Part II

Geometry and Groups

Paper 4, Section I

3F Geometry and Groups

Define the limit set $\Lambda(G)$ of a Kleinian group G. Assuming that G has no finite orbit in $\mathbb{H}^3 \cup S^2_\infty$, and that $\Lambda(G) \neq \emptyset$, prove that if $E \subset \mathbb{C} \cup \{\infty\}$ is any non-empty closed set which is invariant under G, then $\Lambda(G) \subset E$.

Paper 3, Section I

3F Geometry and Groups

Let $\mathbb{H}^2$ denote the hyperbolic plane, and $T \subset \mathbb{H}^2$ be a non-degenerate triangle, i.e. the bounded region enclosed by three finite-length geodesic arcs. Prove that the three angle bisectors of T meet at a point.

Must the three vertices of T lie on a hyperbolic circle? Justify your answer.

Paper 2, Section I

3F Geometry and Groups

Let g, h be non-identity Möbius transformations. Prove that g and h commute if and only if one of the following holds:

1. $\mathrm{Fix}(g) = \mathrm{Fix}(h)$;
2. g, h are involutions each of which exchanges the other's fixed points.

Give an example to show that the second case can occur.

Paper 1, Section I

3F Geometry and Groups

Let $G \leqslant SO(3)$ be a finite group. Suppose G does not preserve any plane in $\mathbb{R}^3$. Show that for any point p in the unit sphere $S^2 \subset \mathbb{R}^3$, the stabiliser $\mathrm{Stab}_G(p)$ contains at most 5 elements.

Paper 1, Section II

11F Geometry and Groups

Prove that an orientation-preserving isometry of the ball-model of hyperbolic space $\mathbb{H}^3$ which fixes the origin is an element of $SO(3)$. Hence, or otherwise, prove that a finite subgroup of the group of orientation-preserving isometries of hyperbolic space $\mathbb{H}^3$ has a common fixed point.

Can an infinite non-cyclic subgroup of the isometry group of $\mathbb{H}^3$ have a common fixed point? Can any such group be a Kleinian group? Justify your answers.

Paper 4, Section II

12F Geometry and Groups

Define the s-dimensional *Hausdorff measure* $\mathcal{H}^s(F)$ of a set $F \subset \mathbb{R}^N$. Explain briefly how properties of this measure may be used to define the *Hausdorff dimension* $\dim_H(F)$ of such a set.

Prove that the limit sets of conjugate Kleinian groups have equal Hausdorff dimension. Hence, or otherwise, prove that there is no subgroup of $\mathbb{P}SL(2,\mathbb{R})$ which is conjugate in $\mathbb{P}SL(2,\mathbb{C})$ to $\mathbb{P}SL(2,\mathbb{Z}\oplus\mathbb{Z}i)$.

Paper 4, Section I

3G Geometry and Groups

Let Δ_1, Δ_2 be two disjoint closed discs in the Riemann sphere with bounding circles Γ_1, Γ_2 respectively. Let J_k be inversion in the circle Γ_k and let T be the Möbius transformation $J_2 \circ J_1$.

Show that, if $w \notin \Delta_1$, then $T(w) \in \Delta_2$ and so $T^n(w) \in \Delta_2$ for $n = 1, 2, 3, \ldots$. Deduce that T has a fixed point in Δ_2 and a second in Δ_1.

Deduce that there is a Möbius transformation A with

$$A(\Delta_1) = \{z : |z| \leqslant 1\} \qquad \text{and} \qquad A(\Delta_2) = \{z : |z| \geqslant R\}$$

for some $R > 1$.

Paper 3, Section I

3G Geometry and Groups

Let Λ be a rank 2 lattice in the Euclidean plane. Show that the group G of all Euclidean isometries of the plane that map Λ onto itself is a discrete group. List the possible sizes of the point groups for G and give examples to show that point groups of these sizes do arise.

[You may quote any standard results without proof.]

Paper 2, Section I

3G Geometry and Groups

Let ℓ_1, ℓ_2 be two straight lines in Euclidean 3-space. Show that there is a rotation about some axis through an angle π that maps ℓ_1 onto ℓ_2. Is this rotation unique?

