Part II

Linear Analysis

Paper 1, Section II

22F Linear Analysis

(a) State the open mapping theorem and the closed graph theorem, and prove that the former implies the latter.

(b) Let V be a Banach space. Give the definition of the *dual space* V^*, and prove that V^* is a Banach space.

(c) Let V be a Banach space over the real field, and let $T : V \to V^*, v \mapsto T_v$ be a linear map between these two Banach spaces that satisfies $T_v(v) \geqslant 0$ for all $v \in V$. Prove that T is continuous.

Paper 2, Section II

22F Linear Analysis

(a) Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$, and $v, w \in V$. Define

$$S_1^{vw} := \left\{ z \in V \ : \ \|z - v\| = \|z - w\| = \frac{1}{2}\|v - w\| \right\}$$

and then inductively, for $n \geqslant 2$,

$$S_n^{vw} := \left\{ z \in S_{n-1}^{vw} \ : \ \forall\, \tilde{z} \in S_{n-1}^{vw}, \ \|z - \tilde{z}\| \leqslant \frac{1}{2}\mathrm{diam}(S_{n-1}^{vw}) \right\},$$

with the definition $\mathrm{diam}(S) := \sup_{z,\tilde{z}\in S} \|z - \tilde{z}\|$. Prove that $\cap_{n\geqslant 1} S_n^{vw} = \{\frac{v+w}{2}\}$.

(b) Let $(V, \|\cdot\|_V)$ and $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ be normed vector spaces over $\mathbb{R}$, and $u : V \to \widetilde{V}$ an isometry, i.e. a map with the property that $\|u(v) - u(w)\|_{\widetilde{V}} = \|v - w\|_V$. Using part (a), prove that $u(\frac{v+w}{2}) = \frac{u(v)+u(w)}{2}$ for all $v, w \in V$.

(c) Assume furthermore that the isometry $u : V \to \widetilde{V}$ satisfies $u(0) = 0$. Prove that u is linear.

Paper 3, Section II

21F Linear Analysis

Recall that a topological space X is called *normal* if for any pair of non-empty disjoint closed subsets $A, B \subset X$, there is a pair of disjoint open subsets $U_1, U_2 \subset X$ so that $A \subset U_1$ and $B \subset U_2$. Also recall that the Urysohn lemma states that in a normal topological space X, for any pair of non-empty disjoint closed subsets $A, B \subset X$, there is an $f : X \to [0,1]$ continuous so that $f = 0$ on A and $f = 1$ on B.

(a) State and prove the Tietze extension theorem. [You may use the Urysohn lemma.]

(b) Consider a normal topological space X, and $A \subset X$ a non-empty closed subset that can be realised as a countable intersection of open sets. Show that there exists $f : X \to [0,1]$ continuous so that f vanishes on A and on A only.

(c) Consider a normal topological space X, and $A, B \subset X$ a pair of non-empty disjoint closed subsets that can both be realised as countable intersections of open sets. Show that there exists $f : X \to [0,1]$ continuous so that f vanishes on A and on A only, and is equal to 1 on B and on B only.

Paper 4, Section II

22F Linear Analysis

Below, H denotes a Hilbert space over $\mathbb{C}$.

(a) Consider a sequence (x_n) in H with the property that there exists an $x \in H$ such that for any $y \in H$, $\langle x_n, y\rangle$ converges to $\langle x, y\rangle$ in $\mathbb{C}$. Prove that the sequence (x_n) is bounded. [The uniform boundedness principle may be used without proof, provided it is properly stated.]

(b) With (x_n) and x as above, prove that there exists another sequence $(\tilde{x}_k)$ in H such that $\|\tilde{x}_k - x\|_H \to 0$ and such that each $\tilde{x}_k$ is a convex combination of terms in (x_n).

(c) Deduce that if $C \subset H$ is closed and convex, and (x_n) is a sequence in C as in part (a), i.e. with the property that there exists $x \in H$ such that for any $y \in H$, $\langle x_n, y\rangle \to \langle x, y\rangle$, then in fact $x \in C$.

(d) Is the statement in part (c) still true when C is closed but not necessarily convex? [You must either provide a proof if true or a detailed counterexample if untrue.]

Paper 1, Section II
22G Linear Analysis

Let ℓ^∞ denote the space of bounded real sequences and let ℓ^1 denote the space of summable real sequences. Suppose that $\varphi : \ell^\infty \to \mathbb{R}$ is linear and continuous, that φ is non-negative on non-negative sequences, that $\varphi((x_n)_{n\geqslant 1}) = \varphi((x_{n+1})_{n\geqslant 1})$, and that φ maps the constant sequence equal to one to one.

(a) Prove that $\liminf_{n\to\infty} x_n \leqslant \varphi((x_n)_{n\geqslant 1}) \leqslant \limsup_{n\to\infty} x_n$ for all $(x_n)_{n\geqslant 1} \in \ell^\infty$.

(b) Is there $(y_n)_{n\geqslant 1} \in \ell^1$ so that $\varphi((x_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} x_n y_n$ for all $(x_n)_{n\geqslant 1} \in \ell^\infty$?

(c) Give an example of $(x_n)_{n\geqslant 1} \in \ell^\infty$ that does not converge but for which all φ defined as above give the same value.

(d) Let $y \in \mathbb{R}$. Assume $(x_n)_{n\geqslant 1} \in \ell^\infty$ satisfies $\dfrac{x_{n+1} + x_{n+2} + \cdots + x_{n+p}}{p} \to y$ as $p \to \infty$ uniformly in $n \geqslant 1$. Prove that $\varphi((x_n)_{n\geqslant 1}) = y$.

Paper 2, Section II
22G Linear Analysis

(a) Given a complex Banach space $(V, \|\cdot\|)$, prove that the space of bounded linear maps $(\mathcal{B}(V,V), |||\cdot|||)$ endowed with the norm

$$|||T||| = \sup_{v\in V,\ \|v\|=1} \|Tv\|$$

is a Banach space.

(b) Assume $(V, \|\cdot\|)$ is a complex Hilbert space. State the definitions of a *compact operator* $T : V \to V$ and of a *Hilbertian basis*. Suppose $T \in \mathcal{B}(V,V)$ and V has a Hilbertian basis $(e_n)_{n\geqslant 1}$ such that $T(e_n) = \lambda_n e_n$ for complex numbers λ_n, $n \geqslant 1$. Prove that T is compact if and only if $\lambda_n \to 0$.

(c) Given a complex Hilbert space $(V, \|\cdot\|)$ and $(e_n)_{n\geqslant 1}$ a Hilbertian basis of V, consider $\mathcal{H}(V,V)$, the set of linear operators T such that $\sum_{n\geqslant 1} \|Te_n\|^2 < +\infty$. Prove that operators in $\mathcal{H}(V,V)$ are bounded and compact, and that $(\mathcal{H}(V,V), |||\cdot|||_*)$ with

$$|||T|||_* = \left(\sum_{n\geqslant 1} \|Te_n\|^2 \right)^{1/2}$$

is a Hilbert space. Are $|||\cdot|||$ and $|||\cdot|||_*$ equivalent norms on $\mathcal{H}(V,V)$?

Paper 3, Section II
21G Linear Analysis

(a) Prove that any metric space (X, d) is normal for the induced topology.

(b) State the *Urysohn lemma* and the *Tietze extension theorem.*

(c) Prove that a metric space (X, d) is compact if and only if all continuous functions from X to $\mathbb{R}$ are bounded.

Paper 4, Section II
22G Linear Analysis

(a) Define what it means for a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ to be *equi-continuous* on $[0,1]$. State the *Arzelà–Ascoli theorem.*

(b) Given a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we can inductively define functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n \geqslant 0$ by $f_{n+1}(t) = \int_0^t \varphi(f_n(s))\,\mathrm{d}s$, and $f_0(t) = 0$ for all $t \in \mathbb{R}$. Show that there exists $T_1 > 0$ so that the sequence $(f_n)_{n\geqslant 1}$ is equi-bounded and equi-continuous on $[0, T_1]$.

(c) Deduce the existence of $T_2 \in (0, T_1]$ and a continuously differentiable function $f : [0, T_2] \to \mathbb{R}$ such that $f(0) = 0$ and $f'(t) = \varphi(f(t))$ on $[0, T_2]$. [*Hint: Prove that if* $T_2 \in (0, T_1]$ *is small enough,* $R_n(t) = f_{n+1}(t) - f_n(t) \to 0$ *uniformly on* $[0, T_2]$.]

Paper 1, Section II
22H Linear Analysis

Let H be a separable Hilbert space and $\{e_i\}$ be a Hilbertian (orthonormal) basis of H. Given a sequence (x_n) of elements of H and $x_\infty \in H$, we say that x_n weakly converges to x_∞, denoted $x_n \rightharpoonup x_\infty$, if $\forall h \in H$, $\lim_{n\to\infty}\langle x_n, h\rangle = \langle x_\infty, h\rangle$.

(a) Given a sequence (x_n) of elements of H, prove that the following two statements are equivalent:

(i) $\exists x_\infty \in H$ such that $x_n \rightharpoonup x_\infty$;

(ii) the sequence (x_n) is bounded in H and $\forall i \geqslant 1$, the sequence $(\langle x_n, e_i\rangle)$ is convergent.

