Part II

—

General Relativity

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UNIVERSITY OF CAMBRIDGE

Paper 1, Section II
38B General Relativity

A Klein-Gordon scalar field ϕ satisfies the equation of motion $\nabla^\alpha \nabla_\alpha \phi = m^2 \phi$ where m is a constant. Its stress-energy tensor takes the form:

$$T_{\mu\nu} = \frac{1}{2}\left[\nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}\left(A \nabla_\rho \phi \nabla^\rho \phi + B\phi^2\right)\right]. \qquad (*)$$

(a) Using the fact that the stress-energy tensor is covariantly conserved, determine the value of the parameters A and B.

(b) Using the Einstein equation, write an expression for the Ricci curvature $R_{\mu\nu}$ in terms of ϕ and its derivatives, in a $D > 2$ dimensional spacetime. Simplify your answer as much as possible.

(c) Now consider a general stress-energy tensor of the form $(*)$, with A and B not necessarily given by the values you have found above. The stress-energy tensor is said to satisfy the *weak energy condition* if

$$T_{\mu\nu} X^\mu X^\nu \geqslant 0$$

for all timelike vectors X^μ. Find the most general constraints on A and B such that $(*)$ satisfies the weak energy condition, and show that your answer to part (a) satisfies these constraints.
[Hint: you may find it useful to work in normal coordinates and furthermore to choose these coordinates such that $X^\mu = \delta_0{}^\mu$.]

Paper 2, Section II
38B General Relativity

Consider the geometry of 2-dimensional hyperbolic space:

$$ds^2 = a^2(dr^2 + \sinh^2 r \, d\phi^2)$$

where a is a constant. The coordinates have ranges $0 \leqslant r$ and $0 \leqslant \phi < 2\pi$.

(a) For a general metric with components $g_{\alpha\beta}$, give an expression for the Christoffel symbols, $\Gamma^\alpha_{\beta\gamma}$, in terms of the metric components and their derivatives. Use this formula to calculate the Christoffel symbols for the metric above.

(b) Using the geodesic equation, show that lines of constant ϕ are always geodesics, but circles of constant $r > 0$ never are.

(c) Calculate both of the nonzero components of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$.

[You may use: $R^\alpha{}_{\beta\gamma\delta} := \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta}.]

(d) Show that the Ricci scalar R is constant.

Paper 3, Section II

37B General Relativity

(a) Let M be the mass of a star and consider a photon with impact parameter b which passes near the star. In this problem, by following the steps below, you will derive the general relativistic formula for the total angle $\delta\phi$ by which the photon bends.

The general relativistic formulae for equatorial null orbits in the Schwarzschild metric (in units where $c = G = 1$) are:

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2, \qquad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2},$$

where dot is derivative with respect to proper time, and $L = r^2\dot{\phi}$ is the angular momentum.

   (i) Write down the geodesic equation for the trajectory of the photon, parameterized by the ϕ coordinate. Switch to an inverse radial coordinate $y = 1/r$. By differentiating the geodesic equation by ϕ, show that $y'' + y = 3My^2$. Here $'$ denotes $d/d\phi$.

  (ii) Solve this equation in the flat space regime ($M = 0$), for a trajectory for which $r \to \infty$ at $\phi = 0, \pi$.

 (iii) Using perturbation theory in M identify a differential equation for Δy, the first order perturbation of y due to nonzero M.

 (iv) Find the homogeneous and particular solutions for Δy.

  (v) Taking $r \to \infty$ at $\phi = 0$, show that the leading order result for the bending of the light ray is:
$$|\delta\phi| \approx \frac{4M}{b}.$$

(b) In Nordström's theory of gravitation, the metric is required to take the form

$$g_{\mu\nu} = \phi^2\eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\phi > 0$ is a dynamical scalar field which approaches the value 1 far from any isolated gravitating system.

Write down the equation satisfied by an affinely parameterised geodesic of the metric $g_{\mu\nu}$. What can you deduce about the bending of light rays around a star of mass M in Nordström's theory? Is this result compatible with observations?

Paper 4, Section II
37B General Relativity

(a) Consider a linearized gravitational plane wave of the form

$$\bar{h}_{\mu\nu} = H_{\mu\nu}e^{ik_\rho x^\rho}$$

where $H_{\mu\nu}$ is independent of x^α, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ is the trace-reversed perturbation to the Minkowski metric $\eta_{\mu\nu}$, and we are using Lorentz gauge $\partial^\mu\bar{h}_{\mu\nu} = 0$.

(i) What restrictions are there on k^μ and $H_{\mu\nu}$? Justify your answers.

(ii) Derive the residual gauge symmetry remaining in $H_{\mu\nu}$, even after imposing Lorentz gauge.

[*You may use:* $G_{\mu\nu} = -\frac{1}{2}\partial^\rho\partial_\rho\bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma}$.]

(b) Suppose that LIGO detects the merger of two black holes, each of which is about 30 solar masses, from an event which takes place approximately a few billion lightyears away.

(i) Estimate the frequency (in Hz) of the gravitational wave source, from the perspective of a hypothetical observer close to the binary system and at rest with respect to it, during the last orbit of the black holes before they merge. In solving this problem you may use the (Newtonian) Kepler's law:

$$T^2 = \frac{4\pi^2}{GM}r^3.$$

Here T is the period and for purposes of estimation you may take $r = 6MG/c^2$, the general relativistic formula for the inner-most stable circular orbit for a test particle in a Schwarzschild geometry. As these assumptions are inexact, do not keep more than one significant figure.

[*You may use:* $c \approx 3.0 \times 10^8$ m/s, $G \approx 6.7 \times 10^{-11}$ m^3/(kg s^2), and the solar mass $M_\odot \approx 2.0 \times 10^{30}$ kg.]

(ii) Write down a Big Bang metric suitable for calculations in our universe, which is spatially flat. You may leave the scale factor $a(t)$ as an undetermined function (where t is the proper time).

Let t_e be the time of emission, and t_o be the time of detection. Write down a formula for the frequency of the gravitational wave as it is observed by LIGO, from the perspective of Earth's local reference frame. [For purposes of solving this problem, you may treat the Earth and the binary black hole system as both being at rest relative to the cosmological frame of reference.]

Paper 1, Section II
38D General Relativity

A *Milne universe* is an isotropic, homogeneous model of cosmology which has negative spatial curvature, $k = -1$, and an expanding scale factor, $\dot{a}(t) > 0$, even though there is no matter or radiation ($T_{\alpha\beta} = 0$) and no cosmological constant ($\Lambda = 0$).

(a) Write down the FLRW metric for this cosmological model. Calculate the scale factor $a(t)$ as an explicit function of the proper time t of a stationary observer.

(b) Verify that the singularity as $a \to 0$ is a coordinate singularity by calculating the Kretschmann scalar. [*Hint: You may find it useful to relate the Riemann tensor to the Ricci tensor.*]

(c) By constructing an appropriate coordinate transformation, show that the Milne universe is equivalent to the interior of the future light-cone of a point p in Minkowski space-time. What do the spatial isometries of the hyperbolic $t =$ const. slices correspond to in this Minkowski space-time?

[*Hint: You may wish to use the following formulae:*

$$3\frac{\dot{a} + k}{a^2} - \Lambda = 8\pi\rho\,, \qquad \text{(Friedmann I)}$$

$$2a\ddot{a} + \dot{a}^2 + ka^2 - \Lambda = -8\pi P\,. \qquad \text{(Friedmann II)}$$

Riemann tensor in normal coordinates:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\beta\partial_\nu g_{\alpha\mu})\,.\,]$$

Paper 2, Section II
38D General Relativity
(a) Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \; .$$

  (i) Show that lines of constant longitude ($\phi = $ constant) are geodesics, and that the only line of constant latitude ($\theta = $ constant) that is a geodesic is the equator ($\theta = \pi/2$).

  (ii) Take a vector with components $V^\mu = (1, 0)$ in these coordinates, and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as functions of θ?

(b) In units where $8\pi G = 1$, the Einstein equation states that $T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$. Solve for $R_{\alpha\beta}$ in terms of $T_{\alpha\beta}$ and $T = g^{\alpha\beta}T_{\alpha\beta}$, in general space-time dimension $n > 2$.

(c) Using the symmetries of the Riemann curvature tensor, show that in $n = 2$ dimensions, $R_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta}R$. [*Hint: Since this is a tensor equation, it only needs to be proved in one particular coordinate system.*] Explain the implications of this if we try to define General Relativity in $n = 2$ space-time dimensions.

Paper 3, Section II

37D General Relativity

Recall that the Schwarzschild metric is

$$ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,,$$

in units where $c = G = 1$. An advanced alien civilization builds a static, spherically-symmetrical space station surrounding a non-rotating black hole of mass M. The station itself has mass $M_{\mathrm{st}} \ll M$ and is located at a radius $r_{\mathrm{st}} > 2M$ (in Schwarzschild coordinates). It occupies a very thin shell of width $\delta r \ll r_{\mathrm{st}}$.

(a) Some sodium lamps, which emit photons at a characteristic wavelength λ, are attached to the space station. In terms of r_{st}, what is the wavelength of these photons as seen by an observer at radius $r \gg r_{\mathrm{st}}$? What happens in the limit that r_{st} approaches the event horizon?

(b) What is the magnitude and direction of the proper acceleration of the space station (*i.e.* the acceleration in its own instantaneous rest frame)? Verify that in the limit $r_{\mathrm{st}} \to \infty$, the magnitude is equal to the acceleration due to Newtonian gravity.

Now suppose we wish to take into account the gravitational effects of the space station itself, even though $M_{\mathrm{st}} \ll M$. The space station has a mass per unit area of ρ as measured in its own local frame of reference. However, its effective gravitational energy is reduced by the fact that it is in a gravitational potential.

(c) What is an appropriate metric to use outside of the space station? Your answer should indicate how the metric depends on ρ. Why is this justified? [*Hint: You do not need to explicitly solve the Einstein equation in order to answer this problem.*]

Paper 4, Section II
37D General Relativity

(a) Determine whether each of the following spaces is, or is not, a manifold. Justify your answers.

    (i) $\mathbb{R}^3$ with points identified if they are related by the transformation $(x, y, z) \to (-x, -y, -z)$.

    (ii) $\mathbb{R}^3$, except that the closed ball of all points with $x^2 + y^2 + z^2 \leqslant 1$ is removed.

(b) Let a tensor $\mathbf{S}$ at point $p \in \mathcal{M}$ be defined as a linear map

$$\mathbf{S} : T_p^*(\mathcal{M}) \to T_p(\mathcal{M}) \times T_p(\mathcal{M}) \,,$$

where T_p is tangent space and T_p^* is cotangent space.

    (i) What is the rank of $\mathbf{S}$? Use $\binom{r}{s}$ notation.

    (ii) What is the rank of $\mathbf{S} \otimes \nabla\mathbf{S}$, where $\otimes$ is an outer product and ∇ is the covariant derivative?

Consider a spacelike geodesic which goes from point p to point q. As a geodesic, this curve minimizes the action

$$\mathcal{S} = \int_0^1 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\lambda \,,$$

where $x = x(\lambda)$ with $x(0) = p, x(1) = q$ and $\dot{x}^\mu = dx^\mu/d\lambda$. Show using the Euler-Lagrange equations that

$$\frac{d^2 x^\beta}{ds^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \,,$$

where s is the proper distance along the geodesic and $\Gamma^\beta_{\mu\nu}$ is the Levi-Civita connection.

Paper 1, Section II

38C General Relativity

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ may be defined (in $n = 4$ spacetime dimensions) as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\right) + \frac{1}{6}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R,$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor and R is the Ricci scalar.

(a) Show that $C^{\alpha}{}_{\beta\alpha\delta} = 0$ and deduce that all other contractions vanish.