Paper 1, Section I

3G Geometry and Groups

Show that any pair of lines in hyperbolic 3-space that does not have a common endpoint must have a common normal. Is this still true when the pair of lines does have a common endpoint?

Paper 1, Section II

11G Geometry and Groups

Define the *modular group* Γ acting on the upper half-plane.

Describe the set S of points z in the upper half-plane that have $\mathrm{Im}(T(z)) \leqslant \mathrm{Im}(z)$ for each $T \in \Gamma$. Hence find a fundamental set for Γ acting on the upper half-plane.

Let A and J be the two Möbius transformations

$$A : z \mapsto z+1 \qquad \text{and} \qquad J : z \mapsto -1/z \ .$$

When is $\mathrm{Im}(J(z)) > \mathrm{Im}(z)$?

For any point z in the upper half-plane, show that either $z \in S$ or else there is an integer k with

$$\mathrm{Im}(J(A^k(z))) > \mathrm{Im}(z) \ .$$

Deduce that the modular group is generated by A and J.

Paper 4, Section II

12G Geometry and Groups

Define the *limit set* for a Kleinian group. If your definition of the limit set requires an arbitrary choice of a base point, you should prove that the limit set does not depend on this choice.

Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be the four discs $\{z \in \mathbb{C} : |z - c| \leqslant 1\}$ where c is the point $1+i, 1-i, -1-i, -1+i$ respectively. Show that there is a parabolic Möbius transformation A that maps the interior of Δ_1 onto the exterior of Δ_2 and fixes the point where Δ_1 and Δ_2 touch. Show further that we can choose A so that it maps the unit disc onto itself.

Let B be the similar parabolic transformation that maps the interior of Δ_3 onto the exterior of Δ_4, fixes the point where Δ_3 and Δ_4 touch, and maps the unit disc onto itself. Explain why the group generated by A and B is a Kleinian group G. Find the limit set for the group G and justify your answer.

Paper 4, Section I

3G Geometry and Groups

Explain briefly how to extend a Möbius transformation

$$T : z \mapsto \frac{az+b}{cz+d} \qquad \text{with } ad-bc=1$$

from the boundary of the upper half-space $\mathbb{R}^3_+$ to give a hyperbolic isometry $\widetilde{T}$ of the upper half-space. Write down explicitly the extension of the transformation $z \mapsto \lambda^2 z$ for any constant $\lambda \in \mathbb{C} \setminus \{0\}$.

Show that, if $\widetilde{T}$ has an *axis*, which is a hyperbolic line that is mapped onto itself by $\widetilde{T}$ with the orientation preserved, then $\widetilde{T}$ moves each point of this axis by the same hyperbolic distance, ℓ say. Prove that

$$\ell = 2\left|\log\left|\tfrac{1}{2}\left(a+d+\sqrt{(a+d)^2-4}\right)\right|\right| .$$

Paper 3, Section I

3G Geometry and Groups

Let A be a Möbius transformation acting on the Riemann sphere. Show that, if A is not loxodromic, then there is a disc Δ in the Riemann sphere with $A(\Delta) = \Delta$. Describe all such discs for each Möbius transformation A.

Hence, or otherwise, show that the group G of Möbius transformations generated by

$$A : z \mapsto iz \qquad \text{and} \qquad B : z \mapsto 2z$$

does not map any disc onto itself.

Describe the set of points of the Riemann sphere at which G acts discontinuously. What is the quotient of this set by the action of G?

Paper 2, Section I

3G Geometry and Groups

Define the *modular group* acting on the upper half-plane. Explain briefly why it acts discontinuously and describe a fundamental domain. You should prove that the region which you describe is a fundamental domain.

Paper 1, Section I

3G Geometry and Groups

Let G be a crystallographic group of the Euclidean plane. Define the *lattice* and the *point group* of G. Suppose that the lattice for G is $\{(k, 0) : k \in \mathbb{Z}\}$. Show that there are five different possibilities for the point group. Show that at least one of these point groups can arise from two groups G that are not conjugate in the group of all isometries of the Euclidean plane.

