Part II

Integrable Systems

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Paper 1, Section II

33E Integrable Systems

Let $q = q(x,t)$ and $r = r(x,t)$ be complex valued functions and consider the matrices (U, V) defined by

$$U(\lambda) = \begin{pmatrix} i\lambda & iq \\ ir & -i\lambda \end{pmatrix}, \quad V(\lambda) = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix}.$$

Derive the zero curvature equation as the consistency condition for the system of equations

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi$$

and show that it holds precisely when q, r satisfy a system of the form

$$ir_t + r_{xx} + aqr^2 = 0, \tag{1}$$
$$iq_t - q_{xx} - arq^2 = 0, \tag{2}$$

where a is a real number which you should determine. Show that if $r = \overline{q}$ this system reduces to the *nonlinear Schrödinger equation*

$$ir_t + r_{xx} + a|r|^2 r = 0, \tag{NLS1}$$

and find a similar reduction to the equation

$$ir_t + r_{xx} - a|r|^2 r = 0. \tag{NLS2}$$

Write these equations in Hamiltonian form. Search for solutions to (NLS1) and (NLS2) of the form $e^{-iEt} f(x)$ with real constant E and smooth, rapidly decreasing real-valued f. In each case either find such a solution explicitly, or explain briefly why it is not expected to exist.

[*Hint: you may use without derivation the indefinite integral*

$$\int \frac{dy}{\sqrt{\lambda^2 y^2 - y^4}} = -\frac{1}{\lambda} \operatorname{sech}^{-1} \frac{y}{\lambda}\,.]$$

Paper 2, Section II
34E Integrable Systems

Assume $\phi = \phi(x,t)$ is a solution of

$$-\phi_{xx} + u(x,t)\phi = \lambda(t)\phi, \qquad -\infty < x < \infty\,, \tag{S}$$

where $u = u(x,t)$ is smooth. Define $Q = Q(x,t)$ by $Q = \phi_t + u_x\phi - 2(u+2\lambda)\phi_x$ and show that there exists a number α, which you should find, such that

$$\partial_x(\phi_x Q - \phi Q_x) = \phi^2\Big(\dot\lambda + \alpha(u_t + u_{xxx} - 6uu_x)\Big) \tag{$*$}$$

where $\dot\lambda = \frac{d\lambda}{dt}$.

Now let $u = u(x,t)$ be a smooth solution of the KdV equation $u_t + u_{xxx} - 6uu_x = 0$, which is rapidly decreasing in x, and consider the case when $\phi = \varphi_n$ is the discrete eigenfunction of (S) corresponding to eigenvalue $\lambda_n = -\kappa_n^2 < 0$. Deduce from $(*)$ that $\lambda_n(t) = \lambda_n(0)$. [You may assume that $\kappa_n > 0$ and φ_n is normalized, i.e., $\int_{-\infty}^{\infty}\varphi_n(x,t)^2dx = 1$ for all times t.]

Deduce further that in this case $Q(x,t) = h_n(t)\varphi_n(x,t)$ for some function $h_n = h_n(t)$ and, by multiplying by φ_n, making use of (S) and integrating, show that $h_n(t) = 0$ and $Q = 0$. Finally, derive from this the time evolution of the discrete normalization $c_n(t)$ which is defined by the asymptotic relation

$$\varphi_n(x,t) \approx c_n(t)e^{-\kappa_n x} \quad \text{as} \quad x \to +\infty\,.$$

[You may assume the differentiated version of this relation also holds.]

Paper 3, Section II
32E Integrable Systems

(a) Compute the group of transformations generated by the vector field

$$V = t\partial_t + x\partial_x\,,$$

and hence, or otherwise, calculate the second prolongation of the vector field V and show that V generates a group of Lie symmetries of the wave equation $u_{tt} - u_{xx} = 0$.

Use the group of symmetries you have just found for the equation $u_{tt} - u_{xx} = 0$ to obtain a group invariant solution for this equation.

(b) Compute the group of transformations generated by the vector field

$$4t^2\partial_t + 4tx\partial_x - (x^2+2t)\partial_u$$

and verify that they give rise to a group of Lie symmetries of the equation $u_t = u_{xx} + u_x^2$.

Paper 1, Section II
33E Integrable Systems

(a) Show that if L is a symmetric $n \times n$ matrix ($L = L^T$) and B is a skew-symmetric $n \times n$ matrix ($B = -B^T$) then $[B, L] = BL - LB$ is symmetric. If L evolves in time according to

$$\frac{dL}{dt} = [B, L]\,,$$

show that the eigenvalues of L are constant in time.

Write the harmonic oscillator equation $\ddot{q} + \omega^2 q = 0$ in Hamiltonian form. (The frequency ω is a fixed real number). Starting with the symmetric matrix

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}$$

find a Lax pair formulation for the harmonic oscillator and use this formulation to obtain the conservation of energy for the oscillator.

(b) Consider the Airy partial differential equation, given for $-\infty < x < \infty$ and $t \geqslant 0$ by

$$q_t + q_{xxx} = 0. \tag{1}$$

Show that this is a compatibility condition for the pair of linear equations

$$\psi_x - ik\psi = q \tag{2}$$
$$\psi_t - ik^3\psi = -q_{xx} - ikq_x + k^2 q \tag{3}$$

for a function $\psi = \psi(x, t, k) \in \mathbb{C}$. Show that for each t, equation (2) has a solution ψ_+ which is defined for $\operatorname{Im} k \geqslant 0$, analytic in k for $\operatorname{Im} k > 0$, and satisfies

$$\lim_{x \to +\infty} e^{-ikx}\psi_+(x, t, k) = \hat{q}(k, t) = \int_{-\infty}^{+\infty} e^{-ikx} q(x, t)dx\,.$$

Deduce from this and equation (3) that $\hat{q}(k, t)$ evolves in time according to

$$\hat{q}_t - ik^3\hat{q} = 0$$

and hence obtain a representation for the solution of the Airy equation (1).

[*You may assume that q is a smooth function whose derivatives are rapidly decreasing in x.*]

Paper 2, Section II
34E Integrable Systems

It is possible to obtain solutions of the partial differential equation

$$u_{XT} = \sin u\,, \tag{1}$$

at time T from certain discrete scattering data $\{\lambda_m(T), c_m(T)\}_{m=1}^N$ and corresponding eigenfunctions $\psi_m(X,T)$ for an associated linear problem by means of the formula

$$u_X(T,X) = -4\sum_m c_m \psi_m^{(1)}(X,T)e^{i\lambda_m X},$$

where $\psi_m = \begin{pmatrix}\psi_m^{(1)}\\ \psi_m^{(2)}\end{pmatrix}$ and $\tilde{\psi}_m = \begin{pmatrix}-\overline{\psi_m^{(2)}}\\ \overline{\psi_m^{(1)}}\end{pmatrix}$ solve

$$\tilde{\psi}_n(X,T)e^{i\overline{\lambda_n(T)}X} - \begin{pmatrix}0\\1\end{pmatrix} = \sum_m \frac{c_m(T)\psi_m(X,T)}{(\overline{\lambda_n(T)} - \lambda_m(T))}e^{i\lambda_m(T)X}\,.$$

Given the fact that the discrete scattering data $\{\lambda_m(T), c_m(T)\}_{m=1}^N$ evolve according to $\lambda_m(T) = \lambda_m(0) = \lambda_m$ and $c_m(T) = c_m(0)e^{-\frac{iT}{2\lambda_n}}$, obtain the solution in the case $N=1$ with $\lambda_1(T) = il$ purely imaginary and $c_1(0) = c = 2l > 0$. Show that there is a unique *positive* value of l for which the solution is of the form $F(X+T)$ for some function F, which you should give.

Show that

$$g^s : \begin{pmatrix}X\\T\\u\end{pmatrix} \mapsto \begin{pmatrix}e^sX\\e^{-s}T\\u\end{pmatrix} \tag{2}$$

defines a group of Lie point symmetries of (1). Show that all the solutions to (1) you obtained for $N=1$ transform under (2) into $F(X+T)$, with F as above.

In the case $N=2$ and $\lambda_1 = il+m$, $\lambda_2 = il-m$ with real $l>0, m>0$ there is a solution of (1) given by

$$u(T,X) = 4\arctan\frac{l\sin\left(2mX - \frac{2mT}{4(l^2+m^2)}\right)}{m\cosh\left(\frac{2lT}{4(l^2+m^2)} + 2lX\right)}\,. \tag{3}$$

Show that if $l^2+m^2 = \frac{1}{4}$ then this solution is periodic in $t = T-X$ for fixed $x = X+T$; find the period.

Show that for arbitrary l^2+m^2 the solutions (3) may be transformed by (2) into the case $l^2+m^2 = \frac{1}{4}$.

Paper 3, Section II
32E Integrable Systems

Explain what it means for a vector field $V = V_1(x,u)\partial_x + \phi(x,u)\partial_u$ to generate a *Lie symmetry* for a differential equation $\Delta(x, u, \partial_x u, \ldots, \partial_x^n u) = 0$. State a condition for this to hold in terms of the n^{th} *prolongation* of V, $\mathrm{pr}^{(n)}V$, giving also a definition of this latter concept.

Calculate the second prolongation of the vector field V, and hence show that if V generates an infinitesimal Lie symmetry for the equation

$$u'' = \frac{(u')^2}{u} - u^2 \tag{1}$$

then V_1 must be of the form

$$V_1(x,u) = F(x)\ln|u| + G(x)$$

for some functions F, G.

Show that if c and d are arbitrary real numbers then

$$V = (cx+d)\partial_x - 2cu\partial_u$$

is an infinitesimal Lie symmetry for equation (1), and give the form of the group of symmetries that it generates.