(b) A conformally flat metric takes the form

$$g_{\alpha\beta} = e^{2\omega}\eta_{\alpha\beta},$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and ω is a scalar function. Calculate the Weyl tensor at a given point p. [You may assume that $\partial_{\alpha}\omega = 0$ at p.]

(c) The Schwarzschild metric outside a spherically symmetric mass (such as the Sun, Earth or Moon) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2.$$

(i) Calculate the leading-order contribution to the Weyl component C_{trtr} valid at large distances, $r \gg 2M$, beyond the central spherical mass.

(ii) What physical phenomenon, known from ancient times, can be attributed to this component of the Weyl tensor at the location of the Earth? [This is after subtracting off the Earth's own gravitational field, and neglecting the Earth's motion within the solar system.] Briefly explain why your answer is consistent with the Einstein equivalence principle.

Paper 2, Section II

38C General Relativity

Consider the following metric for a 3-dimensional, static and rotationally symmetric Lorentzian manifold:

$$ds^2 = r^{-2}(-dt^2 + dr^2) + r^2 d\theta^2 \,.$$

(a) Write down a Lagrangian $\mathcal{L}$ for arbitrary geodesics in this metric, if the geodesic is affinely parameterized with respect to λ. What condition may be imposed to distinguish spacelike, timelike, and null geodesics?

(b) Find the three constants of motion for any geodesic.

(c) Two observation stations are sitting at radii $r = R$ and $r = 2R$ respectively, and at the same angular coordinate. Each is accelerating so as to remain stationary with respect to time translations. At $t = 0$ a photon is emitted from the naked singularity at $r = 0$.

    (i) At what time t_1 does the photon reach the inner station?

    (ii) Express the frequency ν_2 of the photon at the outer station in terms of the frequency ν_1 at the inner station. Explain whether the photon is redshifted or blueshifted as it travels.

(d) Consider a complete (i.e. infinite in both directions) spacelike geodesic on a constant-t slice with impact parameter $b = r_{\min} > 0$. What is the angle $\Delta\theta$ between the two asymptotes of the geodesic at $r = \infty$? [You need not be concerned with the sign of $\Delta\theta$ or the periodicity of the θ coordinate.]

[*Hint: You may find integration by substitution useful.*]

Paper 3, Section II

37C General Relativity

(a) Determine the signature of the metric tensor $g_{\mu\nu}$ given by

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Is it Riemannian, Lorentzian, or neither?

(b) Consider a stationary black hole with the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

These coordinates break down at the horizon $r = 2M$. By making a change of coordinates, show that this metric can be converted to infalling Eddington–Finkelstein coordinates.

(c) A spherically symmetric, narrow pulse of radiation with total energy E falls radially inwards at the speed of light from infinity, towards the origin of a spherically symmetric spacetime that is otherwise empty. Assume that the radial width λ of the pulse is very small compared to the energy ($\lambda \ll E$), and the pulse can therefore be treated as instantaneous.

  (i) Write down a metric for the region outside the pulse, which is free from coordinate singularities. Briefly justify your answer. For what range of coordinates is this metric valid?

  (ii) Write down a metric for the region inside the pulse. Briefly justify your answer. For what range of coordinates is this metric valid?

  (iii) What is the final state of the system?

Paper 4, Section II

37C General Relativity

(a) A flat ($k=0$), isotropic and homogeneous universe has metric $g_{\alpha\beta}$ given by

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) . \qquad (\dagger)$$

(i) Show that the non-vanishing Christoffel symbols and Ricci tensor components are

$$\Gamma^0_{ii} = a\,\dot{a}\,, \quad \Gamma^i_{0i} = \Gamma^i_{i0} = \frac{\dot{a}}{a}\,, \qquad R_{00} = -3\frac{\ddot{a}}{a}\,, \quad R_{ii} = a\,\ddot{a} + 2\dot{a}^2\,,$$

where dots are time derivatives and $i \in \{1,2,3\}$ (no summation assumed).

(ii) Derive the first-order Friedmann equation from the Einstein equations, $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$.

(b) Consider a flat universe described by ($\dagger$) with $\Lambda = 0$ in which late-time acceleration is driven by "phantom" dark energy obeying an equation of state with pressure $P_{\rm ph} = w\rho_{\rm ph}$, where $w < -1$ and the energy density $\rho_{\rm ph} > 0$. The remaining matter is dust, so we have $\rho = \rho_{\rm ph} + \rho_{\rm dust}$ with each component separately obeying $\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)$.

(i) Calculate an approximate solution for the scale factor $a(t)$ that is valid at late times. Show that the asymptotic behaviour is given by a Big Rip, that is, a singularity in which $a \to \infty$ at some finite time t^*.

(ii) Sketch a diagram of the scale factor a as a function of t for a convenient choice of w, ensuring that it includes (1) the Big Bang, (2) matter domination, (3) phantom-energy domination, and (4) the Big Rip. Label these epochs and mark them on the axes.

(iii) Most reasonable classical matter fields obey the null energy condition, which states that the energy–momentum tensor everywhere satisfies $T_{\alpha\beta}\,V^\alpha V^\beta \geqslant 0$ for any null vector V^α. Determine if this applies to phantom energy.

$$\left[\textit{The energy–momentum tensor for a perfect fluid is } \; T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + P g_{\alpha\beta} \right]$$

Paper 1, Section II

38D General Relativity

Let $(\mathcal{M}, \boldsymbol{g})$ be a four-dimensional manifold with metric $g_{\alpha\beta}$ of Lorentzian signature. The Riemann tensor $\boldsymbol{R}$ is defined through its action on three vector fields $\boldsymbol{X}, \boldsymbol{V}, \boldsymbol{W}$ by

$$\boldsymbol{R}(\boldsymbol{X}, \boldsymbol{V})\boldsymbol{W} = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{V}}\boldsymbol{W} - \nabla_{\boldsymbol{V}}\nabla_{\boldsymbol{X}}\boldsymbol{W} - \nabla_{[\boldsymbol{X},\boldsymbol{V}]}\boldsymbol{W},$$

and the Ricci identity is given by

$$\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma = R^\gamma{}_{\rho\alpha\beta} V^\rho.$$

(i) Show that for two arbitrary vector fields $\boldsymbol{V}, \boldsymbol{W}$, the commutator obeys

$$[\boldsymbol{V}, \boldsymbol{W}]^\alpha = V^\mu \nabla_\mu W^\alpha - W^\mu \nabla_\mu V^\alpha.$$

(ii) Let $\gamma : I \times I' \to \mathcal{M}, \quad I, I' \subset \mathbb{R}, \quad (s, t) \mapsto \gamma(s, t)$ be a one-parameter family of affinely parametrized geodesics. Let $\boldsymbol{T}$ be the tangent vector to the geodesic $\gamma(s = \text{const}, t)$ and $\boldsymbol{S}$ be the tangent vector to the curves $\gamma(s, t = \text{const})$. Derive the equation for geodesic deviation,

$$\nabla_{\boldsymbol{T}}\nabla_{\boldsymbol{T}}\boldsymbol{S} = \boldsymbol{R}(\boldsymbol{T}, \boldsymbol{S})\boldsymbol{T}.$$

(iii) Let X^α be a unit timelike vector field $(X^\mu X_\mu = -1)$ that satisfies the geodesic equation $\nabla_{\boldsymbol{X}}\boldsymbol{X} = 0$ at every point of $\mathcal{M}$. Define

$$B_{\alpha\beta} := \nabla_\beta X_\alpha, \qquad h_{\alpha\beta} := g_{\alpha\beta} + X_\alpha X_\beta,$$

$$\Theta := B^{\alpha\beta} h_{\alpha\beta}, \qquad \sigma_{\alpha\beta} := B_{(\alpha\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}, \qquad \omega_{\alpha\beta} := B_{[\alpha\beta]}.$$

Show that

$$B_{\alpha\beta} X^\alpha = B_{\alpha\beta} X^\beta = h_{\alpha\beta} X^\alpha = h_{\alpha\beta} X^\beta = 0,$$

$$B_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \qquad g^{\alpha\beta} \sigma_{\alpha\beta} = 0.$$

(iv) Let $\boldsymbol{S}$ denote the geodesic deviation vector, as defined in (ii), of the family of geodesics defined by the vector field X^α. Show that $\boldsymbol{S}$ satisfies

$$X^\mu \nabla_\mu S^\alpha = B^\alpha{}_\mu S^\mu.$$

(v) Show that

$$X^\mu \nabla_\mu B_{\alpha\beta} = -B^\mu{}_\beta B_{\alpha\mu} + R_{\mu\beta\alpha}{}^\nu X^\mu X_\nu.$$

Paper 2, Section II

37D General Relativity

The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,.$$

(i) Show that geodesics in the Schwarzschild spacetime obey the equation

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2\,, \quad \text{where} \quad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - Q\right)\,,$$

where E, L, Q are constants and the dot denotes differentiation with respect to a suitably chosen affine parameter λ.

(ii) Consider the following three observers located in one and the same plane in the Schwarzschild spacetime which also passes through the centre of the black hole:

- Observer $\mathcal{O}_1$ is on board a spacecraft (to be modeled as a pointlike object moving on a geodesic) on a circular orbit of radius $r > 3M$ around the central mass M.

- Observer $\mathcal{O}_2$ starts at the same position as $\mathcal{O}_1$ but, instead of orbiting, stays fixed at the initial coordinate position by using rocket propulsion to counteract the gravitational pull.

- Observer $\mathcal{O}_3$ is also located at a fixed position but at large distance $r \to \infty$ from the central mass and is assumed to be able to see $\mathcal{O}_1$ whenever the two are at the same azimuthal angle ϕ.

Show that the proper time intervals $\Delta\tau_1$, $\Delta\tau_2$, $\Delta\tau_3$, that are measured by the three observers during the completion of one full orbit of observer $\mathcal{O}_1$, are given by

$$\Delta\tau_i = 2\pi\sqrt{\frac{r^2(r - \alpha_i M)}{M}}\,, \quad i = 1,\ 2,\ 3\,,$$

where α_1, α_2 and α_3 are numerical constants that you should determine.

(iii) Briefly interpret the result by arranging the $\Delta\tau_i$ in ascending order.

Paper 3, Section II

37D General Relativity

(a) Let $(\mathcal{M}, \boldsymbol{g})$ be a four-dimensional spacetime and let $\boldsymbol{T}$ denote the rank $\binom{1}{1}$ tensor defined by

$$\boldsymbol{T} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}, \quad (\boldsymbol{\eta}, \boldsymbol{V}) \mapsto \boldsymbol{\eta}(\boldsymbol{V}), \quad \forall\, \boldsymbol{\eta} \in \mathcal{T}_p^*(\mathcal{M}),\ \boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})\,.$$

Determine the components of the tensor $\boldsymbol{T}$ and use the general law for the transformation of tensor components under a change of coordinates to show that the components of $\boldsymbol{T}$ are the same in any coordinate system.

(b) In Cartesian coordinates (t, x, y, z) the Minkowski metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,.$$

Spheroidal coordinates (r, θ, ϕ) are defined through

$$\begin{aligned}
x &= \sqrt{r^2 + a^2}\, \sin\theta\, \cos\phi\,, \\
y &= \sqrt{r^2 + a^2}\, \sin\theta\, \sin\phi\,, \\
z &= r\cos\theta\,,
\end{aligned}$$

where $a \geqslant 0$ is a real constant.

(i) Show that the Minkowski metric in coordinates (t, r, θ, ϕ) is given by

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\, dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2\,. \qquad (\dagger)$$

(ii) Transform the metric $(\dagger)$ to null coordinates given by $u = t - r$, $R = r$ and show that $\partial/\partial R$ is not a null vector field for $a > 0$.