Paper 1, Section II

11G Geometry and Groups

Define the *axis* of a loxodromic Möbius transformation acting on hyperbolic 3-space.

When do two loxodromic transformations commute? Justify your answer.

Let G be a Kleinian group that contains a loxodromic transformation. Show that the fixed point of any loxodromic transformation in G lies in the limit set of G. Prove that the set of such fixed points is dense in the limit set. Give examples to show that the set of such fixed points can be equal to the limit set or a proper subset.

Paper 4, Section II

12G Geometry and Groups

Define the Hausdorff dimension of a subset of the Euclidean plane.

Let Δ be a closed disc of radius r_0 in the Euclidean plane. Define a sequence of sets $K_n \subseteq \Delta$, $n = 1, 2, \ldots$, as follows: $K_1 = \Delta$ and for each $n \geqslant 1$ a subset $K_{n+1} \subset K_n$ is produced by replacing each component disc Γ of K_n by three disjoint, closed discs inside Γ with radius at most c_n times the radius of Γ. Let K be the intersection of the sets K_n. Show that if the factors c_n converge to a limit c with $0 < c < 1$, then the Hausdorff dimension of K is at most $\log \frac{1}{3} / \log c$.

Paper 1, Section I

3G Geometry and Groups

Let G be a finite subgroup of SO(3) and let Ω be the set of unit vectors that are fixed by some non-identity element of G. Show that the group G permutes the unit vectors in Ω and that Ω has at most three orbits. Describe these orbits when G is the group of orientation-preserving symmetries of a regular dodecahedron.

Paper 2, Section I

3G Geometry and Groups

Let A and B be two rotations of the Euclidean plane $\mathbb{E}^2$ about centres a and b respectively. Show that the conjugate ABA^{-1} is also a rotation and find its fixed point. When do A and B commute? Show that the commutator $ABA^{-1}B^{-1}$ is a translation.

Deduce that any group of orientation-preserving isometries of the Euclidean plane either fixes a point or is infinite.

Paper 3, Section I

3G Geometry and Groups

Define a *Kleinian group.*

Give an example of a Kleinian group that is a free group on two generators and explain why it has this property.

Paper 4, Section I

3G Geometry and Groups

Define inversion in a circle Γ on the Riemann sphere. You should show from your definition that inversion in Γ exists and is unique.

Prove that the composition of an even number of inversions is a Möbius transformation of the Riemann sphere and that every Möbius transformation is the composition of an even number of inversions.

Paper 1, Section II

11G Geometry and Groups

Prove that a group of Möbius transformations is discrete if, and only if, it acts discontinuously on hyperbolic 3-space.

Let G be the set of Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$ with

$$a, b, c, d \in \mathbb{Z}[i] = \{u + iv : u, v \in \mathbb{Z}\} \qquad \text{and} \qquad ad - bc = 1 \ .$$

Show that G is a group and that it acts discontinuously on hyperbolic 3-space. Show that G contains transformations that are elliptic, parabolic, hyperbolic and loxodromic.

Paper 4, Section II

12G Geometry and Groups

Define a *lattice* in $\mathbb{R}^2$ and the *rank* of such a lattice.

Let Λ be a rank 2 lattice in $\mathbb{R}^2$. Choose a vector $\boldsymbol{w}_1 \in \Lambda \setminus \{\boldsymbol{0}\}$ with $||\boldsymbol{w}_1||$ as small as possible. Then choose $\boldsymbol{w}_2 \in \Lambda \setminus \mathbb{Z}\boldsymbol{w}_1$ with $||\boldsymbol{w}_2||$ as small as possible. Show that $\Lambda = \mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$.

Suppose that $\boldsymbol{w}_1$ is the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Draw the region of possible values for $\boldsymbol{w}_2$.

Suppose that Λ also equals $\mathbb{Z}\boldsymbol{v}_1 + \mathbb{Z}\boldsymbol{v}_2$. Prove that

$$\boldsymbol{v}_1 = a\boldsymbol{w}_1 + b\boldsymbol{w}_2 \qquad \text{and} \qquad \boldsymbol{v}_2 = c\boldsymbol{w}_1 + d\boldsymbol{w}_2 \,,$$

for some integers a, b, c, d with $ad - bc = \pm 1$.