[*Assume $u > 0$ throughout.*]

Paper 1, Section II
33D Integrable Systems

(a) Let $U(z,\bar{z},\lambda)$ and $V(z,\bar{z},\lambda)$ be matrix-valued functions, whilst $\psi(z,\bar{z},\lambda)$ is a vector-valued function. Show that the linear system

$$\partial_z \psi = U\psi\,, \qquad \partial_{\bar{z}} \psi = V\psi$$

is over-determined and derive a consistency condition on U, V that is necessary for there to be non-trivial solutions.

(b) Suppose that

$$U = \frac{1}{2\lambda}\begin{pmatrix} \lambda\,\partial_z u & e^{-u} \\ e^{u} & -\lambda\,\partial_z u \end{pmatrix} \qquad \text{and} \qquad V = \frac{1}{2}\begin{pmatrix} -\partial_{\bar{z}} u & \lambda e^{u} \\ \lambda e^{-u} & \partial_{\bar{z}} u \end{pmatrix}\,,$$

where $u(z,\bar{z})$ is a scalar function. Obtain a partial differential equation for u that is equivalent to your consistency condition from part (a).

(c) Now let $z = x + iy$ and suppose u is independent of y. Show that the trace of $(U-V)^n$ is constant for all positive integers n. Hence, or otherwise, construct a non-trivial first integral of the equation

$$\frac{d^2\phi}{dx^2} = 4\sinh\phi\,, \qquad \text{where} \quad \phi = \phi(x)\,.$$

Paper 2, Section II
34D Integrable Systems

(a) Explain briefly how the linear operators $L = -\partial_x^2 + u(x,t)$ and $A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u$ can be used to give a Lax-pair formulation of the KdV equation $u_t + u_{xxx} - 6uu_x = 0$.

(b) Give a brief definition of the *scattering data*

$$\mathcal{S}_{u(t)} = \big\{\{R(k,t)\}_{k\in\mathbb{R}}\,,\ \{-\kappa_n(t)^2, c_n(t)\}_{n=1}^N\big\}$$

attached to a smooth solution $u = u(x,t)$ of the KdV equation at time t. [You may assume $u(x,t)$ to be rapidly decreasing in x.] State the time dependence of $\kappa_n(t)$ and $c_n(t)$, and derive the time dependence of $R(k,t)$ from the Lax-pair formulation.

(c) Show that

$$F(x,t) = \sum_{n=1}^N c_n(t)^2\, e^{-\kappa_n(t)x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\, e^{ikx}\, dk$$

satisfies $\partial_t F + 8\partial_x^3 F = 0$. Now let $K(x,y,t)$ be the solution of the equation

$$K(x,y,t) + F(x+y,t) + \int_x^\infty K(x,z,t)F(z+y,t)\,dz = 0$$

and let $u(x,t) = -2\partial_x\phi(x,t)$, where $\phi(x,t) = K(x,x,t)$. Defining $G(x,y,t)$ by $G = \big(\partial_x^2 - \partial_y^2 - u(x,t)\big)K(x,y,t)$, show that

$$G(x,y,t) + \int_x^\infty G(x,z,t)F(z+y,t)\,dz = 0\,.$$

(d) Given that $K(x,y,t)$ obeys the equations

$$(\partial_x^2 - \partial_y^2)K - uK = 0\,,$$
$$(\partial_t + 4\partial_x^3 + 4\partial_y^3)K - 3(\partial_x u)K - 6u\,\partial_x K = 0\,,$$

where $u = u(x,t)$, deduce that

$$\partial_t K + (\partial_x + \partial_y)^3 K - 3u\,(\partial_x + \partial_y)K = 0\,,$$

and hence that u solves the KdV equation.

Paper 3, Section II
32D Integrable Systems

(a) Consider the group of transformations of $\mathbb{R}^2$ given by $g_1^s : (t,x) \mapsto (\tilde{t},\tilde{x}) = (t, x+st)$, where $s \in \mathbb{R}$. Show that this acts as a group of Lie symmetries for the equation $d^2x/dt^2 = 0$.

(b) Let $(\psi_1,\psi_2) \in \mathbb{R}^2$ and define $\psi = \psi_1 + i\psi_2$. Show that the vector field $\psi_1\partial_{\psi_2} - \psi_2\partial_{\psi_1}$ generates the group of phase rotations $g_2^s : \psi \to e^{is}\psi\,$.

(c) Show that the transformations of $\mathbb{R}^2 \times \mathbb{C}$ defined by

$$g^s : (t,x,\psi) \mapsto (\tilde{t},\tilde{x},\tilde{\psi}) = (t, x+st, \psi\, e^{isx+is^2t/2})$$

form a one-parameter group generated by the vector field

$$V = t\partial_x + x(\psi_1\partial_{\psi_2} - \psi_2\partial_{\psi_1}) = t\partial_x + ix(\psi\partial_\psi - \psi^*\partial_{\psi^*})\,,$$

and find the second prolongation $\mathrm{Pr}^{(2)}g^s$ of the action of $\{g^s\}$. Hence find the coefficients η^0 and η^{11} in the second prolongation of V,

$$\mathrm{pr}^{(2)}V = t\partial_x + \Big(ix\psi\partial_\psi + \eta^0\partial_{\psi_t} + \eta^1\partial_{\psi_x} + \eta^{00}\partial_{\psi_{tt}} + \eta^{01}\partial_{\psi_{xt}} + \eta^{11}\partial_{\psi_{xx}} + \text{complex conjugate}\Big).$$

(d) Show that the group $\{g^s\}$ of transformations in part (c) acts as a group of Lie symmetries for the nonlinear Schrödinger equation $i\partial_t\psi + \frac{1}{2}\partial_x^2\psi + |\psi|^2\psi = 0$. Given that $ae^{ia^2t/2}\,\mathrm{sech}(ax)$ solves the nonlinear Schrödinger equation for any $a \in \mathbb{R}$, find a solution which describes a solitary wave travelling at arbitrary speed $s \in \mathbb{R}$.

Paper 1, Section II

33C Integrable Systems

(a) Show that if L is a symmetric matrix ($L = L^T$) and B is skew-symmetric ($B = -B^T$) then $[B, L] = BL - LB$ is symmetric.

(b) Consider the real $n \times n$ symmetric matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & \dots & \dots & 0 \\ a_1 & 0 & a_2 & 0 & \dots & \dots & \dots & 0 \\ 0 & a_2 & 0 & a_3 & \dots & \dots & \dots & 0 \\ 0 & 0 & a_3 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & a_{n-2} & 0 \\ 0 & \dots & \dots & \dots & \dots & a_{n-2} & 0 & a_{n-1} \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{n-1} & 0 \end{pmatrix}$$

(i.e. $L_{i,i+1} = L_{i+1,i} = a_i$ for $1 \leqslant i \leqslant n-1$, all other entries being zero) and the real $n \times n$ skew-symmetric matrix

$$B = \begin{pmatrix} 0 & 0 & a_1 a_2 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_2 a_3 & \dots & \dots & \dots & 0 \\ -a_1 a_2 & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & -a_2 a_3 & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{n-2} a_{n-1} \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & -a_{n-2} a_{n-1} & 0 & 0 \end{pmatrix}$$

(i.e. $B_{i,i+2} = -B_{i+2,i} = a_i a_{i+1}$ for $1 \leqslant i \leqslant n-2$, all other entries being zero).

(i) Compute $[B, L]$.

(ii) Assume that the a_j are smooth functions of time t so the matrix $L = L(t)$ also depends smoothly on t. Show that the equation $\frac{dL}{dt} = [B, L]$ implies that

$$\frac{da_j}{dt} = f(a_{j-1}, a_j, a_{j+1})$$

for some function f which you should find explicitly.

(iii) Using the transformation $a_j = \frac{1}{2}\exp[\frac{1}{2}u_j]$ show that

$$\frac{du_j}{dt} = \frac{1}{2}\Big(e^{u_{j+1}} - e^{u_{j-1}}\Big) \qquad (\dagger)$$

for $j = 1, \dots n-1$. [Use the convention $u_0 = -\infty, a_0 = 0, u_n = -\infty, a_n = 0$.]

(iv) Deduce that given a solution of equation ($\dagger$), there exist matrices $\{U(t)\}_{t\in\mathbb{R}}$ depending on time such that $L(t) = U(t)L(0)U(t)^{-1}$, and explain how to obtain first integrals for ($\dagger$) from this.

Paper 2, Section II

33C Integrable Systems

(i) Explain how the inverse scattering method can be used to solve the initial value problem for the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0\,, \qquad u(x,0) = u_0(x)\,,$$

including a description of the scattering data associated to the operator $L_u = -\partial_x^2 + u(x,t)$, its time dependence, and the reconstruction of u via the inverse scattering problem.

(ii) Solve the inverse scattering problem for the *reflectionless* case, in which the reflection coefficient $R(k)$ is identically zero and the discrete scattering data consists of a single bound state, and hence derive the 1-soliton solution of KdV.

(iii) Consider the direct and inverse scattering problems in the case of a small potential $u(x) = \epsilon q(x)$, with ϵ arbitrarily small: $0 < \epsilon \ll 1$. Show that the reflection coefficient is given by

$$R(k) = \epsilon \int_{-\infty}^{\infty} \frac{e^{-2ikz}}{2ik} q(z)\, dz + O(\epsilon^2)$$

and verify that the solution of the inverse scattering problem applied to this reflection coefficient does indeed lead back to the potential $u = \epsilon q$ when calculated to first order in ϵ. [*Hint: you may make use of the Fourier inversion theorem.*]

Paper 3, Section II

32C Integrable Systems

(a) Given a smooth vector field

$$V = V_1(x,u)\frac{\partial}{\partial x} + \phi(x,u)\frac{\partial}{\partial u}$$

on $\mathbb{R}^2$ define the *prolongation* of V of arbitrary order N.