(iii) Determine a new azimuthal angle $\varphi = \phi - F(R)$ such that in the new coordinate system (u, R, θ, φ), the vector field $\partial/\partial R$ is null for any $a \geqslant 0$. Write down the Minkowski metric in this new coordinate system.

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Paper 4, Section II

37D General Relativity

In linearized general relativity, we consider spacetime metrics that are perturbatively close to Minkowski, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1$. In the Lorenz gauge, the Einstein tensor, at linear order, is given by

$$G_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu}\,, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,h\,, \tag{\dagger}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $h = \eta^{\mu\nu}h_{\mu\nu}$.

(i) Show that the (fully nonlinear) Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ can be equivalently written in terms of the Ricci tensor $R_{\alpha\beta}$ as

$$R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\,T \right)\,, \qquad T = g^{\mu\nu}T_{\mu\nu}\,.$$

Show likewise that equation $(\dagger)$ can be written as

$$\Box h_{\mu\nu} = -16\pi \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,T \right)\,. \tag{$*$}$$

(ii) In the Newtonian limit we consider matter sources with small velocities $v \ll 1$ such that time derivatives $\partial/\partial t \sim v\,\partial/\partial x^i$ can be neglected relative to spatial derivatives, and the only non-negligible component of the energy-momentum tensor is the energy density $T_{00} = \rho$. Show that in this limit, we recover from equation $(*)$ the Poisson equation $\vec{\nabla}^2 \Phi = 4\pi\rho$ of Newtonian gravity if we identify $h_{00} = -2\Phi$.

(iii) A point particle of mass M is modelled by the energy density $\rho = M\,\delta(\boldsymbol{r})$. Derive the Newtonian potential Φ for this point particle by solving the Poisson equation.

[You can assume the solution of $\vec{\nabla}^2\varphi = f(\boldsymbol{r})$ is $\varphi(\boldsymbol{r}) = -\displaystyle\int \frac{f(\boldsymbol{r}')}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}d^3r'$.]

(iv) Now consider the Einstein equations with a small positive cosmological constant, $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$, $\Lambda = \mathcal{O}(\epsilon) > 0$. Repeat the steps of questions (i)-(iii), again identifying $h_{00} = -2\Phi$, to show that the Newtonian limit is now described by the Poisson equation $\vec{\nabla}^2\Phi = 4\pi\rho - \Lambda$, and that a solution for the potential of a point particle is given by

$$\Phi = -\frac{M}{r} - Br^2\,,$$

where B is a constant you should determine. Briefly discuss the effect of the Br^2 term and determine for which range of the radius r the weak-field limit is a justified approximation. [*Hint: Absorb the term $\Lambda g_{\alpha\beta}$ as part of the energy-momentum tensor. Note also that in spherical symmetry $\vec{\nabla}^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf)$.*]

Paper 4, Section II

36D General Relativity

(a) Consider the spherically symmetric spacetime metric

$$ds^2 = -\lambda^2 dt^2 + \mu^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \,, \tag{†}$$

where λ and μ are functions of t and r. Use the Euler-Lagrange equations for the geodesics of the spacetime to compute all non-vanishing Christoffel symbols for this metric.

(b) Consider the static limit of the line element (†) where λ and μ are functions of the radius r only, and let the matter coupled to gravity be a spherically symmetric fluid with energy momentum tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} \,, \qquad u^\mu = [\lambda^{-1}, \, 0, \, 0, \, 0] \,,$$

where the pressure P and energy density ρ are also functions of the radius r. For these Tolman-Oppenheimer-Volkoff stellar models, the Einstein and matter equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and $\nabla_\mu T^\mu{}_\nu = 0$ reduce to

$$\frac{\partial_r \lambda}{\lambda} = \frac{\mu^2 - 1}{2r} + 4\pi r \mu^2 P \,,$$

$$\partial_r m = 4\pi r^2 \rho \,, \quad \text{where} \quad m(r) = \frac{r}{2}\left(1 - \frac{1}{\mu^2}\right) \,,$$

$$\partial_r P = -(\rho + P)\left(\frac{\mu^2 - 1}{2r} + 4\pi r \mu^2 P\right) \,.$$

Consider now a constant density solution to the above Einstein and matter equations, where ρ takes the non-zero constant value ρ_0 out to a radius R and $\rho = 0$ for $r > R$. Show that for such a star,

$$\partial_r P = \frac{4\pi r}{1 - \frac{8}{3}\pi \rho_0 r^2}\left(P + \frac{1}{3}\rho_0\right)(P + \rho_0) \,,$$

and that the pressure at the centre of the star is

$$P(0) = -\rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1} \,, \quad \text{with} \quad M = \frac{4}{3}\pi \rho_0 R^3 \,.$$

Show that $P(0)$ diverges if $M = 4R/9$. $\left[\textit{Hint: at the surface of the star the pressure vanishes: } P(R) = 0 \,.\right]$

Paper 2, Section II

36D General Relativity

Consider the spacetime metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \quad \text{with} \quad f(r) = 1 - \frac{2m}{r} - H^2 r^2\,,$$

where $H > 0$ and $m > 0$ are constants.

(a) Write down the Lagrangian for geodesics in this spacetime, determine three independent constants of motion and show that geodesics obey the equation

$$\dot{r}^2 + V(r) = E^2\,,$$

where E is constant, the overdot denotes differentiation with respect to an affine parameter and $V(r)$ is a potential function to be determined.

(b) Sketch the potential $V(r)$ for the case of null geodesics, find any circular null geodesics of this spacetime, and determine whether they are stable or unstable.

(c) Show that $f(r)$ has two positive roots r_- and r_+ if $mH < 1/\sqrt{27}$ and that these satisfy the relation $r_- < 1/(\sqrt{3}H) < r_+$.

(d) Describe in one sentence the physical significance of those points where $f(r) = 0$.

Paper 3, Section II

37D General Relativity

(a) Let $\mathcal{M}$ be a manifold with coordinates x^μ. The commutator of two vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ is defined as

$$[\boldsymbol{V}, \boldsymbol{W}]^\alpha = V^\nu \partial_\nu W^\alpha - W^\nu \partial_\nu V^\alpha \,.$$

(i) Show that $[\boldsymbol{V}, \boldsymbol{W}]$ transforms like a vector field under a change of coordinates from x^μ to $\tilde{x}^\mu$.

(ii) Show that the commutator of any two basis vectors vanishes, i.e.

$$\left[\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right] = 0 \,.$$

(iii) Show that if $\boldsymbol{V}$ and $\boldsymbol{W}$ are linear combinations (not necessarily with constant coefficients) of n vector fields $\boldsymbol{Z}_{(a)}$, $a = 1, \dots, n$ that all commute with one another, then the commutator $[\boldsymbol{V}, \boldsymbol{W}]$ is a linear combination of the same n fields $\boldsymbol{Z}_{(a)}$.

[You may use without proof the following relations which hold for any vector fields $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ and any function f:

$$[\boldsymbol{V}_1, \boldsymbol{V}_2] = -[\boldsymbol{V}_2, \boldsymbol{V}_1] \,, \tag{1}$$

$$[\boldsymbol{V}_1, \boldsymbol{V}_2 + \boldsymbol{V}_3] = [\boldsymbol{V}_1, \boldsymbol{V}_2] + [\boldsymbol{V}_1, \boldsymbol{V}_3] \,, \tag{2}$$

$$[\boldsymbol{V}_1, f\boldsymbol{V}_2] = f[\boldsymbol{V}_1, \boldsymbol{V}_2] + \boldsymbol{V}_1(f)\,\boldsymbol{V}_2 \,, \tag{3}$$

but you should clearly indicate each time relation (1), (2), or (3) is used.]

(b) Consider the 2-dimensional manifold $\mathbb{R}^2$ with Cartesian coordinates $(x^1, x^2) = (x, y)$ carrying the Euclidean metric $g_{\alpha\beta} = \delta_{\alpha\beta}$.

(i) Express the coordinate basis vectors ∂_r and ∂_θ, where r and θ denote the usual polar coordinates, in terms of their Cartesian counterparts.

(ii) Define the unit vectors

$$\hat{\boldsymbol{r}} = \frac{\partial_r}{||\partial_r||} \,, \qquad \hat{\boldsymbol{\theta}} = \frac{\partial_\theta}{||\partial_\theta||}$$

and show that $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}})$ are *not* a coordinate basis, i.e. there exist no coordinates z^α such that $\hat{\boldsymbol{r}} = \partial/\partial z^1$ and $\hat{\boldsymbol{\theta}} = \partial/\partial z^2$.

Paper 1, Section II

37D General Relativity

Let $(\mathcal{M}, \boldsymbol{g})$ be a spacetime and $\boldsymbol{\Gamma}$ the Levi-Civita connection of the metric $\boldsymbol{g}$. The Riemann tensor of this spacetime is given in terms of the connection by

$$R^{\gamma}{}_{\rho\alpha\beta} = \partial_{\alpha}\Gamma^{\gamma}_{\rho\beta} - \partial_{\beta}\Gamma^{\gamma}_{\rho\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\gamma}_{\mu\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\gamma}_{\mu\beta}\,.$$

The contracted Bianchi identities ensure that the Einstein tensor satisfies

$$\nabla^{\mu}G_{\mu\nu} = 0\,.$$

(a) Show that the Riemann tensor obeys the symmetry

$$R^{\mu}{}_{\rho\alpha\beta} + R^{\mu}{}_{\beta\rho\alpha} + R^{\mu}{}_{\alpha\beta\rho} = 0\,.$$

(b) Show that a vector field V^{α} satisfies the Ricci identity

$$2\nabla_{[\alpha}\nabla_{\beta]}V^{\gamma} = \nabla_{\alpha}\nabla_{\beta}V^{\gamma} - \nabla_{\beta}\nabla_{\alpha}V^{\gamma} = R^{\gamma}{}_{\rho\alpha\beta}V^{\rho}\,.$$

Calculate the analogous expression for a rank $\binom{2}{0}$ tensor $T^{\mu\nu}$, i.e. calculate $\nabla_{[\alpha}\nabla_{\beta]}T^{\mu\nu}$ in terms of the Riemann tensor.

(c) Let K^{α} be a vector that satisfies the Killing equation

$$\nabla_{\alpha}K_{\beta} + \nabla_{\beta}K_{\alpha} = 0\,.$$

Use the symmetry relation of part (a) to show that

$$\nabla_{\nu}\nabla_{\mu}K^{\alpha} = R^{\alpha}{}_{\mu\nu\beta}K^{\beta}\,,$$
$$\nabla^{\mu}\nabla_{\mu}K^{\alpha} = -R^{\alpha}{}_{\beta}K^{\beta}\,,$$

where $R_{\alpha\beta}$ is the Ricci tensor.

(d) Show that

$$K^{\alpha}\nabla_{\alpha}R = 2\nabla^{[\mu}\nabla^{\lambda]}\nabla_{[\mu}K_{\lambda]}\,,$$

and use the result of part (b) to show that the right hand side evaluates to zero, hence showing that $K^{\alpha}\nabla_{\alpha}R = 0$.

Paper 1, Section II

37E General Relativity

Consider the de Sitter metric

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2)\,,$$

where $H > 0$ is a constant.

(a) Write down the Lagrangian governing the geodesics of this metric. Use the Euler–Lagrange equations to determine all non-vanishing Christoffel symbols.

