Part II

Graph Theory

Paper 1, Section II

17H Graph Theory

(a) By considering the random graph $G(n,p)$, with $p = (1/2)n^{-2/3}$, show that for every $k \geqslant 1$ there exists a graph G that contains no $K_{3,3}$ and has $\chi(G) > k$.

(b) (i) For $p = n^{-2/3}\log n$, let $G \sim G(n,p)$. Show that

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 copies of } K_4) = 1.$$

(ii) Let $\{H_1, \ldots, H_r\}$ be vertex disjoint copies of K_4, in a graph G. (Recall that this means that $V(H_i) \cap V(H_j) = \emptyset$, for all $i \neq j$.)

Show that

$$\lim_{n\to\infty} \mathbb{P}(G \text{ contains at least 100 vertex disjoint copies of } K_4) = 1.$$

[*Hint: you may wish to consider the number of K_4's in the graph and also the number of pairs of K_4's that are not disjoint.*]

Paper 2, Section II

17H Graph Theory

(a) Define the *chromatic polynomial* $P_G(t)$ of a graph G and prove that it is a polynomial.

(b) Let $f(t) = t^4 - 4t^3 + 5t^2 - 2t$. Explain why f is not the chromatic polynomial of a bipartite graph. Is f the chromatic polynomial of some graph? Justify your answers.

(c) Let G be a connected graph, let $A = A(G)$ be its adjacency matrix and let $d(G)$ denote the diameter of G. Show that the matrices $I, A, A^2, \ldots, A^{d(G)}$ are linearly independent.

(d) Give an infinite family of connected graphs $\{G_n\}$ with adjacency matrices $\{A_n\}$ so that

$$I, A_n, A_n^2, \ldots, A_n^{d(G_n)+1}$$

are linearly dependent, for each n.

Paper 3, Section II

17H Graph Theory

(a) Let $r \geqslant 2$. Prove Turán's theorem in the form: if G is an n vertex graph that does not contain a K_{r+1} then

$$e(G) \leqslant \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

(b) For $t \leqslant n-1$, show that if G is a connected n vertex graph with $\delta(G) \geqslant t/2$ then G contains a path P_t of length t.

(c) For graphs G, H define the *Ramsey number* $r(G,H)$ to be the minimum n such that every red-blue colouring of the edges of K_n contains either a red copy of G or a blue copy of H. For $s \geqslant 2, t \geqslant 1$, show that

$$r(K_s, P_t) \geqslant (s-1)t + 1.$$

(d) Show further that for $s \geqslant 2, t \geqslant 1$ we have

$$r(K_s, P_t) = (s-1)t + 1.$$

Paper 4, Section II

17H Graph Theory

(a) State Menger's theorem relating the size of x-y separators in a graph to the number of independent x-y paths.

(b) State Hall's theorem and, assuming Menger's theorem, prove Hall's theorem.

(c) Let $k \geqslant 1$ and let $[0,1]^3 = A_1 \cup \cdots \cup A_k$ and $[0,1]^3 = B_1 \cup \cdots \cup B_k$ be two partitions of the unit cube into sets of equal volume. Show there is a permutation σ of $[k]$ so that $A_i \cap B_{\sigma(i)} \neq \emptyset$, for all $i \in [k]$.

(d) Let G be a $2k$-connected graph that contains a K_{2k} and let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be distinct vertices in G. Show that there exist paths $P_1, \ldots, P_k$ for which

$$V(P_i) \cap V(P_j) = \emptyset,$$

for all $i \neq j$ where, for each i, P_i is an x_i-y_i path.

[*In parts (c) and (d) you may assume results from the course provided they are stated clearly.*]

Paper 1, Section II

17F Graph Theory

(a) Define a *proper k-colouring* of a graph G. Define the *chromatic number* $\chi(G)$ of a graph G. Prove that $\chi(G) \leqslant \Delta(G) + 1$ for all graphs G. Do there exist graphs G for which $\chi(G) = \Delta(G) + 1$ for each $\Delta(G) = 0, 1, 2, \ldots$?

(b) What does it mean for a graph to be *k-connected*? If G is a non-complete 3-connected graph, show that $\chi(G) \leqslant \Delta(G)$.

(c) State *Euler's formula.* If G is a triangle-free planar graph, prove that $\chi(G) \leqslant 4$.

(d) Define the *edge-chromatic number* $\chi'(G)$ of a graph G. State *Hall's theorem.* If G is a 4-regular bipartite graph, determine $\chi'(G)$.

Paper 2, Section II

17F Graph Theory

(a) For a graph H and a positive integer n, define $ex(n, H)$. Prove that $ex(n, K_3) \leqslant n^2/4$. [You may not assume Turan's theorem without proof.]

(b) For a fixed $\delta > 0$, suppose that G is a graph on n vertices with $e(G) > (1+\delta)n^2/4$. Prove that G must contain at least ϵn^3 triangles, where $\epsilon > 0$ is a constant that does not depend on n or G.

(c) Prove that $ex(n, K_{3,2}) < cn^{3/2}$, for some constant $c > 0$.

(d) Let $x_1, \ldots, x_n$ be distinct points in $\mathbb{R}^2$. Show that there exists a constant $c > 0$ such that at most $cn^{3/2}$ of the ordered pairs (x_i, x_j) can satisfy $|x_i - x_j| = 1$.

Paper 3, Section II

17F Graph Theory

(a) Let G be a graph. Show that G contains a subgraph H with $\chi(H) \leqslant 3$ and

$$e(H) = \lfloor (2/3)e(G) \rfloor.$$

Show that the constant 2/3 is sharp, in the following sense: for any $\epsilon > 0$ there exists a graph G (with $e(G) > 0$) such that every subgraph H of G with $\chi(H) \leqslant 3$ has $e(H) \leqslant (2/3 + \epsilon)e(G)$.

(b) An *unfriendly partition* of a graph $G = (V, E)$ is a partition $V = A \cup B$, where every $v \in A$ has $|N(v) \cap B| \geqslant |N(v) \cap A|$ and every vertex $v \in B$ has $|N(v) \cap A| \geqslant |N(v) \cap B|$. Show that every finite graph G has an unfriendly partition. [*Hint: Consider a partition $A \cup B = V$ maximizing the number of edges with one end in A and one end in B.*]

(c) Let $G = (\mathbb{N}, E)$ be a countably infinite graph in which all the vertices have finite degree. Show that G has an unfriendly partition.

(d) Let $G = (\mathbb{N}, E)$ be a countably infinite graph in which all the vertices have *infinite* degree. Show that G has an unfriendly partition. (In other words, in this infinite degree case, we want each vertex $v \in A$ to have $N(v) \cap B$ infinite and each $v \in B$ to have $N(v) \cap A$ infinite.)

Paper 4, Section II

17F Graph Theory

Define the *binomial random graph* $G(n,p)$, where $n \in \mathbb{N}$ and $p \in [0,1]$.

Let $G_n \sim G(n,p)$ and let E_n be the event that $\delta(G_n) > 0$. Show that for every $\varepsilon > 0$, if $p = p(n)$ satisfies $p \geqslant (1+\varepsilon)n^{-1}\log n$ then $\mathbb{P}(E_n) \to 1$.

State *Chebyshev's inequality* and show that for every $\varepsilon > 0$, if p is such that $p \leqslant (1-\varepsilon)n^{-1}\log n$ then $\mathbb{P}(E_n) \to 0$.

