{x\in\mathbb{R}}(u_0(x))}\right).$$

Paper 2, Section II

31B Partial Differential Equations

Consider the elliptic Dirichlet problem on $\Omega \subset \mathbb{R}^n$, Ω bounded with a smooth boundary:

$$\Delta u - e^u = f \text{ in } \Omega, \qquad u = u_D \text{ on } \partial\Omega.$$

Assume that $u_D \in L^\infty(\partial\Omega)$ and $f \in L^\infty(\Omega)$.

(i) State the strong Minimum-Maximum Principle for uniformly elliptic operators.

(ii) Prove that there exists at most one classical solution of the boundary value problem.

(iii) Assuming further that $f \geqslant 0$ in Ω, use the maximum principle to obtain an upper bound on the solution (assuming that it exists).

Paper 1, Section II

30A Partial Differential Equations

Let $H = H(x, v)$, $x, v \in \mathbb{R}^n$, be a smooth real-valued function which maps $\mathbb{R}^{2n}$ into $\mathbb{R}$. Consider the initial value problem for the equation

$$f_t + \nabla_v H \cdot \nabla_x f - \nabla_x H \cdot \nabla_v f = 0, \quad x, v \in \mathbb{R}^n,\ t > 0\,,$$
$$f(x, v, t = 0) = f_I(x, v), \quad x, v \in \mathbb{R}^n\,,$$

for the unknown function $f = f(x, v, t)$.

(i) Use the method of characteristics to solve the initial value problem, locally in time.

(ii) Let $f_I \geqslant 0$ on $\mathbb{R}^{2n}$. Use the method of characteristics to prove that f remains non-negative (as long as it exists).

(iii) Let $F : \mathbb{R} \to \mathbb{R}$ be smooth. Prove that

$$\int_{\mathbb{R}^{2n}} F(f(x, v, t))\, dx\, dv = \int_{\mathbb{R}^{2n}} F(f_I(x, v))\, dx\, dv\,,$$

as long as the solution exists.

(iv) Let H be independent of x, namely $H(x, v) = a(v)$, where a is smooth and real-valued. Give the explicit solution of the initial value problem.

Paper 2, Section II

31A Partial Differential Equations

Consider the Schrödinger equation

$$i\partial_t \psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x)\ , \qquad x \in \mathbb{R}^n,\ t > 0\ ,$$
$$\psi(t=0,x) = \psi_I(x)\ , \qquad x \in \mathbb{R}^n\ ,$$

where V is a smooth real-valued function.

Prove that, for smooth solutions, the following equations are valid for all $t > 0$:

(i)

$$\int_{\mathbb{R}^n} |\psi(t,x)|^2\, dx = \int_{\mathbb{R}^n} |\psi_I(x)|^2\, dx\ .$$

(ii)

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi(t,x)|^2\, dx + \int_{\mathbb{R}^n} V(x)|\psi(t,x)|^2\, dx$$
$$= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi_I(x)|^2\, dx + \int_{\mathbb{R}^n} V(x)|\psi_I(x)|^2\, dx\ .$$

Paper 3, Section II

30A Partial Differential Equations

(a) State the local existence theorem of a classical solution of the Cauchy problem

$$a(x_1,x_2,u)\frac{\partial u}{\partial x_1} + b(x_1,x_2,u)\frac{\partial u}{\partial x_2} = c(x_1,x_2,u)\,,$$
$$u|_\Gamma = u_0\,,$$

where Γ is a smooth curve in $\mathbb{R}^2$.

(b) Solve, by using the method of characteristics,

$$2x_1\frac{\partial u}{\partial x_1} + 4x_2\frac{\partial u}{\partial x_2} = u^2\,,$$
$$u(x_1,2) = h\,,$$

where $h > 0$ is a constant. What is the maximal domain of existence in which u is a solution of the Cauchy problem?