(b) Let $\mathcal{C}$ be a timelike geodesic parametrized by proper time τ with initial conditions at $\tau = 0$,

$$t = 0\,, \quad x = y = z = 0\,, \quad \dot{x} = v_0 > 0\,, \quad \dot{y} = \dot{z} = 0\,,$$

where the dot denotes differentiation with respect to τ and v_0 is a constant. Assuming both t and τ to be future oriented, show that at $\tau = 0$,

$$\dot{t} = \sqrt{1 + v_0^2}\,.$$

(c) Find a relation between τ and t along the geodesic of part (b) and show that $t \to -\infty$ for a finite value of τ. [You may use without proof that

$$\int \frac{1}{\sqrt{1 + ae^{-bu}}}\, du = \frac{1}{b} \ln \frac{\sqrt{1 + ae^{-bu}} + 1}{\sqrt{1 + ae^{-bu}} - 1} + \text{constant}\,, \qquad a,\, b > 0.]$$

(d) Briefly interpret this result.

Paper 2, Section II

37E General Relativity

The Friedmann equations and the conservation of energy-momentum for a spatially homogeneous and isotropic universe are given by:

$$3\frac{\dot{a}^2 + k}{a^2} - \Lambda = 8\pi\rho\,, \qquad \frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} - \Lambda = -8\pi P\,, \qquad \dot{\rho} = -3\frac{\dot{a}}{a}(P + \rho)\,,$$

where a is the scale factor, ρ the energy density, P the pressure, Λ the cosmological constant and $k = +1,\ 0,\ -1$.

(a) Show that for an equation of state $P = w\rho$, $w = $ constant, the energy density obeys $\rho = \frac{3\mu}{8\pi}a^{-3(1+w)}$, for some constant μ.

(b) Consider the case of a matter dominated universe, $w = 0$, with $\Lambda = 0$. Write the equation of motion for the scale factor a in the form of an effective potential equation,

$$\dot{a}^2 + V(a) = C\,,$$

where you should determine the constant C and the potential $V(a)$. Sketch the potential $V(a)$ together with the possible values of C and qualitatively discuss the long-term dynamics of an initially small and expanding universe for the cases $k = +1,\ 0,\ -1$.

(c) Repeat the analysis of part (b), again assuming $w = 0$, for the cases:

(i) $\Lambda > 0$, $k = -1$,

(ii) $\Lambda < 0$, $k = 0$,

(iii) $\Lambda > 0$, $k = 1$.

Discuss all qualitatively different possibilities for the dynamics of the universe in each case.

Paper 4, Section II

37E   General Relativity

(a) In the Newtonian weak-field limit, we can write the spacetime metric in the form

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\delta_{ij}\, dx^i\, dx^j \,, \qquad (*)$$

where $\delta_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2$ and the potential $\Phi(t,x,y,z)$, as well as the velocity v of particles moving in the gravitational field are assumed to be small, i.e.,

$$\Phi,\ \partial_t\Phi,\ \partial_{x^i}\Phi,\ v^2 \ll 1.$$

Use the geodesic equation for this metric to derive the equation of motion for a massive point particle in the Newtonian limit.

(b) The far-field limit of the Schwarzschild metric is a special case of $(*)$ given, in spherical coordinates, by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right),$$

where now $M/r \ll 1$. For the following questions, state your results to first order in M/r, *i.e.* neglecting terms of $\mathcal{O}((M/r)^2)$.

(i) Let $r_1, r_2 \gg M$. Calculate the proper length S along the radial curve from r_1 to r_2 at fixed t, θ, φ.

(ii) Consider a massless particle moving radially from $r = r_1$ to $r = r_2$. According to an observer at rest at r_2, what time T elapses during this motion?

(iii) The *effective velocity* of the particle as seen by the observer at r_2 is defined as $v_{\text{eff}} := S/T$. Evaluate v_{eff} and then take the limit of this result as $r_1 \to r_2$. Briefly discuss the value of v_{eff} in this limit.

Paper 3, Section II

38E   General Relativity

The Schwarzschild metric in isotropic coordinates $\bar{x}^{\bar{\alpha}} = (\bar{t}, \bar{x}, \bar{y}, \bar{z})$, $\bar{\alpha} = 0, \ldots, 3$, is given by:

$$ds^2 = \bar{g}_{\bar{\alpha}\bar{\beta}}d\bar{x}^{\bar{\alpha}}d\bar{x}^{\bar{\beta}} = -\frac{(1-A)^2}{(1+A)^2}d\bar{t}^2 + (1+A)^4(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)$$

where

$$A = \frac{m}{2\bar{r}}, \qquad \bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}\,,$$

and m is the mass of the black hole.

(a) Let $x^\mu = (t, x, y, z)$, $\mu = 0, \ldots, 3$, denote a coordinate system related to $\bar{x}^{\bar{\alpha}}$ by

$$\bar{t} = \gamma(t - vx), \quad \bar{x} = \gamma(x - vt), \quad \bar{y} = y, \quad \bar{z} = z\,,$$

where $\gamma = 1/\sqrt{1 - v^2}$ and $-1 < v < 1$. Write down the transformation matrix $\partial\bar{x}^{\bar{\alpha}}/\partial x^\mu$, briefly explain its physical meaning and show that the inverse transformation is of the same form, but with $v \to -v$.

(b) Using the coordinate transformation matrix of part (a), or otherwise, show that the components $g_{\mu\nu}$ of the metric in coordinates x^μ are given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(A)(-dt^2 + dx^2 + dy^2 + dz^2) + \gamma^2 g(A)(dt - v\,dx)^2\,,$$

where f and g are functions of A that you should determine. You should also express A in terms of the coordinates (t, x, y, z).

(c) Consider the limit $v \to 1$ with $p = m\gamma$ held constant. Show that for points $x \neq t$ the function $A \to 0$, while $\gamma^2 A$ tends to a finite value, which you should determine. Hence determine the metric components $g_{\mu\nu}$ at points $x \neq t$ in this limit.

Paper 2, Section II
35D General Relativity
(a) The Friedmann–Robertson–Walker metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right],$$

where $k = -1, 0, +1$ and $a(t)$ is the scale factor.

For $k = +1$, show that this metric can be written in the form

$$ds^2 = -dt^2 + \gamma_{ij}dx^i dx^j = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Calculate the equatorial circumference ($\theta = \pi/2$) of the submanifold defined by constant t and χ.

Calculate the proper volume, defined by $\int \sqrt{\det\gamma}\, d^3x$, of the hypersurface defined by constant t.

(b) The Friedmann equations are

$$3\left(\frac{\dot{a}^2 + k}{a^2}\right) - \Lambda = 8\pi\rho,$$

$$\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} - \Lambda = -8\pi P,$$

where $\rho(t)$ is the energy density, $P(t)$ is the pressure, Λ is the cosmological constant and dot denotes d/dt.

The Einstein static universe has vanishing pressure, $P(t) = 0$. Determine a, k and Λ as a function of the density ρ.

The Einstein static universe with $a = a_0$ and $\rho = \rho_0$ is perturbed by radiation such that

$$a = a_0 + \delta a(t), \qquad \rho = \rho_0 + \delta\rho(t), \qquad P = \frac{1}{3}\delta\rho(t),$$

where $\delta a \ll a_0$ and $\delta\rho \ll \rho_0$. Show that the Einstein static universe is unstable to this perturbation.

Paper 1, Section II
36D General Relativity
A static black hole in a five-dimensional spacetime is described by the metric

$$ds^2 = -\left(1 - \frac{\mu}{r^2}\right) dt^2 + \left(1 - \frac{\mu}{r^2}\right)^{-1} dr^2 + r^2[d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)]\,,$$

where $\mu > 0$ is a constant.

A geodesic lies in the plane $\theta = \psi = \pi/2$ and has affine parameter λ. Show that

$$E = \left(1 - \frac{\mu}{r^2}\right)\frac{dt}{d\lambda} \quad \text{and} \quad L = r^2\frac{d\phi}{d\lambda}$$

are both constants of motion. Write down a third constant of motion.

Show that timelike and null geodesics satisfy the equation

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2$$

for some potential $V(r)$ which you should determine.

Circular geodesics satisfy the equation $V'(r) = 0$. Calculate the values of r for which circular null geodesics exist and for which circular timelike geodesics exist. Which are stable and which are unstable? Briefly describe how this compares to circular geodesics in the four-dimensional Schwarzschild geometry.

Paper 3, Section II
36D General Relativity
Let $\mathcal{M}$ be a two-dimensional manifold with metric $\boldsymbol{g}$ of signature $-+$.

(i) Let $p \in \mathcal{M}$. Use normal coordinates at the point p to show that one can choose two null vectors $\mathbf{V}$, $\mathbf{W}$ that form a basis of the vector space $\mathcal{T}_p(\mathcal{M})$.

(ii) Consider the interval $I \subset \mathbb{R}$. Let $\gamma : I \to \mathcal{M}$ be a null curve through p and $\mathbf{U} \neq 0$ be the tangent vector to γ at p. Show that the vector $\mathbf{U}$ is either parallel to $\mathbf{V}$ or parallel to $\mathbf{W}$.

(iii) Show that every null curve in $\mathcal{M}$ is a null geodesic.
    [*Hint: You may wish to consider the acceleration $a^\alpha = U^\beta \nabla_\beta U^\alpha$.*]

(iv) By providing an example, show that not every null curve in four-dimensional Minkowski spacetime is a null geodesic.

Paper 4, Section II
36D General Relativity

(a) In the transverse traceless gauge, a plane gravitational wave propagating in the z direction is described by a perturbation $h_{\alpha\beta}$ of the Minkowski metric $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ in Cartesian coordinates $x^\alpha = (t, x, y, z)$, where

$$h_{\alpha\beta} = H_{\alpha\beta} e^{ik_\mu x^\mu}, \quad \text{where} \quad k^\mu = \omega(1, 0, 0, 1),$$

and $H_{\alpha\beta}$ is a constant matrix. Spacetime indices in this question are raised or lowered with the Minkowski metric.

The energy-momentum tensor of a gravitational wave is defined to be

$$\tau_{\mu\nu} = \frac{1}{32\pi}(\partial_\mu h^{\alpha\beta})(\partial_\nu h_{\alpha\beta}).$$

Show that $\partial^\nu \tau_{\mu\nu} = \frac{1}{2}\partial_\mu \tau^\nu{}_\nu$ and hence, or otherwise, show that energy and momentum are conserved.

(b) A point mass m undergoes harmonic motion along the z-axis with frequency ω and amplitude L. Compute the energy flux emitted in gravitational radiation.

[*Hint: The quadrupole formula for time-averaged energy flux radiated in gravitational waves is*

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{5}\langle \dddot{Q}_{ij} \dddot{Q}_{ij}\rangle$$

where Q_{ij} is the reduced quadrupole tensor.]

Paper 4, Section II

35D General Relativity

A spherically symmetric static spacetime has metric

$$ds^2 = -\left(1 + r^2/b^2\right) dt^2 + \frac{dr^2}{1 + r^2/b^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

where $-\infty < t < \infty$, $r \geqslant 0$, b is a positive constant, and units such that $c = 1$ are used.

(a) Explain why a time-like geodesic may be assumed, without loss of generality, to lie in the equatorial plane $\theta = \pi/2$. For such a geodesic, show that the quantities

$$E = (1 + r^2/b^2)\,\dot{t} \qquad \text{and} \qquad h = r^2\dot{\phi}$$

are constants of the motion, where a dot denotes differentiation with respect to proper time, τ. Hence find a first-order differential equation for $r(\tau)$.

(b) Consider a massive particle fired from the origin, $r = 0$. Show that the particle will return to the origin and find the proper time taken.

(c) Show that circular orbits $r = a$ are possible for any $a > 0$ and determine whether such orbits are stable. Show that on any such orbit a clock measures coordinate time.

Paper 1, Section II

35D General Relativity

Consider a family of geodesics with s an affine parameter and V^a the tangent vector on each curve. The *equation of geodesic deviation for a vector field W^a* is

$$\frac{D^2 W^a}{Ds^2} = R^a{}_{bcd} V^b V^c W^d, \qquad (*)$$

where $\dfrac{D}{Ds}$ denotes the directional covariant derivative $V^b \nabla_b$.