Paper 1, Section I

3F Geometry of Group Actions

Explain what it means to say that G is a *crystallographic group* of isometries of the Euclidean plane and that $\overline{G}$ is its *point group*. Prove the *crystallographic restriction:* a rotation in such a point group $\overline{G}$ must have order $1, 2, 3, 4$ or 6.

Paper 2, Section I

3F Geometry of Group Actions

Show that a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry for the Euclidean metric on the plane $\mathbb{R}^2$ if and only if there is a vector $\boldsymbol{v} \in \mathbb{R}^2$ and an orthogonal linear map $B \in \mathrm{O}(2)$ with

$$T(\boldsymbol{x}) = B(\boldsymbol{x}) + \boldsymbol{v} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^2 .$$

When T is an isometry with $\det B = -1$, show that T is either a reflection or a glide reflection.

Paper 3, Section I

3F Geometry of Group Actions

Let U be a "triangular" region in the unit disc $\mathbb{D}$ bounded by three hyperbolic geodesics $\gamma_1, \gamma_2, \gamma_3$ that do not meet in $\mathbb{D}$ nor on its boundary. Let J_k be inversion in γ_k and set

$$A = J_2 \circ J_1 \, ; \qquad B = J_3 \circ J_2 \, .$$

Let G be the group generated by the Möbius transformations A and B. Describe briefly a fundamental set for the group G acting on $\mathbb{D}$.

Prove that G is a free group on the two generators A and B. Describe the quotient surface $\mathbb{D}/G$.

Paper 4, Section I

3F Geometry of Group Actions

Define *loxodromic transformations* and explain how to determine when a Möbius transformation

$$T : z \mapsto \frac{az+b}{cz+d} \qquad \text{with} \quad ad-bc = 1$$

is loxodromic.

Show that any Möbius transformation that maps a disc Δ onto itself cannot be loxodromic.

Paper 1, Section II

11F Geometry of Group Actions

For which circles Γ does inversion in Γ interchange 0 and ∞?

Let Γ be a circle that lies entirely within the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let K be inversion in this circle Γ, let J be inversion in the unit circle, and let T be the Möbius transformation $K \circ J$. Show that, if z_0 is a fixed point of T, then

$$J(z_0) = K(z_0)$$

and this point is another fixed point of T.

By applying a suitable isometry of the hyperbolic plane $\mathbb{D}$, or otherwise, show that Γ is the set of points at a fixed hyperbolic distance from some point of $\mathbb{D}$.

Paper 4, Section II

12F Geometry of Group Actions

Explain briefly how Möbius transformations of the Riemann sphere are extended to give isometries of the unit ball $B^3 \subset \mathbb{R}^3$ for the hyperbolic metric.

Which Möbius transformations have extensions that fix the origin in B^3?

For which Möbius transformations T can we find a hyperbolic line in B^3 that T maps onto itself? For which of these Möbius transformations is there only one such hyperbolic line?

Paper 1, Section I

3F Geometry of Group Actions

Explain what is meant by *stereographic projection* from the 2-dimensional sphere to the complex plane.

Prove that u and v are the images under stereographic projection of antipodal points on the sphere if and only if $u\bar{v} = -1$.

Paper 2, Section I

3F Geometry of Group Actions

Describe the geodesics in the hyperbolic plane (in a model of your choice).

Let l_1 and l_2 be geodesics in the hyperbolic plane which do not meet either in the plane or at infinity. By considering the action on a suitable third geodesic, or otherwise, prove that the composite $R_{l_1} \circ R_{l_2}$ of the reflections in the two geodesics has infinite order.

Paper 3, Section I

3F Geometry of Group Actions

Explain why there are discrete subgroups of the Möbius group $\mathbb{P}SL_2(\mathbb{C})$ which abstractly are free groups of rank 2.