For $G_n \sim G(n,p)$, let F_n be the event that G_n is connected. Prove that for every $\varepsilon > 0$, if $p \geqslant (1+\varepsilon)n^{-1}\log n$ then $\mathbb{P}(F_n) \to 1$ as $n \to \infty$ and if $p \leqslant (1-\varepsilon)n^{-1}\log n$ then $\mathbb{P}(F_n) \to 0$ as $n \to \infty$. [You may wish to consider separately the case when there is a component of size at most say $n\varepsilon/10$ and the case when there is not.]

[*You may use, without proof, the fact that* $1 - x \leqslant e^{-x}$ *for all* $x \in [0,1]$, *and also that for any fixed* $\delta \in (0,1)$ *we have* $1 - x \geqslant e^{-(1+2\delta)x}$ *for all* $x \in [0,\delta)$. *All logarithms in this question are natural logarithms.*]

Paper 1, Section II
17G Graph Theory

Define the *binomial random graph* $G(n,p)$, where $n \in \mathbb{N}$ and $p \in (0,1)$.

(a) Let $G_n \sim G(n,p)$ and let E_t be the event that G_n contains a copy of the complete graph K_t. Show that if $p = p(n)$ is such that $p \cdot n^{2/(t-1)} \to 0$ then $\mathbb{P}(E_t) \to 0$ as $n \to \infty$.

(b) State Chebyshev's inequality. Show that if $p \cdot n \to \infty$ then $\mathbb{P}(E_3) \to 1$.

(c) Let H be a triangle with an added leaf vertex, that is

$$H = (\{x_1, \ldots, x_4\}, \{x_1x_2, x_2x_3, x_3x_1, x_1x_4\}),$$

where $x_1, \ldots, x_4$ are distinct. Let F be the event that $G_n \sim G(n,p)$ contains a copy of H. Show that if $p = n^{-0.9}$ then $\mathbb{P}(F) \to 1$.

Paper 2, Section II
17G Graph Theory

(a) Define a *tree* and what it means for a graph to be *acyclic*. Show that if G is an acyclic graph on n vertices then $e(G) \leqslant n-1$. [You may use the fact that a spanning tree on n vertices has $n-1$ edges.]

(b) Show that any 3-regular graph on n vertices contains a cycle of length $\leqslant 100 \log n$. Hence show that there exists n_0 such that every 3-regular graph on more than n_0 vertices must contain two cycles C_1, C_2 with disjoint vertex sets.

(c) An *unfriendly partition* of a graph $G = (V,E)$ is a partition $V = A \cup B$, where $A, B \neq \emptyset$, such that every vertex $v \in A$ has $|N(v) \cap B| \geqslant |N(v) \cap A|$ and every $v \in B$ has $|N(v) \cap A| \geqslant |N(v) \cap B|$. Show that every graph G with $|G| \geqslant 2$ has an unfriendly partition.

(d) A *friendly partition* of a graph $G = (V,E)$ is a partition $V = S \cup T$, where $S, T \neq \emptyset$, such that every vertex $v \in S$ has $|N(v) \cap S| \geqslant |N(v) \cap T|$ and every $v \in T$ has $|N(v) \cap T| \geqslant |N(v) \cap S|$. Give an example of a 3-regular graph (on at least 1 vertex) that does not have a friendly partition. Using part (b), show that for large enough n_0 every 3-regular graph G with $|G| \geqslant n_0$ has a friendly partition.

Paper 3, Section II

17G Graph Theory

(a) Define the *Ramsey number* $R(k)$ and show that $R(k) \leqslant 4^k$.

Show that every 2-coloured complete graph K_n with $n \geqslant 2$ contains a monochromatic spanning tree. Is the same true if K_n is coloured with 3 colours? Give a proof or counterexample.

(b) Let $G = (V, E)$ be a graph. Show that the number of paths of length 2 in G is

$$\sum_{x \in V} d(x)\big(d(x) - 1\big).$$

Now consider a 2-coloured complete graph K_n with $n \geqslant 3$. Show that the number of monochromatic triangles in K_n is

$$\frac{1}{2} \sum_{x} \left\{ \binom{d_r(x)}{2} + \binom{d_b(x)}{2} \right\} - \frac{1}{2}\binom{n}{3},$$

where $d_r(x)$ denotes the number of red edges incident with a vertex x and $d_b(x) = (n-1) - d_r(x)$ denotes the number of blue edges incident with x. [*Hint: Count paths of length 2 in two different ways.*]

Paper 4, Section II

17G Graph Theory

State and prove Hall's theorem, giving any definitions required by the proof (e.g. of an M-alternating path).

Let $G = (V, E)$ be a (not necessarily bipartite) graph, and let $\gamma(G)$ be the size of the largest matching in G. Let $\beta(G)$ be the smallest k for which there exist k vertices $v_1, \ldots, v_k \in V$ such that every edge in G is incident with at least one of $v_1, \ldots, v_k$. Show that $\gamma(G) \leqslant \beta(G)$ and that $\beta(G) \leqslant 2\gamma(G)$. For each positive integer k, find a graph G with $\beta(G) = 2k$ and $\gamma(G) = k$. Determine $\beta(G)$ and $\gamma(G)$ when G is the Turan graph $T_3(30)$ on 30 vertices.

By using Hall's theorem, or otherwise, show that if G is a bipartite graph then $\gamma(G) = \beta(G)$.

Define the *chromatic index* $\chi'(G)$ of a graph G. Prove that if $n = 2^r$ with $r \geqslant 1$ then $\chi'(K_n) = n - 1$.

Paper 1, Section II

17G Graph Theory

(a) The complement of a graph is defined as having the same vertex set as the graph, with vertices being adjacent in the complement if and only if they are not adjacent in the graph.

Show that no planar graph of order greater than 10 has a planar complement.

What is the maximum order of a bipartite graph that has a bipartite complement?

(b) For the remainder of this question, let G be a connected bridgeless planar graph with $n \geqslant 4$ vertices, e edges, and containing no circuit of length 4. Suppose that it is drawn with f faces, of which t are 3-sided.

Show that $2e \geqslant 3t + 5(f - t)$. Show further that $e \geqslant 3t$, and hence $f \leqslant 8e/15$.

Deduce that $e \leqslant 15(n-2)/7$. Is there some n and some G for which equality holds? [*Hint: consider "slicing the corners off" a dodecahedron.*]

Paper 2, Section II

17G Graph Theory

(i) Define the *local connectivity* $\kappa(a, b; G)$ for two non-adjacent vertices a and b in a graph G. Prove Menger's theorem, that G contains a set of $\kappa(a, b; G)$ vertex-disjoint a–b paths.

(ii) Recall that a subdivision TK_r of K_r is any graph obtained from K_r by replacing its edges by vertex-disjoint paths. Let G be a 3-connected graph. Show that G contains a TK_3. Show further that G contains a TK_4. Must G contain a TK_5?

Paper 3, Section II

17G Graph Theory

(i) State and prove Turán's theorem.

(ii) Let G be a graph of order $2n \geqslant 4$ with $n^2 + 1$ edges. Show that G must contain a triangle, and that if $n = 2$ then G contains two triangles.

(iii) Show that if every edge of G lies in a triangle then G contains at least $(n^2+1)/3$ triangles.