Paper 4, Section II

30A Partial Differential Equations

Consider the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega F(u,x)\, dx\,,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $F : \mathbb{R} \times \Omega \to \mathbb{R}$ is smooth. Assume that $F(u,x)$ is convex in u for all $x \in \Omega$ and that there is a $K > 0$ such that

$$-K \leqslant F(v,x) \leqslant K\left(|v|^2 + 1\right) \qquad \forall v \in \mathbb{R},\, x \in \Omega\,.$$

(i) Prove that E is well-defined on $H_0^1(\Omega)$, bounded from below and strictly convex. Assume without proof that E is weakly lower-semicontinuous. State this property. Conclude the existence of a unique minimizer of E.

(ii) Which elliptic boundary value problem does the minimizer solve?

Paper 1, Section II

30E Partial Differential Equations

(a) Solve by using the method of characteristics

$$x_1 \frac{\partial}{\partial x_1} u + 2\, x_2 \frac{\partial}{\partial x_2} u \;=\; 5\, u\,, \qquad u(x_1, 1) \;=\; g(x_1)\,,$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. What is the maximal domain in $\mathbb{R}^2$ in which u is a solution of the Cauchy problem?

(b) Prove that the function

$$u(x,t) \;=\; \begin{cases} 0\,, & x < 0\,,\, t > 0\,, \\ x/t\,, & 0 < x < t\,,\, t > 0\,, \\ 1\,, & x > t > 0\,, \end{cases}$$

is a weak solution of the Burgers equation

$$\frac{\partial}{\partial t} u + \frac{1}{2} \frac{\partial}{\partial x} u^2 \;=\; 0\,, \qquad x \in \mathbb{R},\, t > 0\,, \qquad (*)$$

with initial data

$$u(x,0) = \begin{cases} 0\,, & x < 0\,, \\ 1\,, & x > 0\,. \end{cases}$$

(c) Let $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$ be a piecewise C^1-function with a jump discontinuity along the curve

$$\Gamma : \, x \;=\; s(t)$$

and let u solve the Burgers equation $(*)$ on both sides of Γ. Prove that u is a weak solution of (1) if and only if

$$\dot{s}(t) \;=\; \frac{1}{2}\left(u_l(t) + u_r(t)\right)$$

holds, where $u_l(t)$, $u_r(t)$ are the one-sided limits

$$u_l(t) \;=\; \lim_{x \nearrow s(t)^-} u(x,t)\,, \qquad u_r(t) \;=\; \lim_{x \searrow s(t)^+} u(x,t)\,.$$

[*Hint: Multiply the equation by a test function* $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, *split the integral appropriately and integrate by parts. Consider how the unit normal vector along* Γ *can be expressed in terms of* $\dot{s}$.]

Paper 2, Section II

31E Partial Differential Equations

(a) State the Lax-Milgram lemma. Use it to prove that there exists a unique function u in the space

$$H^2_\partial(\Omega) = \left\{u \in H^2(\Omega)\,; u|_{\partial\Omega} = \partial u/\partial\gamma|_{\partial\Omega} = 0\right\} ,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and γ its outwards unit normal vector, which is the weak solution of the equations

$$\begin{aligned} \Delta^2 u &= f \quad \text{in} \quad \Omega\,, \\ u = \frac{\partial u}{\partial \gamma} &= 0 \quad \text{on} \quad \partial\,\Omega\,, \end{aligned}$$

for $f \in L^2(\Omega)$, Δ the Laplacian and $\Delta^2 = \Delta\Delta$.

[*Hint: Use regularity of the solution of the Dirichlet problem for the Poisson equation.*]

(b) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $u \in H^1(\Omega)$ and denote

$$\bar{u} = \int_\Omega u\, d^n x \Big/ \int_\Omega d^n x\ .$$

The following Poincaré-type inequality is known to hold

$$\|u - \bar{u}\|_{L^2} \leqslant C \|\nabla u\|_{L^2}\,,$$

where C only depends on Ω. Use the Lax-Milgram lemma and this Poincaré-type inequality to prove that the Neumann problem

$$\begin{aligned} \Delta u &= f \quad \text{in} \quad \Omega\,, \\ \frac{\partial u}{\partial \gamma} &= 0 \quad \text{on} \quad \partial\,\Omega\,, \end{aligned}$$

has a unique weak solution in the space

$$H^1_-(\Omega) = H^1(\Omega) \cap \{u : \Omega \to \mathbb{R}\,; \bar{u} = 0\}$$

if and only if $\bar{f} = 0$.