(b) Let (x_n) be a bounded sequence of elements of H. Show that there exists $x_\infty \in H$ and a subsequence $(x_{\phi(n)})$ such that $x_{\phi(n)} \rightharpoonup x_\infty$ in H.

(c) Let (x_n) be a sequence of elements of H and $x_\infty \in H$ be such that $x_n \rightharpoonup x_\infty$. Show that the following three statements are equivalent:

(i) $\lim_{n\to\infty} \|x_n - x_\infty\| = 0$;

(ii) $\lim_{n\to\infty} \|x_n\| = \|x_\infty\|$;

(iii) $\forall \epsilon > 0$, $\exists I(\epsilon)$ such that $\forall n \geqslant 1$, $\sum_{i\geqslant I(\epsilon)} |\langle x_n, e_i\rangle|^2 < \epsilon$.

Paper 2, Section II
22H Linear Analysis

(a) Let V be a real normed vector space. Show that any proper subspace of V has empty interior.

Assuming V to be infinite-dimensional and complete, prove that any algebraic basis of V is uncountable. [The Baire category theorem can be used if stated properly.] Deduce that the vector space of polynomials with real coefficients cannot be equipped with a complete norm, i.e. a norm that makes it complete.

(b) Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a vector space V such that $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ are both complete. Prove that if there exists $C_1 > 0$ such that $\|x\|_2 \leqslant C_1\|x\|_1$ for all $x \in V$, then there exists $C_2 > 0$ such that $\|x\|_1 \leqslant C_2\|x\|_2$ for all $x \in V$. Is this still true without the assumption that $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ are both complete? Justify your answer.

(c) Let V be a real normed vector space (not necessarily complete) and V^* be the set of linear continuous forms $f : V \to \mathbb{R}$. Let $(x_n)_{n\geqslant 1}$ be a sequence in V such that $\sum_{n\geqslant 1} |f(x_n)| < \infty$ for all $f \in V^*$. Prove that

$$\sup_{\|f\|_{V^*}\leqslant 1} \sum_{n\geqslant 1} |f(x_n)| < \infty\,.$$

Paper 3, Section II
21H Linear Analysis

(a) State the Arzela–Ascoli theorem, including the definition of *equicontinuity*.

(b) Consider a sequence (f_n) of continuous real-valued functions on $\mathbb{R}$ such that for all $x \in \mathbb{R}$, $\big(f_n(x)\big)$ is bounded and the sequence is equicontinuous at x. Prove that there exists $f \in C(\mathbb{R})$ and a subsequence $(f_{\varphi(n)})$ such that $f_{\varphi(n)} \to f$ uniformly on any closed bounded interval.

(c) Let K be a Hausdorff compact topological space, and $C(K)$ the real-valued continuous functions on K. Let $\mathcal{K} \subset C(K)$ be a compact subset of $C(K)$. Prove that the collection of functions $\mathcal{K}$ is equicontinuous.

(d) We say that a Hausdorff topological space X is *locally compact* if every point has a compact neighbourhood. Let X be such a space, $K \subset X$ compact and $U \subset X$ open such that $K \subset U$. Prove that there exists $f : X \to \mathbb{R}$ continuous with compact support contained in U and equal to 1 on K. [*Hint: Construct an open set V such that $K \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact, and use Urysohn's lemma to construct a function in $\overline{V}$ and then extend it by zero.*]

Paper 4, Section II
22H Linear Analysis

(a) Let $(H_1, \langle \cdot,\cdot \rangle_1)$, $(H_2, \langle \cdot,\cdot \rangle_2)$ be two Hilbert spaces, and $T : H_1 \to H_2$ be a bounded linear operator. Show that there exists a unique bounded linear operator $T^* : H_2 \to H_1$ such that

$$\langle Tx_1, x_2 \rangle_2 = \langle x_1, T^* x_2 \rangle_1 \,, \quad \forall\, x_1 \in H_1, x_2 \in H_2 \,.$$

(b) Let H be a separable Hilbert space. We say that a sequence (e_i) is a *frame* of H if there exists $A, B > 0$ such that

$$\forall x \in H, \quad A\|x\|^2 \leqslant \sum_{i \geqslant 1} |\langle x, e_i \rangle|^2 \leqslant B\|x\|^2.$$

State briefly why such a frame exists. From now on, let (e_i) be a frame of H. Show that $\mathrm{Span}\{e_i\}$ is dense in H.

(c) Show that the linear map $U : H \to \ell^2$ given by $U(x) = \big(\langle x, e_i \rangle\big)_{i \geqslant 1}$ is bounded and compute its adjoint U^*.

(d) Assume now that (e_i) is a Hilbertian (orthonormal) basis of H and let $a \in H$. Show that the Hilbert cube $\mathcal{C}_a = \big\{ x \in H \text{ such that } \forall i \geqslant 1,\ |\langle x, e_i \rangle| \leqslant |\langle a, e_i \rangle| \big\}$ is a compact subset of H.

Paper 1, Section II

22I Linear Analysis

(a) Define the dual space X^* of a (real) normed space $(X, \|\cdot\|)$. Define what it means for two normed spaces to be isometrically isomorphic. Prove that $(l_1)^*$ is isometrically isomorphic to l_∞.

(b) Let $p \in (1, \infty)$. [In this question, you may use without proof the fact that $(l_p)^*$ is isometrically isomorphic to l_q where $\frac{1}{p} + \frac{1}{q} = 1$.]

(i) Show that if $\{\phi_m\}_{m=1}^\infty$ is a countable dense subset of $(l_p)^*$, then the function

$$d(x, y) := \sum_{m=1}^{\infty} 2^{-m} \frac{|\phi_m(x-y)|}{1 + |\phi_m(x-y)|}$$

defines a metric on the closed unit ball $B \subset l_p$. Show further that for a sequence $\{x^{(n)}\}_{n=1}^\infty$ of elements $x^{(n)} \in B$, we have

$$\phi(x^{(n)}) \to \phi(x) \quad \forall\, \phi \in (l_p)^* \quad \Leftrightarrow \quad d(x^{(n)}, x) \to 0.$$

Deduce that (B, d) is a compact metric space.

(ii) Give an example to show that for a sequence $\{x^{(n)}\}_{n=1}^\infty$ of elements $x^{(n)} \in B$ and $x \in B$,

$$\phi(x^{(n)}) \to \phi(x) \quad \forall\, \phi \in (l_p)^* \quad \not\Rightarrow \quad \left\|x^{(n)} - x\right\|_{l_p} \to 0.$$

Paper 2, Section II

22I Linear Analysis

(a) State and prove the Baire Category theorem.

Let $p > 1$. Apply the Baire Category theorem to show that $\bigcup_{1 \leqslant q < p} l_q \neq l_p$. Give an explicit element of $l_p \setminus \bigcup_{1 \leqslant q < p} l_q$.

(b) Use the Baire Category theorem to prove that $C([0,1])$ contains a function which is nowhere differentiable.

(c) Let $(X, \|\cdot\|)$ be a real Banach space. Verify that the map sending x to the function $e_x : \phi \mapsto \phi(x)$ is a continuous linear map of X into $(X^*)^*$ where X^* denotes the dual space of $(X, \|\cdot\|)$. Taking for granted the fact that this map is an isometry regardless of the norm on X, prove that if $\|\cdot\|'$ is another norm on the vector space X which is not equivalent to $\|\cdot\|$, then there is a linear function $\psi : X \to \mathbb{R}$ which is continuous with respect to one of the two norms $\|\cdot\|, \|\cdot\|'$ and not continuous with respect to the other.

Paper 3, Section II

21I Linear Analysis

Let H be a separable complex Hilbert space.

(a) For an operator $T : H \to H$, define the *spectrum* and *point spectrum*. Define what it means for T to be: (i) a *compact operator*; (ii) a *self-adjoint operator* and (iii) a *finite rank operator*.

(b) Suppose $T : H \to H$ is compact. Prove that given any $\delta > 0$, there exists a finite-dimensional subspace $E \subset H$ such that $\|T(e_n) - P_E T(e_n)\| < \delta$ for each n, where $\{e_1, e_2, e_3, \ldots\}$ is an orthonormal basis for H and P_E denotes the orthogonal projection onto E. Deduce that a compact operator is the operator norm limit of finite rank operators.

(c) Suppose that $S : H \to H$ has finite rank and $\lambda \in \mathbb{C} \setminus \{0\}$ is not an eigenvalue of S. Prove that $S - \lambda I$ is surjective. [You may wish to consider the action of $S(S - \lambda I)$ on $\ker(S)^{\perp}$.]

(d) Suppose $T : H \to H$ is compact and $\lambda \in \mathbb{C} \setminus \{0\}$ is not an eigenvalue of T. Prove that the image of $T - \lambda I$ is dense in H.

Prove also that $T - \lambda I$ is bounded below, i.e. prove also that there exists a constant $c > 0$ such that $\|(T - \lambda I)x\| \geqslant c\|x\|$ for all $x \in H$. Deduce that $T - \lambda I$ is surjective.