(iv) Suppose that G has some edge uv contained in no triangles. Show that $\Gamma(u) \cap \Gamma(v) = \emptyset$, and that if $|\Gamma(u)| + |\Gamma(v)| = 2n$ then $\Gamma(u)$ and $\Gamma(v)$ are not both independent sets.

By induction on n, or otherwise, show that every graph of order $2n \geqslant 4$ with $n^2 + 1$ edges contains at least n triangles. [*Hint: If uv is an edge that is contained in no triangles, consider $G - u - v$.*]

Paper 4, Section II

17G Graph Theory

State and prove Vizing's theorem about the chromatic index $\chi'(G)$ of a graph G.

Let $K_{m,n}$ be the complete bipartite graph with class sizes m and n. By first considering $\chi'(K_{n,n})$, find $\chi'(K_{m,n})$ for all m and n.

Let G be the graph of order $2n+1$ obtained by subdividing a single edge of $K_{n,n}$ by a new vertex. Show that $\chi'(G) = \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Paper 4, Section II

17G Graph Theory

State and prove *Hall's theorem.*

Let n be an even positive integer. Let $X = \{A : A \subset [n]\}$ be the power set of $[n] = \{1, 2, \dots, n\}$. For $1 \leqslant i \leqslant n$, let $X_i = \{A \in X : |A| = i\}$. Let Q be the graph with vertex set X where A, $B \in X$ are adjacent if and only if $|A \triangle B| = 1$. [Here, $A \triangle B$ denotes the *symmetric difference* of A and B, given by $A \triangle B := (A \cup B) \setminus (A \cap B)$.]

Let $1 \leqslant i \leqslant \frac{n}{2}$. Why is the induced subgraph $Q[X_i \cup X_{i-1}]$ bipartite? Show that it contains a matching from X_{i-1} to X_i.

A *chain* in X is a subset $\mathcal{C} \subset X$ such that whenever A, $B \in \mathcal{C}$ we have $A \subset B$ or $B \subset A$. What is the least positive integer k such that X can be partitioned into k pairwise disjoint chains? Justify your answer.

Paper 3, Section II

17G Graph Theory

(a) What does it mean to say that a graph is *bipartite*?

(b) Show that G is bipartite if and only if it contains no cycles of odd length.

(c) Show that if G is bipartite then

$$\frac{\mathrm{ex}\,(n; G)}{\binom{n}{2}} \quad \to \quad 0$$

as $n \to \infty$.
[You may use without proof the Erdős–Stone theorem provided it is stated precisely.]

(d) Let G be a graph of order n with m edges. Let U be a random subset of $V(G)$ containing each vertex of G independently with probability $\frac{1}{2}$. Let X be the number of edges with precisely one vertex in U. Find, with justification, $\mathbb{E}(X)$, and deduce that G contains a bipartite subgraph with at least $\frac{m}{2}$ edges.

By using another method of choosing a random subset of $V(G)$, or otherwise, show that if n is even then G contains a bipartite subgraph with at least $\frac{mn}{2(n-1)}$ edges.

Paper 2, Section II

17G Graph Theory

(a) Suppose that the edges of the complete graph K_6 are coloured blue and yellow. Show that it must contain a monochromatic triangle. Does this remain true if K_6 is replaced by K_5?

(b) Let $t \geqslant 1$. Suppose that the edges of the complete graph K_{3t-1} are coloured blue and yellow. Show that it must contain t edges of the same colour with no two sharing a vertex. Is there any $t \geqslant 1$ for which this remains true if K_{3t-1} is replaced by K_{3t-2}?

(c) Now let $t \geqslant 2$. Suppose that the edges of the complete graph K_n are coloured blue and yellow in such a way that there are a blue triangle and a yellow triangle with no vertices in common. Show that there are also a blue triangle and a yellow triangle that *do* have a vertex in common. Hence, or otherwise, show that whenever the edges of the complete graph K_{5t} are coloured blue and yellow it must contain t monochromatic triangles, all of the same colour, with no two sharing a vertex. Is there any $t \geqslant 2$ for which this remains true if K_{5t} is replaced by K_{5t-1}? [You may assume that whenever the edges of the complete graph K_{10} are coloured blue and yellow it must contain two monochromatic triangles of the same colour with no vertices in common.]

Paper 1, Section II

17G Graph Theory

Let G be a connected d-regular graph.

(a) Show that d is an eigenvalue of G with multiplicity 1 and eigenvector

$$e = (1\,1\,\ldots\,1)^T.$$

(b) Suppose that G is strongly regular. Show that G has at most three distinct eigenvalues.

(c) Conversely, suppose that G has precisely three distinct eigenvalues d, λ and μ. Let A be the adjacency matrix of G and let

$$B = A^2 - (\lambda + \mu)A + \lambda\mu I.$$

Show that if v is an eigenvector of G that is not a scalar multiple of e then $Bv = 0$. Deduce that B is a scalar multiple of the matrix J whose entries are all equal to one. Hence show that, for $i \neq j$, $(A^2)_{ij}$ depends only on whether or not vertices i and j are adjacent, and so G is strongly regular.

(d) Which connected d-regular graphs have precisely two eigenvalues? Justify your answer.

Paper 4, Section II

17I Graph Theory

Let $s \geqslant 3$. Define the *Ramsey number* $R(s)$. Show that $R(s)$ exists and that $R(s) \leqslant 4^s$.

Show that $R(3) = 6$. Show that (up to relabelling the vertices) there is a unique way to colour the edges of the complete graph K_5 blue and yellow with no monochromatic triangle.

What is the least positive integer n such that the edges of the complete graph K_6 can be coloured blue and yellow in such a way that there are precisely n monochromatic triangles?

Paper 3, Section II

17I Graph Theory

What does it mean to say that a graph G has a *k-colouring*? What are the *chromatic number* $\chi(G)$ and the *independence number* $\alpha(G)$ of a graph G? For each $r \geqslant 3$, give an example of a graph G such that $\chi(G) > r$ but $K_r \not\subset G$.

Let g, $k \geqslant 3$. Show that there exists a graph G containing no cycle of length $\leqslant g$ with $\chi(G) \geqslant k$.

Show also that if n is sufficiently large then there is a triangle-free G of order n with $\alpha(G) < n^{0.7}$.

Paper 2, Section II

17I Graph Theory

Let G be a graph and A, $B \subset V(G)$. Show that if every AB-separator in G has order at least k then there exist k vertex-disjoint AB-paths in G.

Let $k \geqslant 3$ and assume that G is k-connected. Show that G must contain a cycle of length at least k.

Assume further that $|G| \geqslant 2k$. Must G contain a cycle of length at least $2k$? Justify your answer.

What is the largest integer n such that any 3-connected graph G with $|G| \geqslant n$ must contain a cycle of length at least n?

[*No form of Menger's theorem or of the max-flow-min-cut theorem may be assumed without proof.*]

Paper 1, Section II

17I Graph Theory

(a) Define $\mathrm{ex}\,(n, H)$ where H is a graph with at least one edge and $n \geqslant |H|$. Show that, for any such H, the limit $\lim_{n\to\infty} \mathrm{ex}\,(n, H)/\binom{n}{2}$ exists.

[You may not assume the Erdős–Stone theorem.]