Paper 3, Section II

30E Partial Differential Equations

Consider the Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n\,,\ t>0\,,$$

for complex-valued solutions $\Psi(x,t)$ and where Δ is the Laplacian.

(a) Derive, by using a Fourier transform and its inversion, the fundamental solution of the Schrödinger equation. Obtain the solution of the initial value problem

$$\begin{aligned} i\partial_t \Psi &= -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n,\quad t>0\ , \\ \Psi(x,0) &= f(x)\,, \qquad x \in \mathbb{R}^n\,, \end{aligned}$$

as a convolution.

(b) Consider the Wigner-transform of the solution of the Schrödinger equation

$$w(x,\xi,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \Psi(x+\tfrac{1}{2}\,y,\ t)\ \bar{\Psi}(x-\tfrac{1}{2}\,y,\ t)\ e^{-iy\cdot\xi}\ \mathrm{d}^n y\,,$$

defined for $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$, $t>0$. Derive an evolution equation for w by using the Schrödinger equation. Write down the solution of this evolution equation for given initial data $w(x,\xi,0)=g(x,\xi)$.

Paper 4, Section II

30E Partial Differential Equations

a) Solve the Dirichlet problem for the Laplace equation in a disc in $\mathbb{R}^2$

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad G=\{x^2+y^2<R^2\}\subseteq \mathbb{R}^2\ ,\ R>0\,, \\ u = u_D \quad &\text{on} \quad \partial G\,, \end{aligned}$$

using polar coordinates (r,φ) and separation of variables, $u(x,y)=R(r)\Theta(\varphi)$. Then use the ansatz $R(r)=r^\alpha$ for the radial function.

b) Solve the Dirichlet problem for the Laplace equation in a square in $\mathbb{R}^2$

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad G=[0,a]\times[0,a]\,, \\ &u(x,0)=f_1(x)\,, \qquad u(x,a)=f_2(x)\,, \qquad u(0,y)=f_3(y)\,, \qquad u(a,y)=f_4(y)\,. \end{aligned}$$

Paper 1, Section II

30B Partial Differential Equations

Consider the initial value problem for the so-called Liouville equation

$$f_t + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = 0,\ (x,v) \in \mathbb{R}^{2d},\ t \in \mathbb{R},$$

$$f(x,v,t=0) = f_I(x,v),$$

for the function $f = f(x,v,t)$ on $\mathbb{R}^{2d} \times \mathbb{R}$. Assume that $V = V(x)$ is a given function with V, $\nabla_x V$ Lipschitz continuous on $\mathbb{R}^d$.

(i) Let $f_I(x,v) = \delta(x - x_0, v - v_0)$, for $x_0, v_0 \in \mathbb{R}^d$ given. Show that a solution f is given by

$$f(x,v,t) = \delta(x - \hat{x}(t,x_0,v_0), v - \hat{v}(t,x_0,v_0)),$$

where $(\hat{x},\hat{v})$ solve the Newtonian system

$$\begin{aligned} \dot{\hat{x}} &= \hat{v}, & \hat{x}(t=0) &= x_0, \\ \dot{\hat{v}} &= -\nabla V(\hat{x}), & \hat{v}(t=0) &= v_0. \end{aligned}$$

(ii) Let $f_I \in L^1_{loc}(\mathbb{R}^{2d})$, $f_I \geqslant 0$. Prove (by using characteristics) that f remains non-negative (as long as it exists).

(iii) Let $f_I \in L^p(\mathbb{R}^{2d})$, $f_I \geqslant 0$ on $\mathbb{R}^{2d}$. Show (by a formal argument) that

$$\|f(\cdot,\cdot,t)\|_{L^p(\mathbb{R}^{2d})} = \|f_I\|_{L^p(\mathbb{R}^{2d})}$$

for all $t \in \mathbb{R}$, $1 \leqslant p < \infty$.

(iv) Let $V(x) = \frac{1}{2}|x|^2$. Use the method of characteristics to solve the initial value problem for general initial data.