Paper 4, Section II

22I Linear Analysis

(a) For K a compact Hausdorff space, what does it mean to say that a set $S \subset C(K)$ is *equicontinuous*. State and prove the Arzelà–Ascoli theorem.

(b) Suppose K is a compact Hausdorff space for which $C(K)$ is a countable union of equicontinuous sets. Prove that K is finite.

(c) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded, continuous function and let $x_0 \in \mathbb{R}^n$. Consider the problem of finding a differentiable function $x : [0,1] \to \mathbb{R}^n$ with

$$x(0) = x_0 \qquad \text{and} \qquad x'(t) = F(x(t)) \qquad (*)$$

for all $t \in [0,1]$. For each $k = 1, 2, 3, \ldots$, let $x_k : [0,1] \to \mathbb{R}^n$ be defined by setting $x_k(0) = x_0$ and

$$x_k(t) = x_0 + \int_0^t F(y_k(s))\, ds$$

for $t \in [0,1]$, where

$$y_k(t) = x_k\left(\frac{j}{k}\right)$$

for $t \in (\frac{j}{k}, \frac{j+1}{k}]$ and $j \in \{0, 1, \ldots, k-1\}$.

(i) Verify that x_k is well-defined and continuous on $[0,1]$ for each k.

(ii) Prove that there exists a differentiable function $x : [0,1] \to \mathbb{R}^n$ solving $(*)$ for $t \in [0,1]$.

Paper 3, Section II
21H Linear Analysis

(a) Let X be a Banach space and consider the open unit ball $B = \{x \in X : \|x\| < 1\}$. Let $T : X \to X$ be a bounded operator. Prove that $\overline{T(B)} \supset B$ implies $T(B) \supset B$.

(b) Let P be the vector space of all polynomials in one variable with real coefficients. Let $\|\cdot\|$ be any norm on P. Show that $(P, \|\cdot\|)$ is not complete.

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be entire, and assume that for every $z \in \mathbb{C}$ there is n such that $f^{(n)}(z) = 0$ where $f^{(n)}$ is the n-th derivative of f. Prove that f is a polynomial.

[You may use that an entire function vanishing on an open subset of $\mathbb{C}$ must vanish everywhere.]

(d) A Banach space X is said to be uniformly convex if for every $\varepsilon \in (0, 2]$ there is $\delta > 0$ such that for all $x, y \in X$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geqslant \varepsilon$, one has $\|(x + y)/2\| \leqslant 1 - \delta$. Prove that ℓ^2 is uniformly convex.

Paper 4, Section II
22H Linear Analysis

(a) State and prove the *Riesz representation theorem* for a real Hilbert space H.

[You may use that if H is a real Hilbert space and $Y \subset H$ is a closed subspace, then $H = Y \oplus Y^{\perp}$.]

(b) Let H be a real Hilbert space and $T : H \to H$ a bounded linear operator. Show that T is invertible if and only if both T and T^* are bounded below. [Recall that an operator $S : H \to H$ is bounded below if there is $c > 0$ such that $\|Sx\| \geqslant c\|x\|$ for all $x \in H$.]

(c) Consider the complex Hilbert space of two-sided sequences,

$$X = \{(x_n)_{n \in \mathbb{Z}} : x_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty\}$$

with norm $\|x\| = (\sum_n |x_n|^2)^{1/2}$. Define $T : X \to X$ by $(Tx)_n = x_{n+1}$. Show that T is unitary and find the point spectrum and the approximate point spectrum of T.

Paper 2, Section II
22H Linear Analysis

(a) State the real version of the *Stone–Weierstrass theorem* and state the *Urysohn–Tietze extension theorem.*

(b) In this part, you may assume that there is a sequence of polynomials P_i such that $\sup_{x\in[0,1]} |P_i(x) - \sqrt{x}| \to 0$ as $i \to \infty$.

Let $f : [0,1] \to \mathbb{R}$ be a continuous piecewise linear function which is linear on $[0,1/2]$ and on $[1/2,1]$. Using the polynomials P_i mentioned above (but not assuming any form of the Stone-Weierstrass theorem), prove that there are polynomials Q_i such that $\sup_{x\in[0,1]} |Q_i(x) - f(x)| \to 0$ as $i \to \infty$.

(d) Which of the following families of functions are relatively compact in $C[0,1]$ with the supremum norm? Justify your answer.

$$\begin{aligned}\mathcal{F}_1 &= \{x \mapsto \frac{\sin(\pi n x)}{n} : n \in \mathbb{N}\}\\ \mathcal{F}_2 &= \{x \mapsto \frac{\sin(\pi n x)}{n^{1/2}} : n \in \mathbb{N}\}\\ \mathcal{F}_3 &= \{x \mapsto \sin(\pi n x) : n \in \mathbb{N}\}\end{aligned}$$

[*In this question* $\mathbb{N}$ *denotes the set of positive integers.*]

Paper 1, Section II
22H Linear Analysis

Let F be the space of real-valued sequences with only finitely many nonzero terms.

(a) For any $p \in [1,\infty)$, show that F is dense in ℓ^p. Is F dense in ℓ^∞? Justify your answer.

(b) Let $p \in [1,\infty)$, and let $T : F \to F$ be an operator that is bounded in the $\|\cdot\|_p$-norm, i.e., there exists a C such that $\|Tx\|_p \leqslant C\|x\|_p$ for all $x \in F$. Show that there is a unique bounded operator $\widetilde{T} : \ell^p \to \ell^p$ satisfying $\widetilde{T}|_F = T$, and that $\|\widetilde{T}\|_p \leqslant C$.

(c) For each $p \in [1,\infty]$ and for each $i = 1,\ldots,5$ determine if there is a bounded operator from ℓ^p to ℓ^p (in the $\|\cdot\|_p$ norm) whose restriction to F is given by T_i:

$$(T_1x)_n = nx_n, \quad (T_2x)_n = n(x_n - x_{n+1}), \quad (T_3x)_n = \frac{x_n}{n},$$
$$(T_4x)_n = \frac{x_1}{n^{1/2}}, \quad (T_5x)_n = \frac{\sum_{j=1}^n x_j}{2^n}.$$

(d) Let X be a normed vector space such that the closed unit ball $\overline{B_1(0)}$ is compact. Prove that X is finite dimensional.

Paper 3, Section II

21F Linear Analysis

(a) Let X be a normed vector space and let Y be a Banach space. Show that the space of bounded linear operators $\mathcal{B}(X,Y)$ is a Banach space.

(b) Let X and Y be Banach spaces, and let $D \subset X$ be a dense linear subspace. Prove that a bounded linear map $T : D \to Y$ can be extended uniquely to a bounded linear map $T : X \to Y$ with the same operator norm. Is the claim also true if one of X and Y is not complete?

(c) Let X be a normed vector space. Let (x_n) be a sequence in X such that

$$\sum_{n=1}^{\infty} |f(x_n)| < \infty \qquad \forall f \in X^*.$$

Prove that there is a constant C such that

$$\sum_{n=1}^{\infty} |f(x_n)| \leqslant C\|f\| \qquad \forall f \in X^*.$$

Paper 1, Section II

22F Linear Analysis

Let K be a compact Hausdorff space.

(a) State the Arzelà–Ascoli theorem, and state both the real and complex versions of the Stone–Weierstraß theorem. Give an example of a compact space K and a bounded set of functions in $C(K)$ that is not relatively compact.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Show that there exists a sequence of polynomials (p_i) in n variables such that

$$B \subset \mathbb{R}^n \text{ compact} \quad \Rightarrow \quad p_i|_B \to f|_B \text{ uniformly.}$$

Characterize the set of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ for which there exists a sequence of polynomials (p_i) such that $p_i \to f$ uniformly on $\mathbb{R}^n$.

(c) Prove that if $C(K)$ is equicontinuous then K is finite. Does this implication remain true if we drop the requirement that K be compact? Justify your answer.

Paper 2, Section II

22F Linear Analysis

Let X, Y be Banach spaces and let $\mathcal{B}(X, Y)$ denote the space of bounded linear operators $T : X \to Y$.

(a) Define what it means for a bounded linear operator $T : X \to Y$ to be *compact*. Let $T_i : X \to Y$ be linear operators with finite rank, i.e., $T_i(X)$ is finite-dimensional. Assume that the sequence T_i converges to T in $\mathcal{B}(X, Y)$. Show that T is compact.

(b) Let $T : X \to Y$ be compact. Show that the dual map $T^* : Y^* \to X^*$ is compact. [*Hint: You may use the Arzelà–Ascoli theorem.*]

(c) Let X be a Hilbert space and let $T : X \to X$ be a compact operator. Let (λ_j) be an infinite sequence of eigenvalues of T with eigenvectors x_j. Assume that the eigenvectors are orthogonal to each other. Show that $\lambda_j \to 0$.