(i) Show that if

$$V^b \frac{\partial W^a}{\partial x^b} = W^b \frac{\partial V^a}{\partial x^b}$$

then W^a satisfies $(*)$.

(ii) Show that V^a and sV^a satisfy $(*)$.

(iii) Show that if W^a is a Killing vector field, meaning that $\nabla_b W_a + \nabla_a W_b = 0$, then W^a satisfies $(*)$.

(iv) Show that if $W^a = wU^a$ satisfies $(*)$, where w is a scalar field and U^a is a time-like unit vector field, then

$$\frac{d^2 w}{ds^2} = (\Omega^2 - K)w,$$

where $\quad \Omega^2 = -\dfrac{DU^a}{Ds}\dfrac{DU_a}{Ds} \quad$ and $\quad K = R_{abcd} U^a V^b V^c U^d.$

[*You may use*: $\nabla_b \nabla_c X^a - \nabla_c \nabla_b X^a = R^a{}_{dbc} X^d$ *for any vector field X^a.*]

Paper 2, Section II

35D General Relativity

The Kasner (vacuum) cosmological model is defined by the line element

$$ds^2 = -c^2 dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \qquad \text{with} \qquad t > 0,$$

where p_1, p_2, p_3 are constants with $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$ and $0 < p_1 < 1$. Show that $p_2\, p_3 < 0$.

Write down four equations that determine the null geodesics of the Kasner model.

If k^a is the tangent vector to the trajectory of a photon and u^a is the four-velocity of a comoving observer (i.e., an observer at rest in the (t, x, y, z) coordinate system above), what is the physical interpretation of $k_a u^a$?

Let O be a comoving observer at the origin, $x = y = z = 0$, and let S be a comoving source of photons located on one of the spatial coordinate axes.

   (i) Show that photons emitted by S and observed by O can be either red-shifted or blue-shifted, depending on the location of S.

   (ii) Given any fixed time $t = T$, show that there are locations for S on each coordinate axis from which no photons reach O for $t \leqslant T$.

Now suppose that $p_1 = 1$ and $p_2 = p_3 = 0$. Does the property in (ii) still hold?

Paper 3, Section II

35D General Relativity

For a spacetime that is nearly flat, the metric g_{ab} can be expressed in the form

$$g_{ab} = \eta_{ab} + h_{ab},$$

where η_{ab} is a flat metric (not necessarily diagonal) with constant components, and the components of h_{ab} and their derivatives are small. Show that

$$2R_{bd} \approx h_d{}^a{}_{,ba} + h_b{}^a{}_{,da} - h^a{}_{a,bd} - h_{bd,ac}\eta^{ac},$$

where indices are raised and lowered using η_{ab}.

[You may assume that $R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}$.]

For the line element

$$ds^2 = 2du\, dv + dx^2 + dy^2 + H(u, x, y)\, du^2,$$

where H and its derivatives are small, show that the linearised vacuum field equations reduce to $\nabla^2 H = 0$, where ∇^2 is the two-dimensional Laplacian operator in x and y.

Paper 4, Section II

34D General Relativity

In static spherically symmetric coordinates, the metric g_{ab} for de Sitter space is given by

$$ds^2 = -(1 - r^2/a^2)dt^2 + (1 - r^2/a^2)^{-1}dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and a is a constant.

(a) Let $u = t - a\tanh^{-1}(r/a)$ for $r \leqslant a$. Use the (u, r, θ, ϕ) coordinates to show that the surface $r = a$ is non-singular. Is $r = 0$ a space-time singularity?

(b) Show that the vector field $g^{ab}u_{,a}$ is null.

(c) Show that the radial null geodesics must obey either

$$\frac{du}{dr} = 0 \qquad \text{or} \qquad \frac{du}{dr} = -\frac{2}{1 - r^2/a^2}.$$

Which of these families of geodesics is outgoing $(dr/dt > 0)$?

Sketch these geodesics in the (u, r) plane for $0 \leqslant r \leqslant a$, where the r-axis is horizontal and lines of constant u are inclined at $45°$ to the horizontal.

(d) Show, by giving an explicit example, that an observer moving on a timelike geodesic starting at $r = 0$ can cross the surface $r = a$ within a finite proper time.

Paper 2, Section II
34D General Relativity

(a) The Schwarzschild metric is

$$ds^2 = -(1 - r_s/r)dt^2 + (1 - r_s/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(in units for which the speed of light $c = 1$). Show that a timelike geodesic in the equatorial plane obeys

$$\tfrac{1}{2}\dot{r}^2 + V(r) = \tfrac{1}{2}E^2,$$

where

$$2V(r) = \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{h^2}{r^2}\right)$$

and E and h are constants.

(b) For a circular orbit of radius r, show that

$$h^2 = \frac{r^2 r_s}{2r - 3r_s}.$$

Given that the orbit is stable, show that $r > 3r_s$.

(c) Alice lives on a small planet that is in a stable circular orbit of radius r around a (non-rotating) black hole of radius r_s. Bob lives on a spacecraft in deep space far from the black hole and at rest relative to it. Bob is ageing k times faster than Alice. Find an expression for k^2 in terms of r and r_s and show that $k < \sqrt{2}$.

Paper 3, Section II

35D General Relativity

Let $\Gamma^a{}_{bc}$ be the Levi-Civita connection and $R^a{}_{bcd}$ the Riemann tensor corresponding to a metric $g_{ab}(x)$, and let $\widetilde{\Gamma}^a{}_{bc}$ be the Levi-Civita connection and $\widetilde{R}^a{}_{bcd}$ the Riemann tensor corresponding to a metric $\tilde{g}_{ab}(x)$. Let $T^a{}_{bc} = \widetilde{\Gamma}^a{}_{bc} - \Gamma^a{}_{bc}$.

(a) Show that $T^a{}_{bc}$ is a tensor.

(b) Using local inertial coordinates for the metric g_{ab}, or otherwise, show that

$$\widetilde{R}^a{}_{bcd} - R^a{}_{bcd} = 2T^a{}_{b[d;c]} - 2T^a{}_{e[d}T^e{}_{c]b}$$

holds in all coordinate systems, where the semi-colon denotes covariant differentiation using the connection $\Gamma^a{}_{bc}$. [You may assume that $R^a{}_{bcd} = 2\Gamma^a{}_{b[d,c]} - 2\Gamma^a{}_{e[d}\Gamma^e{}_{c]b}$.]

(c) In the case that $T^a{}_{bc} = \ell^a g_{bc}$ for some vector field ℓ^a, show that $\widetilde{R}_{bd} = R_{bd}$ if and only if

$$\ell_{b;d} + \ell_b \ell_d = 0.$$

(d) Using the result that $\ell_{[a;b]} = 0$ if and only if $\ell_a = \phi_{,a}$ for some scalar field ϕ, show that the condition on ℓ_a in part (c) can be written as

$$k_{a;b} = 0$$

for a certain covector field k_a, which you should define.

Paper 1, Section II

35D General Relativity

A vector field ξ^a is said to be a *conformal Killing vector field* of the metric g_{ab} if

$$\xi_{(a;b)} = \tfrac{1}{2}\psi g_{ab} \qquad\qquad (*)$$

for some scalar field ψ. It is a *Killing vector field* if $\psi = 0$.

(a) Show that $(*)$ is equivalent to

$$\xi^c g_{ab,c} + \xi^c{}_{,a}\, g_{bc} + \xi^c{}_{,b}\, g_{ac} = \psi\, g_{ab}\,.$$

(b) Show that if ξ^a is a conformal Killing vector field of the metric g_{ab}, then ξ^a is a Killing vector field of the metric $\mathrm{e}^{2\phi}g_{ab}$, where ϕ is any function that obeys

$$2\xi^c\phi_{,c} + \psi = 0\,.$$

(c) Use part (b) to find an example of a metric with coordinates t, x, y and z (where $t > 0$) for which (t, x, y, z) are the contravariant components of a Killing vector field. [*Hint: You may wish to start by considering what happens in Minkowski space.*]

Paper 4, Section II

36E General Relativity

A plane-wave spacetime has line element

$$ds^2 = Hdu^2 + 2du\, dv + dx^2 + dy^2,$$

where $H = x^2 - y^2$. Show that the line element is unchanged by the coordinate transformation

$$u = \bar{u}, \quad v = \bar{v} + \bar{x}e^{\bar{u}} - \tfrac{1}{2}e^{2\bar{u}}, \quad x = \bar{x} - e^{\bar{u}}, \quad y = \bar{y}. \qquad (*)$$

Show more generally that the line element is unchanged by coordinate transformations of the form

$$u = \bar{u} + a, \quad v = \bar{v} + b\bar{x} + c, \quad x = \bar{x} + p, \quad y = \bar{y},$$

where a, b, c and p are functions of $\bar{u}$, which you should determine and which depend in total on four parameters (arbitrary constants of integration).

Deduce (without further calculation) that the line element is unchanged by a 6-parameter family of coordinate transformations, of which a 5-parameter family leave invariant the surfaces $u = $ constant.

For a general coordinate transformation $x^a = x^a(\bar{x}^b)$, give an expression for the transformed Ricci tensor $\bar{R}_{cd}$ in terms of the Ricci tensor R_{ab} and the transformation matrices $\dfrac{\partial x^a}{\partial \bar{x}^c}$. Calculate $\bar{R}_{\bar{x}\bar{x}}$ when the transformation is given by $(*)$ and deduce that $R_{vv} = R_{vx}$.

Paper 2, Section II

36E General Relativity

Show how the geodesic equations and hence the Christoffel symbols $\Gamma^a{}_{bc}$ can be obtained from a Lagrangian.

In units with $c = 1$, the FLRW spacetime line element is

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).$$

Show that $\Gamma^1{}_{01} = \dot{a}/a$.

You are given that, for the above metric, $G_0{}^0 = -3\dot{a}^2/a^2$ and $G_1{}^1 = -2\ddot{a}/a - \dot{a}^2/a^2$, where $G_a{}^b$ is the Einstein tensor, which is diagonal. Verify by direct calculation that $\nabla_b G_a{}^b = 0$.

Solve the vacuum Einstein equations in the presence of a cosmological constant to determine the form of $a(t)$.

Paper 3, Section II

37E General Relativity

The vector field V^a is the normalised ($V_a V^a = -c^2$) tangent to a congruence of timelike geodesics, and $B_{ab} = \nabla_b V_a$.

Show that:

(i) $V^a B_{ab} = V^b B_{ab} = 0$;

(ii) $V^c \nabla_c B_{ab} = -B^c{}_b B_{ac} - R^d{}_{acb} V^c V_d$.

[You may use the Ricci identity $\nabla_c \nabla_b X_a = \nabla_b \nabla_c X_a - R^d{}_{acb} X_d$.]

Now assume that B_{ab} is symmetric and let $\theta = B_a{}^a$. By writing $B_{ab} = \widetilde{B}_{ab} + \frac{1}{4}\theta g_{ab}$, or otherwise, show that

$$\frac{d\theta}{d\tau} \leqslant -\tfrac{1}{4}\theta^2 - R_{00} \,,$$

where $R_{00} = R_{ab} V^a V^b$ and $\dfrac{d\theta}{d\tau} \equiv V^a \nabla_a \theta$. [You may use without proof the result that $\widetilde{B}_{ab} \widetilde{B}^{ab} \geqslant 0$.]

Assume, in addition, that the stress-energy tensor T_{ab} takes the perfect-fluid form $(\rho + p/c^2)V_a V_b + p g_{ab}$ and that $\rho c^2 + 3p > 0$. Show that

$$\frac{d\theta^{-1}}{d\tau} > \frac{1}{4} \,,$$

and deduce that, if $\theta(0) < 0$, then $|\theta(\tau)|$ will become unbounded for some value of τ less than $4/|\theta(0)|$.