Calculate the prolongation of order two for the group $SO(2)$ of transformations of $\mathbb{R}^2$ given for $s \in \mathbb{R}$ by

$$g^s \begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} u\cos s - x\sin s \\ u\sin s + x\cos s \end{pmatrix},$$

and hence, or otherwise, calculate the prolongation of order two of the vector field $V = -x\partial_u + u\partial_x$. Show that both of the equations $u_{xx} = 0$ and $u_{xx} = (1+u_x^2)^{\frac{3}{2}}$ are invariant under this action of $SO(2)$, and interpret this geometrically.

(b) Show that the sine-Gordon equation

$$\frac{\partial^2 u}{\partial X \partial T} = \sin u$$

admits the group $\{g^s\}_{s\in\mathbb{R}}$, where

$$g^s : \begin{pmatrix} X \\ T \\ u \end{pmatrix} \mapsto \begin{pmatrix} e^s X \\ e^{-s} T \\ u \end{pmatrix}$$

as a group of Lie point symmetries. Show that there is a group invariant solution of the form $u(X,T) = F(z)$ where z is an invariant formed from the independent variables, and hence obtain a second order equation for $w = w(z)$ where $\exp[iF] = w$.

Paper 3, Section II

32C Integrable Systems

Suppose $\psi^s : (x,u) \mapsto (\tilde{x},\tilde{u})$ is a smooth one-parameter group of transformations acting on $\mathbb{R}^2$, with infinitesimal generator

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}.$$

(a) Define the n^{th} *prolongation* $\mathrm{Pr}^{(n)}\, V$ of V, and show that

$$\mathrm{Pr}^{(n)}\, V = V + \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial u^{(i)}},$$

where you should give an explicit formula to determine the η_i recursively in terms of ξ and η.

(b) Find the n^{th} prolongation of each of the following generators:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = x\frac{\partial}{\partial x}, \qquad V_3 = x^2\frac{\partial}{\partial x}.$$

(c) Given a smooth, real-valued, function $u = u(x)$, the Schwarzian derivative is defined by,

$$S = S[u] := \frac{u_x u_{xxx} - \frac{3}{2}u_{xx}^2}{u_x^2}.$$

Show that,

$$\mathrm{Pr}^{(3)}\, V_i\,(S) = c_i S,$$

for $i = 1,2,3$ where c_i are real functions which you should determine. What can you deduce about the symmetries of the equations:

(i) $S[u] = 0$,

(ii) $S[u] = 1$,

(iii) $S[u] = \frac{1}{x^2}$?

Paper 2, Section II

32C Integrable Systems

Suppose $p = p(x)$ is a smooth, real-valued, function of $x \in \mathbb{R}$ which satisfies $p(x) > 0$ for all x and $p(x) \to 1$, $p_x(x), p_{xx}(x) \to 0$ as $|x| \to \infty$. Consider the Sturm-Liouville operator:

$$L\psi := -\frac{d}{dx}\left(p^2 \frac{d\psi}{dx}\right),$$

which acts on smooth, complex-valued, functions $\psi = \psi(x)$. You may assume that for any $k > 0$ there exists a unique function $\varphi_k(x)$ which satisfies:

$$L\varphi_k = k^2 \varphi_k,$$

and has the asymptotic behaviour:

$$\varphi_k(x) \sim \begin{cases} e^{-ikx} & \text{as } x \to -\infty, \\ a(k)e^{-ikx} + b(k)e^{ikx} & \text{as } x \to +\infty. \end{cases}$$

(a) By analogy with the standard Schrödinger scattering problem, define the reflection and transmission coefficients: $R(k), T(k)$. Show that $|R(k)|^2 + |T(k)|^2 = 1$. [*Hint: You may wish to consider* $W(x) = p(x)^2 \left[\psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)\right]$ *for suitable functions* ψ_1 *and* ψ_2.]

(b) Show that, if $\kappa > 0$, there exists no non-trivial normalizable solution ψ to the equation

$$L\psi = -\kappa^2 \psi.$$

Assume now that $p = p(x,t)$, such that $p(x,t) > 0$ and $p(x,t) \to 1$, $p_x(x,t), p_{xx}(x,t) \to 0$ as $|x| \to \infty$. You are given that the operator A defined by:

$$A\psi := -4p^3 \frac{d^3\psi}{dx^3} - 18p^2 p_x \frac{d^2\psi}{dx^2} - (12pp_x^2 + 6p^2 p_{xx})\frac{d\psi}{dx},$$

satisfies:

$$(LA - AL)\psi = -\frac{d}{dx}\left(2p^4 p_{xxx} \frac{d\psi}{dx}\right).$$

(c) Show that L, A form a Lax pair if the Harry Dym equation,

$$p_t = p^3 p_{xxx}$$

is satisfied. [You may assume $L = L^\dagger$, $A = -A^\dagger$.]

(d) Assuming that p solves the Harry Dym equation, find how the transmission and reflection amplitudes evolve as functions of t.

Paper 1, Section II

32C Integrable Systems

Let $M = \mathbb{R}^{2n} = \{(\mathbf{q}, \mathbf{p}) | \mathbf{q}, \mathbf{p} \in \mathbb{R}^n\}$ be equipped with its standard Poisson bracket.

(a) Given a Hamiltonian function $H = H(\mathbf{q}, \mathbf{p})$, write down *Hamilton's equations* for (M, H). Define a *first integral* of the system and state what it means that the system is *integrable*.

(b) Show that if $n = 1$ then every Hamiltonian system is integrable whenever

$$\left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) \neq \mathbf{0}.$$

Let $\tilde{M} = \mathbb{R}^{2m} = \{(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) | \tilde{\mathbf{q}}, \tilde{\mathbf{p}} \in \mathbb{R}^m\}$ be another phase space, equipped with its standard Poisson bracket. Suppose that $\tilde{H} = \tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ is a Hamiltonian function for $\tilde{M}$. Define $\mathbf{Q} = (q_1, \ldots, q_n, \tilde{q}_1, \ldots, \tilde{q}_m)$, $\mathbf{P} = (p_1, \ldots, p_n, \tilde{p}_1, \ldots, \tilde{p}_m)$ and let the combined phase space $\mathcal{M} = \mathbb{R}^{2(n+m)} = \{(\mathbf{Q}, \mathbf{P})\}$ be equipped with the standard Poisson bracket.

(c) Show that if (M, H) and $(\tilde{M}, \tilde{H})$ are both integrable, then so is $(\mathcal{M}, \mathcal{H})$, where the combined Hamiltonian is given by:

$$\mathcal{H}(\mathbf{Q}, \mathbf{P}) = H(\mathbf{q}, \mathbf{p}) + \tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}).$$

(d) Consider the $n-$dimensional simple harmonic oscillator with phase space M and Hamiltonian H given by:

$$H = \frac{1}{2}p_1^2 + \ldots + \frac{1}{2}p_n^2 + \frac{1}{2}\omega_1^2 q_1^2 + \ldots + \frac{1}{2}\omega_n^2 q_n^2,$$

where $\omega_i > 0$. Using the results above, or otherwise, show that (M, H) is integrable for $(\mathbf{q}, \mathbf{p}) \neq \mathbf{0}$.

(e) Is it true that every bounded orbit of an integrable system is necessarily periodic? You should justify your answer.

Paper 1, Section II

32A Integrable Systems

Let $M = \mathbb{R}^{2n} = \{(\mathbf{q}, \mathbf{p}) | \mathbf{q}, \mathbf{p} \in \mathbb{R}^n\}$ be equipped with the standard symplectic form so that the Poisson bracket is given by:

$$\{f, g\} = \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j},$$

for f, g real-valued functions on M. Let $H = H(\mathbf{q}, \mathbf{p})$ be a Hamiltonian function.

(a) Write down *Hamilton's equations* for (M, H), define a *first integral* of the system and state what it means that the system is *integrable.*

(b) State the Arnol'd–Liouville theorem.

(c) Define complex coordinates z_j by $z_j = q_j + ip_j$, and show that if f, g are real-valued functions on M then:

$$\{f, g\} = -2i\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \overline{z_j}} + 2i\frac{\partial g}{\partial z_j}\frac{\partial f}{\partial \overline{z_j}}.$$

(d) For an $n \times n$ anti-Hermitian matrix A with components A_{jk}, let $I_A := \frac{1}{2i}\overline{z_j}A_{jk}z_k$. Show that:

$$\{I_A, I_B\} = -I_{[A,B]},$$

where $[A, B] = AB - BA$ is the usual matrix commutator.

(e) Consider the Hamiltonian:

$$H = \frac{1}{2}\overline{z_j}z_j.$$

Show that (M, H) is integrable and describe the invariant tori.

[*In this question $j, k = 1, \ldots, n$, and the summation convention is understood for these indices.*]

Paper 2, Section II

33A Integrable Systems

(a) Let $\mathcal{L}, \mathcal{A}$ be two families of linear operators, depending on a parameter t, which act on a Hilbert space H with inner product $(,)$. Suppose further that for each t, $\mathcal{L}$ is self-adjoint and that $\mathcal{A}$ is anti-self-adjoint. State *Lax's equation* for the pair $\mathcal{L}, \mathcal{A}$, and show that if it holds then the eigenvalues of $\mathcal{L}$ are independent of t.