Paper 4, Section II

22F Linear Analysis

(a) Let X be a separable normed space. For any sequence $(f_n)_{n\in\mathbb{N}} \subset X^*$ with $\|f_n\| \leqslant 1$ for all n, show that there is $f \in X^*$ and a subsequence $\Lambda \subset \mathbb{N}$ such that $f_n(x) \to f(x)$ for all $x \in X$ as $n \in \Lambda$, $n \to \infty$. [You may use without proof the fact that X^* is complete and that any bounded linear map $f : D \to \mathbb{R}$, where $D \subset X$ is a dense linear subspace, can be extended uniquely to an element $f \in X^*$.]

(b) Let H be a Hilbert space and $U : H \to H$ a unitary map. Let

$$I = \{x \in H : Ux = x\}, \qquad W = \{Ux - x : x \in H\}.$$

Prove that I and W are orthogonal, $H = I \oplus \overline{W}$, and that for every $x \in H$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i x = Px,$$

where P is the orthogonal projection onto the closed subspace I.

(c) Let $T : C(S^1) \to C(S^1)$ be a linear map, where $S^1 = \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$ is the unit circle, induced by a homeomorphism $\tau : S^1 \to S^1$ by $(Tf)e^{i\theta} = f(\tau(e^{i\theta}))$. Prove that there exists $\mu \in C(S^1)^*$ with $\mu(1_{S^1}) = 1$ such that $\mu(Tf) = \mu(f)$ for all $f \in C(S^1)$. (Here 1_{S^1} denotes the function on S^1 which returns 1 identically.) If T is not the identity map, does it follow that μ as above is necessarily unique? Justify your answer.

Paper 3, Section II

19F Linear Analysis

Let K be a non-empty compact Hausdorff space and let $C(K)$ be the space of real-valued continuous functions on K.

(i) State the real version of the Stone–Weierstrass theorem.

(ii) Let A be a closed subalgebra of $C(K)$. Prove that $f \in A$ and $g \in A$ implies that $m \in A$ where the function $m : K \to \mathbb{R}$ is defined by $m(x) = \max\{f(x), g(x)\}$. [You may use without proof that $f \in A$ implies $|f| \in A$.]

(iii) Prove that K is normal and state Urysohn's Lemma.

(iv) For any $x \in K$, define $\delta_x \in C(K)^*$ by $\delta_x(f) = f(x)$ for $f \in C(K)$. Justifying your answer carefully, find

$$\inf_{x \neq y} \|\delta_x - \delta_y\|.$$

Paper 2, Section II

20F Linear Analysis

(a) Let X be a normed vector space and $Y \subset X$ a closed subspace with $Y \neq X$. Show that Y is nowhere dense in X.

(b) State any version of the Baire Category theorem.

(c) Let X be an infinite-dimensional Banach space. Show that X cannot have a countable algebraic basis, i.e. there is no countable subset $(x_k)_{k \in \mathbb{N}} \subset X$ such that every $x \in X$ can be written as a finite linear combination of elements of (x_k).

Paper 1, Section II

21F Linear Analysis

Let X be a normed vector space over the real numbers.

(a) Define the *dual space* X^* of X and prove that X^* is a Banach space. [You may use without proof that X^* is a vector space.]

(b) The Hahn–Banach theorem states the following. Let X be a real vector space, and let $p : X \to \mathbb{R}$ be sublinear, i.e., $p(x + y) \leqslant p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in X$ and all $\lambda > 0$. Let $Y \subset X$ be a linear subspace, and let $g : Y \to \mathbb{R}$ be linear and satisfy $g(y) \leqslant p(y)$ for all $y \in Y$. Then there exists a linear functional $f : X \to \mathbb{R}$ such that $f(x) \leqslant p(x)$ for all $x \in X$ and $f|_Y = g$.

Using the Hahn–Banach theorem, prove that for any non-zero $x_0 \in X$ there exists $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

(c) Show that X can be embedded isometrically into a Banach space, i.e. find a Banach space Y and a linear map $\Phi : X \to Y$ with $\|\Phi(x)\| = \|x\|$ for all $x \in X$.

Paper 4, Section II

21F Linear Analysis

Let H be a complex Hilbert space with inner product $(\cdot, \cdot)$ and let $T : H \to H$ be a bounded linear map.

(i) Define the *spectrum* $\sigma(T)$, the *point spectrum* $\sigma_p(T)$, the *continuous spectrum* $\sigma_c(T)$, and the *residual spectrum* $\sigma_r(T)$.

(ii) Show that T^*T is self-adjoint and that $\sigma(T^*T) \subset [0, \infty)$. Show that if T is compact then so is T^*T.

(iii) Assume that T is compact. Prove that T has a singular value decomposition: for $N < \infty$ or $N = \infty$, there exist orthonormal systems $(u_i)_{i=1}^N \subset H$ and $(v_i)_{i=1}^N \subset H$ and $(\lambda_i)_{i=1}^N \subset [0, \infty)$ such that, for any $x \in H$,

$$Tx = \sum_{i=1}^{N} \lambda_i (u_i, x) v_i.$$

[You may use the spectral theorem for compact self-adjoint linear operators.]

Paper 3, Section II

19I Linear Analysis

(a) Define *Banach spaces* and *Euclidean spaces* over $\mathbb{R}$. [You may assume the definitions of vector spaces and inner products.]

(b) Let X be the space of sequences of real numbers with finitely many non-zero entries. Does there exist a norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is a Banach space? Does there exist a norm such that $(X, \|\cdot\|)$ is Euclidean? Justify your answers.

(c) Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$ satisfying the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$. Show that $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ is an inner product on X. [*You may use without proof the fact that the vector space operations* $+$ *and* $\cdot$ *are continuous with respect to* $\|\cdot\|$. *To verify the identity* $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$, *you may find it helpful to consider the parallelogram law for the pairs* $(a+c, b)$, $(b+c, a)$, $(a-c, b)$ *and* $(b-c, a)$.]

(d) Let $(X, \|\cdot\|_X)$ be an incomplete normed vector space over $\mathbb{R}$ which is not a Euclidean space, and let $(X^*, \|\cdot\|_{X^*})$ be its dual space with the dual norm. Is $(X^*, \|\cdot\|_{X^*})$ a Banach space? Is it a Euclidean space? Justify your answers.

Paper 2, Section II

20I Linear Analysis

(a) Let K be a topological space and let $C_{\mathbb{R}}(K)$ denote the normed vector space of bounded continuous real-valued functions on K with the norm $\|f\|_{C_{\mathbb{R}}(K)} = \sup_{x\in K} |f(x)|$. Define the terms *uniformly bounded*, *equicontinuous* and *relatively compact* as applied to subsets $S \subset C_{\mathbb{R}}(K)$.

(b) The Arzela–Ascoli theorem [which you need not prove] states in particular that if K is compact and $S \subset C_{\mathbb{R}}(K)$ is uniformly bounded and equicontinuous, then S is relatively compact. Show by examples that each of the compactness of K, uniform boundedness of S, and equicontinuity of S are necessary conditions for this conclusion.

(c) Let L be a topological space. Assume that there exists a sequence of compact subsets K_n of L such that $K_1 \subset K_2 \subset K_3 \subset \cdots \subset L$ and $\bigcup_{n=1}^{\infty} K_n = L$. Suppose $S \subset C_{\mathbb{R}}(L)$ is uniformly bounded and equicontinuous and moreover satisfies the condition that, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|f(x)| < \epsilon$ for every $x \in L \setminus K_n$ and for every $f \in S$. Show that S is relatively compact.

Paper 1, Section II

21I Linear Analysis

(a) State the *closed graph theorem.*

(b) Prove the closed graph theorem assuming the inverse mapping theorem.

(c) Let X, Y, Z be Banach spaces and $T : X \to Y$, $S : Y \to Z$ be linear maps. Suppose that $S \circ T$ is bounded and S is both bounded and injective. Show that T is bounded.

Paper 4, Section II

21I Linear Analysis

Let H be a complex Hilbert space.

(a) Let $T : H \to H$ be a bounded linear map. Show that the spectrum of T is a subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|T\|_{\mathcal{B}(H)}\}$.

(b) Let $T : H \to H$ be a bounded self-adjoint linear map. For $\lambda, \mu \in \mathbb{C}$, let $E_\lambda := \{x \in H : Tx = \lambda x\}$ and $E_\mu := \{x \in H : Tx = \mu x\}$. If $\lambda \neq \mu$, show that $E_\lambda \perp E_\mu$.

(c) Let $T : H \to H$ be a compact self-adjoint linear map. For $\lambda \neq 0$, show that $E_\lambda := \{x \in H : Tx = \lambda x\}$ is finite-dimensional.

(d) Let $H_1 \subset H$ be a closed, proper, non-trivial subspace. Let P be the orthogonal projection to H_1.

(i) Prove that P is self-adjoint.

(ii) Determine the spectrum $\sigma(P)$ and the point spectrum $\sigma_p(P)$ of P.

(iii) Find a necessary and sufficient condition on H_1 for P to be compact.

Paper 3, Section II

18G Linear Analysis

State and prove the Baire Category Theorem. [Choose any version you like.]