Paper 2, Section II

31B Partial Differential Equations

(a) Solve the initial value problem for the Burgers equation

$$u_t + \frac{1}{2}(u^2)_x = 0, \quad x \in \mathbb{R},\, t > 0,$$

$$u(x, t = 0) = u_I(x),$$

where

$$u_I(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Use the method of characteristics. What is the maximal time interval in which this (weak) solution is well defined? What is the regularity of this solution?

(b) Apply the method of characteristics to the Burgers equation subject to the initial condition

$$u_I(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

In $\{(x,t) \mid 0 < x < t\}$ use the ansatz $u(x,t) = f(\frac{x}{t})$ and determine f.

(c) Using the method of characteristics show that the initial value problem for the Burgers equation has a classical solution defined for all $t > 0$ if u_I is continuously differentiable and

$$\frac{du_I}{dx}(x) > 0$$

for all $x \in \mathbb{R}$.

Paper 3, Section II

30B Partial Differential Equations

(a) Consider the nonlinear elliptic problem

$$\begin{cases} \Delta u = f(u,x), & x \in \Omega \subseteq \mathbb{R}^d, \\ u = u_D, & x \in \partial\Omega. \end{cases}$$

Let $\frac{\partial f}{\partial u}(y,x) \geqslant 0$ for all $y \in \mathbb{R}$, $x \in \Omega$. Prove that there exists at most one classical solution.

[*Hint: Use the weak maximum principle.*]

(b) Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a radial function. Prove that the Fourier transform of φ is radial too.

(c) Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a radial function. Solve

$$-\Delta u + u = \varphi(x), \quad x \in \mathbb{R}^n$$

by Fourier transformation and prove that u is a radial function.

(d) State the Lax–Milgram lemma and explain its use in proving the existence and uniqueness of a weak solution of

$$-\Delta u + a(x)u = f(x), \; x \in \Omega,$$

$$u = 0 \;\text{ on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^d$ bounded, $0 \leqslant \underline{a} \leqslant a(x) \leqslant \overline{a} < \infty$ for all $x \in \Omega$ and $f \in L^2(\Omega)$.

Paper 4, Section II

30B Partial Differential Equations

Consider the two-dimensional domain

$$G = \{(x, y) \mid R_1^2 < x^2 + y^2 < R_2^2\},$$

where $0 < R_1 < R_2 < \infty$. Solve the Dirichlet boundary value problem for the Laplace equation

$$\Delta u = 0 \text{ in } G,$$
$$u = u_1(\varphi),\ r = R_1,$$
$$u = u_2(\varphi),\ r = R_2,$$

where (r, φ) are polar coordinates. Assume that u_1, u_2 are 2π-periodic functions on the real line and $u_1, u_2 \in L^2_{loc}(\mathbb{R})$.

[*Hint: Use separation of variables in polar coordinates,* $u = R(r)\Phi(\varphi)$*, with periodic boundary conditions for the function* Φ *of the angle variable. Use an ansatz of the form* $R(r) = r^\alpha$ *for the radial function.*]

1/II/29C Partial Differential Equations

(i) State the local existence theorem for the first order quasi-linear partial differential equation

$$\sum_{j=1}^{n} a_j(x,u)\,\frac{\partial u}{\partial x_j} \;=\; b(x,u),$$

which is to be solved for a real-valued function with data specified on a hypersurface S. Include a definition of "non-characteristic" in your answer.

(ii) Consider the linear constant-coefficient case (that is, when all the functions $a_1,\ldots,a_n$ are real constants and $b(x,u)=cx+d$ for some $c=(c_1,\ldots,c_n)$ with $c_1,\ldots,c_n$ real and d real) and with the hypersurface S taken to be the hyperplane $\mathbf{x}\cdot\mathbf{n}=0$. Explain carefully the relevance of the non-characteristic condition in obtaining a solution via the method of characteristics.

(iii) Solve the equation

$$\frac{\partial u}{\partial y} + u\,\frac{\partial u}{\partial x} \;=\; 0,$$

with initial data $u(0,y)=-y$ prescribed on $x=0$, for a real-valued function $u(x,y)$. Describe the domain on which your solution is C^1 and comment on this in relation to the theorem stated in (i).

