## Part II

## Fluid Dynamics II

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## Paper 1, Section II

## 39C Fluid Dynamics II

(a) In an incompressible Stokes flow, show that the Laplacian of the vorticity is zero. If the flow is also two-dimensional, deduce the equation satisfied by the streamfunction $\psi(x, y)$.
(b) A stationary two-dimensional rigid disk of radius $a$, centred at the origin, is subject to an external shear flow $\mathbf{u}_{\infty}=\gamma y \mathbf{e}_{x}$, where $\gamma$ is a constant and $\mathbf{e}_{x}$ is the unit vector in the $x$-direction.
(i) What are the equations and boundary conditions satisfied by $\psi$ for the flow outside the disk?
(ii) Solve for $\psi$, ensuring that your solution satisfies all the required boundary conditions.
(iii) Compute the hydrodynamic torque exerted by the shear flow on the disk.
[ Hint: in polar coordinates,

$$
\begin{gathered}
(\nabla \times \mathbf{u}) \cdot \mathbf{e}_{z}=\frac{1}{r} \frac{\partial}{\partial r}\left(r u_{\theta}\right)-\frac{1}{r} \frac{\partial u_{r}}{\partial \theta} \\
\nabla^{2} f(r, \theta)=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} f}{\partial \theta^{2}} \\
\left.\sigma_{r \theta}=\mu\left[r \frac{\partial}{\partial r}\left(\frac{u_{\theta}}{r}\right)+\frac{1}{r} \frac{\partial u_{r}}{\partial \theta}\right] .\right]
\end{gathered}
$$

## Paper 2, Section II

## 39C Fluid Dynamics II

(a) Consider the incompressible flow of a Newtonian fluid with constant viscosity $\mu$ and constant density $\rho$ subject to a body force per unit mass f. Derive the equation for rate of change of kinetic energy in a finite volume $\Omega$ with boundary $\partial \Omega$ and give the physical interpretation for each term.
(b) Explain, justifying your arguments with appropriate order-of-magnitude estimates, how the energy balance can be used to estimate the drag on a steadily moving bubble of fixed shape in fluid at rest at infinity, when the Reynolds number is large, without having to solve for the details of the boundary layer around the bubble. [You may ignore all contributions from body forces.]
(c) A two-dimensional circular bubble of radius $a$, in fluid that is at rest far from the bubble, is moving steadily with velocity $U \mathbf{e}_{x}$, where $\mathbf{e}_{x}$ is the unit vector in the $x$-direction. The flow occurs at high Reynolds number and is assumed to be irrotational outside the boundary layer. [Again you may ignore the effect of any body forces.]
(i) Solve for the irrotational flow outside the boundary layer.
(ii) Using the method in part (b), or otherwise, estimate the drag force exerted on the bubble.
(iii) Give brief reasons why the same approach could not be applied to a rigid body.
[Hint: In polar coordinates the rate-of-strain tensor has components

$$
\left.e_{r r}=\frac{\partial u_{r}}{\partial r}, \quad e_{r \theta}=\frac{1}{2}\left[r \frac{\partial}{\partial r}\left(\frac{u_{\theta}}{r}\right)+\frac{1}{r} \frac{\partial u_{r}}{\partial \theta}\right], \quad e_{\theta \theta}=\frac{u_{r}}{r}+\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} .\right]
$$

## Paper 3, Section II

## 38C Fluid Dynamics II

A two-dimensional lubrication flow occurs between two rigid surfaces in a fluid that has otherwise uniform pressure $p_{0}$. The bottom surface at $y=0$ moves in the horizontal direction with velocity $\mathbf{u}=U \mathbf{e}_{x}$ while the top surface at $y=h(x)$ moves towards $y=0$ with velocity $\mathbf{u}=-V \mathbf{e}_{y}$, with $\mathbf{e}_{x}$ and $\mathbf{e}_{y}$ being unit vectors in the $x$ - and $y$-directions respectively. Both surfaces are of length $L$ in the $x$ direction. Consider the instant when both occupy the region $0<x<L$.
(a) State all conditions involving $h, U, V$ and $L$ ensuring that the flow between the two surfaces is in the lubrication limit.
(b) Solve for the flow in the $x$ direction between the two surfaces.
(c) Use conservation of mass to derive an expression for the pressure gradient between the two surfaces as a function of $x$.
[Hint: You may find it convenient to introduce the notation $\langle f\rangle$ to denote the mean value of a function $f$ over the range $0 \leqslant x \leqslant L$.]
(d) In the particular case $U=0$, show that the pressure gradient is necessarily zero somewhere between the two surfaces.
(e) Find the value of $U$ such that the force in the $x$ direction on the bottom surface is zero at the instant considered.