An *isometry* from a metric space (M,d) to another metric space (N,e) is a function $\varphi\colon M \to N$ such that $e(\varphi(x),\varphi(y)) = d(x,y)$ for all $x,y \in M$. Prove that there exists no isometry from the Euclidean plane ℓ_2^2 to the Banach space c_0 of sequences converging to 0. [*Hint: Assume $\varphi\colon \ell_2^2 \to \mathrm{c}_0$ is an isometry. For $n \in \mathbb{N}$ and $x \in \ell_2^2$ let $\varphi_n(x)$ denote the n^{th} coordinate of $\varphi(x)$. Consider the sets F_n consisting of all pairs (x,y) with $\|x\|_2 = \|y\|_2 = 1$ and $\|\varphi(x) - \varphi(y)\|_\infty = |\varphi_n(x) - \varphi_n(y)|$.*]

Show that for each $n \in \mathbb{N}$ there is a linear isometry $\ell_1^n \to \mathrm{c}_0$.

Paper 4, Section II

19G Linear Analysis

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Define what is meant by an *adjoint* of T and prove that it exists, it is linear and bounded, and that it is unique. [You may use the Riesz Representation Theorem without proof.]

What does it mean to say that T is a *normal operator*? Give an example of a bounded linear map on ℓ_2 that is not normal.

Show that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Prove that if T is normal, then $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, that is, that every element of the spectrum of T is an approximate eigenvalue of T.

Paper 2, Section II
19G Linear Analysis

(a) Let $T\colon X \to Y$ be a linear map between normed spaces. What does it mean to say that T is *bounded*? Show that T is bounded if and only if T is continuous. Define the *operator norm* of T and show that the set $\mathcal{B}(X,Y)$ of all bounded, linear maps from X to Y is a normed space in the operator norm.

(b) For each of the following linear maps T, determine if T is bounded. When T is bounded, compute its operator norm and establish whether T is compact. Justify your answers. Here $\|f\|_\infty = \sup_{t\in[0,1]}|f(t)|$ for $f \in C[0,1]$ and $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0,1]$.

(i) $T\colon \big(C^1[0,1], \|\cdot\|_\infty\big) \to \big(C^1[0,1], \|\cdot\|\big)$, $T(f) = f$.

(ii) $T\colon \big(C^1[0,1], \|\cdot\|\big) \to \big(C[0,1], \|\cdot\|_\infty\big)$, $T(f) = f$.

(iii) $T\colon \big(C^1[0,1], \|\cdot\|\big) \to \big(C[0,1], \|\cdot\|_\infty\big)$, $T(f) = f'$.

(iv) $T\colon \big(C[0,1], \|\cdot\|_\infty\big) \to \mathbb{R}$, $T(f) = \displaystyle\int_0^1 f(t)h(t)\,dt$, where h is a given element of $C[0,1]$. [*Hint: Consider first the case that* $h(x) \neq 0$ *for every* $x \in [0,1]$, *and apply* T *to a suitable function. In the general case apply* T *to a suitable sequence of functions.*]

Paper 1, Section II
19G Linear Analysis

(a) Let $(e_n)_{n=1}^\infty$ be an orthonormal basis of an inner product space X. Show that for all $x \in X$ there is a unique sequence $(a_n)_{n=1}^\infty$ of scalars such that $x = \sum_{n=1}^\infty a_n e_n$.

Assume now that X is a Hilbert space and that $(f_n)_{n=1}^\infty$ is another orthonormal basis of X. Prove that there is a unique bounded linear map $U\colon X \to X$ such that $U(e_n) = f_n$ for all $n \in \mathbb{N}$. Prove that this map U is unitary.

(b) Let $1 \leqslant p < \infty$ with $p \neq 2$. Show that no subspace of ℓ_2 is isomorphic to ℓ_p. [*Hint: Apply the generalized parallelogram law to suitable vectors.*]

Paper 3, Section II

21G Linear Analysis

(i) State carefully the theorems of Stone–Weierstrass and Arzelá–Ascoli (work with real scalars only).

(ii) Let $\mathcal{F}$ denote the family of functions on $[0,1]$ of the form

$$f(x)=\sum_{n=1}^{\infty} a_n \sin(nx),$$

where the a_n are real and $|a_n| \leqslant 1/n^3$ for all $n \in \mathbb{N}$. Prove that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on $[0,1]$.

(iii) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0)=0$ and $f'(0)$ exists. Show that for each $\varepsilon>0$ there exists a real polynomial p having only odd powers, i.e. of the form

$$p(x)=a_1x+a_3x^3+\cdots+a_{2m-1}x^{2m-1}\ ,$$

such that $\sup_{x\in[0,1]}|f(x)-p(x)|<\varepsilon$. Show that the same holds without the assumption that f is differentiable at 0.

Paper 1, Section II

22G Linear Analysis

Let X and Y be normed spaces. What is an *isomorphism* between X and Y? Show that a bounded linear map $T\colon X \to Y$ is an isomorphism if and only if T is surjective and there is a constant $c>0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in X$. Show that if there is an isomorphism $T\colon X \to Y$ and X is complete, then Y is complete.

Show that two normed spaces of the same finite dimension are isomorphic. [You may assume without proof that any two norms on a finite-dimensional space are equivalent.] *Briefly* explain why this implies that every finite-dimensional space is complete, and every closed and bounded subset of a finite-dimensional space is compact.

Let Z and F be subspaces of a normed space X with $Z \cap F=\{0\}$. Assume that Z is closed in X and F is finite-dimensional. Prove that $Z+F$ is closed in X. [*Hint: First show that the function* $x \mapsto d(x,Z)=\inf\{\|x-z\| : z \in Z\}$ *restricted to the unit sphere of* F *achieves its minimum.*]

Paper 2, Section II

22G Linear Analysis

(a) Let X and Y be Banach spaces, and let $T\colon X \to Y$ be a surjective linear map. Assume that there is a constant $c > 0$ such that $\|Tx\| \geqslant c\|x\|$ for all $x \in X$. Show that T is continuous. [You may use any standard result from general Banach space theory provided you clearly state it.] Give an example to show that the assumption that X and Y are complete is necessary.

(b) Let C be a closed subset of a Banach space X such that

(i) $x_1 + x_2 \in C$ for each $x_1, x_2 \in C$;

(ii) $\lambda x \in C$ for each $x \in C$ and $\lambda > 0$;

(iii) for each $x \in X$, there exist $x_1, x_2 \in C$ such that $x = x_1 - x_2$.

Prove that, for some $M > 0$, the unit ball of X is contained in the closure of the set

$$\{x_1 - x_2 : x_i \in C,\ \ \|x_i\| \leqslant M\ (i = 1, 2)\}\ .$$

[You may use without proof any version of the Baire Category Theorem.] Deduce that, for some $K > 0$, every $x \in X$ can be written as $x = x_1 - x_2$ with $x_i \in C$ and $\|x_i\| \leqslant K\|x\|$ $(i = 1, 2)$.

Paper 4, Section II

22G Linear Analysis

Define the *spectrum* $\sigma(T)$ and the *approximate point spectrum* $\sigma_{\mathrm{ap}}(T)$ of a bounded linear operator T on a Banach space. Prove that $\sigma_{\mathrm{ap}}(T) \subset \sigma(T)$ and that $\sigma(T)$ is a closed and bounded subset of $\mathbb{C}$. [You may assume without proof that the set of invertible operators is open.]

Let T be a hermitian operator on a non-zero Hilbert space. Prove that $\sigma(T)$ is not empty.

Let K be a non-empty, compact subset of $\mathbb{C}$. Show that there is a bounded linear operator $T\colon \ell_2 \to \ell_2$ with $\sigma(T) = K$. [You may assume without proof that a compact metric space is separable.]

Paper 3, Section II

21F Linear Analysis

State the Stone–Weierstrass Theorem for real-valued functions.

State Riesz's Lemma.

Let K be a compact, Hausdorff space and let A be a subalgebra of $C(K)$ separating the points of K and containing the constant functions. Fix two disjoint, non-empty, closed subsets E and F of K.

(i) If $x \in E$ show that there exists $g \in A$ such that $g(x) = 0$, $0 \leqslant g < 1$ on K, and $g > 0$ on F. Explain *briefly* why there is $M \in \mathbb{N}$ such that $g \geqslant \frac{2}{M}$ on F.

(ii) Show that there is an open cover $U_1, U_2, \ldots, U_m$ of E, elements $g_1, g_2, \ldots, g_m$ of A and positive integers $M_1, M_2, \ldots, M_m$ such that

$$0 \leqslant g_r < 1 \ \text{ on } K, \quad g_r \geqslant \tfrac{2}{M_r} \ \text{ on } F, \quad g_r < \tfrac{1}{2M_r} \ \text{ on } U_r$$

for each $r = 1, 2, \ldots, m$.

(iii) Using the inequality

$$1 - Nt \leqslant (1-t)^N \leqslant \frac{1}{Nt} \qquad (0 < t < 1,\ N \in \mathbb{N}),$$

show that for sufficiently large positive integers $n_1, n_2, \ldots, n_m$, the element

$$h_r = 1 - (1 - g_r^{n_r})^{M_r^{n_r}}$$

of A satisfies

$$0 \leqslant h_r \leqslant 1 \ \text{ on } K, \quad h_r \leqslant \tfrac{1}{4} \ \text{ on } U_r, \quad h_r \geqslant \left(\tfrac{3}{4}\right)^{\frac{1}{m}} \ \text{ on } F$$

for each $r = 1, 2, \ldots, m$.