(b) For $\psi, \phi : \mathbb{R} \to \mathbb{C}$, define the inner product:

$$(\psi, \phi) := \int_{-\infty}^{\infty} \overline{\psi(x)}\phi(x)dx.$$

Let L, A be the operators:

$$L\psi := i\frac{d^3\psi}{dx^3} - i\left(q\frac{d\psi}{dx} + \frac{d}{dx}(q\psi)\right) + p\psi,$$

$$A\psi := 3i\frac{d^2\psi}{dx^2} - 4iq\psi,$$

where $p = p(x,t), q = q(x,t)$ are smooth, real-valued functions. You may assume that the normalised eigenfunctions of L are smooth functions of x, t, which decay rapidly as $|x| \to \infty$ for all t.

(i) Show that if ψ, ϕ are smooth and rapidly decaying towards infinity then:

$$(L\psi, \phi) = (\psi, L\phi), \qquad (A\psi, \phi) = -(\psi, A\phi).$$

Deduce that the eigenvalues of L are real.

(ii) Show that if Lax's equation holds for L, A, then q must satisfy the Boussinesq equation:

$$q_{tt} = aq_{xxxx} + b(q^2)_{xx},$$

where a, b are constants whose values you should determine. [You may assume without proof that the identity:

$$LA\psi = AL\psi - 3i\left(p_x\frac{d\psi}{dx} + \frac{d}{dx}(p_x\psi)\right) + \left[q_{xxx} - 4(q^2)_x\right]\psi,$$

holds for smooth, rapidly decaying ψ.]

Paper 3, Section II

33A Integrable Systems

Suppose $\psi^s : (x,u) \mapsto (\tilde{x},\tilde{u})$ is a smooth one-parameter group of transformations acting on $\mathbb{R}^2$.

(a) Define the *generator* of the transformation,

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u},$$

where you should specify ξ and η in terms of ψ^s.

(b) Define the n^{th} *prolongation* of V, $\mathrm{Pr}^{(n)}\, V$ and explicitly compute $\mathrm{Pr}^{(1)}\, V$ in terms of ξ, η.

Recall that if ψ^s is a Lie point symmetry of the ordinary differential equation:

$$\Delta\left(x, u, \frac{du}{dx}, \ldots, \frac{d^n u}{dx^n}\right) = 0,$$

then it follows that $\mathrm{Pr}^{(n)}\, V\,[\Delta] = 0$ whenever $\Delta = 0$.

(c) Consider the ordinary differential equation:

$$\frac{du}{dx} = F\,(x,u)\,,$$

for F a smooth function. Show that if V generates a Lie point symmetry of this equation, then:

$$0 = \eta_x + (\eta_u - \xi_x - F\xi_u)\,F - \xi F_x - \eta F_u.$$

(d) Find all the Lie point symmetries of the equation:

$$\frac{du}{dx} = xG\left(\frac{u}{x^2}\right),$$

where G is an arbitrary smooth function.

Paper 1, Section II

31A Integrable Systems

Define a *Lie point symmetry* of the first order ordinary differential equation $\Delta[t, \mathbf{x}, \dot{\mathbf{x}}] = 0$. Describe such a Lie point symmetry in terms of the vector field that generates it.

Consider the $2n$-dimensional Hamiltonian system (M, H) governed by the differential equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = J \frac{\partial H}{\partial \mathbf{x}}. \qquad (\star)$$

Define the *Poisson bracket* $\{\cdot, \cdot\}$. For smooth functions $f, g : M \to \mathbf{R}$ show that the associated Hamiltonian vector fields V_f, V_g satisfy

$$[V_f, V_g] = -V_{\{f,g\}}.$$

If $F : M \to \mathbf{R}$ is a first integral of (M, H), show that the Hamiltonian vector field V_F generates a Lie point symmetry of $(\star)$. Prove the converse is also true if $(\star)$ has a fixed point, i.e. a solution of the form $\mathbf{x}(t) = \mathbf{x}_0$.

Paper 2, Section II

31A Integrable Systems

Let U and V be non-singular $N \times N$ matrices depending on (x, t, λ) which are periodic in x with period 2π. Consider the associated linear problem

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi\,,$$

for the vector $\Psi = \Psi(x, t; \lambda)$. On the assumption that these equations are compatible, derive the zero curvature equation for (U, V).

Let $W = W(x, t, \lambda)$ denote the $N \times N$ matrix satisfying

$$W_x = UW, \quad W(0, t, \lambda) = I_N,$$

where I_N is the $N \times N$ identity matrix. You should assume W is unique. By considering $(W_t - VW)_x$, show that the matrix $w(t, \lambda) = W(2\pi, t, \lambda)$ satisfies the Lax equation

$$w_t = [v, w], \quad v(t, \lambda) \equiv V(2\pi, t, \lambda).$$

Deduce that $\left\{\operatorname{tr}\left(w^k\right)\right\}_{k \geqslant 1}$ are first integrals.

By considering the matrices

$$\frac{1}{2\mathrm{i}\lambda}\begin{bmatrix} \cos u & -\mathrm{i}\sin u \\ \mathrm{i}\sin u & -\cos u \end{bmatrix}, \quad \frac{\mathrm{i}}{2}\begin{bmatrix} 2\lambda & u_x \\ u_x & -2\lambda \end{bmatrix},$$

show that the periodic Sine-Gordon equation $u_{xt} = \sin u$ has infinitely many first integrals. [You need not prove anything about independence.]

Paper 3, Section II

31A Integrable Systems

Let $u = u(x,t)$ be a smooth solution to the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0$$

which decays rapidly as $|x| \to \infty$ and let $L = -\partial_x^2 + u$ be the associated Schrödinger operator. You may assume L and $A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)$ constitute a Lax pair for KdV.

Consider a solution to $L\varphi = k^2\varphi$ which has the asymptotic form

$$\varphi(x,k,t) = \begin{cases} e^{-\mathrm{i}kx}, & \text{as } x \to -\infty, \\ a(k,t)e^{-\mathrm{i}kx} + b(k,t)e^{\mathrm{i}kx}, & \text{as } x \to +\infty. \end{cases}$$

Find evolution equations for a and b. Deduce that $a(k,t)$ is t-independent.

By writing φ in the form

$$\varphi(x,k,t) = \exp\left[-\mathrm{i}kx + \int_{-\infty}^{x} S(y,k,t)\,\mathrm{d}y\right], \quad S(x,k,t) = \sum_{n=1}^{\infty} \frac{S_n(x,t)}{(2\mathrm{i}k)^n},$$

show that

$$a(k,t) = \exp\left[\int_{-\infty}^{\infty} S(x,k,t)\,\mathrm{d}x\right].$$

Deduce that $\{\int_{-\infty}^{\infty} S_n(x,t)\,\mathrm{d}x\}_{n=1}^{\infty}$ are first integrals of KdV.

By writing a differential equation for $S = X + \mathrm{i}Y$ (with X, Y real), show that these first integrals are trivial when n is even.

Paper 3, Section II

30D Integrable Systems

What is meant by an *auto-Bäcklund* transformation?

The sine-Gordon equation in light-cone coordinates is

$$\frac{\partial^2 \varphi}{\partial \xi \partial \tau} = \sin \varphi, \tag{1}$$

where $\xi = \frac{1}{2}(x - t)$, $\tau = \frac{1}{2}(x + t)$ and φ is to be understood modulo 2π. Show that the pair of equations

$$\partial_\xi(\varphi_1 - \varphi_0) = 2\epsilon \sin\left(\frac{\varphi_1 + \varphi_0}{2}\right), \quad \partial_\tau(\varphi_1 + \varphi_0) = \frac{2}{\epsilon}\sin\left(\frac{\varphi_1 - \varphi_0}{2}\right) \tag{2}$$

constitute an auto-Bäcklund transformation for (1).

By noting that $\varphi = 0$ is a solution to (1), use the transformation (2) to derive the soliton (or 'kink') solution to the sine-Gordon equation. Show that this solution can be expressed as

$$\varphi(x,t) \;=\; 4 \arctan\left[\exp\left(\pm\frac{x - ct}{\sqrt{1 - c^2}} + x_0\right)\right],$$

for appropriate constants c and x_0.

[*Hint: You may use the fact that* $\int \operatorname{cosec} x \, \mathrm{d}x = \log \tan(x/2) + \text{const.}$]

The following function is a solution to the sine-Gordon equation:

$$\varphi(x,t) \;=\; 4 \arctan\left[c\,\frac{\sinh(x/\sqrt{1-c^2})}{\cosh(ct/\sqrt{1-c^2})}\right] \quad (c > 0).$$

Verify that this represents two solitons travelling towards each other at the same speed by considering $x \pm ct = \text{constant}$ and taking an appropriate limit.

Paper 1, Section II

30D Integrable Systems

What does it mean for an evolution equation $u_t = K(x, u, u_x, \ldots)$ to be in *Hamiltonian form*? Define the associated Poisson bracket.

An evolution equation $u_t = K(x, u, u_x, \ldots)$ is said to be *bi-Hamiltonian* if it can be written in Hamiltonian form in two distinct ways, i.e.

$$K = \mathcal{J}\,\delta H_0 = \mathcal{E}\,\delta H_1$$

for Hamiltonian operators $\mathcal{J}, \mathcal{E}$ and functionals H_0, H_1. By considering the sequence $\{H_m\}_{m\geqslant 0}$ defined by the recurrence relation

$$\mathcal{E}\,\delta H_{m+1} = \mathcal{J}\,\delta H_m\,, \qquad (*)$$

show that bi-Hamiltonian systems possess infinitely many first integrals in involution. [You may assume that $(*)$ can always be solved for H_{m+1}, given H_m.]