2/II/30C Partial Differential Equations

(i) Define the concept of "fundamental solution" of a linear constant-coefficient partial differential operator and write down the fundamental solution for the operator $-\Delta$ on $\mathbb{R}^3$.

(ii) State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

(iii) Let $u\in C^2(\mathbb{R}^3)$ be a harmonic function which satisfies $u(p)\geqslant 0$ at every point p in an open set $\Omega\subset\mathbb{R}^3$. Show that if $B(z,r)\subset B(w,R)\subset\Omega$, then

$$u(w) \;\geqslant\; \left(\frac{r}{R}\right)^3 u\,(z)\,.$$

Assume that $B(x,4r)\subset\Omega$. Deduce, by choosing $R=3r$ and w,z appropriately, that

$$\inf_{B(x,r)} u \;\geqslant\; 3^{-3} \sup_{B(x,r)} u\,.$$

[In (iii), $B(z,\rho)=\{x\in\mathbb{R}^3 : \|x-z\|<\rho\}$ is the ball of radius $\rho>0$ centred at $z\in\mathbb{R}^3$.]

3/II/29C Partial Differential Equations

Let $C^{\infty}_{per} = \{u \in C^{\infty}(\mathbb{R}) : u(x + 2\pi) = u(x)\}$ be the space of smooth 2π-periodic functions of one variable.

(i) For $f \in C^{\infty}_{per}$ show that there exists a unique $u_f \in C^{\infty}_{per}$ such that

$$-\frac{d^2 u_f}{dx^2} + u_f \;=\; f.$$

(ii) Show that $I_f[u_f + \phi] > I_f[u_f]$ for every $\phi \in C^{\infty}_{per}$ which is not identically zero, where $I_f : C^{\infty}_{per} \to \mathbb{R}$ is defined by

$$I_f[u] \;=\; \frac{1}{2}\int_{-\pi}^{+\pi} \left[\left(\frac{\partial u}{\partial x}\right)^2 + u^2 - 2f(x)\,u \right] dx.$$

(iii) Show that the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \;=\; f(x),$$

with initial data $u(0,x) = u_0(x) \in C^{\infty}_{per}$ has, for $t > 0$, a smooth solution $u(t,x)$ such that $u(t,\cdot) \in C^{\infty}_{per}$ for each fixed $t > 0$. Give a representation of this solution as a Fourier series in x. Calculate $\lim_{t \to +\infty} u(t,x)$ and comment on your answer in relation to (i).

(iv) Show that $I_f[u(t,\cdot)] \leqslant I_f[u(s,\cdot)]$ for $t > s > 0$, and that $I_f[u(t,\cdot)] \to I_f[u_f]$ as $t \to +\infty$.

4/II/30C **Partial Differential Equations**

(i) Define the Fourier transform $\hat{f} = \mathcal{F}(f)$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, and also of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) From your definition, compute the Fourier transform of the distribution $W_t \in \mathcal{S}'(\mathbb{R}^3)$ given by

$$W_t(\psi) \;=< W_t, \psi >= \frac{1}{4\pi t} \int_{\|y\|=t} \psi(y)\, d\Sigma(y)$$

for every Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^3)$. Here $d\Sigma(y) = t^2 d\Omega(y)$ is the integration element on the sphere of radius t.

Hence deduce the formula of Kirchoff for the solution of the initial value problem for the wave equation in three space dimensions,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u \;=\; 0,$$

with initial data $u(0,x) = 0$ and $\frac{\partial u}{\partial t}(0,x) = g(x)$, $x \in \mathbb{R}^3$, where $g \in \mathcal{S}(\mathbb{R}^3)$. Explain briefly why the formula is also valid for arbitrary smooth $g \in C^\infty(\mathbb{R}^3)$.

(iii) Show that any C^2 solution of the initial value problem in (ii) is given by the formula derived in (ii) (uniqueness).

(iv) Show that any two C^2 solutions of the initial value problem for

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u \;=\; 0\,,$$

with the same initial data as in (ii), also agree for any $t > 0$.