(b) State the Erdős–Stone theorem. Use it to deduce that if H is bipartite then $\lim_{n\to\infty} \mathrm{ex}\,(n, H)/\binom{n}{2} = 0$.

(c) Let $t \geqslant 2$. Show that $\mathrm{ex}\,(n, K_{t,t}) = O\left(n^{2-\frac{1}{t}}\right)$.

We say $A \subset \mathbb{Z}_n$ is *nice* if whenever a, b, c, $d \in A$ with $a + b = c + d$ then either $a = c$, $b = d$ or $a = d$, $b = c$. Let $f(n) = \max\{|A| : A \subset \mathbb{Z}_n, A \text{ is nice}\}$. Show that $f(n) = O(\sqrt{n})$.

[$\mathbb{Z}_n$ denotes the set of integers modulo n, *i.e.* $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ with addition modulo n.]

Paper 3, Section II

15H Graph Theory

Define the *Ramsey numbers* $R(s,t)$ for integers $s,t \geqslant 2$. Show that $R(s,t)$ exists for all $s,t \geqslant 2$. Show also that $R(s,s) \leqslant 4^s$ for all $s \geqslant 2$.

Let $t \geqslant 2$ be fixed. Give a red-blue colouring of the edges of K_{2t-2} for which there is no red K_t and no blue odd cycle. Show, however, that for any red-blue colouring of the edges of K_{2t-1} there must exist either a red K_t or a blue odd cycle.

Paper 2, Section II

15H Graph Theory

State and prove Hall's theorem about matchings in bipartite graphs.

Let $A = (a_{ij})$ be an $n \times n$ matrix, with all entries non-negative reals, such that every row sum and every column sum is 1. By applying Hall's theorem, show that there is a permutation σ of $\{1, \ldots, n\}$ such that $a_{i\sigma(i)} > 0$ for all i.

Paper 1, Section II

16H Graph Theory

Let G be a graph of order $n \geqslant 3$ satisfying $\delta(G) \geqslant \frac{n}{2}$. Show that G is Hamiltonian.

Give an example of a planar graph G, with $\chi(G) = 4$, that is Hamiltonian, and also an example of a planar graph G, with $\chi(G) = 4$, that is not Hamiltonian.

Let G be a planar graph with the property that the boundary of the unbounded face is a Hamilton cycle of G. Prove that $\chi(G) \leqslant 3$.

Paper 4, Section II

16H Graph Theory

Let G be a graph of maximum degree Δ. Show the following:

(i) Every eigenvalue λ of G satisfies $|\lambda| \leqslant \Delta$.

(ii) If G is regular then Δ is an eigenvalue.

(iii) If G is regular and connected then the multiplicity of Δ as an eigenvalue is 1.

(iv) If G is regular and not connected then the multiplicity of Δ as an eigenvalue is greater than 1.

Let A be the adjacency matrix of the Petersen graph. Explain why $A^2 + A - 2I = J$, where I is the identity matrix and J is the all-1 matrix. Find, with multiplicities, the eigenvalues of the Petersen graph.

Paper 3, Section II
15G Graph Theory

Define the *chromatic polynomial* $p_G(t)$ of a graph G. Show that if G has n vertices and m edges then

$$p_G(t) \;=\; a_n t^n - a_{n-1}t^{n-1} + a_{n-2}t^{n-2} - \ldots + (-1)^n a_0$$

where $a_n = 1$, $a_{n-1} = m$ and $a_i \geqslant 0$ for all i. [You may assume the deletion-contraction relation, provided that you state it clearly.]

Show that for every graph G (with $n > 0$) we have $a_0 = 0$. Show also that $a_1 = 0$ if and only if G is disconnected.

Explain why $t^4 - 2t^3 + 3t^2 - t$ is not the chromatic polynomial of any graph.

Paper 2, Section II
15G Graph Theory

Define the *Turán graph* $T_r(n)$, where r and n are positive integers with $n \geqslant r$. For which r and n is $T_r(n)$ regular? For which r and n does $T_r(n)$ contain $T_4(8)$ as a subgraph?

State and prove Turán's theorem.

Let $x_1, \ldots, x_n$ be unit vectors in the plane. Prove that the number of pairs $i < j$ for which $x_i + x_j$ has length less than 1 is at most $\lfloor n^2/4 \rfloor$.

Paper 4, Section II
16G Graph Theory

State Menger's theorem in both the vertex form and the edge form. Explain briefly how the edge form of Menger's theorem may be deduced from the vertex form.

(a) Show that if G is 3-connected then G contains a cycle of even length.

(b) Let G be a connected graph with all degrees even. Prove that $\lambda(G)$ is even. [*Hint: if S is a minimal set of edges whose removal disconnects G, let H be a component of $G-S$ and consider the degrees of the vertices of H in the graph $G-S$.*] Give an example to show that $\kappa(G)$ can be odd.

Paper 1, Section II

16G Graph Theory

(a) Show that if G is a planar graph then $\chi(G) \leqslant 5$. [You may assume Euler's formula, provided that you state it precisely.]

(b) (i) Prove that if G is a triangle-free planar graph then $\chi(G) \leqslant 4$.

(ii) Prove that if G is a planar graph of girth at least 6 then $\chi(G) \leqslant 3$.

(iii) Does there exist a constant g such that, if G is a planar graph of girth at least g, then $\chi(G) \leqslant 2$? Justify your answer.

Paper 1, Section II
14I Graph Theory

(a) What does it mean to say that a graph G is *strongly regular with parameters* (k, a, b)?

(b) Let G be an incomplete, strongly regular graph with parameters (k, a, b) and of order n. Suppose $b \geqslant 1$. Show that the numbers

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}}\right)$$

are integers.

(c) Suppose now that G is an incomplete, strongly regular graph with parameters $(k, 0, 3)$. Show that $|G| \in \{6, 162\}$.

Paper 2, Section II
14I Graph Theory

(a) Define the *Ramsey numbers* $R(s,t)$ and $R(s)$ for integers $s,\ t \geqslant 2$. Show that $R(s,t)$ exists for all $s,\ t \geqslant 2$ and that if $s,\ t \geqslant 3$ then $R(s,t) \leqslant R(s-1,t) + R(s,t-1)$.

(b) Show that, as $s \to \infty$, we have $R(s) = O(4^s)$ and $R(s) = \Omega(2^{s/2})$.

(c) Show that, as $t \to \infty$, we have $R(3,t) = O(t^2)$ and $R(3,t) = \Omega\left(\left(\frac{t}{\log t}\right)^{3/2}\right)$.

[*Hint: For the lower bound in* (c), *you may wish to begin by modifying a random graph to show that for all n and p we have*

$$R(3,t) > n - \binom{n}{3}p^3 - \binom{n}{t}(1-p)^{\binom{t}{2}}. \quad]$$

Paper 3, Section II

14I Graph Theory

(a) Let G be a graph. What is a *Hamilton cycle* in G? What does it mean to say that G is *Hamiltonian*?

(b) Let G be a graph of order $n \geqslant 3$ satisfying $\delta(G) \geqslant \frac{n}{2}$. Show that G is Hamiltonian. For each $n \geqslant 3$, exhibit a non-Hamiltonian graph G_n of order n with $\delta(G_n) = \left\lceil \frac{n}{2} \right\rceil - 1$.