## Paper 4, Section II

## 38C Fluid Dynamics II

Consider a two-dimensional wake of constant width $2 h$ in an otherwise uniform horizontal flow of speed $U$. The unperturbed velocity $\mathbf{u}=u \mathbf{e}_{x}$, where $\mathbf{e}_{x}$ is a unit vector in the $x$-direction, is given by

$$
u= \begin{cases}U, & y>h \\ 0, & -h<y<h \\ U, & y<-h\end{cases}
$$

The two shear layers at $y= \pm h$ are perturbed symmetrically so that at time $t$ their location is $y= \pm[h+\eta(x, t)]$. The flow is assumed to be irrotational everywhere, the fluid is inviscid and the effects of gravity may be ignored.
(a) Sketch the unperturbed flow and the shape of the deformed shear layers.
(b) State the equation satisfied by the velocity potential $\phi$ and all the boundary conditions applicable to the three fluid domains $(y<-h-\eta(x, t),-h-\eta(x, t)<y<$ $h+\eta(x, t)$ and $y>h+\eta(x, t))$.
(c) What are the conditions on $\eta$ and $\partial \eta / \partial x$ necessary in order to linearise the equations and boundary conditions? State the linearised versions of the boundary conditions on $\phi$ and its derivatives valid under those conditions.
(d) Justify why a full description of the linearised problem is provided by considering solutions of the form

$$
\eta(x, t)=\operatorname{Re}\left\{\eta_{0} \exp (i k x+\sigma t)\right\}
$$

where Re is the real part.
(e) Solve for the dispersion relation linking $\sigma$ and $k$ with the parameters of the problem. [Hint: The specified symmetry of the perturbation may allow simplification of the algebra.] Deduce the conditions on $k$ under which the wake flow is unstable.
(f) In the limit $h k \gg 1$, interpret the result for $\sigma$ in light of what you know about the Kelvin-Helmholtz instability.

## Paper 1, Section II

## 39C Fluid Dynamics II

A viscous fluid of viscosity $\mu$ and density $\rho$ is located in the annulus confined between two long co-axial cylinders of radii $R$ and $\alpha R$ with $\alpha<1$. The ends of the annular space are open to the atmosphere. The axes of the cylinders are aligned in the vertical direction. We use cylindrical coordinates $(r, \theta, z)$ with unit vector $\mathbf{e}_{z}$ in the downward vertical direction. There is a gravitational force $g$ per unit mass acting on the fluid in the downward direction. In the following you may consider the flow in the long central region of the annulus, far from the ends, and neglect any details of the flow near the ends.

The outer cylinder is fixed and stationary. The inner cylinder steadily translates along its axis with velocity $V \mathbf{e}_{z}$. The fluid flow between the two cylinders may be assumed to be steady and unidirectional.
(a) Explain why we expect the velocity $\mathbf{u}$ to be of the form $\mathbf{u}=u(r) \mathbf{e}_{z}$.
(b) Derive the equation satisfied by $u(r)$ and state the corresponding boundary conditions.
(c) Show that the pressure gradient in the $z$-direction is constant and compute its value.
(d) Solve for the flow $u(r)$ in the annular gap and sketch it for $V=0$, and for two further values of $V$, one positive and one negative.
(e) Calculate the force per unit length acting on the inner cylinder and the corresponding force per unit length acting on the outer cylinder. Comment on the sum of these forces.
[Hint: in cylindrical coordinates $(r, \theta, z)$ with velocity components $\left(u_{r}, u_{\theta}, u_{z}\right)$ we have

$$
\nabla^{2} u_{z}=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_{z}}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} u_{z}}{\partial \theta^{2}}+\frac{\partial^{2} u_{z}}{\partial z^{2}}
$$

The rz-component of the rate-of-strain tensor is $\left.e_{r z}=\frac{1}{2}\left(\frac{\partial u_{r}}{\partial z}+\frac{\partial u_{z}}{\partial r}\right).\right]$