Paper 4, Section I

3F Geometry of Group Actions

For every $k \in \mathbb{R}$, show that there is a closed bounded totally disconnected subset X of some Euclidean space, such that X has Hausdorff dimension at least k. [Standard properties of Hausdorff dimension may be quoted without proof if carefully stated.]

Paper 1, Section II

11F Geometry of Group Actions

Define *frieze group* and *crystallographic group* and give three examples of each, identifying them as abstract groups as well as geometrically.

Let G be a discrete group of isometries of the Euclidean plane which contains a translation. Prove that G contains no element of order 5.

Paper 4, Section II

12F Geometry of Group Actions

Define *three-dimensional hyperbolic space*, the *translation length* of an isometry of hyperbolic 3-space, and the *axis* of a hyperbolic isometry. Briefly explain how and why the latter two concepts are related.

Find the translation length of the isometries defined by (i) $z \mapsto kz$, $k \in \mathbb{C} \setminus \{0\}$ and (ii) $z \mapsto \dfrac{3z+2}{7z+5}$.

1/I/3G **Geometry of Group Actions**

Prove that an isometry of Euclidean space $\mathbb{R}^3$ is an affine transformation.

Deduce that a finite group of isometries of $\mathbb{R}^3$ has a common fixed point.

1/II/11G **Geometry of Group Actions**

What is meant by an *inversion* in a circle in $\mathbb{C} \cup \{\infty\}$? Show that a composition of two inversions is a Möbius transformation.

Hence, or otherwise, show that if C^+ and C^- are two disjoint circles in $\mathbb{C}$, then the composition of the inversions in C^+ and C^- has two fixed points.

2/I/3G **Geometry of Group Actions**

State a theorem classifying lattices in $\mathbb{R}^2$. Define a frieze group.

Show there is a frieze group which is isomorphic to $\mathbb{Z}$ but is not generated by a translation, and draw a picture whose symmetries are this group.

3/I/3G **Geometry of Group Actions**

Let $\dim_H$ denote the Hausdorff dimension of a set in $\mathbb{R}^n$. Prove that if $\dim_H(F) < 1$ then F is totally disconnected.

[You may assume that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map then

$$\dim_H(f(F)) \leqslant \dim_H(F).]$$

4/I/3G **Geometry of Group Actions**

Define the hyperbolic metric (in the sense of metric spaces) on the 3-ball.

Given a finite set in hyperbolic 3-space, show there is at least one closed ball of minimal radius containing that set.

4/II/12G **Geometry of Group Actions**

What does it mean for a subgroup G of the Möbius group to be discrete?

Show that a discrete group necessarily acts properly discontinuously in hyperbolic 3-space.

[You may assume that a discrete subgroup of a matrix group is a closed subset.]

1/I/3G Geometry of Group Actions

Show that there are two ways to embed a regular tetrahedron in a cube C so that the vertices of the tetrahedron are also vertices of C. Show that the symmetry group of C permutes these tetrahedra and deduce that the symmetry group of C is isomorphic to the Cartesian product $S_4 \times C_2$ of the symmetric group S_4 and the cyclic group C_2.

1/II/12G Geometry of Group Actions

Define the Hausdorff d-dimensional measure $\mathcal{H}^d(C)$ and the Hausdorff dimension of a subset C of $\mathbb{R}$.

Set $s = \log 2/\log 3$. Define the Cantor set C and show that its Hausdorff s-dimensional measure is at most 1.

Let (X_n) be independent Bernoulli random variables that take the values 0 and 2, each with probability $\frac{1}{2}$. Define

$$\xi = \sum_{n=1}^{\infty} \frac{X_n}{3^n} .$$

Show that ξ is a random variable that takes values in the Cantor set C.

Let U be a subset of $\mathbb{R}$ with $3^{-(k+1)} \leqslant \mathrm{diam}(U) < 3^{-k}$. Show that $\mathbb{P}(\xi \in U) \leqslant 2^{-k}$ and deduce that, for any set $U \subset \mathbb{R}$, we have

$$\mathbb{P}(\xi \in U) \leqslant 2(\mathrm{diam}(U))^s .$$

Hence, or otherwise, prove that $\mathcal{H}^s(C) \geqslant \frac{1}{2}$ and that the Cantor set has Hausdorff dimension s.