Paper 1, Section II

37E General Relativity

For a timelike geodesic in the equatorial plane ($\theta = \frac{1}{2}\pi$) of the Schwarzschild space-time with line element

$$ds^2 = -(1 - r_s/r)c^2 dt^2 + (1 - r_s/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

derive the equation

$$\tfrac{1}{2}\dot{r}^2 + V(r) = \tfrac{1}{2}(E/c)^2\,,$$

where

$$\frac{2V(r)}{c^2} = 1 - \frac{r_s}{r} + \frac{h^2}{c^2 r^2} - \frac{h^2 r_s}{c^2 r^3}$$

and h and E are constants. The dot denotes the derivative with respect to an affine parameter τ satisfying $c^2 d\tau^2 = -ds^2$.

Given that there is a stable circular orbit at $r = R$, show that

$$\frac{h^2}{c^2} = \frac{R^2 \epsilon}{2 - 3\epsilon}\,,$$

where $\epsilon = r_s/R$.

Compute Ω, the orbital angular frequency (with respect to τ).

Show that the angular frequency ω of small radial perturbations is given by

$$\frac{\omega^2 R^2}{c^2} = \frac{\epsilon(1 - 3\epsilon)}{2 - 3\epsilon}\,.$$

Deduce that the rate of precession of the perihelion of the Earth's orbit, $\Omega - \omega$, is approximately $3\Omega^3 T^2$, where T is the time taken for light to travel from the Sun to the Earth. [You should assume that the Earth's orbit is approximately circular, with $r_s/R \ll 1$ and $E \simeq c^2$.]

36D General Relativity

Consider the metric describing the interior of a star,

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

defined for $0 \leqslant r \leqslant r_0$ by

$$e^{\alpha(r)} = \frac{3}{2}e^{-\beta_0} - \frac{1}{2}e^{-\beta(r)},$$

with

$$e^{-2\beta(r)} = 1 - Ar^2.$$

Here $A = 2M/r_0^3$, where M is the mass of the star, $\beta_0 = \beta(r_0)$, and we have taken units in which we have set $G = c = 1$.

(i) The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p + \rho)u_a u_b + p\, g_{ab}.$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star $(0 \leqslant r \leqslant r_0)$ and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that they may be written as

$$R_{ab} = 8\pi(p + \rho)u_a u_b + 4\pi(\rho - p)g_{ab}.$$

(ii) Using the formulae given below, or otherwise, show that for $0 \leqslant r \leqslant r_0$, one has

$$4\pi(\rho + p) = \frac{(\alpha' + \beta')}{r}e^{-2\beta(r)},$$

$$4\pi(\rho - p) = \left(\frac{\beta' - \alpha'}{r} - \frac{1}{r^2}\right)e^{-2\beta(r)} + \frac{1}{r^2},$$

where primes denote differentiation with respect to r. Hence show that

$$\rho = \frac{3A}{8\pi}, \quad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\beta(r)} - e^{-\beta_0}}{3e^{-\beta_0} - e^{-\beta(r)}}\right).$$

[The non-zero components of the Ricci tensor are

$$R_{00} = e^{2\alpha - 2\beta}\left(\alpha'' - \alpha'\beta' + \alpha'^2 + \frac{2\alpha'}{r}\right)$$

$$R_{11} = -\alpha'' + \alpha'\beta' - \alpha'^2 + \frac{2\beta'}{r}$$

$$R_{22} = 1 + e^{-2\beta}\left[(\beta' - \alpha')r - 1\right]$$

$$R_{33} = \sin^2\theta R_{22}.$$

Note that

$$\alpha' = \frac{1}{2}A\, r\, e^{\beta - \alpha}, \quad \beta' = A\, r\, e^{2\beta}.]$$

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Paper 2, Section II

36D General Relativity

A spacetime contains a one-parameter family of geodesics $x^a = x^a(\lambda, \mu)$, where λ is a parameter along each geodesic, and μ labels the geodesics. The tangent to the geodesics is $T^a = \partial x^a/\partial\lambda$, and $N^a = \partial x^a/\partial\mu$ is a connecting vector. Prove that

$$\nabla_\mu T^a = \nabla_\lambda N^a \,,$$

and hence derive the equation of geodesic deviation:

$$\nabla_\lambda^2 N^a + R^a{}_{bcd} T^b N^c T^d = 0 \,.$$

[You may assume $R^a{}_{bcd} = -R^a{}_{bdc}$ and the Ricci identity in the form

$$(\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) T^a = R^a{}_{bcd} T^b T^c N^d \,. \quad]$$

Consider the two-dimensional space consisting of the sphere of radius r with line element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \,.$$

Show that one may choose $T^a = (1,0)$, $N^a = (0,1)$, and that

$$\nabla_\theta N^a = \cot\theta \, N^a \,.$$

Hence show that $R = 2/r^2$, using the geodesic deviation equation and the identity in any two-dimensional space

$$R^a{}_{bcd} = \frac{1}{2} R(\delta^a_c \, g_{bd} - \delta^a_d \, g_{bc}) \,,$$

where R is the Ricci scalar.

Verify your answer by direct computation of R.

[You may assume that the only non-zero connection components are

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$$

and

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \,.$$

You may also use the definition

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \,. \quad]$$

Paper 3, Section II

37D General Relativity

The Schwarzschild metric for a spherically symmetric black hole is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) ,$$

where we have taken units in which we set $G = c = 1$. Consider a photon moving within the equatorial plane $\theta = \frac{\pi}{2}$, along a path $x^a(\lambda)$ with affine parameter λ. Using a variational principle with Lagrangian

$$L = g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} ,$$

or otherwise, show that

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right) = E \quad \text{and} \quad r^2 \left(\frac{d\phi}{d\lambda}\right) = h ,$$

for constants E and h. Deduce that

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right) . \qquad (*)$$

Assume now that the photon approaches from infinity. Show that the impact parameter (distance of closest approach) is given by

$$b = \frac{h}{E} .$$

Denote the right hand side of equation $(*)$ as $f(r)$. By sketching $f(r)$ in each of the cases below, or otherwise, show that:

(a) if $b^2 > 27M^2$, the photon is deflected but not captured by the black hole;

(b) if $b^2 < 27M^2$, the photon is captured;

(c) if $b^2 = 27M^2$, the photon orbit has a particular form, which should be described.

Paper 1, Section II

37D General Relativity

The curve γ, $x^a = x^a(\lambda)$, is a geodesic with affine parameter λ. Write down the geodesic equation satisfied by $x^a(\lambda)$.

Suppose the parameter is changed to $\mu(\lambda)$, where $d\mu/d\lambda > 0$. Obtain the corresponding equation and find the condition for μ to be affine. Deduce that, whatever parametrization ν is used along the curve γ, the tangent vector K^a to γ satisfies

$$(\nabla_\nu K)^{[a} K^{b]} = 0 \,.$$

Now consider a spacetime with metric g_{ab}, and conformal transformation

$$\tilde{g}_{ab} = \Omega^2(x^c) g_{ab} \,.$$

The curve γ is a geodesic of the metric connection of g_{ab}. What further restriction has to be placed on γ so that it is also a geodesic of the metric connection of $\tilde{g}_{ab}$? Justify your answer.

Paper 4, Section II

36B General Relativity

The metric for a homogenous isotropic universe, in comoving coordinates, can be written as

$$ds^2 = -dt^2 + a^2\{dr^2 + f^2[d\theta^2 + \sin^2\theta\, d\phi^2]\}\,,$$

where $a = a(t)$ and $f = f(r)$ are some functions.

Write down expressions for the Hubble parameter H and the deceleration parameter q in terms of $a(\eta)$ and $h \equiv d\log a/d\eta$, where η is conformal time, defined by $d\eta = a^{-1}dt$.

The universe is composed of a perfect fluid of density ρ and pressure $p = (\gamma - 1)\rho$, where γ is a constant. Defining $\Omega = \rho/\rho_c$, where $\rho_c = 3H^2/8\pi G$, show that

$$\frac{k}{h^2} = \Omega - 1\,, \quad q = \alpha\Omega\,, \quad \frac{d\Omega}{d\eta} = 2qh(\Omega - 1)\,,$$

where k is the curvature parameter ($k = +1$, 0 or -1) and $\alpha \equiv \frac{1}{2}(3\gamma - 2)$. Hence deduce that

$$\frac{d\Omega}{da} = \frac{2\alpha}{a}\Omega(\Omega - 1)$$

and

$$\Omega = \frac{1}{1 - Aa^{2\alpha}}\,,$$

where A is a constant. Given that $A = \dfrac{k}{2GM}$, sketch curves of Ω against a in the case when $\gamma > 2/3$.

[You may assume an Einstein equation, for the given metric, in the form

$$\frac{h^2}{a^2} + \frac{k}{a^2} = \frac{8}{3}\pi G\rho$$

and the energy conservation equation

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0\,.]$$

Paper 2, Section II
36B General Relativity

The metric of any two-dimensional rotationally-symmetric curved space can be written in terms of polar coordinates, (r, θ), with $0 \leqslant \theta < 2\pi, r \geqslant 0$, as

$$ds^2 = e^{2\phi}(dr^2 + r^2 d\theta^2)\,,$$

where $\phi = \phi(r)$. Show that the Christoffel symbols $\Gamma^r_{r\theta}$, Γ^θ_{rr} and $\Gamma^\theta_{\theta\theta}$ are each zero, and compute Γ^r_{rr}, $\Gamma^r_{\theta\theta}$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r}$.

The Ricci tensor is defined by

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^d_{ac}\Gamma^c_{bd}$$

where a comma here denotes partial derivative. Prove that $R_{r\theta} = 0$ and that

$$R_{rr} = -\phi'' - \frac{\phi'}{r}\,, \qquad R_{\theta\theta} = r^2 R_{rr}\,.$$

Suppose now that, in this space, the Ricci scalar takes the constant value -2. Find a differential equation for $\phi(r)$.

By a suitable coordinate transformation $r \to \chi(r)$, θ unchanged, this space of constant Ricci scalar can be described by the metric

$$ds^2 = d\chi^2 + \sinh^2 \chi \, d\theta^2\,.$$

From this coordinate transformation, find $\cosh \chi$ and $\sinh \chi$ in terms of r. Deduce that

$$e^{\phi(r)} = \frac{2A}{1 - A^2 r^2}\,,$$

where $0 \leqslant Ar < 1$, and A is a positive constant.

[You may use

$$\int \frac{d\chi}{\sinh \chi} = \frac{1}{2}\log(\cosh \chi - 1) - \frac{1}{2}\log(\cosh \chi + 1) + \text{constant}\,.]$$

Paper 3, Section II

37B General Relativity

(i) The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Consider a time-like geodesic $x^a(\tau)$, where τ is the proper time, lying in the plane $\theta = \pi/2$. Use the Lagrangian $L = g_{ab}\dot{x}^a\dot{x}^b$ to derive the equations governing the geodesic, showing that

$$r^2\dot{\phi} = h,$$

with h constant, and hence demonstrate that

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + 3Mu^2,$$

where $u = 1/r$. State which term in this equation makes it different from an analogous equation in Newtonian theory.

(ii) Now consider Kruskal coordinates, in which the Schwarzschild t and r are replaced by U and V, defined for $r > 2M$ by

$$U \equiv \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/(4M)} \cosh\left(\frac{t}{4M}\right)$$

$$V \equiv \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/(4M)} \sinh\left(\frac{t}{4M}\right)$$

and for $r < 2M$ by

$$U \equiv \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/(4M)} \sinh\left(\frac{t}{4M}\right)$$

$$V \equiv \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/(4M)} \cosh\left(\frac{t}{4M}\right).$$

Given that the metric in these coordinates is

$$ds^2 = \frac{32M^3}{r} e^{-r/(2M)}(-dV^2 + dU^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $r = r(U, V)$ is defined implicitly by

$$\left(\frac{r}{2M} - 1\right) e^{r/(2M)} = U^2 - V^2,$$

sketch the Kruskal diagram, indicating the positions of the singularity at $r = 0$, the event horizon at $r = 2M$, and general lines of constant r and of constant t.