The Harry Dym equation for the function $u = u(x,t)$ is

$$u_t = \frac{\partial^3}{\partial x^3}\left(u^{-1/2}\right).$$

This equation can be written in Hamiltonian form $u_t = \mathcal{E}\delta H_1$ with

$$\mathcal{E} = 2u\frac{\partial}{\partial x} + u_x\,, \quad H_1[u] = \frac{1}{8}\int u^{-5/2}u_x^2\,\mathrm{d}x\,.$$

Show that the Harry Dym equation possesses infinitely many first integrals in involution. [You need not verify the Jacobi identity if your argument involves a Hamiltonian operator.]

Paper 2, Section II

31D Integrable Systems

What does it mean for $g^\epsilon : (x,u) \mapsto (\tilde{x},\tilde{u})$ to describe a *1-parameter group of transformations*? Explain how to compute the vector field

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} \qquad (*)$$

that generates such a 1-parameter group of transformations.

Suppose now $u = u(x)$. Define the nth prolongation, $\mathrm{pr}^{(n)}g^\epsilon$, of g^ϵ and the vector field which generates it. If V is defined by $(*)$ show that

$$\mathrm{pr}^{(n)}V \;=\; V \;+\; \sum_{k=1}^{n} \eta_k \, \frac{\partial}{\partial u^{(k)}}\,,$$

where $u^{(k)} = \mathrm{d}^k u/\mathrm{d}x^k$ and η_k are functions to be determined.

The curvature of the curve $u = u(x)$ in the (x,u)-plane is given by

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}\,.$$

Rotations in the (x,u)-plane are generated by the vector field

$$W = x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}\,.$$

Show that the curvature κ at a point along a plane curve is invariant under such rotations. Find two further transformations that leave κ invariant.

Paper 1, Section II

29D Integrable Systems

Let $u_t = K(x, u, u_x, \ldots)$ be an evolution equation for the function $u = u(x,t)$. Assume u and all its derivatives decay rapidly as $|x| \to \infty$. What does it mean to say that the evolution equation for u can be written in *Hamiltonian form*?

The modified KdV (mKdV) equation for u is

$$u_t + u_{xxx} - 6u^2 u_x = 0\,.$$

Show that small amplitude solutions to this equation are dispersive.

Demonstrate that the mKdV equation can be written in Hamiltonian form and define the associated Poisson bracket $\{\ ,\ \}$ on the space of functionals of u. Verify that the Poisson bracket is linear in each argument and anti-symmetric.

Show that a functional $I = I[u]$ is a first integral of the mKdV equation if and only if $\{I, H\} = 0$, where $H = H[u]$ is the Hamiltonian.

Show that if u satisfies the mKdV equation then

$$\frac{\partial}{\partial t}(u^2) + \frac{\partial}{\partial x}\left(2uu_{xx} - u_x^2 - 3u^4\right) = 0\,.$$

Using this equation, show that the functional

$$I[u] = \int u^2\,dx$$

Poisson-commutes with the Hamiltonian.

Paper 2, Section II

29D Integrable Systems

(a) Explain how a vector field

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$$

generates a 1-parameter group of transformations $g^\epsilon : (x,u) \mapsto (\tilde{x},\tilde{u})$ in terms of the solution to an appropriate differential equation. [You may assume the solution to the relevant equation exists and is unique.]

(b) Suppose now that $u = u(x)$. Define what is meant by a *Lie-point symmetry* of the ordinary differential equation

$$\Delta\big[x, u, u^{(1)}, \ldots, u^{(n)}\big] = 0\,, \quad \text{where} \quad u^{(k)} \equiv \frac{d^k u}{dx^k}\,, \quad k = 1, \ldots, n\,.$$

(c) Prove that every homogeneous, linear ordinary differential equation for $u = u(x)$ admits a Lie-point symmetry generated by the vector field

$$V = u\frac{\partial}{\partial u}\,.$$

By introducing new coordinates

$$s = s(x,u), \quad t = t(x,u)$$

which satisfy $V(s) = 1$ and $V(t) = 0$, show that every differential equation of the form

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0$$

can be reduced to a first-order differential equation for an appropriate function.

Paper 3, Section II

29D Integrable Systems

Let $L = L(t)$ and $A = A(t)$ be real $N \times N$ matrices, with L symmetric and A antisymmetric. Suppose that

$$\frac{dL}{dt} = LA - AL\,.$$

Show that all eigenvalues of the matrix $L(t)$ are t-independent. Deduce that the coefficients of the polynomial

$$P(x) = \det(xI - L(t))$$

are first integrals of the system.

What does it mean for a $2n$-dimensional Hamiltonian system to be *integrable*? Consider the *Toda system* with coordinates (q_1, q_2, q_3) obeying

$$\frac{d^2 q_i}{dt^2} = \mathrm{e}^{q_{i-1}-q_i} - \mathrm{e}^{q_i - q_{i+1}}\,, \quad i = 1,\, 2,\, 3$$

where here and throughout the subscripts are to be determined modulo 3 so that $q_4 \equiv q_1$ and $q_0 \equiv q_3$. Show that

$$H(q_i, p_i) \;=\; \frac{1}{2}\sum_{i=1}^{3} p_i^2 \;+\; \sum_{i=1}^{3} \mathrm{e}^{\,q_i - q_{i+1}}$$

is a Hamiltonian for the Toda system.

Set $a_i = \frac{1}{2}\exp\left(\frac{q_i - q_{i+1}}{2}\right)$ and $b_i = -\frac{1}{2}p_i$. Show that

$$\frac{da_i}{dt} = (b_{i+1} - b_i)\, a_i\,, \quad \frac{db_i}{dt} = 2\left(a_i^2 - a_{i-1}^2\right), \quad i = 1,\, 2,\, 3\,.$$

Is this coordinate transformation canonical?

By considering the matrices

$$L = \begin{pmatrix} b_1 & a_1 & a_3 \\ a_1 & b_2 & a_2 \\ a_3 & a_2 & b_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -a_1 & a_3 \\ a_1 & 0 & -a_2 \\ -a_3 & a_2 & 0 \end{pmatrix},$$

or otherwise, compute three independent first integrals of the Toda system. [Proof of independence is not required.]

Paper 3, Section II

32D Integrable Systems

What does it mean to say that a finite-dimensional Hamiltonian system is *integrable*? State the Arnold–Liouville theorem.

A six-dimensional dynamical system with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ is governed by the differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j(y_i - y_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j(x_i - x_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}$$

for $i = 1, 2, 3$, where $\{\Gamma_i\}_{i=1}^3$ are positive constants. Show that these equations can be written in the form

$$\Gamma_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial F}{\partial y_i}, \quad \Gamma_i \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3$$

for an appropriate function F. By introducing the coordinates

$$\mathbf{q} = (x_1, x_2, x_3)\,, \quad \mathbf{p} = (\Gamma_1 y_1, \Gamma_2 y_2, \Gamma_3 y_3)\,,$$

show that the system can be written in Hamiltonian form

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}\,, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}}$$

for some Hamiltonian $H = H(\mathbf{q}, \mathbf{p})$ which you should determine.

Show that the three functions

$$A = \sum_{i=1}^{3}\Gamma_i x_i, \quad B = \sum_{i=1}^{3}\Gamma_i y_i, \quad C = \sum_{i=1}^{3}\Gamma_i\left(x_i^2 + y_i^2\right)$$

are first integrals of the Hamiltonian system.

Making use of the fundamental Poisson brackets $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$, show that

$$\{A, C\} = 2B, \quad \{B, C\} = -2A.$$

Hence show that the Hamiltonian system is integrable.

Paper 2, Section II

32D Integrable Systems

Let $u = u(x)$ be a smooth function that decays rapidly as $|x| \to \infty$ and let $L = -\partial_x^2 + u(x)$ denote the associated Schrödinger operator. Explain very briefly each of the terms appearing in the scattering data

$$S = \left\{ \{\chi_n, c_n\}_{n=1}^N, R(k) \right\},$$

associated with the operator L. What does it mean to say $u(x)$ is *reflectionless*?

Given S, define the function

$$F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} R(k)\, \mathrm{d}k \,.$$

If $K = K(x, y)$ is the unique solution to the GLM equation

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,z) F(z+y)\, \mathrm{d}z = 0\,,$$

what is the relationship between $u(x)$ and $K(x, x)$?

Now suppose that $u = u(x, t)$ is time dependent and that it solves the KdV equation $u_t + u_{xxx} - 6uu_x = 0$. Show that $L = -\partial_x^2 + u(x, t)$ obeys the Lax equation

$$L_t = [L, A], \quad \text{where } A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)\,.$$

Show that the discrete eigenvalues of L are time independent.

In what follows you may assume the time-dependent scattering data take the form

$$S(t) = \left\{ \left\{ \chi_n, c_n e^{4\chi_n^3 t} \right\}_{n=1}^N, R(k,0) e^{8ik^3 t} \right\}.$$

Show that if $u(x, 0)$ is reflectionless, then the solution to the KdV equation takes the form

$$u(x,t) = -2 \frac{\partial^2}{\partial x^2} \log\left[\det A(x,t)\right]\,,$$

where A is an $N \times N$ matrix which you should determine.

Assume further that $R(k, 0) = k^2 f(k)$, where f is smooth and decays rapidly at infinity. Show that, for any fixed x,

$$\int_{-\infty}^{\infty} e^{ikx} R(k,0)\, e^{8ik^3 t}\, \mathrm{d}k = O(t^{-1}) \quad \text{as } t \to \infty\,.$$

Comment briefly on the significance of this result.

[*You may assume* $\dfrac{1}{\det A} \dfrac{\mathrm{d}}{\mathrm{d}x}(\det A) = \operatorname{tr}\left(A^{-1} \dfrac{\mathrm{d}A}{\mathrm{d}x}\right)$ *for a non-singular matrix* $A(x)$.]