(c) Let H be a bipartite graph with $n \geqslant 2$ vertices in each class satisfying $\delta(H) > \frac{n}{2}$. Show that H is Hamiltonian. For each $n \geqslant 2$, exhibit a non-Hamiltonian bipartite graph H_n with n vertices in each class and $\delta(H_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Paper 4, Section II

14I Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G contains a *matching from X to Y*? State and prove Hall's Marriage Theorem.

Suppose now that every $x \in X$ has $d(x) \geqslant 1$, and that if $x \in X$ and $y \in Y$ with $xy \in E(G)$ then $d(x) \geqslant d(y)$. Show that G contains a matching from X to Y.

Paper 4, Section II

17I Graph Theory

Define the *Ramsey number* $R^{(r)}(s,t)$. What is the value of $R^{(1)}(s,t)$? Prove that $R^{(r)}(s,t) \leqslant 1+R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1))$ holds for $r \geqslant 2$ and deduce that $R^{(r)}(s,t)$ exists.

Show that $R^{(2)}(3,3) = 6$ and that $R^{(2)}(3,4)$=9.

Show that $7 \leqslant R^{(3)}(4,4) \leqslant 19$. [*Hint: For the lower bound, choose a suitable subset U and colour e red if $|U \cap e|$ is odd.*]

Paper 3, Section II

17I Graph Theory

Prove that $\chi(G) \leqslant \Delta(G) + 1$ for every graph G. Prove further that, if $\kappa(G) \geqslant 3$, then $\chi(G) \leqslant \Delta(G)$ unless G is complete.

Let $k \geqslant 2$. A graph G is said to be *k-critical* if $\chi(G) = k+1$, but $\chi(G-v) = k$ for every vertex v of G. Show that, if G is k-critical, then $\kappa(G) \geqslant 2$.

Let $k \geqslant 2$, and let H be the graph K_{k+1} with an edge removed. Show that H has the following property: it has two vertices which receive the same colour in every k-colouring of H. By considering two copies of H, construct a k-colourable graph G of order $2k+1$ with the following property: it has three vertices which receive the same colour in every k-colouring of G.

Construct, for all integers $k \geqslant 2$ and $\ell \geqslant 2$, a k-critical graph G of order $\ell k+1$ with $\kappa(G) = 2$.

Paper 2, Section II

17I Graph Theory

Let k and n be integers with $1 \leqslant k < n$. Show that every connected graph of order n, in which $d(u) + d(v) \geqslant k$ for every pair u, v of non-adjacent vertices, contains a path of length k.

Let k and n be integers with $1 \leqslant k \leqslant n$. Show that a graph of order n that contains no path of length k has at most $(k-1)n/2$ edges, and that this value is achieved only if k divides n and G is the union of n/k disjoint copies of K_k. [*Hint: Proceed by induction on n and consider a vertex of minimum degree.*]

Paper 1, Section II

17I Graph Theory

Show that a graph is bipartite if and only if all of its cycles are of even length.

Show that a bridgeless plane graph is bipartite if and only if all of its faces are of even length.

Let G be an Eulerian plane graph. Show that the faces of G can be coloured with two colours so that no two contiguous faces have the same colour. Deduce that it is possible to assign a direction to each edge of G in such a way that the edges around each face form a directed cycle.

Paper 4, Section II

17F Graph Theory

Define the *maximum degree* $\Delta(G)$ and the *chromatic index* $\chi'(G)$ of the graph G.

State and prove Vizing's theorem relating $\Delta(G)$ and $\chi'(G)$.

Let G be a connected graph such that $\chi'(G) = \Delta(G) + 1$ but, for every subgraph H of G, $\chi'(H) = \Delta(H)$ holds. Show that G is a circuit of odd length.

Paper 3, Section II

17F Graph Theory

Let G be a graph of order n and average degree d. Let A be the adjacency matrix of G and let $x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$ be its characteristic polynomial. Show that $c_1 = 0$ and $c_2 = -nd/2$. Show also that $-c_3$ is twice the number of triangles in G.

The eigenvalues of A are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Prove that $\lambda_1 \geqslant d$.

Evaluate $\lambda_1 + \cdots + \lambda_n$. Show that $\lambda_1^2 + \cdots + \lambda_n^2 = nd$ and infer that $\lambda_1 \leqslant \sqrt{d(n-1)}$. Does there exist, for each n, a graph G with $d > 0$ for which $\lambda_2 = \cdots = \lambda_n$?

Paper 2, Section II

17F Graph Theory

Let G be a graph with $|G| \geqslant 3$. State and prove a necessary and sufficient condition for G to be Eulerian (that is, for G to have an Eulerian circuit).

Prove that if $\delta(G) \geqslant |G|/2$ then G is Hamiltonian (that is, G has a Hamiltonian circuit).

The *line graph* $L(G)$ of G has vertex set $V(L(G)) = E(G)$ and edge set

$$E(L(G)) = \{ef : e, f \in E(G),\ e \text{ and } f \text{ are incident}\}.$$

Show that $L(G)$ is Eulerian if G is regular and connected.

Must $L(G)$ be Hamiltonian if G is Eulerian? Must G be Eulerian if $L(G)$ is Hamiltonian? Justify your answers.

Paper 1, Section II

17F Graph Theory

State and prove Hall's theorem about matchings in bipartite graphs.

Show that a regular bipartite graph has a matching meeting every vertex.

A graph is *almost r-regular* if each vertex has degree $r-1$ or r. Show that, if $r \geqslant 2$, an almost r-regular graph G must contain an almost $(r-1)$-regular graph H with $V(H) = V(G)$.

[*Hint: First, if possible, remove edges from G whilst keeping it almost r-regular.*]

Paper 4, Section II

17F Graph Theory

(a) Show that every finite tree of order at least 2 has a leaf. Hence, or otherwise, show that a tree of order $n \geqslant 1$ must have precisely $n-1$ edges.

(b) Let G be a graph. Explain briefly why $|G|/\alpha(G) \leqslant \chi(G) \leqslant \Delta(G)+1$.

Let $k=\chi(G)$, and assume $k \geqslant 2$. By induction on $|G|$, or otherwise, show that G has a subgraph H with $\delta(H) \geqslant k-1$. Hence, or otherwise, show that if T is a tree of order k then $T \subseteq G$.

(c) Let s, $t \geqslant 2$ be integers, let $n=(s-1)(t-1)+1$ and let T be a tree of order t. Show that whenever the edges of the complete graph K_n are coloured blue and yellow then it must contain either a blue K_s or a yellow T.

Does this remain true if K_n is replaced by K_{n-1}? Justify your answer.

[*The independence number $\alpha(G)$ of a graph G is the size of the largest set $W \subseteq V(G)$ of vertices such that no edge of G joins two points of W. Recall that $\chi(G)$ is the chromatic number and $\delta(G), \Delta(G)$ are respectively the minimal/maximal degrees of vertices in G.*]

Paper 3, Section II

17F Graph Theory

Let H be a graph with at least one edge. Define $\mathrm{ex}\,(n;H)$, where n is an integer with $n \geqslant |H|$. Without assuming the Erdős–Stone theorem, show that the sequence $\mathrm{ex}\,(n;H)/\binom{n}{2}$ converges as $n \to \infty$.

State precisely the Erdős–Stone theorem. Hence determine, with justification, $\lim_{n\to\infty} \mathrm{ex}\,(n;H)/\binom{n}{2}$.