2/I/3G Geometry of Group Actions

Explain what is meant by a lattice in the Euclidean plane $\mathbb{R}^2$. Prove that such a lattice is either $\mathbb{Z}\boldsymbol{w}$ for some vector $\boldsymbol{w} \in \mathbb{R}^2$ or else $\mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2$ for two linearly independent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ in $\mathbb{R}^2$.

3/I/3G Geometry of Group Actions

Let G be a 2-dimensional Euclidean crystallographic group. Define the lattice and point group corresponding to G.

Prove that any non-trivial rotation in the point group of G must have order $2, 3, 4$ or 6.

4/I/3G **Geometry of Group Actions**

Let Γ be a circle on the Riemann sphere. Explain what it means to say that two points of the sphere are inverse points for the circle Γ. Show that, for each point z on the Riemann sphere, there is a unique point z' with z, z' inverse points. Define inversion in Γ.

Prove that the composition of an even number of inversions is a Möbius transformation.

4/II/12G **Geometry of Group Actions**

Explain what it means to say that a group G is a Kleinian group. What is the definition of the limit set for the group G? Prove that a fixed point of a parabolic element in G must lie in the limit set.

Show that the matrix $\begin{pmatrix} 1+aw & -aw^2 \\ a & 1-aw \end{pmatrix}$ represents a parabolic transformation for any non-zero choice of the complex numbers a and w. Find its fixed point.

The *Gaussian integers* are $\mathbb{Z}[i] = \{m + in : m, n \in \mathbb{Z}\}$. Let G be the set of Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{Z}[i]$ and $ad - bc = 1$. Prove that G is a Kleinian group. For each point $w = \dfrac{p+iq}{r}$ with p, q, r non-zero integers, find a parabolic transformation $T \in G$ that fixes w. Deduce that the limit set for G is all of the Riemann sphere.

1/I/3F Geometry and Groups

Suppose $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with contraction factor $c_i \in (0,1)$ for $1 \leqslant i \leqslant k$. Let X be the unique non-empty compact invariant set for the S_i's. State a formula for the Hausdorff dimension of X, under an assumption on the S_i's you should state. Hence compute the Hausdorff dimension of the subset X of the square $[0,1]^2$ defined by dividing the square into a 5×5 array of squares, removing the open middle square $(2/5, 3/5)^2$, then removing the middle 1/25th of each of the remaining 24 squares, and so on.

1/II/12F Geometry and Groups

Compute the area of the ball of radius r around a point in the hyperbolic plane. Deduce that, for any tessellation of the hyperbolic plane by congruent, compact tiles, the number of tiles which are at most n "steps" away from a given tile grows exponentially in n. Give an explicit example of a tessellation of the hyperbolic plane.

2/I/3F Geometry and Groups

Determine whether the following elements of $\mathrm{PSL}_2(\mathbb{R})$ are elliptic, parabolic, or hyperbolic. Justify your answers.

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}, \qquad \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}.$$

In the case of the first of these transformations find the fixed points.

3/I/3F Geometry and Groups

Let G be a discrete subgroup of the Möbius group. Define the limit set of G in S^2. If G contains two loxodromic elements whose fixed point sets in S^2 are different, show that the limit set of G contains no isolated points.

4/I/3F Geometry and Groups

What is a crystallographic group in the Euclidean plane? Prove that, if G is crystallographic and g is a nontrivial rotation in G, then g has order 2, 3, 4, or 6.

4/II/12F **Geometry and Groups**

Let G be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Show that G is countable. Let $G = \{g_1, g_2, \ldots\}$ be some enumeration of the elements of G. Show that for any point p in hyperbolic 3-space $\mathbb{H}^3$, the distance $d_{hyp}(p, g_n(p))$ tends to infinity. Deduce that a subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ is discrete if and only if it acts properly discontinuously on $\mathbb{H}^3$.