Paper 1, Section II

32D Integrable Systems

Consider the coordinate transformation

$$g^{\epsilon} : (x, u) \mapsto (\tilde{x}, \tilde{u}) = (x \cos \epsilon - u \sin \epsilon, x \sin \epsilon + u \cos \epsilon)\,.$$

Show that g^{ϵ} defines a one-parameter group of transformations. Define what is meant by the *generator* V of a one-parameter group of transformations and compute it for the above case.

Now suppose $u = u(x)$. Explain what is meant by the *first prolongation* $\mathrm{pr}^{(1)} g^{\epsilon}$ of g^{ϵ}. Compute $\mathrm{pr}^{(1)} g^{\epsilon}$ in this case and deduce that

$$\mathrm{pr}^{(1)} V = V + (1 + u_x^2) \frac{\partial}{\partial u_x}\,. \qquad (\star)$$

Similarly find $\mathrm{pr}^{(2)} V$.

Define what is meant by a *Lie point symmetry* of the first-order differential equation $\Delta[x, u, u_x] = 0$. Describe this condition in terms of the vector field that generates the Lie point symmetry. Consider the case

$$\Delta[x, u, u_x] \equiv u_x - \frac{u + x f(x^2 + u^2)}{x - u f(x^2 + u^2)}\,,$$

where f is an arbitrary smooth function of one variable. Using $(\star)$, show that g^{ϵ} generates a Lie point symmetry of the corresponding differential equation.

Paper 3, Section II

32C Integrable Systems

Let $U = U(x,y)$ and $V = V(x,y)$ be two $n \times n$ complex-valued matrix functions, smoothly differentiable in their variables. We wish to explore the solution of the overdetermined linear system

$$\frac{\partial \mathbf{v}}{\partial y} = U(x,y)\mathbf{v}, \qquad \frac{\partial \mathbf{v}}{\partial x} = V(x,y)\mathbf{v}\ ,$$

for some twice smoothly differentiable vector function $\mathbf{v}(x,y)$.

Prove that, if the overdetermined system holds, then the functions U and V obey the zero curvature representation

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} + UV - VU = 0\,.$$

Let $u = u(x,y)$ and

$$U = \begin{bmatrix} i\lambda & i\bar{u} \\ iu & -i\lambda \end{bmatrix}, \qquad V = \begin{bmatrix} 2i\lambda^2 - i|u|^2 & 2i\lambda\bar{u} + \bar{u}_y \\ 2i\lambda u - u_y & -2i\lambda^2 + i|u|^2 \end{bmatrix},$$

where subscripts denote derivatives, $\bar{u}$ is the complex conjugate of u and λ is a constant. Find the compatibility condition on the function u so that U and V obey the zero curvature representation.

Paper 2, Section II

32C Integrable Systems

Consider the Hamiltonian system

$$\mathbf{p}' = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \mathbf{q}' = \frac{\partial H}{\partial \mathbf{p}}\,,$$

where $H = H(\mathbf{p},\mathbf{q})$.

When is the transformation $\mathbf{P} = \mathbf{P}(\mathbf{p},\mathbf{q})$, $\mathbf{Q} = \mathbf{Q}(\mathbf{p},\mathbf{q})$ canonical?

Prove that, if the transformation is canonical, then the equations in the new variables $(\mathbf{P},\mathbf{Q})$ are also Hamiltonian, with the same Hamiltonian function H.

Let $\mathbf{P} = C^{-1}\mathbf{p} + Bq$, $\mathbf{Q} = C\mathbf{q}$, where C is a symmetric nonsingular matrix. Determine necessary and sufficient conditions on C for the transformation to be canonical.

Paper 1, Section II

32C Integrable Systems

Quoting carefully all necessary results, use the theory of inverse scattering to derive the 1-soliton solution of the KdV equation

$$u_t = 6uu_x - u_{xxx}\,.$$

Paper 3, Section II

32D Integrable Systems

Consider a one-parameter group of transformations acting on $\mathbb{R}^4$

$$(x, y, t, u) \longrightarrow (\exp(\epsilon\alpha)x, \exp(\epsilon\beta)y, \exp(\epsilon\gamma)t, \exp(\epsilon\delta)u) \,, \tag{1}$$

where ϵ is a group parameter and $(\alpha, \beta, \gamma, \delta)$ are constants.

(a) Find a vector field W which generates this group.

(b) Find two independent Lie point symmetries S_1 and S_2 of the PDE

$$(u_t - uu_x)_x = u_{yy}, \quad u = u(x, y, t)\,, \tag{2}$$

which are of the form (1).

(c) Find three functionally-independent invariants of S_1, and do the same for S_2. Find a non-constant function $G = G(x, y, t, u)$ which is invariant under both S_1 and S_2.

(d) Explain why all the solutions of (2) that are invariant under a two-parameter group of transformations generated by vector fields

$$W = u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}, \quad V = \frac{\partial}{\partial y}\,,$$

are of the form $u = xF(t)$, where F is a function of one variable. Find an ODE for F characterising these group-invariant solutions.

Paper 2, Section II

32D Integrable Systems

Consider the KdV equation for the function $u(x,t)$

$$u_t = 6uu_x - u_{xxx} \,. \tag{1}$$

(a) Write equation (1) in the Hamiltonian form

$$u_t = \frac{\partial}{\partial x}\frac{\delta H[u]}{\delta u} \,,$$

where the functional $H[u]$ should be given. Use equation (1), together with the boundary conditions $u \to 0$ and $u_x \to 0$ as $|x| \to \infty$, to show that $\int_{\mathbb{R}} u^2 dx$ is independent of t.

(b) Use the Gelfand–Levitan–Marchenko equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(z+y)dz = 0 \tag{2}$$

to find the one soliton solution of the KdV equation, i.e.

$$u(x,t) = -\frac{4\beta\chi \exp\left(-2\chi x\right)}{\left[1 + \frac{\beta}{2\chi}\exp\left(-2\chi x\right)\right]^2} \,.$$

[*Hint. Consider* $F(x) = \beta \exp\left(-\chi x\right)$*, with* $\beta = \beta_0 \exp\left(8\chi^3 t\right)$*, where* β_0, χ *are constants, and* t *should be regarded as a parameter in equation (2). You may use any facts about the Inverse Scattering Transform without proof.*]

Paper 1, Section II

32D Integrable Systems

State the Arnold–Liouville theorem.

Consider an integrable system with six-dimensional phase space, and assume that $\nabla \wedge \mathbf{p} = 0$ on any Liouville tori $p_i = p_i(q_j, c_j)$, where $\nabla = (\partial/\partial q_1, \partial/\partial q_2, \partial/\partial q_3)$.

(a) Define the action variables and use Stokes' theorem to show that the actions are independent of the choice of the cycles.

(b) Define the generating function, and show that the angle coordinates are periodic with period 2π.

Paper 1, Section II

32A Integrable Systems

Define a finite-dimensional integrable system and state the Arnold–Liouville theorem.

Consider a four-dimensional phase space with coordinates (q_1, q_2, p_1, p_2), where $q_2 > 0$ and q_1 is periodic with period 2π. Let the Hamiltonian be

$$H = \frac{(p_1)^2}{2(q_2)^2} + \frac{(p_2)^2}{2} - \frac{k}{q_2}, \quad \text{where } k > 0\,.$$

Show that the corresponding Hamilton equations form an integrable system.

Determine the sign of the constant E so that the motion is periodic on the surface $H = E$. Demonstrate that in this case, the action variables are given by

$$I_1 = p_1\,, \quad I_2 = \gamma \int_{\alpha}^{\beta} \frac{\sqrt{(q_2 - \alpha)(\beta - q_2)}}{q_2}\, dq_2\,,$$

where α, β, γ are positive constants which you should determine.

Paper 2, Section II

32A Integrable Systems

Consider the Poisson structure

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} \, dx \,, \tag{1}$$

where F, G are polynomial functionals of $u, u_x, u_{xx}, \ldots$. Assume that $u, u_x, u_{xx}, \ldots$ tend to zero as $|x| \to \infty$.

(i) Show that $\{F, G\} = -\{G, F\}$.

(ii) Write down Hamilton's equations for $u = u(x, t)$ corresponding to the following Hamiltonians:

$$H_0[u] = \int_{\mathbb{R}} \frac{1}{2} u^2 \, dx \,, \quad H[u] = \int_{\mathbb{R}} \left(\frac{1}{2} {u_x}^2 + u^3 + u u_x \right) dx \,.$$

(iii) Calculate the Poisson bracket $\{H_0, H\}$, and hence or otherwise deduce that the following overdetermined system of partial differential equations for $u = u(x, t_0, t)$ is compatible:

$$u_{t_0} = u_x \,, \tag{2}$$

$$u_t = 6 u u_x - u_{xxx} \,. \tag{3}$$

[You may assume that the Jacobi identity holds for (1).]

(iv) Find a symmetry of (3) generated by $X = \partial/\partial u + \alpha t \partial/\partial x$ for some constant $\alpha \in \mathbb{R}$ which should be determined. Construct a vector field Y corresponding to the one–parameter group

$$x \to \beta x \,, \quad t \to \gamma t \,, \quad u \to \delta u \,,$$

where (β, γ, δ) should be determined from the symmetry requirement. Find the Lie algebra generated by the vector fields (X, Y).

Paper 3, Section II

32A Integrable Systems

Let $U(\rho, \tau, \lambda)$ and $V(\rho, \tau, \lambda)$ be matrix-valued functions. Consider the following system of overdetermined linear partial differential equations:

$$\frac{\partial}{\partial \rho}\psi = U\psi\,, \quad \frac{\partial}{\partial \tau}\psi = V\psi\,,$$

where ψ is a column vector whose components depend on (ρ, τ, λ). Using the consistency condition of this system, derive the associated zero curvature representation (ZCR)

$$\frac{\partial}{\partial \tau}U - \frac{\partial}{\partial \rho}V + [U, V] = 0\,, \qquad (*)$$

where $[\cdot\,,\,\cdot]$ denotes the usual matrix commutator.