Let K be another graph with at least one edge. For each integer n such that $n \geqslant \max\{|H|,|K|\}$, let

$$f(n)=\max\{e(G) : |G|=n; H \not\subset G \text{ and } K \not\subset G\}$$

and let

$$g(n)=\max\{e(G) : |G|=n; H \not\subset G \text{ or } K \not\subset G\}.$$

Find, with justification, $\lim_{n\to\infty} f(n)/\binom{n}{2}$ and $\lim_{n\to\infty} g(n)/\binom{n}{2}$.

Paper 2, Section II

17F Graph Theory

Let G be a k-connected graph ($k \geqslant 2$). Let $v \in G$ and let $U \subset V(G) \backslash \{v\}$ with $|U| \geqslant k$. Show that G contains k paths from v to U with any two having only the vertex v in common.

[No form of Menger's theorem or of the Max-Flow-Min-Cut theorem may be assumed without proof.]

Deduce that G must contain a cycle of length at least k.

Suppose further that G has no independent set of vertices of size $> k$. Show that G is Hamiltonian.

[*Hint. If not, let C be a cycle of maximum length in G and let $v \in V(G) \backslash V(C)$; consider the set of vertices on C immediately preceding the endvertices of a collection of k paths from v to C that have only the vertex v in common.*]

Paper 1, Section II

17F Graph Theory

State Markov's inequality and Chebyshev's inequality.

Let $\mathcal{G}^{(2)}(n,p)$ denote the probability space of bipartite graphs with vertex classes $U = \{1, 2, \ldots, n\}$ and $V = \{-1, -2, \ldots, -n\}$, with each possible edge uv ($u \in U$, $v \in V$) present, independently, with probability p. Let X be the number of subgraphs of $G \in \mathcal{G}^{(2)}(n,p)$ that are isomorphic to the complete bipartite graph $K_{2,2}$. Write down $\mathbb{E}X$ and $\operatorname{Var}(X)$. Hence show that $p = 1/n$ is a threshold for $G \in \mathcal{G}^{(2)}(n,p)$ to contain $K_{2,2}$, in the sense that if $np \to \infty$ then a. e. $G \in \mathcal{G}^{(2)}(n,p)$ contains a $K_{2,2}$, whereas if $np \to 0$ then a. e. $G \in \mathcal{G}^{(2)}(n,p)$ does not contain a $K_{2,2}$.

By modifying a random $G \in \mathcal{G}^{(2)}(n,p)$ for suitably chosen p, show that, for each n, there exists a bipartite graph H with n vertices in each class such that $K_{2,2} \not\subset H$ but $e(H) \geqslant \frac{3}{4}\left(\frac{n}{\sqrt[3]{n-1}}\right)^2$.

Paper 1, Section II

17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What is a *matching* from X to Y?

Show that if $|\Gamma(A)| \geqslant |A|$ for all $A \subset X$ then G contains a matching from X to Y.

Let d be a positive integer. Show that if $|\Gamma(A)| \geqslant |A| - d$ for all $A \subset X$ then G contains a set of $|X| - d$ independent edges.

Show that if 0 is not an eigenvalue of G then G contains a matching from X to Y.

Suppose now that $|X| = |Y| \geqslant 1$ and that G does contain a matching from X to Y. Must it be the case that 0 is not an eigenvalue of G? Justify your answer.

Paper 2, Section II

17F Graph Theory

What does it mean to say that a graph G is *k-colourable?* Define the *chromatic number* $\chi(G)$ of a graph G, and the *chromatic number* $\chi(S)$ of a closed surface S.

State the Euler–Poincaré formula relating the numbers of vertices, edges and faces in a drawing of a graph G on a closed surface S of Euler characteristic E. Show that if $E \leqslant 0$ then

$$\chi(S) \leqslant \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor .$$

Find, with justification, the chromatic number of the Klein bottle N_2. Show that if G is a triangle-free graph which can be drawn on the Klein bottle then $\chi(G) \leqslant 4$.

[You may assume that the Klein bottle has Euler characteristic 0, and that K_6 can be drawn on the Klein bottle but K_7 cannot. You may use Brooks's theorem.]

Paper 3, Section II

17F Graph Theory

Define the *Turán graph* $T_r(n)$. State and prove Turán's theorem. Hence, or otherwise, find $\mathrm{ex}(K_3; n)$.

Let G be a bipartite graph with n vertices in each class. Let k be an integer, $1 \leqslant k \leqslant n$, and assume $e(G) > (k-1)n$. Show that G contains a set of k independent edges.

[*Hint: Suppose G contains a set D of a independent edges but no set of $a+1$ independent edges. Let U be the set of vertices of the edges in D and let F be the set of edges in G with precisely one vertex in U; consider $|F|$.*]

Hence, or otherwise, show that if H is a triangle-free tripartite graph with n vertices in each class then $e(H) \leqslant 2n^2$.

Paper 4, Section II

17F Graph Theory

(i) Given a positive integer k, show that there exists a positive integer n such that, whenever the edges of the complete graph K_n are coloured with k colours, there exists a monochromatic triangle.

Denote the least such n by $f(k)$. Show that $f(k) \leqslant 3 \cdot k!$ for all k.

(ii) You may now assume that $f(2) = 6$ and $f(3) = 17$.

Let H denote the graph of order 4 consisting of a triangle together with one extra edge. Given a positive integer k, let $g(k)$ denote the least positive integer n such that, whenever the edges of the complete graph K_n are coloured with k colours, there exists a monochromatic copy of H. By considering the edges from one vertex of a monochromatic triangle in K_7, or otherwise, show that $g(2) \leqslant 7$. By exhibiting a blue-yellow colouring of the edges of K_6 with no monochromatic copy of H, show that in fact $g(2) = 7$.

What is $g(3)$? Justify your answer.

Paper 1, Section II

17F Graph Theory

(a) Define the *Ramsey number* $R(s)$. Show that for all integers $s \geqslant 2$ the Ramsey number $R(s)$ exists and that $R(s) \leqslant 4^s$.

(b) For any graph G, let $R(G)$ denote the least positive integer n such that in any red-blue colouring of the edges of the complete graph K_n there must be a monochromatic copy of G.

(i) How do we know that $R(G)$ exists for every graph G?

(ii) Let s be a positive integer. Show that, whenever the edge of K_{2s} are red-blue coloured, there must be a monochromatic copy of the complete bipartite graph $K_{1,s}$.

(iii) Suppose s is odd. By exhibiting a suitable colouring of K_{2s-1}, show that $R(K_{1,s}) = 2s$.

(iv) Suppose instead s is even. What is $R(K_{1,s})$? Justify your answer.

Paper 2, Section II

17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G contains a *matching* from X to Y?

State and prove Hall's Marriage Theorem, giving a necessary and sufficient condition for G to contain a matching from X to Y.

Now assume that G does contain a matching (from X to Y). For a subset $A \subset X$, $\Gamma(A)$ denotes the set of vertices adjacent to some vertex in A.

(i) Suppose $|\Gamma(A)| > |A|$ for every $A \subset X$ with $A \neq \emptyset$, X. Show that every edge of G is contained in a matching.

(ii) Suppose that every edge of G is contained in a matching and that G is connected. Show that $|\Gamma(A)| > |A|$ for every $A \subset X$ with $A \neq \emptyset$, X.