1/II/29A **Partial Differential Equations**

(i) Consider the problem of solving the equation

$$\sum_{j=1}^{n} a_j(\mathbf{x}) \frac{\partial u}{\partial x_j} = b(\mathbf{x}, u)$$

for a C^1 function $u = u(\mathbf{x}) = u(x_1, \ldots, x_n)$, with data specified on a C^1 hypersurface $\mathcal{S} \subset \mathbb{R}^n$

$$u(\mathbf{x}) = \phi(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathcal{S}.$$

Assume that $a_1, \ldots, a_n, \phi, b$ are C^1 functions. Define the characteristic curves and explain what it means for the non-characteristic condition to hold at a point on $\mathcal{S}$. State a local existence and uniqueness theorem for the problem.

(ii) Consider the case $n = 2$ and the equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = x_2 u$$

with data $u(x_1, 0) = \phi(x_1, 0) = f(x_1)$ specified on the axis $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}$. Obtain a formula for the solution.

(iii) Consider next the case $n = 2$ and the equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = 0$$

with data $u(\mathbf{g}(s)) = \phi(\mathbf{g}(s)) = f(s)$ specified on the hypersurface $\mathcal{S}$, which is given parametrically as $\mathcal{S} \equiv \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{g}(s)\}$ where $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\mathbf{g}(s) = (s, 0), \qquad s < 0,$$

$$\mathbf{g}(s) = (s, s^2), \qquad s \geqslant 0.$$

Find the solution u and show that it is a global solution. (Here "global" means u is C^1 on all of $\mathbb{R}^2$.)

(iv) Consider next the equation

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0$$

to be solved with the same data given on the same hypersurface as in (iii). Explain, with reference to the characteristic curves, why there is generally no global C^1 solution. Discuss the existence of local solutions defined in some neighbourhood of a given point $\mathbf{y} \in \mathcal{S}$ for various $\mathbf{y}$. [*You need not give formulae for the solutions.*]

2/II/30A **Partial Differential Equations**

Define (i) the Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$, and (ii) the convolution $T * g$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^3)$. Give a formula for the Fourier transform of $T * g$ ("convolution theorem").

Let $t > 0$. Compute the Fourier transform of the tempered distribution $A_t \in \mathcal{S}'(\mathbb{R}^3)$ defined by

$$\langle A_t, \phi \rangle = \int_{\|y\|=t} \phi(y) d\Sigma(y), \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^3),$$

and deduce the Kirchhoff formula for the solution $u(t,x)$ of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

$$u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)\ .$$

Prove, by consideration of the quantities $e = \frac{1}{2}(u_t^2 + |\nabla u|^2)$ and $p = -u_t \nabla u$, that any C^2 solution is also given by the Kirchhoff formula (uniqueness).

Prove a corresponding uniqueness statement for the initial value problem

$$\frac{\partial^2 w}{\partial t^2} - \Delta w + V(x) w = 0,$$

$$w(0,x) = 0, \qquad \frac{\partial w}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)$$

where V is a smooth positive real-valued function of $x \in \mathbb{R}^3$ only.

3/II/29A **Partial Differential Equations**

Write down the formula for the solution $u = u(t, x)$ for $t > 0$ of the initial value problem for the heat equation in one space dimension

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\,,$$

$$u(0, x) = g(x)\,,$$

for $g : \mathbb{R} \to \mathbb{C}$ a given smooth bounded function.

Define the distributional derivative of a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$. Define a fundamental solution of a constant-coefficient linear differential operator P, and show that the distribution defined by the function $\frac{1}{2}e^{-|x|}$ is a fundamental solution for the operator $P = -\frac{d^2}{dx^2} + 1$.

For the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = e^t \phi(x), \tag{$*$}$$

where $\phi \in \mathcal{S}(\mathbb{R})$, prove that there is a unique solution of the form $e^t v(x)$ with $v \in \mathcal{S}(\mathbb{R})$. Hence write down the solution of $(*)$ with general initial data $u(0, x) = f(x)$ and describe the large time behaviour.

4/II/30A **Partial Differential Equations**

State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

Obtain a generalization of the mean value property for sub-harmonic functions on $\mathbb{R}^3$, i.e. C^2 functions for which

$$-\Delta u(x) \leqslant 0$$

for all $x \in \mathbb{R}^3$.