(i) Let

$$U = \frac{i}{2}\begin{pmatrix} 2\lambda & \partial_\rho \phi \\ \partial_\rho \phi & -2\lambda \end{pmatrix}, \qquad V = \frac{1}{4i\lambda}\begin{pmatrix} \cos\phi & -i\sin\phi \\ i\sin\phi & -\cos\phi \end{pmatrix}.$$

Find a partial differential equation for $\phi = \phi(\rho, \tau)$ which is equivalent to the ZCR $(*)$.

(ii) Assuming that U and V in $(*)$ do not depend on $t := \rho - \tau$, show that the trace of $(U - V)^p$ does not depend on $x := \rho + \tau$, where p is any positive integer. Use this fact to construct a first integral of the ordinary differential equation

$$\phi'' = \sin\phi\,, \qquad \text{where} \quad \phi = \phi(x)\,.$$

Paper 1, Section II

32E Integrable Systems

Define a Poisson structure on an open set $U \subset \mathbb{R}^n$ in terms of an anti-symmetric matrix $\omega^{ab} : U \longrightarrow \mathbb{R}$, where $a, b = 1, \cdots, n$. By considering the Poisson brackets of the coordinate functions x^a show that

$$\sum_{d=1}^{n} \left(\omega^{dc} \frac{\partial \omega^{ab}}{\partial x^d} + \omega^{db} \frac{\partial \omega^{ca}}{\partial x^d} + \omega^{da} \frac{\partial \omega^{bc}}{\partial x^d} \right) = 0 .$$

Now set $n = 3$ and consider $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc} x^c$, where ε^{abc} is the totally antisymmetric symbol on $\mathbb{R}^3$ with $\varepsilon^{123} = 1$. Find a non–constant function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$\{f, x^a\} = 0 , \qquad a = 1, 2, 3 .$$

Consider the Hamiltonian

$$H(x^1, x^2, x^3) = \frac{1}{2} \sum_{a,b=1}^{3} M^{ab} x^a x^b ,$$

where M^{ab} is a constant symmetric matrix and show that the Hamilton equations of motion with $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc} x^c$ are of the form

$$\dot{x}^a = \sum_{b,c=1}^{3} Q^{abc} x^b x^c ,$$

where the constants Q^{abc} should be determined in terms of M^{ab}.

Paper 2, Section II

32E Integrable Systems

Consider the Gelfand–Levitan–Marchenko (GLM) integral equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(z+y)\,dz \;=\; 0\,,$$

with $F(x) = \sum_1^N \beta_n\, e^{-c_n x}$, where $c_1,\ldots,c_N$ are positive constants and $\beta_1,\ldots,\beta_N$ are constants. Consider separable solutions of the form

$$K(x,y) \;=\; \sum_{n=1}^{N} K_n(x)\, e^{-c_n y}\,,$$

and reduce the GLM equation to a linear system

$$\sum_{m=1}^{N} A_{nm}(x) K_m(x) \;=\; B_n(x)\,,$$

where the matrix $A_{nm}(x)$ and the vector $B_n(x)$ should be determined.

How is K related to solutions of the KdV equation?

Set $N=1$, $c_1 = c$, $\beta_1 = \beta \exp(8c^3 t)$ where c,β are constants. Show that the corresponding one–soliton solution of the KdV equation is given by

$$u(x,t) \;=\; -\,\frac{4\,\beta_1 c\, e^{-2\,cx}}{(1+(\beta_1/2c)\,e^{-2\,cx})^2}\,.$$

[You may use any facts about the Inverse Scattering Transform without proof.]

Paper 3, Section II

32E Integrable Systems

Consider a vector field

$$V = \alpha\, x\,\frac{\partial}{\partial x} + \beta\, t\,\frac{\partial}{\partial t} + \gamma\, v\,\frac{\partial}{\partial v}\,,$$

on $\mathbb{R}^3$, where α, β and γ are constants. Find the one-parameter group of transformations generated by this vector field.

Find the values of the constants (α, β, γ) such that V generates a Lie point symmetry of the modified KdV equation (mKdV)

$$v_t - 6\,v^2\,v_x + v_{xxx} = 0\,, \qquad \text{where} \quad v = v(x,t)\,.$$

Show that the function $u = u(x,t)$ given by $u = v^2 + v_x$ satisfies the KdV equation and find a Lie point symmetry of KdV corresponding to the Lie point symmetry of mKdV which you have determined from V.

Paper 1, Section II

32B Integrable Systems

Let H be a smooth function on a $2n$–dimensional phase space with local coordinates (p_j, q_j). Write down the Hamilton equations with the Hamiltonian given by H and state the Arnold–Liouville theorem.

By establishing the existence of sufficiently many first integrals demonstrate that the system of n coupled harmonic oscillators with the Hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{n}(p_k^2 + \omega_k^2 q_k^2),$$

where $\omega_1, \ldots, \omega_n$ are constants, is completely integrable. Find the action variables for this system.

Paper 2, Section II

32B Integrable Systems

Let $L = -\partial_x^2 + u(x,t)$ be a Schrödinger operator and let A be another differential operator which does not contain derivatives with respect to t and such that

$$L_t = [L, A].$$

Show that the eigenvalues of L are independent of t, and deduce that if f is an eigenfunction of L then so is $f_t + Af$. [You may assume that L is self–adjoint.]

Let f be an eigenfunction of L corresponding to an eigenvalue λ which is non-degenerate. Show that there exists a function $\hat{f} = \hat{f}(x,t,\lambda)$ such that

$$L\hat{f} = \lambda\hat{f}, \qquad \hat{f}_t + A\hat{f} = 0. \qquad (*)$$

Assume

$$A = \partial_x^3 + a_1\partial_x + a_0,$$

where $a_k = a_k(x,t)$, $k = 0, 1$ are functions. Show that the system $(*)$ is equivalent to a pair of first order matrix PDEs

$$\partial_x F = UF, \qquad \partial_t F = VF,$$

where $F = (\hat{f}, \partial_x\hat{f})^T$ and U, V are $2{\times}2$ matrices which should be determined.

Paper 3, Section II

32B Integrable Systems

Consider the partial differential equation

$$\frac{\partial u}{\partial t} = u^n \frac{\partial u}{\partial x} + \frac{\partial^{2k+1} u}{\partial x^{2k+1}}, \qquad (*)$$

where $u = u(x,t)$ and k, n are non-negative integers.

(i) Find a Lie point symmetry of $(*)$ of the form

$$(x, t, u) \longrightarrow (\alpha x, \beta t, \gamma u), \qquad (**)$$

where (α, β, γ) are non-zero constants, and find a vector field generating this symmetry. Find two more vector fields generating Lie point symmetries of $(*)$ which are not of the form $(**)$ and verify that the three vector fields you have found form a Lie algebra.

(ii) Put $(*)$ in a Hamiltonian form.

1/II/31C **Integrable Systems**

Define an integrable system in the context of Hamiltonian mechanics with a finite number of degrees of freedom and state the Arnold–Liouville theorem.

Consider a six-dimensional phase space with its canonical coordinates (p_j, q_j), $j = 1, 2, 3$, and the Hamiltonian

$$\frac{1}{2}\sum_{j=1}^{3} {p_j}^2 + F(r),$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$ and where F is an arbitrary function. Show that both $M_1 = q_2p_3 - q_3p_2$ and $M_2 = q_3p_1 - q_1p_3$ are first integrals.

State the Jacobi identity and deduce that the Poisson bracket

$$M_3 = \{M_1, M_2\}$$

is also a first integral. Construct a suitable expression out of M_1, M_2, M_3 to demonstrate that the system admits three first integrals in involution and thus satisfies the hypothesis of the Arnold–Liouville theorem.

2/II/31C **Integrable Systems**

Describe the inverse scattering transform for the KdV equation, paying particular attention to the Lax representation and the evolution of the scattering data.

[*Hint: you may find it helpful to consider the operator*

$$A = 4\frac{d^3}{dx^3} - 3\Big(u\frac{d}{dx} + \frac{d}{dx}u\Big).]$$

3/II/31C **Integrable Systems**

Let $U(\lambda)$ and $V(\lambda)$ be matrix-valued functions of (x, y) depending on the auxiliary parameter λ. Consider a system of linear PDEs

$$\frac{\partial}{\partial x}\Phi = U(\lambda)\Phi, \quad \frac{\partial}{\partial y}\Phi = V(\lambda)\Phi \tag{1}$$

where Φ is a column vector whose components depend on (x, y, λ). Derive the zero curvature representation as the compatibility conditions for this system.

Assume that

$$U(\lambda) = -\begin{pmatrix} u_x & 0 & \lambda \\ 1 & -u_x & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V(\lambda) = -\begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ \lambda^{-1}e^u & 0 & 0 \end{pmatrix}$$

and show that (1) is compatible if the function $u = u(x, y)$ satisfies the PDE

$$\frac{\partial^2 u}{\partial x \partial y} = F(u) \tag{2}$$

for some $F(u)$ which should be determined.

Show that the transformation

$$(x, y) \longrightarrow (cx, c^{-1}y), \qquad c \in \mathbb{R} \setminus \{0\}$$

forms a symmetry group of the PDE (2) and find the vector field generating this group.

Find the ODE characterising the group-invariant solutions of (2).