## Paper 2, Section II

## 39C Fluid Dynamics II

(a) A fluid has kinematic viscosity $\nu>0$. In flow over a stationary rigid boundary with length scale $\mathcal{L}$, the fluid velocity far from the boundary has typical magnitude $\mathcal{U}$. Define the Reynolds number. Explain why even if the Reynolds number is large the effects of viscosity cannot be neglected and explain briefly how boundary layer theory provides a useful approximate approach to including these effects.
(b) A steady high-Reynolds number flow is induced in a semi-infinite fluid otherwise at rest, in the region $y>0$, by the in-plane motion of an extensible sheet lying along $x \geqslant 0, y=0$. Points on the sheet move with velocity $\mathbf{V}=\alpha x \mathbf{e}_{x}$, where $\alpha$ is the prescribed constant rate of extension and $\mathbf{e}_{x}$ is the unit vector in the $x$-direction.
(i) What should be chosen for the typical flow speed $U(x)$ in the boundary layer? Give an estimate of the corresponding $x$-dependent Reynolds number and deduce that, for $x$ sufficiently large, the flow is described by the boundary layer equations. Derive the fundamental boundary-layer scaling relating $U(x)$ and the thickness $\delta(x)$ of the boundary layer and deduce the scaling for $\delta(x)$ as a function of $x$.
(ii) State the two-dimensional boundary layer equations and their boundary conditions for this problem in terms of a streamfunction $\psi(x, y)$.
(iii) Seek a similarity solution to the boundary layer equations using

$$
\psi(x, y)=U(x) \delta(x) f(\eta)
$$

where $\eta \equiv \frac{y}{\delta(x)}$. Derive the ODE and boundary conditions satisfied by $f(\eta)$.
(iv) Show that the ODE satisfied by $f$ has a solution of the form $A+B \exp (-C \eta)$ and determine the values of the constants $A, B$ and $C$.
(i) Comment on the behaviour of $f$ as $\eta \rightarrow \infty$. What are the implications for the flow external to the boundary layer?

## Paper 3, Section II <br> 38C Fluid Dynamics II

A uniform rod in the shape of an elongated cylinder falls through a viscous fluid under the action of gravity. The motion is sufficiently slow that the fluid flow is described by the Stokes equations.
(a) Show that when the long axis of the rod is initially aligned with the horizontal direction the rod falls vertically.
(b) Show that for any initial orientation of the rod the motion of the rod occurs with no rotation.
(c) Denoting by $\mathbf{F}$ the hydrodynamic force exerted on the rod and $\mathbf{U}$ its translation speed, explain why we expect a linear relationship of the form $\mathbf{F}=-\mathbf{R} \cdot \mathbf{U}$, where $\mathbf{R}$ is a matrix.
(d) State the reciprocal theorem of Stokes flows. Show that it implies that $\mathbf{R}$ is symmetric.
(e) Use the energy equation, as applied to this steady flow problem, to deduce that the matrix $\mathbf{R}$ is also positive definite.
(f) We denote by $\mathbf{t}$ the unit tangent vector along the rod and by $\theta$ the angle between $\mathbf{t}$ and the vertical. Writing $\mathbf{R}=c_{1} \mathbf{t t}+c_{2}(\mathbf{1}-\mathbf{t t})$ with $c_{2} \geqslant c_{1}>0$, compute the value of $\cos \alpha$ where $\alpha$ is the angle between the vertical and the direction of motion of the rod. Check the case where $c_{1}=c_{2}$ and comment.

## Paper 4, Section II

38C Fluid Dynamics II
A thin layer of fluid is flowing down an inclined plane due to the action of gravity. The gravitational acceleration is $g$, the viscosity of the fluid is $\mu$ and the density of the fluid is $\rho$. The angle between the plane and the horizontal is denoted by $\alpha$. Cartesian coordinates are defined with $x$ along the plane in the downward direction and $y$ perpendicular to the plane. All quantities may be assumed to be constant in the in-plane direction perpendicular to the slope. The thickness of the fluid layer is denoted by $h(x, t)$.
(a) Assume that the dynamics of the layer is described by the lubrication equations and hence estimate the order of magnitude for the flow speed $u$ in the film. Deduce the two conditions involving $h, \partial h / \partial x$ and the other parameters of the problem that are required for the assumption of the lubrication limit to be self-consistent.
(b) State the momentum equations in the $(x, y)$ coordinates under the lubricationlimit assumption. What are the boundary conditions for the velocity and the pressure?
(c) Solve for the pressure in the fluid and deduce the flow velocity along the plane.
(d) Applying conservation of mass, deduce the partial differential equation satisfied by $h(x, t)$.
(e) Seek a travelling-wave solution $h(x, t)=f(x-c t)$ and hence derive a first-order ODE (containing an unknown constant of integration) satisfied by the function $f$.