(iv) Show that the element $h = h_1 \cdot h_2 \cdot \cdots \cdot h_m - \frac{1}{2}$ of A satisfies

$$-\tfrac{1}{2} \leqslant h \leqslant \tfrac{1}{2} \ \text{ on } K, \quad h \leqslant -\tfrac{1}{4} \ \text{ on } E, \quad h \geqslant \tfrac{1}{4} \ \text{ on } F.$$

Now let $f \in C(K)$ with $\|f\| \leqslant 1$. By considering the sets $\{x \in K : f(x) \leqslant -\frac{1}{4}\}$ and $\{x \in K : f(x) \geqslant \frac{1}{4}\}$, show that there exists $h \in A$ such that $\|f - h\| \leqslant \frac{3}{4}$. Deduce that A is dense in $C(K)$.

Paper 4, Section II

22F Linear Analysis

Let $T \colon X \to X$ be a bounded linear operator on a complex Banach space X. Define the *spectrum* $\sigma(T)$ of T. What is an *approximate eigenvalue* of T? What does it mean to say that T is *compact*?

Assume now that T is compact. Show that if λ is in the boundary of $\sigma(T)$ and $\lambda \neq 0$, then λ is an eigenvalue of T. [You may use without proof the result that every λ in the boundary of $\sigma(T)$ is an approximate eigenvalue of T.]

Let $T \colon H \to H$ be a compact Hermitian operator on a complex Hilbert space H. Prove the following:

(a) If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then λ is an eigenvalue of T.

(b) $\sigma(T)$ is countable.

Paper 2, Section II

22F Linear Analysis

Let X be a Banach space. Let $T \colon X \to \ell_\infty$ be a bounded linear operator. Show that there is a bounded sequence $(f_n)_{n=1}^\infty$ in X^* such that $Tx = (f_n x)_{n=1}^\infty$ for all $x \in X$.

Fix $1 < p < \infty$. Define the Banach space ℓ_p and *briefly* explain why it is separable. Show that for $x \in \ell_p$ there exists $f \in \ell_p^*$ such that $\|f\| = 1$ and $f(x) = \|x\|_p$. [You may use Hölder's inequality without proof.]

Deduce that ℓ_p embeds isometrically into ℓ_∞.

Paper 1, Section II

22F Linear Analysis

State and prove the Closed Graph Theorem. [You may assume any version of the Baire Category Theorem provided it is clearly stated. If you use any other result from the course, then you must prove it.]

Let X be a closed subspace of ℓ_∞ such that X is also a subset of ℓ_1. Show that the left-shift $L \colon X \to \ell_1$, given by $L(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$, is bounded when X is equipped with the sup-norm.

Paper 3, Section II

21G Linear Analysis

State the closed graph theorem.

(i) Let X be a Banach space and Y a vector space. Suppose that Y is endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and that there is a constant $c > 0$ such that $\|y\|_2 \geqslant c\|y\|_1$ for all $y \in Y$. Suppose that Y is a Banach space with respect to both norms. Suppose that $T : X \to Y$ is a linear operator, and that it is bounded when Y is endowed with the $\|\cdot\|_1$ norm. Show that it is also bounded when Y is endowed with the $\|\cdot\|_2$ norm.

(ii) Suppose that X is a normed space and that $(x_n)_{n=1}^{\infty} \subseteq X$ is a sequence with $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for all f in the dual space X^*. Show that there is an M such that

$$\sum_{n=1}^{\infty} |f(x_n)| \leqslant M\|f\|$$

for all $f \in X^*$.

(iii) Suppose that X is the space of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the sup norm, and that $Y \subseteq X$ is the subspace of continuously differentiable functions with bounded derivative. Let $T : Y \to X$ be defined by $Tf = f'$. Show that the graph of T is closed, but that T is not bounded.

Paper 4, Section II
22G Linear Analysis

Let X be a Banach space and suppose that $T : X \to X$ is a bounded linear operator. What is an *eigenvalue* of T? What is the *spectrum* $\sigma(T)$ of T?

Let $X = C[0,1]$ be the space of continuous real-valued functions $f : [0,1] \to \mathbb{R}$ endowed with the sup norm. Define an operator $T : X \to X$ by

$$Tf(x) = \int_0^1 G(x,y) f(y)\, dy,$$

where

$$G(x,y) = \begin{cases} y(x-1) & \text{if } \; y \leqslant x, \\ x(y-1) & \text{if } \; x \leqslant y. \end{cases}$$

Prove the following facts about T:

(i) $Tf(0) = Tf(1) = 0$ and the second derivative $(Tf)''(x)$ is equal to $f(x)$ for $x \in (0,1)$;

(ii) T is compact;

(iii) T has infinitely many eigenvalues;

(iv) 0 is not an eigenvalue of T;

(v) $0 \in \sigma(T)$.

[*The Arzelà–Ascoli theorem may be assumed without proof.*]

Paper 2, Section II
22G Linear Analysis

What is meant by a *normal* topological space? State and prove Urysohn's lemma.

Let X be a normal topological space and let $S \subseteq X$ be closed. Show that there is a continuous function $f : X \to [0,1]$ with $f^{-1}(0) = S$ if, and only if, S is a countable intersection of open sets.

[*Hint. If* $S = \bigcap_{n=1}^{\infty} U_n$ *then consider* $\sum_{n=1}^{\infty} 2^{-n} f_n$*, where the functions* $f_n : X \to [0,1]$ *are supplied by an appropriate application of Urysohn's lemma.*]

Paper 1, Section II

22G Linear Analysis

What is meant by the *dual* X^* of a normed space X? Show that X^* is a Banach space.

Let $X = C^1(0,1)$, the space of functions $f : (0,1) \to \mathbb{R}$ possessing a bounded, continuous first derivative. Endow X with the sup norm $\|f\|_\infty = \sup_{x \in (0,1)} |f(x)|$. Which of the following maps $T : X \to \mathbb{R}$ are elements of X^*? Give brief justifications or counterexamples as appropriate.

1. $Tf = f(\frac{1}{2})$;
2. $Tf = \|f\|_\infty$;
3. $Tf = \int_0^1 f(x)\,dx$;
4. $Tf = f'(\frac{1}{2})$.

Now suppose that X is a (real) Hilbert space. State and prove the Riesz representation theorem. Describe the natural map $X \to X^{**}$ and show that it is surjective.

[All normed spaces are over $\mathbb{R}$. You may assume that if Y is a closed subspace of a Hilbert space X then $X = Y \oplus Y^\perp$.]

Paper 1, Section II

22G Linear Analysis

State a version of the Stone–Weierstrass Theorem for real-valued functions on a compact metric space.

Suppose that $K : [0,1]^2 \to \mathbb{R}$ is a continuous function. Show that $K(x,y)$ may be uniformly approximated by functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$ with $f_i, g_i : [0,1] \to \mathbb{R}$ continuous.

Let X, Y be Banach spaces and suppose that $T : X \to Y$ is a bounded linear operator. What does it mean to say that T is finite-rank? What does it mean to say that T is compact? Give an example of a bounded linear operator from $C[0,1]$ to itself which is not compact.

Suppose that $(T_n)_{n=1}^{\infty}$ is a sequence of finite-rank operators and that $T_n \to T$ in the operator norm. Briefly explain why the T_n are compact. Show that T is compact.

Hence, show that the integral operator $T : C[0,1] \to C[0,1]$ defined by

$$Tf(x) = \int_0^1 f(y)K(x,y)\,dy$$

is compact.

Paper 2, Section II

22G Linear Analysis

State and prove the Baire Category Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $x \in \mathbb{R}$, define

$$\omega_f(x) = \inf_{\delta > 0} \sup_{\substack{|y-x| \leqslant \delta \\ |y'-x| \leqslant \delta}} |f(y) - f(y')| \,.$$

Show that f is continuous at x if and only if $\omega_f(x) = 0$.

Show that for any $\epsilon > 0$ the set $\{x \in \mathbb{R} : \omega_f(x) < \epsilon\}$ is open.

Hence show that the set of points at which f is continuous cannot be precisely the set $\mathbb{Q}$ of rationals.

Paper 3, Section II
21G Linear Analysis

Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n=-\infty}^{\infty}$. Let $T : H \to H$ be a bounded linear operator. What is meant by the spectrum $\sigma(T)$ of T?

Define T by setting $T(e_n) = e_{n-1} + e_{n+1}$ for $n \in \mathbb{Z}$. Show that T has a unique extension to a bounded, self-adjoint linear operator on H. Determine the norm $\|T\|$. Exhibit, with proof, an element of $\sigma(T)$.

Show that T has no eigenvectors. Is T compact?

[General results from spectral theory may be used without proof. You may also use the fact that if a sequence (x_n) satisfies a linear recurrence $\lambda x_n = x_{n-1} + x_{n+1}$ with $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 2$, $\lambda \neq 0$, then it has the form $x_n = A\alpha^n \sin(\theta_1 n + \theta_2)$ or $x_n = (A + nB)\alpha^n$, where $A, B, \alpha \in \mathbb{R}$ and $0 \leqslant \theta_1 < \pi$, $|\theta_2| \leqslant \pi/2$.]