1/I/3G **Geometry of Group Actions**

Let G be a subgroup of the group of isometries $\mathrm{Isom}(\mathbb{R}^2)$ of the Euclidean plane. What does it mean to say that G is *discrete*?

Supposing that G is discrete, show that the subgroup G_T of G consisting of all translations in G is generated by translations in at most two linearly independent vectors in $\mathbb{R}^2$. Show that there is a homomorphism $G \to O(2)$ with kernel G_T.

Draw, and briefly explain, pictures which illustrate two different possibilities for G when G_T is isomorphic to the additive group $\mathbb{Z}$.

1/II/12G **Geometry of Group Actions**

What is the *limit set* of a subgroup G of Möbius transformations?

Suppose that G is complicated and has no finite orbit in $\mathbb{C} \cup \{\infty\}$. Prove that the limit set of G is infinite. Can the limit set be countable?

State Jørgensen's inequality, and deduce that not every two-generator subgroup $G = \langle A, B \rangle$ of Möbius transformations is discrete. Briefly describe two examples of discrete two-generator subgroups, one for which the limit set is connected and one for which it is disconnected.

2/I/3G **Geometry of Group Actions**

Describe the *geodesics* in the disc model of the hyperbolic plane $\mathbb{H}^2$.

Define the *area* of a region in $\mathbb{H}^2$. Compute the area $A(r)$ of a hyperbolic circle of radius r from the definition just given. Compute the circumference $C(r)$ of a hyperbolic circle of radius r, and check explicitly that $dA(r)/dr = C(r)$.

How could you define π geometrically if you lived in $\mathbb{H}^2$? Briefly justify your answer.

3/I/3G **Geometry of Group Actions**

By considering fixed points in $\mathbb{C} \cup \{\infty\}$, prove that any complex Möbius transformation is conjugate either to a map of the form $z \mapsto kz$ for some $k \in \mathbb{C}$ or to $z \mapsto z + 1$. Deduce that two Möbius transformations g,h (neither the identity) are conjugate if and only if $\mathrm{tr}^2(g) = \mathrm{tr}^2(h)$.

Does every Möbius transformation g also have a fixed point in $\mathbb{H}^3$? Briefly justify your answer.

4/I/3G **Geometry of Group Actions**

Show that a set $F \subset \mathbb{R}^n$ with Hausdorff dimension strictly less than one is totally disconnected.

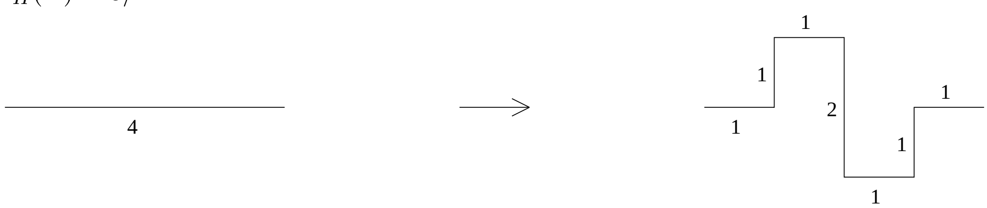
What does it mean for a Möbius transformation to *pair two discs*? By considering a pair of disjoint discs and a pair of tangent discs, or otherwise, explain in words why there is a 2-generator Schottky group with limit set $\Lambda \subset \mathbb{S}^2$ which has Hausdorff dimension at least 1 but which is not homeomorphic to a circle.

4/II/12G **Geometry of Group Actions**

For real $s \geqslant 0$ and $F \subset \mathbb{R}^n$, give a careful definition of the s-dimensional *Hausdorff measure* of F and of the *Hausdorff dimension* $\dim_H(F)$ of F.

For $1 \leqslant i \leqslant k$, suppose $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with contraction factor $c_i \in (0,1)$. Prove there is a unique non-empty compact invariant set I for the $\{S_i\}$. State a formula for the Hausdorff dimension of I, under an assumption on the S_i you should state.

Hence show the Hausdorff dimension of the fractal F given by iterating the scheme below (at each stage replacing each edge by a new copy of the generating template) is $\dim_H(F) = 3/2$.



[Numbers denote lengths]