Let $\phi \in C^2(\mathbb{R}^3; \mathbb{C})$ solve the equation

$$-\Delta \phi + iV(x)\phi = 0\,,$$

where V is a real-valued continuous function. By considering the function $w(x) = |\phi(x)|^2$ show that, on any ball $B(y, R) = \{x : \|x - y\| < R\} \subset \mathbb{R}^3$,

$$\sup_{x \in B(y,R)} |\phi(x)| \leqslant \sup_{\|x-y\|=R} |\phi(x)|.$$

1/II/29A **Partial Differential Equations**

(a) State a local existence theorem for solving first order quasi-linear partial differential equations with data specified on a smooth hypersurface.

(b) Solve the equation

$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

with boundary condition $u(x,0) = f(x)$ where $f \in C^1(\mathbb{R})$, making clear the domain on which your solution is C^1. Comment on this domain with reference to the *non-characteristic condition* for an initial hypersurface (including a definition of this concept).

(c) Solve the equation

$$u^2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

with boundary condition $u(x,0) = x$ and show that your solution is C^1 on some open set containing the initial hypersurface $y = 0$. Comment on the significance of this, again with reference to the non-characteristic condition.

2/II/30A **Partial Differential Equations**

Define a *fundamental solution* of a constant-coefficient linear partial differential operator, and prove that the distribution defined by the function $N : \mathbb{R}^3 \to \mathbb{R}$

$$N(x) = (4\pi|x|)^{-1}$$

is a fundamental solution of the operator $-\Delta$ on $\mathbb{R}^3$.

State and prove the mean value property for harmonic functions on $\mathbb{R}^3$ and deduce that any two smooth solutions of

$$-\Delta u = f\ , \qquad f \in C^\infty(\mathbb{R}^3)$$

which satisfy the condition

$$\lim_{|x|\to\infty} u(x) = 0$$

are in fact equal.

3/II/29A **Partial Differential Equations**

Write down the formula for the solution $u = u(t, x)$ for $t > 0$ of the initial value problem for the n-dimensional heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \ ,$$
$$u(0, x) = g(x) \ ,$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given smooth bounded function.

State and prove the Duhamel principle giving the solution $v(t, x)$ for $t > 0$ to the inhomogeneous initial value problem

$$\frac{\partial v}{\partial t} - \Delta v = f \ ,$$
$$v(0, x) = g(x) \ ,$$

for $f = f(t, x)$ a given smooth bounded function.

For the case $n = 4$ and when $f = f(x)$ is a fixed Schwartz function (independent of t), find $v(t, x)$ and show that $w(x) = \lim_{t\to+\infty} v(t, x)$ is a solution of

$$-\Delta w = f \ .$$

[*Hint: you may use without proof the fact that the fundamental solution of the Laplacian on* $\mathbb{R}^4$ *is* $-1/(4\pi^2|x|^2)$.]

4/II/30A **Partial Differential Equations**

(a) State the Fourier inversion theorem for Schwartz functions $\mathcal{S}(\mathbb{R})$ on the real line. Define the Fourier transform of a tempered distribution and compute the Fourier transform of the distribution defined by the function $F(x) = \frac{1}{2}$ for $-t \leqslant x \leqslant +t$ and $F(x) = 0$ otherwise. (Here t is any positive number.)

Use the Fourier transform in the x variable to deduce a formula for the solution to the one dimensional wave equation

$$u_{tt} - u_{xx} = 0 \ , \qquad \text{with initial data} \qquad u(0, x) = 0 \ , \quad u_t(0, x) = g(x) \ , \qquad (*)$$

for g a Schwartz function. Explain what is meant by "finite propagation speed" and briefly explain why the formula you have derived is in fact valid for arbitrary smooth $g \in C^\infty(\mathbb{R})$.

(b) State a theorem on the representation of a smooth 2π-periodic function g as a Fourier series

$$g(x) = \sum_{\alpha\in\mathbb{Z}} \hat{g}(\alpha) e^{i\alpha x}$$

and derive a representation for solutions to $(*)$ as Fourier series in x.

(c) Verify that the formulae obtained in (a) and (b) agree for the case of smooth 2π-periodic g.