Paper 4, Section II
22G Linear Analysis

State Urysohn's Lemma. State and prove the Tietze Extension Theorem.

Let X, Y be two topological spaces. We say that the extension property holds if, whenever $S \subseteq X$ is a closed subset and $f : S \to Y$ is a continuous map, there is a continuous function $\tilde{f} : X \to Y$ with $\tilde{f}|_S = f$.

For each of the following three statements, say whether it is true or false. Briefly justify your answers.

1. If X is a metric space and $Y = [-1, 1]$ then the extension property holds.
2. If X is a compact Hausdorff space and $Y = \mathbb{R}$ then the extension property holds.
3. If X is an arbitrary topological space and $Y = [-1, 1]$ then the extension property holds.

Paper 1, Section II
22H Linear Analysis

a) State and prove the Banach–Steinhaus Theorem.

[You may use the Baire Category Theorem without proving it.]

b) Let X be a (complex) normed space and $S \subset X$. Prove that if $\{f(x) : x \in S\}$ is a bounded set in $\mathbb{C}$ for every linear functional $f \in X^*$ then there exists $K \geqslant 0$ such that $\|x\| \leqslant K$ for all $x \in S$.

[You may use here the following consequence of the Hahn–Banach Theorem without proving it: for a given $x \in X$, there exists $f \in X^*$ with $\|f\| = 1$ and $|f(x)| = \|x\|$.]

c) Conclude that if two norms $\|.\|_1$ and $\|.\|_2$ on a (complex) vector space V are not equivalent, there exists a linear functional $f : V \to \mathbb{C}$ which is continuous with respect to one of the two norms, and discontinuous with respect to the other.

Paper 2, Section II
22H Linear Analysis

For a sequence $x = (x_1, x_2, \dots)$ with $x_j \in \mathbb{C}$ for all $j \geqslant 1$, let

$$\|x\|_\infty := \sup_{j \geqslant 1} |x_j|$$

and $\ell^\infty = \{x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \geqslant 1 \text{ and } \|x\|_\infty < \infty\}$.

a) Prove that ℓ^∞ is a Banach space.

b) Define

$$c_0 = \{x = (x_1, x_2, \dots) \in \ell^\infty : \lim_{j \to \infty} x_j = 0\}$$

and

$$\ell^1 = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \in \mathbb{N} \text{ and } \|x\|_1 = \sum_{\ell=1}^{\infty} |x_\ell| < \infty \right\}.$$

Show that c_0 is a closed subspace of ℓ^∞. Show that $c_0^* \simeq \ell^1$.

[*Hint: find an isometric isomorphism from ℓ^1 to c_0^*.*]

c) Let

$$c_{00} = \{x = (x_1, x_2, \dots) \in \ell^\infty : x_j = 0 \text{ for all } j \text{ large enough}\}.$$

Is c_{00} a closed subspace of ℓ^∞? If not, what is the closure of c_{00}?

Paper 3, Section II

21H Linear Analysis

State and prove the Stone-Weierstrass theorem for real-valued functions.

[You may use without proof the fact that the function $s \to |s|$ can be uniformly approximated by polynomials on $[-1,1]$.]

Paper 4, Section II

22H Linear Analysis

Let X be a Banach space.

a) What does it mean for a bounded linear map $T : X \to X$ to be compact?

b) Let $\mathcal{B}(X)$ be the Banach space of all bounded linear maps $S : X \to X$. Let $\mathcal{B}_0(X)$ be the subset of $\mathcal{B}(X)$ consisting of all compact operators. Show that $\mathcal{B}_0(X)$ is a closed subspace of $\mathcal{B}(X)$. Show that, if $S \in \mathcal{B}(X)$ and $T \in \mathcal{B}_0(X)$, then $ST, TS \in \mathcal{B}_0(X)$.

c) Let

$$X = \ell^2 = \left\{ x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \quad \text{and} \quad \|x\|_2^2 = \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\},$$

and $T : X \to X$ be defined by

$$(Tx)_k = \frac{x_{k+1}}{k+1}.$$

Is T compact? What is the spectrum of T? Explain your answers.

Paper 1, Section II

22H Linear Analysis

(a) State and prove the Baire category theorem.

(b) Let X be a normed space. Show that every proper linear subspace $V \subset X$ has empty interior.

(c) Let $\mathcal{P}$ be the vector space of all real polynomials in one variable. Using the Baire category theorem and the result from (b), prove that for any norm $\|\cdot\|$ on $\mathcal{P}$, the normed space $(\mathcal{P}, \|\cdot\|)$ is not a Banach space.

Paper 2, Section II

22H Linear Analysis

For $1 \leqslant p < \infty$ and a sequence $x = (x_1, x_2, \dots)$, where $x_j \in \mathbb{C}$ for all $j \geqslant 1$, let $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$.
Let $\ell^p = \{x = (x_1, x_2, \dots) : x_j \in \mathbb{C} \text{ for all } j \geqslant 1 \text{ and } \|x\|_p < \infty\}$.

(a) Let $p, q > 1$ with $1/p + 1/q = 1$, $x = (x_1, x_2, \dots) \in \ell^p$ and $y = (y_1, y_2, \dots) \in \ell^q$. Prove Hölder's inequality:

$$\sum_{j=1}^{\infty} |x_j||y_j| \leqslant \|x\|_p \|y\|_q \,.$$

(b) Use Hölder's inequality to prove the triangle inequality (known, in this case, as the Minkowski inequality):

$$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p \qquad \text{for every } x, y \in \ell^p \quad \text{and every } 1 < p < \infty.$$

(c) Let $2 \leqslant p < \infty$ and let K be a closed, convex subset of ℓ^p. Let $x \in \ell^p$ with $x \notin K$. Prove that there exists $y \in K$ such that

$$\|x - y\| = \inf_{z \in K} \|x - z\|.$$

[You may use without proof the fact that for every $2 \leqslant p < \infty$ and for every $x, y \in \ell^p$,

$$\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2^{p-1}\left(\|x\|_p^p + \|y\|_p^p\right) .]$$

Paper 3, Section II
21H Linear Analysis

(a) State the Arzela–Ascoli theorem, explaining the meaning of all concepts involved.

(b) Prove the Arzela–Ascoli theorem.

(c) Let K be a compact topological space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in the Banach space $C(K)$ of real-valued continuous functions over K equipped with the supremum norm $\|\cdot\|$. Assume that for every $x \in K$, the sequence $f_n(x)$ is monotone increasing and that $f_n(x) \to f(x)$ for some $f \in C(K)$. Show that $\|f_n - f\| \to 0$ as $n \to \infty$.

Paper 4, Section II
22H Linear Analysis

Let X be a Banach space and let $T : X \to X$ be a bounded linear map.

(a) Define the *spectrum* $\sigma(T)$, the *resolvent set* $\rho(T)$ and the *point spectrum* $\sigma_p(T)$ of T.

(b) What does it mean for T to be a *compact operator*?

(c) Show that if T is a compact operator on X and $a > 0$, then T has at most finitely many linearly independent eigenvectors with eigenvalues having modulus larger than a.

[You may use without proof the fact that for any finite dimensional proper subspace Y of a Banach space Z, there exists $x \in Z$ with $\|x\| = 1$ and $\operatorname{dist}(x, Y) = \inf_{y\in Y} \|x - y\| = 1$.]

(d) For a sequence $(\lambda_n)_{n\geqslant 1}$ of complex numbers, let $T : \ell^2 \to \ell^2$ be defined by

$$T(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots).$$

Give necessary and sufficient conditions on the sequence $(\lambda_n)_{n\geqslant 1}$ for T to be compact, and prove your assertion.

1/II/22F Linear Analysis

Suppose p and q are real numbers with $p^{-1}+q^{-1}=1$ and $p,\, q>1$. Show, quoting any results on convexity that you need, that

$$a^{1/p}\,b^{1/q} \leqslant \frac{a}{p}+\frac{b}{q}$$

for all real positive a and b.

Define the space l^p and show that it is a complete normed vector space.

2/II/22F Linear Analysis

State and prove the principle of uniform boundedness.

[You may assume the Baire category theorem.]

Suppose that X, Y and Z are Banach spaces. Suppose that

$$F : X \times Y \to Z$$

is linear and continuous in each variable separately, that is to say that, if y is fixed,

$$F(\cdot, y) : X \to Z$$

is a continuous linear map and, if x is fixed,

$$F(x, \cdot) : Y \to Z$$

is a continuous linear map. Show that there exists an M such that

$$\|F(x,y)\|_Z \leqslant M\|x\|_X\|y\|_Y$$

for all $x \in X$, $y \in Y$. Deduce that F is continuous.

Suppose X, Y, Z and W are Banach spaces. Suppose that

$$G : X \times Y \times W \to Z$$

is linear and continuous in each variable separately. Does it follow that G is continuous? Give reasons.