Paper 1, Section II

37B   General Relativity

(i) Using the condition that the metric tensor g_{ab} is covariantly constant, derive an expression for the Christoffel symbol $\Gamma^a_{bc} = \Gamma^a_{cb}$.

(ii) Show that

$$\Gamma^a_{ba} = \frac{1}{2}g^{ac}g_{ac,b}\,.$$

Hence establish the covariant divergence formula

$$V^a{}_{;a} = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^a}\left(\sqrt{-g}\,V^a\right)\,,$$

where g is the determinant of the metric tensor.

[It may be assumed that $\partial_a(\log\det M) = \text{trace}\,(M^{-1}\partial_a M)$ for any invertible matrix M].

(iii) The Kerr-Newman metric, describing the spacetime outside a rotating black hole of mass M, charge Q and angular momentum per unit mass a, is given in appropriate units by

$$ds^2 = -\,(dt - a\sin^2\theta\,d\phi)^2\frac{\Delta}{\rho^2}$$
$$+\left((r^2+a^2)d\phi - a\,dt\right)^2\frac{\sin^2\theta}{\rho^2} + \left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2\,,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2 + Q^2$. Explain why this metric is stationary, and make a choice of one of the parameters which reduces it to a static metric.

Show that, in the static metric obtained, the equation

$$(g^{ab}\Phi_{,b})_{;a} = 0$$

for a function $\Phi = \Phi(t,r)$ admits solutions of the form

$$\Phi = \sin(\omega t)R(r)\,,$$

where ω is constant and $R(r)$ satisfies an ordinary differential equation which should be found.

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Paper 1, Section II

37D General Relativity

Consider a metric of the form

$$ds^2 = -2\,du\,dv + dx^2 + dy^2 - 2H(u,x,y)du^2\,.$$

Let $x^a(\lambda)$ describe an affinely-parametrised geodesic, where $x^a \equiv (x^1, x^2, x^3, x^4) = (u, v, x, y)$. Write down explicitly the Lagrangian

$$L = g_{ab}\dot{x}^a\dot{x}^b\,,$$

with $\dot{x}^a = dx^a/d\lambda$, using the given metric. Hence derive the four geodesic equations. In particular, show that

$$\ddot{v} + 2\left(\frac{\partial H}{\partial x}\,\dot{x} + \frac{\partial H}{\partial y}\,\dot{y}\right)\dot{u} + \frac{\partial H}{\partial u}\,\dot{u}^2 = 0.$$

By comparing these equations with the standard form of the geodesic equation, show that $\Gamma^2_{13} = \partial H/\partial x$ and derive the other Christoffel symbols.

The Ricci tensor, R_{ab}, is defined by

$$R_{ab} = \Gamma^d_{ab,d} - \Gamma^d_{ad,b} + \Gamma^d_{df}\Gamma^f_{ba} - \Gamma^d_{bf}\Gamma^f_{da}\,.$$

By considering the case $a = 1, b = 1$, show that the vacuum Einstein field equations imply

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0\,.$$

Paper 2, Section II
36D General Relativity

The curvature tensor $R^a{}_{bcd}$ satisfies

$$V_{a;bc} - V_{a;cb} = V_e R^e{}_{abc}$$

for any covariant vector field V_a. Hence express $R^e{}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$$R^e{}_{abc} = -R^e{}_{acb}.$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$$R^e{}_{abc} + R^e{}_{cab} + R^e{}_{bca} = 0.$$

Using local inertial coordinates or otherwise, obtain the Bianchi identities.

Define the Ricci tensor in terms of the curvature tensor and show that it is symmetric. [You may assume that $R_{abcd} = -R_{bacd}$.] Write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geqslant 2$, R_{abcd} takes the form

$$R_{abcd} = K(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}).$$

Obtain the Ricci tensor and curvature scalar. Deduce, under some restriction on n which should be stated, that K is a constant.

Paper 4, Section II

36D General Relativity

The metric of the Schwarzschild solution is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \qquad (*)$$

Show that, for an incoming radial light ray, the quantity

$$v = t + r + 2M \log\left|\frac{r}{2M} - 1\right|$$

is constant.

Express ds^2 in terms of r, v, θ and ϕ. Determine the light-cone structure in these coordinates, and use this to discuss the nature of the apparent singularity at $r = 2M$.

An observer is falling radially inwards in the region $r < 2M$. Assuming that the metric for $r < 2M$ is again given by $(*)$, obtain a bound for $d\tau$, where τ is the proper time of the observer, in terms of dr. Hence, or otherwise, determine the maximum proper time that can elapse between the events at which the observer crosses $r = 2M$ and is torn apart at $r = 0$.

Paper 1, Section II

36B   General Relativity

Consider a spacetime $\mathcal{M}$ with a metric $g_{ab}(x^c)$ and a corresponding connection Γ^a_{bc}. Write down the differential equation satisfied by a geodesic $x^a(\lambda)$, where λ is an affine parameter.

Show how the requirement that

$$\delta \int g_{ab}(x^c) \frac{d}{d\lambda} x^a(\lambda) \frac{d}{d\lambda} x^b(\lambda) \, d\lambda \; = \; 0 \, ,$$

where δ denotes variation of the path, gives the geodesic equation and determines Γ^a_{bc}.

Show that the timelike geodesics for the 2–manifold with line element

$$ds^2 \; = \; t^{-2} \left(dx^2 - dt^2 \right)$$

are given by

$$t^2 \; = \; x^2 + \alpha \, x + \beta \, ,$$

where α and β are constants.

Paper 2, Section II

36B General Relativity

A vector field k^a which satisfies

$$k_{a;b} + k_{b;a} = 0$$

is called a Killing vector field. Prove that k^a is a Killing vector field if and only if

$$k^c g_{ab,c} + k^c{}_{,b}\, g_{ac} + k^c{}_{,a}\, g_{bc} = 0\,.$$

Prove also that if V^a satisfies

$$V^a{}_{;b} V^b = 0\,,$$

then

$$(V^a k_a)_{,b}\, V^b = 0 \tag{$*$}$$

for any Killing vector field k^a.

In the two–dimensional space–time with coordinates $x^a = (u, v)$ and line element

$$ds^2 = -du^2 + u^2 dv^2\,,$$

verify that $(0, 1)$, $e^{-v}(1, u^{-1})$ and $e^v(-1, u^{-1})$ are Killing vector fields. Show, by using $(*)$ with V^a the tangent vector to a geodesic, that geodesics in this space–time are given by

$$\alpha e^v + \beta e^{-v} = 2\,\gamma u^{-1}\,,$$

where α, β and γ are arbitrary real constants.

Paper 4, Section II

36B   General Relativity

The Schwarzschild line element is given by

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right),$$

where $F = 1 - r_s/r$ and r_s is the Schwarzschild radius. Obtain the equation of geodesic motion of photons moving in the equatorial plane, $\theta = \pi/2$, in the form

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \frac{h^2 F}{r^2},$$

where τ is proper time, and E and h are constants whose physical significance should be indicated briefly.

Defining $u = 1/r$ show that light rays are determined by

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{1}{b}\right)^2 - u^2 + r_s \, u^3,$$

where $b = h/E$ and r_s may be taken to be small. Show that, to zeroth order in r_s, a light ray is a straight line passing at distance b from the origin. Show that, to first order in r_s, the light ray is deflected through an angle $2r_s/b$. Comment briefly on some observational evidence for the result.

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Paper 1, Section II

36D General Relativity

Write down the differential equations governing geodesic curves $x^a(\lambda)$ both when λ is an affine parameter and when it is a general one.

A conformal transformation of a spacetime is given by

$$g_{ab} \to \tilde{g}_{ab} = \Omega^2(x)g_{ab}.$$

Obtain a formula for the new Christoffel symbols $\tilde{\Gamma}^a_{bc}$ in terms of the old ones and the derivatives of Ω. Hence show that null geodesics in the metric g_{ab} are also geodesic in the metric $\tilde{g}_{ab}$.

Show that the Riemann tensor has only one independent component in two dimensions, and hence derive

$$R = 2 \det(g^{ab}) R_{0101},$$

where R is the Ricci scalar.

It is given that in a 2-dimensional spacetime M, R transforms as

$$R \to \tilde{R} = \Omega^{-2}(R - 2\square \log \Omega),$$

where $\square \phi = g^{ab}\nabla_a \nabla_b \phi$. Assuming that the equation $\square \phi = \rho(x)$ can always be solved, show that Ω can be chosen to set $\tilde{g}$ to be the metric of 2-dimensional Minkowski spacetime. Hence show that all null curves in M are geodesic.

Discuss the null geodesics if the line element of M is

$$ds^2 = -t^{-1}dt^2 + t\,d\theta^2,$$

where $t \in (-\infty, 0)$ or $(0, \infty)$ and $\theta \in [0, 2\pi]$.

Paper 2, Section II

36D General Relativity

A spacetime has line element

$$ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2,$$

where p_1, p_2 and p_3 are constants. Calculate the Christoffel symbols.

Find the constraints on p_1, p_2 and p_3 for this spacetime to be a solution of the vacuum Einstein equations with zero cosmological constant. For which values is the spacetime flat?

Show that it is not possible to have all of p_1, p_2 and p_3 strictly positive, so that if they are all non-zero, the spacetime expands in at least one direction and contracts in at least one direction.

[The Riemann tensor is given in terms of the Christoffel symbols by

$$R^a_{bcd} = \Gamma^a_{db,c} - \Gamma^a_{cb,d} + \Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb}.]$$

Paper 4, Section II

36D General Relativity

The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where M is the mass in gravitational units. By using the radial component of the geodesic equations, or otherwise, show for a particle moving on a geodesic in the equatorial plane $\theta = \pi/2$ with r constant that

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{M}{r^3}.$$

Show that such an orbit is stable for $r > 6M$.

An astronaut circles the Earth freely for a long time on a circular orbit of radius R, while the astronaut's twin remains motionless on Earth, which is assumed to be spherical, with radius R_0, and non-rotating. Show that, on returning to Earth, the astronaut will be younger than the twin only if $2R < 3R_0$.

1/II/35E   **General Relativity**

For the metric

$$ds^2 = \frac{1}{r^2}\left(-dt^2 + dr^2\right), \qquad r \geqslant 0,$$

obtain the geodesic equations of motion. For a massive particle show that

$$\left(\frac{dr}{dt}\right)^2 = 1 - \frac{1}{k^2 r^2},$$

for some constant k. Show that the particle moves on trajectories

$$r^2 - t^2 = \frac{1}{k^2}, \qquad kr = \sec\tau, \quad kt = \tan\tau,$$

where τ is the proper time, if the origins of t, τ are chosen appropriately.

2/II/35E   **General Relativity**

Let $x^a(\lambda)$ be a path P with tangent vector $T^a = \frac{d}{d\lambda}x^a(\lambda)$. For vectors $X^a(x(\lambda))$ and $Y^a(x(\lambda))$ defined on P let

$$\nabla_T X^a = \frac{d}{d\lambda}X^a + \Gamma^a{}_{bc}(x(\lambda))X^b T^c,$$

where $\Gamma^a{}_{bc}(x)$ is the metric connection for a metric $g_{ab}(x)$. $\nabla_T Y^a$ is defined similarly. Suppose P is geodesic and λ is an affine parameter. Explain why $\nabla_T T^a = 0$. Show that if $\nabla_T X^a = \nabla_T Y^a = 0$ then $g_{ab}(x(\lambda))X^a(x(\lambda))Y^b(x(\lambda))$ is constant along P.