1/II/29C Partial Differential Equations

Consider the equation

$$x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2} + a \frac{\partial u}{\partial x_3} = u, \qquad (*)$$

where $a \in \mathbb{R}$, to be solved for $u = u(x_1, x_2, x_3)$. State clearly what it means for a hypersurface

$$S_\phi = \{(x_1, x_2, x_3) : \ \phi(x_1, x_2, x_3) = 0\},$$

defined by a C^1 function ϕ, to be *non-characteristic for* $(*)$. Does the non-characteristic condition hold when $\phi(x_1, x_2, x_3) = x_3$?

Solve $(*)$ for $a > 0$ with initial condition $u(x_1, x_2, 0) = f(x_1, x_2)$ where $f \in C^1(\mathbb{R}^2)$. For the case $f(x_1, x_2) = x_1^2 + x_2^2$ discuss the limiting behaviour as $a \to 0_+$.

2/II/30C Partial Differential Equations

Define a *fundamental solution* of a linear partial differential operator P. Prove that the function

$$G(x) = \tfrac{1}{2} e^{-|x|}$$

defines a distribution which is a fundamental solution of the operator P given by

$$P\,u = -\frac{d^2 u}{dx^2} + u\,.$$

Hence find a solution u_0 to the equation

$$-\frac{d^2 u_0}{dx^2} + u_0 = V(x)\,,$$

where $V(x) = 0$ for $|x| > 1$ and $V(x) = 1$ for $|x| \leqslant 1$.

Consider the functional

$$I[u] = \int_{\mathbb{R}} \left\{ \frac{1}{2} \left[\left(\frac{du}{dx} \right)^2 + u^2 \right] - V u \right\} dx\,.$$

Show that $I[u_0 + \phi] > I[u_0]$ for all Schwartz functions ϕ that are not identically zero.

3/II/29C **Partial Differential Equations**

Write down a formula for the solution $u = u(t, x)$ of the n-dimensional heat equation

$$w_t(t, x) - \Delta w = 0, \qquad w(0, x) = g(x),$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given Schwartz function; here $w_t = \partial_t w$ and Δ is taken in the variables $x \in \mathbb{R}^n$. Show that

$$w(t, x) \leqslant \frac{\int |g(x)| \, dx}{(4\pi t)^{n/2}}.$$

Consider the equation

$$u_t - \Delta u = e^{it} f(x), \tag{$*$}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a given Schwartz function. Show that $(*)$ has a solution of the form

$$u(t, x) = e^{it} v(x),$$

where v is a Schwartz function.

Prove that the solution $u(t, x)$ of the initial value problem for $(*)$ with initial data $u(0, x) = g(x)$ satisfies

$$\lim_{t \to +\infty} \left| u(t, x) - e^{it} v(x) \right| = 0.$$

4/II/30C **Partial Differential Equations**

Write down the solution of the three-dimensional wave equation

$$u_{tt} - \Delta u = 0, \qquad u(0, x) = 0, \quad u_t(0, x) = g(x),$$

for a Schwartz function g. Here Δ is taken in the variables $x \in \mathbb{R}^3$ and $u_t = \partial u / \partial t$ etc. State the "strong" form of Huygens principle for this solution. Using the method of descent, obtain the solution of the corresponding problem in two dimensions. State the "weak" form of Huygens principle for this solution.

Let $u \in C^2([0, T] \times \mathbb{R}^3)$ be a solution of

$$u_{tt} - \Delta u + |x|^2 u = 0, \qquad u(0, x) = 0, \quad u_t(0, x) = 0. \tag{$*$}$$

Show that

$$\partial_t e + \boldsymbol{\nabla} \cdot \mathbf{p} = 0, \tag{$**$}$$

where

$$e = \tfrac{1}{2}\left(u_t{}^2 + |\boldsymbol{\nabla} u|^2 + |x|^2 u^2\right), \qquad \text{and} \qquad \mathbf{p} = -u_t \boldsymbol{\nabla} u.$$

Hence deduce, by integration of $(**)$ over the region

$$K = \left\{(t, x) : 0 \leqslant t \leqslant t_0 - a \leqslant t_0, \ |x - x_0| \leqslant t_0 - t\right\}$$

or otherwise, that $(*)$ satisfies the weak Huygens principle.