Suppose that X, Y and Z are Banach spaces. Suppose that

$$H : X \times Y \to Z$$

is continuous in each variable separately. Does it follow that H is continuous? Give reasons.

3/II/21F Linear Analysis

State and prove the Stone–Weierstrass theorem for real-valued functions. You may assume that the function $x \mapsto |x|$ can be uniformly approximated by polynomials on any interval $[-k, k]$.

Suppose that $0 < a < b < 1$. Let $\mathcal{F}$ be the set of functions which can be uniformly approximated on $[a, b]$ by polynomials with *integer* coefficients. By making appropriate use of the identity

$$\frac{1}{2} = \frac{x}{1-(1-2x)} = \sum_{n=0}^{\infty} x(1-2x)^n,$$

or otherwise, show that $\mathcal{F} = \mathcal{C}([a, b])$.

Is it true that every continuous function on $[0, b]$ can be uniformly approximated by polynomials with *integer* coefficients?

4/II/22F Linear Analysis

Let H be a Hilbert space. Show that if V is a closed subspace of H then any $f \in H$ can be written as $f = v + w$ with $v \in V$ and $w \perp V$.

Suppose $U : H \to H$ is unitary (that is to say $UU^* = U^*U = I$). Let

$$A_n f = \frac{1}{n} \sum_{k=0}^{n-1} U^k f$$

and consider

$$X = \{g - Ug \,:\, g \in H\}.$$

(i) Show that U is an isometry and $\|A_n\| \leqslant 1$.

(ii) Show that X is a subspace of H and $A_n f \to 0$ as $n \to \infty$ whenever $f \in X$.

(iii) Let V be the closure of X. Show that $A_n v \to 0$ as $n \to \infty$ whenever $v \in V$.

(iv) Show that, if $w \perp X$, then $Uw = w$. Deduce that, if $w \perp V$, then $Uw = w$.

(v) If $f \in H$ show that there is a $w \in H$ such that $A_n f \to w$ as $n \to \infty$.

1/II/22G Linear Analysis

Let X be a normed vector space over $\mathbb{R}$. Define the dual space X^* and show directly that X^* is a Banach space. Show that the map $\phi : X \to X^{**}$ defined by $\phi(x)v = v(x)$, for all $x \in X$, $v \in X^*$, is a linear map. Using the Hahn–Banach theorem, show that ϕ is injective and $|\phi(x)| = |x|$.

Give an example of a Banach space X for which ϕ is not surjective. Justify your answer.

2/II/22G Linear Analysis

Let X be a Banach space, Y a normed vector space, and $T : X \to Y$ a bounded linear map. Assume that $T(X)$ is of second category in Y. Show that T is surjective and $T(\mathcal{U})$ is open whenever $\mathcal{U}$ is open. Show that, if T is also injective, then T^{-1} exists and is bounded.

Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R})$ is of second category in $\mathbb{R}$ but f is not surjective. Give an example of a continuous surjective map $f : \mathbb{R} \to \mathbb{R}$ which does not take open sets to open sets.

3/II/21G Linear Analysis

State and prove the Arzela–Ascoli theorem.

Let N be a positive integer. Consider the subset $\mathcal{S}_N \subset C([0,1])$ consisting of all thrice differentiable solutions to the differential equation

$$f'' = f + (f')^2 \quad \text{with} \quad |f(0)| \leqslant N\,, \quad |f(1)| \leqslant N\,, \quad |f'(0)| \leqslant N\,, \quad |f'(1)| \leqslant N\,.$$

Show that this set is totally bounded as a subset of $C([0,1])$.

[*It may be helpful to consider interior maxima.*]

4/II/22G Linear Analysis

Let X be a Banach space and $T : X \to X$ a bounded linear map. Define the *spectrum* $\sigma(T)$, *point spectrum* $\sigma_p(T)$, *resolvent* $R_T(\lambda)$, and *resolvent set* $\rho(T)$. Show that the spectrum is a closed and bounded subset of $\mathbb{C}$. Is the point spectrum always closed? Justify your answer.

Now suppose H is a Hilbert space, and $T : H \to H$ is self-adjoint. Show that the point spectrum $\sigma_p(T)$ is real.

1/II/22G Linear Analysis

Let U be a vector space. Define what it means for two norms $||\cdot||_1$ and $||\cdot||_2$ on U to be *Lipschitz equivalent*. Give an example of a vector space and two norms which are *not* Lipschitz equivalent.

Show that, if U is finite dimensional, all norms on U are Lipschitz equivalent. Deduce that a finite dimensional subspace of a normed vector space is closed.

Show that a normed vector space W is finite dimensional if and only if W contains a non-empty open set with compact closure.

2/II/22G Linear Analysis

Let X be a metric space. Define what it means for a subset $E \subset X$ to be of *first* or *second category.* State and prove a version of the Baire category theorem. For $1 \leqslant p \leqslant \infty$, show that the set ℓ_p is of first category in the normed space ℓ_r when $r > p$ and ℓ_r is given its standard norm. What about $r = p$?

3/II/21G Linear Analysis

Let X be a complex Banach space. We say a sequence $x^i \in X$ converges to $x \in X$ *weakly* if $\phi(x^i) \to \phi(x)$ for all $\phi \in X^*$. Let $T : X \to Y$ be bounded and linear. Show that if x^i converges to x weakly, then Tx^i converges to Tx weakly.

Now let $X = \ell_2$. Show that for a sequence $x^i \in X$, $i = 1, 2, \ldots$, with $||x^i|| \leqslant 1$, there exists a subsequence x^{i_k} such that x^{i_k} converges weakly to some $x \in X$ with $||x|| \leqslant 1$.

Now let $Y = \ell_1$, and show that $y^i \in Y$ converges to $y \in Y$ weakly if and only if $y^i \to y$ in the usual sense.

Define what it means for a linear operator $T : X \to Y$ to be *compact*, and deduce from the above that any bounded linear $T : \ell_2 \to \ell_1$ is compact.

4/II/22G Linear Analysis

Let H be a complex Hilbert space. Define what it means for a linear operator $T : H \to H$ to be *self-adjoint.* State a version of the spectral theorem for compact self-adjoint operators on a Hilbert space. Give an example of a Hilbert space H and a compact self-adjoint operator on H with infinite dimensional range. Define the notions *spectrum, point spectrum*, and *resolvent set*, and describe these in the case of the operator you wrote down. Justify your answers.

1/II/22F Linear Analysis

Let K be a compact Hausdorff space, and let $C(K)$ denote the Banach space of continuous, complex-valued functions on K, with the supremum norm. Define what it means for a set $S \subset C(K)$ to be *totally bounded*, *uniformly bounded*, and *equicontinuous*.

Show that S is totally bounded if and only if it is both uniformly bounded and equicontinuous.

Give, with justification, an example of a Banach space X and a subset $S \subset X$ such that S is bounded but not totally bounded.

2/II/22F Linear Analysis

Let X and Y be Banach spaces. Define what it means for a linear operator $T : X \to Y$ to be *compact*. For a linear operator $T : X \to X$, define the *spectrum*, *point spectrum*, and *resolvent set* of T.

Now let H be a complex Hilbert space. Define what it means for a linear operator $T : H \to H$ to be *self-adjoint*. Suppose $e_1, e_2, \ldots$ is an orthonormal basis for H. Define a linear operator $T : H \to H$ by setting $Te_i = \frac{1}{i}e_i$. Is T compact? Is T self-adjoint? Justify your answers. Describe, with proof, the spectrum, point spectrum, and resolvent set of T.

3/II/21F Linear Analysis

Let X be a normed vector space. Define the dual X^* of X. Define the normed vector spaces $l^s = l^s(\mathbb{C})$ for all $1 \leqslant s \leqslant \infty$. [*You are* **not** *required to prove that the norms you have given are indeed norms.*]

Now let $1 < p, q < \infty$ be such that $p^{-1} + q^{-1} = 1$. Show that $(l^q)^*$ is isometrically isomorphic to l^p as a normed vector space. [*You may assume any standard inequalities.*]

Show by a similar argument that $(l^1)^*$ is isomorphic to l^∞. Does your argument also show that $(l^\infty)^*$ is isomorphic to l^1? If not, where does it fail?

4/II/22F Linear Analysis

Let X and Y be normed vector spaces. Show that a linear map $T : X \to Y$ is continuous if and only if it is bounded.

Now let X, Y, Z be Banach spaces. We say that a map $F : X \times Y \to Z$ is *bilinear* if

$$F(\alpha x + \beta y, z) = \alpha F(x, z) + \beta F(y, z), \text{ for all scalars } \alpha, \beta \text{ and } x, y \in X, z \in Y$$

$$F(x, \alpha y + \beta z) = \alpha F(x, y) + \beta F(x, z), \text{ for all scalars } \alpha, \beta \text{ and } x \in X, y, z \in Y\,.$$

Suppose that F is bilinear and is continuous in each variable separately. Show that there exists a constant $M \geqslant 0$ such that

$$||F(x, y)|| \leqslant M||x||\ ||y||$$

for all $x \in X$, $y \in Y$.

[*Hint: For each fixed $x \in X$, consider the map $y \mapsto F(x, y)$ from Y to Z.*]