(iii) For each $n \geqslant 2$, give an example of G with $|X| = n$ such that every edge is contained in a matching but $|\Gamma(A)| = |A|$ for some $A \subset X$ with $A \neq \emptyset$, X.

(iv) Suppose that every edge of G is contained in a matching. Must every pair of independent edges in G be contained in a matching? Give a proof or counterexample as appropriate.

[No form of Menger's Theorem or of the Max-Flow-Min-Cut Theorem may be assumed without proof.]

Paper 3, Section II

17F Graph Theory

Let G be a graph of order n. Show that G must contain an independent set of $\left\lceil \sum_{v \in G} \frac{1}{d(v)+1} \right\rceil$ vertices (where $\lceil x \rceil$ denotes the least integer $\geqslant x$).

[*Hint: take a random ordering of the vertices of G, and consider the set of those vertices which are adjacent to no earlier vertex in the ordering.*]

Fix an integer $m < n$ with m dividing n, and suppose that $e(G) = m\binom{n/m}{2}$.

(i) Deduce that G must contain an independent set of m vertices.

(ii) Must G contain an independent set of $m+1$ vertices?

Paper 4, Section II

17F Graph Theory

State Euler's formula relating the number of vertices, edges and faces in a drawing of a connected planar graph. Deduce that every planar graph has chromatic number at most 5.

Show also that any triangle-free planar graph has chromatic number at most 4.

Suppose G is a planar graph which is *minimal* 5-*chromatic;* that is to say, $\chi(G) = 5$ but if H is a subgraph of G with $H \neq G$ then $\chi(H) < 5$. Prove that $\delta(G) \geqslant 5$. Does this remain true if we drop the assumption that G is planar? Justify your answer.

[The Four Colour Theorem may not be assumed.]

Paper 1, Section II

17F Graph Theory

(i) State and prove Hall's theorem concerning matchings in bipartite graphs.

(ii) The *matching number* of a graph G is the maximum size of a family of independent edges (edges without shared vertices) in G. Deduce from Hall's theorem that if G is a k-regular bipartite graph on n vertices (some $k > 0$) then G has matching number $n/2$.

(iii) Now suppose that G is an arbitrary k-regular graph on n vertices (some $k > 0$). Show that G has a matching number at least $\frac{k}{4k-2}n$. [*Hint: Let S be the set of vertices in a maximal set of independent edges. Consider the edges of G with exactly one endpoint in S.*]

For $k = 2$, show that there are infinitely many graphs G for which equality holds.

Paper 2, Section II

17F Graph Theory

(i) Define the Turán graph $T_r(n)$. State and prove Turán's theorem.

(ii) For each value of n and r with $n > r$, exhibit a graph G on n vertices that has fewer edges than $T_{r-1}(n)$ and yet is maximal K_r–free (meaning that G contains no K_r but the addition of any edge to G produces a K_r). In the case $r = 3$, determine the smallest number of edges that such a G can have.

Paper 3, Section II

17F Graph Theory

(a) State Brooks' theorem concerning the chromatic number $\chi(G)$ of a graph G. Prove it in the case when G is 3-connected.

[If you wish to assume that G is regular, you should explain why this assumption is justified.]

(b) State Vizing's theorem concerning the edge-chromatic number $\chi'(G)$ of a graph G.

(c) Are the following statements true or false? Justify your answers.

(1) If G is a connected graph on more than two vertices then $\chi(G) \leqslant \chi'(G)$.

(2) For every ordering of the vertices of a graph G, if we colour G using the greedy algorithm (on this ordering) then the number of colours we use is at most $2\chi(G)$.

(3) For every ordering of the edges of a graph G, if we edge-colour G using the greedy algorithm (on this ordering) then the number of colours we use is at most $2\chi'(G)$.

Paper 4, Section II

17F Graph Theory

Let X denote the number of triangles in a random graph G chosen from $G(n, p)$. Find the mean and variance of X. Hence show that $p = n^{-1}$ is a threshold for the existence of a triangle, in the sense that if $pn \to 0$ then almost surely G does not contain a triangle, while if $pn \to \infty$ then almost surely G does contain a triangle.

Now let $p = n^{-1/2}$, and let Y denote the number of edges of G (chosen as before from $G(n, p)$). By considering the mean of $Y - X$, show that for each $n \geqslant 3$ there exists a graph on n vertices with at least $\frac{1}{6}n^{3/2}$ edges that is triangle-free. Is this within a constant factor of the best-possible answer (meaning the greatest number of edges that a triangle-free graph on n vertices can have)?

1/II/17F **Graph Theory**

State a result of Euler concerning the number of vertices, edges and faces of a connected plane graph. Deduce that if G is a planar graph then $\delta(G) \leqslant 5$. Show that if G is a planar graph then $\chi(G) \leqslant 5$.

Are the following statements true or false? Justify your answers.

[You may quote standard facts about planar and non-planar graphs, provided that they are clearly stated.]

(i) If G is a graph with $\chi(G) \leqslant 4$ then G is planar.

(ii) If G is a connected graph with average degree at most 2.01 then G is planar.

(iii) If G is a connected graph with average degree at most 2 then G is planar.

2/II/17F **Graph Theory**

Prove that every graph G on $n \geqslant 3$ vertices with minimum degree $\delta(G) \geqslant \frac{n}{2}$ is Hamiltonian. For each $n \geqslant 3$, give an example to show that this result does not remain true if we weaken the condition to $\delta(G) \geqslant \frac{n}{2} - 1$ (for n even) or $\delta(G) \geqslant \frac{n-1}{2}$ (for n odd).

For any graph G, let G_k denote the graph formed by adding k new vertices to G, all joined to each other and to all vertices of G. By considering G_1, show that if G is a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant \frac{n-1}{2}$ then G has a Hamilton path (a path passing through all the vertices of G).

For each positive integer k, exhibit a connected graph G such that G_k is not Hamiltonian. Is this still possible if we replace 'connected' with '2-connected'?

3/II/17F Graph Theory

Define the chromatic polynomial $p_G(t)$ of a graph G. Show that if G has n vertices and m edges then

$$p_G(t) = a_n t^n - a_{n-1} t^{n-1} + a_{n-2} t^{n-2} - \ldots + (-1)^n a_0,$$

where $a_n = 1$ and $a_{n-1} = m$ and $a_i \geqslant 0$ for all $0 \leqslant i \leqslant n$. [You may assume the deletion–contraction relation, provided it is clearly stated.]

Show that if G is a tree on n vertices then $p_G(t) = t(t-1)^{n-1}$. Does the converse hold?

[*Hint: if G is disconnected, how is the chromatic polynomial of G related to the chromatic polynomials of its components?*]

Show that if G is a graph on n vertices with the same chromatic polynomial as $T_r(n)$ (the Turán graph on n vertices with r vertex classes) then G must be isomorphic to $T_r(n)$.

4/II/17F Graph Theory

For $s \geqslant 2$, let $R(s)$ be the least integer n such that for every 2-colouring of the edges of K_n there is a monochromatic K_s. Prove that $R(s)$ exists.

For any $k \geqslant 1$ and $s_1, \ldots, s_k \geqslant 2$, define the Ramsey number $R_k(s_1, \ldots, s_k)$, and prove that it exists.

Show that, whenever the positive integers are partitioned into finitely many classes, some class contains x, y, z with $x + y = z$.