Paper 1, Section II

## 39A Fluid Dynamics II

(a) Write down the Stokes equations for the motion of an incompressible viscous fluid with negligible inertia (in the absence of body forces). What does it mean that Stokes flow is linear and reversible?
(b) The region $a<r<b$ between two concentric rigid spheres of radii $a$ and $b$ is filled with fluid of large viscosity $\mu$. The outer sphere is held stationary, while the inner sphere is made to rotate with angular velocity $\boldsymbol{\Omega}$.
(i) Use symmetry and the properties of Stokes flow to deduce that $p=0$, where $p$ is the pressure due to the flow.
(ii) Verify that both solid-body rotation and $\mathbf{u}(\mathbf{x})=\boldsymbol{\Omega} \wedge \boldsymbol{\nabla}(1 / r)$ satisfy the Stokes equations with $p=0$. Hence determine the fluid velocity between the spheres.
(iii) Calculate the stress tensor $\sigma_{i j}$ in the flow.
(iv) Deduce that the couple $\mathbf{G}$ exerted by the fluid in $r<c$ on the fluid in $r>c$, where $a<c<b$, is given by

$$
\mathbf{G}=\frac{8 \pi \mu a^{3} b^{3} \boldsymbol{\Omega}}{b^{3}-a^{3}}
$$

independent of the value of $c$. [Hint: Do not substitute the form of $A$ and $B$ in $A+B r^{-3}$ until the end of the calculation.]
Comment on the form of this result for $a \ll b$ and for $b-a \ll a$.
[You may use $\int_{r=R} n_{i} n_{j} d S=\frac{4}{3} \pi R^{2} \delta_{i j}$, where $\mathbf{n}$ is the normal to $\left.r=R.\right]$

## Paper 2, Section II

## 39A Fluid Dynamics II

(a) Incompressible fluid of viscosity $\mu$ fills the thin, slowly varying gap between rigid boundaries at $z=0$ and $z=h(x, y)>0$. The boundary at $z=0$ translates in its own plane with a constant velocity $\mathbf{U}=(U, 0,0)$, while the other boundary is stationary. If $h$ has typical magnitude $H$ and varies on a lengthscale $L$, state conditions for the lubrication approximation to be appropriate.

Write down the lubrication equations for this problem and show that the horizontal volume flux $\mathbf{q}=\left(q_{x}, q_{y}, 0\right)$ is given by

$$
\mathbf{q}=\frac{\mathbf{U} h}{2}-\frac{h^{3}}{12 \mu} \nabla p
$$

where $p(x, y)$ is the pressure.
Explain why $\mathbf{q}=\nabla \wedge(0,0, \psi)$ for some function $\psi(x, y)$. Deduce that $\psi$ satisfies the equation

$$
\boldsymbol{\nabla} \cdot\left(\frac{1}{h^{3}} \boldsymbol{\nabla} \psi\right)=-\frac{U}{h^{3}} \frac{\partial h}{\partial y} .
$$

(b) Now consider the case $\mathbf{U}=\mathbf{0}, h=h_{0}$ for $r>a$ and $h=h_{1}$ for $r<a$, where $h_{0}, h_{1}$ and $a$ are constants, and $(r, \theta)$ are polar coordinates. A uniform pressure gradient $\nabla p=-G \mathbf{e}_{x}$ is applied at infinity. Show that $\psi \sim \operatorname{Ar} \sin \theta$ as $r \rightarrow \infty$, where the constant $A$ is to be determined.

Given that $a \gg h_{0}, h_{1}$, you may assume that the equations of part (a) apply for $r<a$ and $r>a$, and are subject to conditions that the radial component $q_{r}$ of the volume flux and the pressure $p$ are both continuous across $r=a$. Show that these continuity conditions imply that

$$
\left[\frac{\partial \psi}{\partial \theta}\right]_{-}^{+}=0 \quad \text { and } \quad\left[\frac{1}{h^{3}} \frac{\partial \psi}{\partial r}\right]_{-}^{+}=0
$$

respectively, where []$_{-}^{+}$denotes the jump across $r=a$.
Hence determine $\psi(r, \theta)$ and deduce that the total flux through $r=a$ is given by