If $x^a(\lambda, \mu)$ is a family of geodesics which depend on μ, let $S^a = \frac{\partial}{\partial\mu}x^a$ and define

$$\nabla_S X^a = \frac{\partial}{\partial\mu}X^a + \Gamma^a{}_{bc}(x(\lambda))X^b S^c.$$

Show that $\nabla_T S^a = \nabla_S T^a$ and obtain

$$\nabla_T{}^2 S^a \equiv \nabla_T(\nabla_T S^a) = R^a{}_{bcd}T^b T^c S^d.$$

What is the physical relevance of this equation in general relativity? Describe briefly how this is relevant for an observer moving under gravity.

[You may assume $[\nabla_T, \nabla_S]X^a = R^a{}_{bcd}X^b T^c S^d$.]

4/II/36E   **General Relativity**

A solution of the Einstein equations is given by the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

For an incoming light ray, with constant θ, ϕ, show that

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|,$$

for some fixed v and find a similar solution for an outgoing light ray. For the outgoing case, assuming $r > 2M$, show that in the far past $\frac{r}{2M} - 1 \propto \exp(\frac{t}{2M})$ and in the far future $r \sim t$.

Obtain the transformed metric after the change of variables $(t, r, \theta, \phi) \to (v, r, \theta, \phi)$. With coordinates $\hat{t} = v - r, r$ sketch, for fixed θ, ϕ, the trajectories followed by light rays. What is the significance of the line $r = 2M$?

Show that, whatever path an observer with initial $r = r_0 < 2M$ takes, he must reach $r = 0$ in a finite proper time.

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1/II/35A **General Relativity**

Starting from the Riemann tensor for a metric g_{ab}, define the Ricci tensor R_{ab} and the scalar curvature R.

The Riemann tensor obeys

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0 \,.$$

Deduce that

$$\nabla^a R_{ab} = \tfrac{1}{2}\nabla_b R \,. \tag{$*$}$$

Write down Einstein's field equations in the presence of a matter source, with energy-momentum tensor T_{ab}. How is the relation $(*)$ important for the consistency of Einstein's equations?

Show that, for a scalar function ϕ, one has

$$\nabla^2 \nabla_a \phi = \nabla_a \nabla^2 \phi + R_{ab}\nabla^b \phi \,.$$

Assume that

$$R_{ab} = \nabla_a \nabla_b \phi$$

for a scalar field ϕ. Show that the quantity

$$R + \nabla^a \phi \nabla_a \phi$$

is then a constant.

2/II/35A   **General Relativity**

The symbol ∇_a denotes the covariant derivative defined by the Christoffel connection Γ^a_{bc} for a metric g_{ab}. Explain briefly why

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\phi = 0,$$
$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v_c \neq 0,$$

in general, where ϕ is a scalar field and v_c is a covariant vector field.

A Killing vector field v_a satisfies the equation

$$S_{ab} \equiv \nabla_a v_b + \nabla_b v_a = 0.$$

By considering the quantity $\nabla_a S_{bc} + \nabla_b S_{ac} - \nabla_c S_{ab}$, show that

$$\nabla_a\nabla_b v_c = -R^d{}_{abc}v_d.$$

Find all Killing vector fields v_a in the case of flat Minkowski space-time.

For a metric of the form

$$ds^2 = -f(\mathbf{x})\,dt^2 + g_{ij}(\mathbf{x})\,dx^i dx^j, \qquad i,j = 1,2,3,$$

where $\mathbf{x}$ denotes the coordinates x^i, show that $\Gamma^0_{00} = \Gamma^0_{ij} = 0$ and that $\Gamma^0_{0i} = \Gamma^0_{i0} = \frac{1}{2}(\partial_i f)/f$. Deduce that the vector field $v^a = (1,0,0,0)$ is a Killing vector field.

[*You may assume the standard symmetries of the Riemann tensor.*]

4/II/36A   **General Relativity**

Consider a particle on a trajectory $x^a(\lambda)$. Show that the geodesic equations, with affine parameter λ, coincide with the variational equations obtained by varying the integral

$$I = \int_{\lambda_0}^{\lambda_1} g_{ab}(x) \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \, \mathrm{d}\lambda \,,$$

the end-points being fixed.

In the case that $f(r) = 1 - 2GMu$, show that the space-time metric is given in the form

$$\mathrm{d}s^2 = -f(r) \, \mathrm{d}t^2 + \frac{1}{f(r)} \, \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \,,$$

for a certain function $f(r)$. Assuming the particle motion takes place in the plane $\theta = \dfrac{\pi}{2}$ show that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{h}{r^2} \,, \qquad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{E}{f(r)} \,,$$

for h, E constants. Writing $u = 1/r$, obtain the equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + f(r) \, u^2 = -\frac{k}{h^2} \, f(r) + \frac{E^2}{h^2} \,,$$

where k can be chosen to be 1 or 0, according to whether the particle is massive or massless. In the case that $f(r) = 1 - GMu$, show that

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = k \frac{GM}{h^2} + 3GMu^2 \,.$$

In the massive case, show that there is an approximate solution of the form

$$u = \frac{1}{\ell} \left(1 + e \cos(\alpha\phi)\right),$$

where

$$1 - \alpha = \frac{3GM}{\ell} \,.$$

What is the interpretation of this solution?

1/II/35A   **General Relativity**

Let $\phi(x)$ be a scalar field and ∇_a denote the Levi–Civita covariant derivative operator of a metric tensor g_{ab}. Show that

$$\nabla_a\nabla_b\phi = \nabla_b\nabla_a\phi \ .$$

If the Ricci tensor, R_{ab}, of the metric g_{ab} satisfies

$$R_{ab} = \partial_a\phi\,\partial_b\phi \ ,$$

find the energy momentum tensor T_{ab} and use the contracted Bianchi identity to show that, if $\partial_a\phi \neq 0$, then

$$\nabla_a\nabla^a\phi = 0 \ . \tag{$*$}$$

Show further that $(*)$ implies

$$\partial_a\big(\sqrt{-g}\,g^{ab}\partial_b\phi\big) = 0 \ .$$

2/II/35A   **General Relativity**

The Schwarzschild metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \left(1 - \frac{2M}{r}\right) dt^2 \ .$$

Writing $u = 1/r$, obtain the equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 \ , \qquad\qquad (*)$$

determining the spatial orbit of a null (massless) particle moving in the equatorial plane $\theta = \pi/2$.

Verify that two solutions of $(*)$ are

$$\text{(i)}\qquad u = \frac{1}{3M} \ , \quad \text{and}$$

$$\text{(ii)}\qquad u = \frac{1}{3M} - \frac{1}{M}\frac{1}{\cosh\phi + 1} \ .$$

What is the significance of solution (i)? Sketch solution (ii) and describe its relation to solution (i).

Show that, near $\phi = \cosh^{-1} 2$, one may approximate the solution (ii) by

$$r\sin(\phi - \cosh^{-1} 2) \approx \sqrt{27}M \ ,$$

and hence obtain the impact parameter.

4/II/36A   **General Relativity**

What are local inertial co-ordinates? What is their physical significance and how are they related to the equivalence principle?

If V_a are the components of a covariant vector field, show that

$$\partial_a V_b - \partial_b V_a$$

are the components of an anti-symmetric second rank covariant tensor field.

If K^a are the components of a contravariant vector field and g_{ab} the components of a metric tensor, let

$$Q_{ab} = K^c \partial_c \, g_{ab} + g_{ac} \, \partial_b K^c + g_{cb} \, \partial_a K^c \ .$$

Show that

$$Q_{ab} = 2 \nabla_{(a} K_{b)} \ ,$$

where $K_a = g_{ab} K^b$, and ∇_a is the Levi–Civita covariant derivative operator of the metric g_{ab}.

In a particular co-ordinate system (x^1, x^2, x^3, x^4), it is given that $K^a = (0, 0, 0, 1)$, $Q_{ab} = 0$. Deduce that, in this co-ordinate system, the metric tensor g_{ab} is independent of the co-ordinate x^4. Hence show that

$$\nabla_a K_b = \tfrac{1}{2} \big(\partial_a K_b - \partial_b K_a \big) \ ,$$

and that

$$E = -K_a \frac{dx^a}{d\lambda} \ ,$$

is constant along every geodesic $x^a(\lambda)$ in every co-ordinate system.

What further conditions must one impose on K^a and $dx^a / d\lambda$ to ensure that the metric is stationary and that E is proportional to the energy of a particle moving along the geodesic?

1/II/35C   **General Relativity**

Suppose $(x(\tau), t(\tau))$ is a timelike geodesic of the metric

$$ds^2 = \frac{dx^2}{1 + x^2} - (1 + x^2)\, dt^2\,,$$

where τ is proper time along the world line. Show that $dt/d\tau = E/(1 + x^2)$, where $E > 1$ is a constant whose physical significance should be stated. Setting $a^2 = E^2 - 1$, show that

$$d\tau = \frac{dx}{\sqrt{a^2 - x^2}}\,, \qquad dt = \frac{E\, dx}{(1 + x^2)\sqrt{a^2 - x^2}}\,. \qquad (*)$$

Deduce that x is a periodic function of proper time τ with period 2π. Sketch $dx/d\tau$ as a function of x and superpose on this a sketch of dx/dt as a function of x. Given the identity

$$\int_{-a}^{a} \frac{E\, dx}{(1 + x^2)\sqrt{a^2 - x^2}} = \pi\,,$$

deduce that x is also a periodic function of t with period 2π.

Next consider the family of metrics

$$ds^2 = \frac{[1 + f(x)]^2\, dx^2}{1 + x^2} - (1 + x^2)\, dt^2\,,$$

where f is an odd function of x, $f(-x) = -f(x)$, and $|f(x)| < 1$ for all x. Derive expressions analogous to $(*)$ above. Deduce that x is a periodic function of τ and also that x is a periodic function of t. What are the periods?

2/II/35C   **General Relativity**

State without proof the properties of local inertial coordinates x^a centred on an arbitrary spacetime event p. Explain their physical significance.

Obtain an expression for $\partial_a \Gamma_b{}^c{}_d$ at p in inertial coordinates. Use it to derive the formula

$$R_{abcd} = \tfrac{1}{2}\left(\partial_b\partial_c g_{ad} + \partial_a\partial_d g_{bc} - \partial_b\partial_d g_{ac} - \partial_a\partial_c g_{bd}\right)$$

for the components of the Riemann tensor at p in local inertial coordinates. Hence deduce that at any point in any chart $R_{abcd} = R_{cdab}$.

Consider the metric

$$ds^2 = \frac{\eta_{ab}\, dx^a\, dx^b}{(1 + L^{-2}\eta_{ab}x^a x^b)^2}\,,$$

where $\eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$ is the Minkowski metric tensor and L is a constant. Compute the Ricci scalar $R = R^{ab}{}_{ab}$ at the origin $x^a = 0$.

4/II/36C   **General Relativity**

State clearly, but do not prove, Birkhoff's Theorem about spherically symmetric spacetimes. Let (r, θ, ϕ) be standard spherical polar coordinates and define $F(r) = 1 - 2M/r$, where M is a constant. Consider the metric

$$ds^2 = \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - F(r)\, dt^2.$$

Explain carefully why this is appropriate for the region outside a spherically symmetric star that is collapsing to form a black hole.

By considering radially infalling timelike geodesics $x^a = (r(\tau), 0, 0, t(\tau))$, where τ is proper time along the curve, show that a freely falling observer will reach the event horizon after a finite proper time. Show also that a distant observer would see the horizon crossing only after an infinite time.