[*Hint: given a finite colouring of the positive integers, induce a colouring of the pairs of positive integers by giving the pair ij $(i < j)$ the colour of $j - i$.*]

1/II/17H **Graph Theory**

Let G be a connected cubic graph drawn in the plane with each edge in the boundary of two distinct faces. Show that the associated map is 4-colourable if and only if G is 3-edge colourable.

Is the above statement true if the plane is replaced by the torus and all faces are required to be simply connected? Give a proof or a counterexample.

2/II/17H **Graph Theory**

The Ramsey number $R(G)$ of a graph G is the smallest n such that in any red/blue colouring of the edges of K_n there is a monochromatic copy of G.

Show that $R(K_t) \leqslant \binom{2t-2}{t-1}$ for every $t \geqslant 3$.

Let H be the graph on four vertices obtained by adding an edge to a triangle. Show that $R(H) = 7$.

3/II/17H **Graph Theory**

Let G be a bipartite graph with vertex classes X and Y, each of size n. State and prove Hall's theorem giving a necessary and sufficient condition for G to contain a perfect matching.

A vertex $x \in X$ is *flexible* if every edge from x is contained in a perfect matching. Show that if $|\Gamma(A)| > |A|$ for every subset A of X with $\emptyset \neq A \neq X$, then every $x \in X$ is flexible.

Show that whenever G contains a perfect matching, there is at least one flexible $x \in X$.

Give an example of such a G where no $x \in X$ of minimal degree is flexible.

4/II/17H **Graph Theory**

Let G be a graph with n vertices and m edges. Show that if G contains no C_4, then $m \leqslant \frac{n}{4}(1+\sqrt{4n-3})$.

Let $C_4(G)$ denote the number of subgraphs of G isomorphic to C_4. Show that if $m \geqslant \frac{n(n-1)}{4}$, then G contains at least $\frac{n(n-1)(n-3)}{8}$ paths of length 2. By considering the numbers $r_1, r_2, \ldots, r_{\binom{n}{2}}$ of vertices joined to each pair of vertices of G, deduce that

$$C_4(G) \geqslant \tfrac{1}{2}\binom{n}{2}\binom{(n-3)/4}{2}.$$

Now let $G = G(n, 1/2)$ be the random graph on $\{1, 2, \ldots, n\}$ in which each pair of vertices is joined independently with probability 1/2. Find the expectation $\mathbb{E}(C_4(G))$ of $C_4(G)$. Deduce that if $0 < \epsilon < 1/2$, then

$$\Pr\left(C_4(G) \leqslant (1+2\epsilon)\frac{3}{16}\binom{n}{4}\right) \geqslant \epsilon.$$

1/II/17F Graph Theory

State and prove Euler's formula relating the number of vertices, edges and faces of a connected plane graph.

Deduce that a planar graph of order $n \geqslant 3$ has size at most $3n-6$. What bound can be given if the planar graph contains no triangles?

Without invoking the four colour theorem, prove that a planar graph that contains no triangles is 4-colourable.

2/II/17F Graph Theory

Let G be a bipartite graph with vertex classes X and Y. State Hall's necessary condition for G to have a matching from X to Y, and prove that it is sufficient.

Deduce a necessary and sufficient condition for G to have $|X|-d$ independent edges, where d is a natural number.

Show that the maximum size of a set of independent edges in G is equal to the minimum size of a subset $S \subset V(G)$ such that every edge of G has an end vertex in S.

3/II/17F Graph Theory

Let $R(s)$ be the least integer n such that every colouring of the edges of K_n with two colours contains a monochromatic K_s. Prove that $R(s)$ exists.

Prove that a connected graph of maximum degree $d \geqslant 2$ and order d^k contains two vertices distance at least k apart.

Let $C(s)$ be the least integer n such that every connected graph of order n contains, as an *induced* subgraph, either a complete graph K_s, a star $K_{1,s}$ or a path P_s of length s. Show that $C(s) \leqslant R(s)^s$.

4/II/17F Graph Theory

What is meant by a graph G of order n being *strongly regular* with parameters (d, a, b)? Show that, if such a graph G exists and $b > 0$, then

$$\frac{1}{2}\left\{n-1+\frac{(n-1)(b-a)-2d}{\sqrt{(a-b)^2+4(d-b)}}\right\}$$

is an integer.

Let G be a graph containing no triangles, in which every pair of non-adjacent vertices has exactly three common neighbours. Show that G must be d-regular and $|G| = 1 + d(d+2)/3$ for some $d \in \{1, 3, 21\}$. Show that such a graph exists for $d = 3$.

1/II/17F **Graph Theory**

Show that an acyclic graph has a vertex of degree at most one. Prove that a tree (that is, a connected acyclic graph) of order n has size $n-1$, and deduce that every connected graph of order n and size $n-1$ is a tree.

Let T be a tree of order t. Show that if G is a graph with $\delta(G) \geqslant t-1$ then T is a subgraph of G, but that this need not happen if $\delta(G) \geqslant t-2$.

2/II/17F **Graph Theory**

Brooks' Theorem states that if G is a connected graph then $\chi(G) \leqslant \Delta(G)$ unless G is complete or is an odd cycle. Prove the theorem for 3-connected graphs G.

Let G be a graph, and let $d_1 + d_2 = \Delta(G) - 1$. By considering a partition V_1, V_2 of $V(G)$ that minimizes the quantity $d_2 e(G[V_1]) + d_1 e(G[V_2])$, show that there is a partition with $\Delta(G[V_i]) \leqslant d_i$, $i = 1, 2$.

By taking $d_1 = 3$, show that if a graph G contains no K_4 then $\chi(G) \leqslant \frac{3}{4}\Delta(G) + \frac{3}{2}$.

3/II/17F **Graph Theory**

Let X and Y be disjoint sets of $n \geqslant 6$ vertices each. Let G be a bipartite graph formed by adding edges between X and Y randomly and independently with probability $p = 1/100$. Let $e(U, V)$ be the number of edges of G between the subsets $U \subset X$ and $V \subset Y$. Let $k = \lceil n^{1/2} \rceil$. Consider three events $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, as follows.

$$\begin{array}{rl} \mathcal{A}: & \text{there exist } U \subset X,\ V \subset Y \text{ with } |U| = |V| = k \text{ and } e(U,V) = 0 \\ \mathcal{B}: & \text{there exist } x \in X,\ W \subset Y \text{ with } |W| = n-k \text{ and } e(\{x\}, W) = 0 \\ \mathcal{C}: & \text{there exist } Z \subset X,\ y \in Y \text{ with } |Z| = n-k \text{ and } e(Z, \{y\}) = 0\,. \end{array}$$

Show that $\Pr(\mathcal{A}) \leqslant n^{2k}(1-p)^{k^2}$ and $\Pr(\mathcal{B} \cup \mathcal{C}) \leqslant 2n^{k+1}(1-p)^{n-k}$. Hence show that $\Pr(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) < 3n^{2k}(1-p)^{n/2}$ and so show that, almost surely, none of $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ occur. Deduce that, almost surely, G has a matching from X to Y.

4/II/17F **Graph Theory**

Write an essay on extremal graph theory. Your essay should include the proof of at least one extremal theorem. You should state the Erdős–Stone theorem, as well as describing its proof and showing how it can be applied.