$$
\frac{4 A a h_{1}^{3}}{h_{0}^{3}+h_{1}^{3}}
$$

## Paper 3, Section II

## 38A Fluid Dynamics II

Viscous fluid occupying $z>0$ is bounded by a rigid plane at $z=0$ and is extracted through a small hole at the origin at a constant flow rate $Q=2 \pi A$. Assume that for sufficiently small values of $R=|\mathbf{x}|$ the velocity $\mathbf{u}(\mathbf{x})$ is well-approximated by

$$
\begin{equation*}
\mathbf{u}=-\frac{A \mathbf{x}}{R^{3}} \tag{*}
\end{equation*}
$$

except within a thin axisymmetric boundary layer near $z=0$.
(a) Estimate the Reynolds number of the flow as a function of $R$, and thus give an estimate for how small $R$ needs to be for such a solution to be applicable. Show that the radial pressure gradient is proportional to $R^{-5}$.
(b) In cylindrical polar coordinates $(r, \theta, z)$, the steady axisymmetric boundary-layer equations for the velocity components $(u, 0, w)$ can be written as

$$
u \frac{\partial u}{\partial r}+w \frac{\partial u}{\partial z}=-\frac{1}{\rho} \frac{d P}{d r}+\nu \frac{\partial^{2} u}{\partial z^{2}}, \quad \text { where } \quad u=-\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad w=\frac{1}{r} \frac{\partial \Psi}{\partial r}
$$

and $\Psi(r, z)$ is the Stokes streamfunction. Verify that the condition of incompressibility is satisfied by the use of $\Psi$.

Use scaling arguments to estimate the thickness $\delta(r)$ of the boundary layer near $z=0$ and then to motivate seeking a similarity solution of the form

$$
\Psi=(A \nu r)^{1 / 2} F(\eta), \quad \text { where } \quad \eta=z / \delta(r) .
$$

(c) Obtain the differential equation satisfied by $F$, and state the conditions that would determine its solution. [You are not required to find this solution.]

By considering the flux in the boundary layer, explain why there should be a correction to the approximation $(*)$ of relative magnitude $(\nu R / A)^{1 / 2} \ll 1$.

## Paper 4, Section II

## 38A Fluid Dynamics II

Consider a steady axisymmetric flow with components $(-\alpha r, v(r), 2 \alpha z)$ in cylindrical polar coordinates $(r, \theta, z)$, where $\alpha$ is a positive constant. The fluid has density $\rho$ and kinematic viscosity $\nu$.
(a) Briefly describe the flow and confirm that it is incompressible.
(b) Show that the vorticity has one component $\omega(r)$, in the $z$ direction. Write down the corresponding vorticity equation and derive the solution

$$
\omega=\omega_{0} e^{-\alpha r^{2} /(2 \nu)}
$$

Hence find $v(r)$ and show that it has a maximum at some finite radius $r^{*}$, indicating how $r^{*}$ scales with $\nu$ and $\alpha$.
(c) Find an expression for the net advection of angular momentum, $\rho r v$, into the finite cylinder defined by $r \leqslant r_{0}$ and $-z_{0} \leqslant z \leqslant z_{0}$. Show that this is always positive and asymptotes to the value

$$
\frac{8 \pi \rho z_{0} \omega_{0} \nu^{2}}{\alpha}
$$

as $r_{0} \rightarrow \infty$.
(d) Show that the torque exerted on the cylinder of part (c) by the exterior flow is always negative and demonstrate that it exactly balances the net advection of angular momentum. Comment on why this has to be so.
[You may assume that for a flow $(u, v, w)$ in cylindrical polar coordinates

$$
e_{r \theta}=\frac{r}{2} \frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{2 r} \frac{\partial u}{\partial \theta}, \quad e_{\theta z}=\frac{1}{2 r} \frac{\partial w}{\partial \theta}+\frac{1}{2} \frac{\partial v}{\partial z}, \quad e_{r z}=\frac{1}{2} \frac{\partial u}{\partial z}+\frac{1}{2} \frac{\partial w}{\partial r}
$$

$$
\text { and } \quad \boldsymbol{\omega}=\frac{1}{r}\left|\begin{array}{ccc}
\mathbf{e}_{r} & r \mathbf{e}_{\theta} & \mathbf{e}_{z} \\
\partial / \partial r & \partial / \partial \theta & \partial / \partial z \\
u & r v & w
\end{array}\right|
$$