1/II/31E **Integrable Systems**

(i) Using the Cole–Hopf transformation

$$u = -\frac{2\nu}{\phi}\frac{\partial\phi}{\partial x},$$

map the Burgers equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}$$

to the heat equation

$$\frac{\partial \phi}{\partial t} = \nu\,\frac{\partial^2 \phi}{\partial x^2}\,.$$

(ii) Given that the solution of the heat equation on the infinite line $\mathbb{R}$ with initial condition $\phi(x,0) = \Phi(x)$ is given by

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\Phi(\xi)\,e^{-\frac{(x-\xi)^2}{4\nu t}}\,d\xi\,,$$

show that the solution of the analogous problem for the Burgers equation with initial condition $u(x,0) = U(x)$ is given by

$$u = \frac{\displaystyle\int_{-\infty}^{\infty}\frac{x-\xi}{t}\,e^{-\frac{1}{2\nu}G(x,\xi,t)}\,d\xi}{\displaystyle\int_{-\infty}^{\infty}e^{-\frac{1}{2\nu}G(x,\xi,t)}\,d\xi},$$

where the function G is to be determined in terms of U.

(iii) Determine the ODE characterising the scaling reduction of the spherical modified Korteweg – de Vries equation

$$\frac{\partial u}{\partial t} + 6u^2\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{u}{t} = 0\,.$$

2/II/31E **Integrable Systems**

Solve the following linear singular equation

$$\left(t+t^{-1}\right)\phi(t) + \frac{\left(t-t^{-1}\right)}{\pi i}\,⨍_C \frac{\phi(\tau)}{\tau-t}\,d\tau - \frac{\left(t+t^{-1}\right)}{2\pi i}\oint_C\left(\tau+2\tau^{-1}\right)\phi(\tau)\,d\tau = 2t^{-1}\,,$$

where C denotes the unit circle, $t \in C$ and $⨍_C$ denotes the principal value integral.

3/II/31E **Integrable Systems**

Find a Lax pair formulation for the linearised NLS equation

$$iq_t + q_{xx} = 0\,.$$

Use this Lax pair formulation to show that the initial value problem on the infinite line of the linearised NLS equation is associated with the following Riemann–Hilbert problem

$$\begin{aligned} M^+(x,t,k) &= M^-(x,t,k)\begin{pmatrix} 1 & e^{ikx-ik^2t}\hat{q}_0(k) \\ 0 & 1 \end{pmatrix}, \quad && k \in \mathbb{R}\,, \\ M &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{k}\right), && k \to \infty\,. \end{aligned}$$

By deforming the above problem obtain the Riemann–Hilbert problem and hence the linear integral equation associated with the following system of nonlinear evolution PDEs

$$\begin{aligned} iq_t + q_{xx} - 2\vartheta q^2 &= 0\,, \\ -i\vartheta_t + \vartheta_{xx} - 2\vartheta^2 q &= 0\,. \end{aligned}$$

1/II/31E **Integrable Systems**

(a) Let $q(x,t)$ satisfy the heat equation

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial x^2}.$$

Find the function X, which depends linearly on $\partial q/\partial x$, q, k, such that the heat equation can be written in the form

$$\frac{\partial}{\partial t}\left(e^{-ikx+k^2t}q\right) + \frac{\partial}{\partial x}\left(e^{-ikx+k^2t}X\right) = 0, \quad k \in \mathbb{C}.$$

Use this equation to construct a Lax pair for the heat equation.

(b) Use the above result, as well as the Cole–Hopf transformation, to construct a Lax pair for the Burgers equation

$$\frac{\partial Q}{\partial t} - 2Q\frac{\partial Q}{\partial x} = \frac{\partial^2 Q}{\partial x^2}.$$

(c) Find the second-order ordinary differential equation satisfied by the similarity solution of the so-called cylindrical KdV equation:

$$\frac{\partial q}{\partial t} + \frac{\partial^3 q}{\partial x^3} + q\frac{\partial q}{\partial x} + \frac{q}{3t} = 0, \quad t \neq 0.$$

2/II/31E **Integrable Systems**

Let $\phi(t)$ satisfy the singular integral equation

$$\left(t^4 + t^3 - t^2\right)\frac{\phi(t)}{2} + \frac{(t^4 - t^3 - t^2)}{2\pi i}\oint_C \frac{\phi(\tau)}{\tau - t}\, d\tau = (A-1)t^3 + t^2 \ ,$$

where C denotes the circle of radius 2 centred on the origin, $\oint$ denotes the principal value integral and A is a constant. Derive the associated Riemann–Hilbert problem, and compute the canonical solution of the corresponding homogeneous problem.

Find the value of A such that $\phi(t)$ exists, and compute the unique solution $\phi(t)$ if A takes this value.

3/II/31E **Integrable Systems**

The solution of the initial value problem of the KdV equation is given by

$$q(x,t) = -2i \lim_{k\to\infty} k \frac{\partial N}{\partial x}(x,t,k) \ ,$$

where the scalar function $N(x,t,k)$ can be obtained by solving the following Riemann–Hilbert problem:

$$\frac{M(x,t,k)}{a(k)} = N(x,t,-k) + \frac{b(k)}{a(k)} \exp\left(2ikx + 8ik^3 t\right) N(x,t,k), \quad k \in \mathbb{R},$$

M, N and a are the boundary values of functions of k that are analytic for $\operatorname{Im} k > 0$ and tend to unity as $k \to \infty$. The functions $a(k)$ and $b(k)$ can be determined from the initial condition $q(x,0)$.

Assume that M can be written in the form

$$\frac{M}{a} = \mathcal{M}(x,t,k) + \frac{c \exp\left(-2px + 8p^3 t\right) N(x,t,ip)}{k - ip}, \quad \operatorname{Im} k \geqslant 0,$$

where $\mathcal{M}$ as a function of k is analytic for $\operatorname{Im} k > 0$ and tends to unity as $k \to \infty$; c and p are constants and $p > 0$.

(a) By solving the above Riemann–Hilbert problem find a linear equation relating $N(x,t,k)$ and $N(x,t,ip)$.

(b) By solving this equation explicitly in the case that $b = 0$ and letting $c = 2ipe^{-2x_0}$, compute the one-soliton solution.

(c) Assume that $q(x,0)$ is such that $a(k)$ has a simple zero at $k = ip$. Discuss the dominant form of the solution as $t \to \infty$ and $x/t = O(1)$.

1/II/31D **Integrable Systems**

Let $\phi(t)$ satisfy the linear singular integral equation

$$(t^2+t-1)\phi(t) - \frac{t^2-t-1}{\pi i}\oint_L \frac{\phi(\tau)d\tau}{\tau - t} - \frac{1}{\pi i}\int_L \left(\tau + \frac{1}{\tau}\right)\phi(\tau)d\tau = t-1, \quad t \in L,$$

where $\oint$ denotes the principal value integral and L denotes a counterclockwise smooth closed contour, enclosing the origin but not the points ± 1.

(a) Formulate the associated Riemann–Hilbert problem.

(b) For this Riemann–Hilbert problem, find the index, the homogeneous canonical solution and the solvability condition.

(c) Find $\phi(t)$.

2/II/31C **Integrable Systems**

Suppose $q(x,t)$ satisfies the mKdV equation

$$q_t + q_{xxx} + 6q^2 q_x = 0\,,$$

where $q_t = \partial q/\partial t$ etc.

(a) Find the 1-soliton solution.

[*You may use, without proof, the indefinite integral* $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\mathrm{arcsech}\, x\,.$]

(b) Express the self-similar solution of the mKdV equation in terms of a solution, denoted by $v(z)$, of the Painlevé II equation.

(c) Using the Ansatz

$$\frac{dv}{dz} + iv^2 - \frac{i}{6}z = 0\,,$$

find a particular solution of the mKdV equation in terms of a solution of the Airy equation

$$\frac{d^2\Psi}{dz^2} + \frac{z}{6}\Psi = 0\,.$$

3/II/31A **Integrable Systems**

Let $Q(x,t)$ be an off-diagonal 2×2 matrix. The matrix NLS equation

$$iQ_t - Q_{xx}\sigma_3 + 2Q^3\sigma_3 = 0, \quad \sigma_3 = diag(1,-1),$$

admits the Lax pair

$$\mu_x + ik[\sigma_3, \mu] = Q\mu,$$
$$\mu_t + 2ik^2[\sigma_3, \mu] = (2kQ - iQ^2\sigma_3 - iQ_x\sigma_3)\mu,$$

where $k \in \mathbb{C}$, $\mu(x,t,k)$ is a 2×2 matrix and $[\sigma_3, \mu]$ denotes the matrix commutator.

Let $S(k)$ be a 2×2 matrix-valued function decaying as $|k| \to \infty$. Let $\mu(x,t,k)$ satisfy the 2×2-matrix Riemann–Hilbert problem

$$\mu^+(x,t,k) = \mu^-(x,t,k)e^{-i(kx+2k^2t)\sigma_3}S(k)e^{i(kx+2k^2t)\sigma_3}, \quad k \in \mathbb{R},$$

$$\mu = diag(1,1) + \mathrm{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$

(a) Find expressions for $Q(x,t)$, $A(x,t)$ and $B(x,t)$, in terms of the coefficients in the large k expansion of μ, so that μ solves

$$\mu_x + ik[\sigma_3, \mu] - Q\mu = 0,$$

and

$$\mu_t + 2ik^2[\sigma_3, \mu] - (kA + B)\mu = 0.$$

(b) Use the result of (a) to establish that

$$A = 2Q, \quad B = -i(Q^2 + Q_x)\sigma_3.$$

(c) Show that the above results provide a linearization of the matrix NLS equation. What is the disadvantage of this approach in comparison with the inverse scattering method?