## Paper 1, Section II

## 39B Fluid Dynamics II

A viscous fluid is confined between an inner, impermeable cylinder of radius $a$ with centre at $(x, y)=(0, a)$ and another outer, impermeable cylinder of radius $2 a$ with centre at $(0,2 a)$ (so they touch at the origin and both have their axes in the $z$ direction). The inner cylinder rotates about its axis with angular velocity $\Omega$ and the outer cylinder rotates about its axis with angular velocity $-\Omega / 4$. The fluid motion is two-dimensional and slow enough that the Stokes approximation is appropriate.
(i) Show that the boundary of the inner cylinder is described by the relationship

$$
r=2 a \sin \theta
$$

where $(r, \theta)$ are the usual polar coordinates centred on $(x, y)=(0,0)$. Show also that on this cylinder the boundary condition on the tangential velocity can be written as

$$
u_{r} \cos \theta+u_{\theta} \sin \theta=a \Omega
$$

where $u_{r}$ and $u_{\theta}$ are the components of the velocity in the $r$ and $\theta$ directions respectively. Explain why the boundary condition $\psi=0$ (where $\psi$ is the streamfunction such that $u_{r}=\frac{1}{r} \frac{\partial \psi}{\partial \theta}$ and $u_{\theta}=-\frac{\partial \psi}{\partial r}$ ) can be imposed.
(ii) Write down the boundary conditions to be satisfied on the outer cylinder $r=4 a \sin \theta$, explaining carefully why $\psi=0$ can also be imposed on this cylinder as well.
(iii) It is given that the streamfunction is of the form

$$
\psi=A \sin ^{2} \theta+B r^{2}+C r \sin \theta+D \sin ^{3} \theta / r
$$

where $A, B, C$ and $D$ are constants, which satisfies $\nabla^{4} \psi=0$. Using the fact that $B=0$ due to the symmetry of the problem, show that the streamfunction is

$$
\psi=\frac{\alpha \sin \theta}{r}(r-2 a \sin \theta)(r-4 a \sin \theta)
$$

where the constant $\alpha$ is to be found.
(iv) Sketch the streamline pattern between the cylinders and determine the $(x, y)$ coordinates of the stagnation point in the flow.

## Paper 2, Section II

## 38B Fluid Dynamics II

Consider a two-dimensional flow of a viscous fluid down a plane inclined at an angle $\alpha$ to the horizontal. Initially, the fluid, which has a volume $V$, occupies a region $0 \leqslant x \leqslant x^{*}$ with $x$ increasing down the slope. At large times the flow becomes thin-layer flow.
(i) Write down the two-dimensional Navier-Stokes equations and simplify them using the lubrication approximation. Show that the governing equation for the height of the film, $h=h(x, t)$, is

$$
\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(\frac{g h^{3} \sin \alpha}{3 \nu}\right)=0
$$

where $\nu$ is the kinematic viscosity of the fluid and $g$ is the acceleration due to gravity, being careful to justify why the streamwise pressure gradient has been ignored compared to the gravitational body force.
(ii) Develop a similarity solution to $(\dagger)$ and, using the fact that the volume of fluid is conserved over time, derive an expression for the position and height of the head of the current downstream.
(iii) Fluid is now continuously supplied at $x=0$. By using scaling analysis, estimate the rate at which fluid would have to be supplied for the head height to asymptote to a constant value at large times.

## Paper 3, Section II

## 38B Fluid Dynamics II

(a) Briefly outline the derivation of the boundary layer equation

$$
u u_{x}+v u_{y}=U d U / d x+\nu u_{y y}
$$

explaining the significance of the symbols used and what sets the $x$-direction.
(b) Viscous fluid occupies the sector $0<\theta<\alpha$ in cylindrical coordinates which is bounded by rigid walls and there is a line sink at the origin of strength $\alpha Q$ with $Q / \nu \gg 1$. Assume that vorticity is confined to boundary layers along the rigid walls $\theta=0$ $(x>0, y=0)$ and $\theta=\alpha$.
(i) Find the flow outside the boundary layers and clarify why boundary layers exist at all.
(ii) Show that the boundary layer thickness along the wall $y=0$ is proportional to

$$
\delta:=\left(\frac{\nu}{Q}\right)^{1 / 2} x
$$

(iii) Show that the boundary layer equation admits a similarity solution for the streamfunction $\psi(x, y)$ of the form

$$
\psi=(\nu Q)^{1 / 2} f(\eta)
$$

where $\eta=y / \delta$. You should find the equation and boundary conditions satisfied by $f$.
(iv) Verify that

$$
\frac{d f}{d \eta}=\frac{5-\cosh (\sqrt{2} \eta+c)}{1+\cosh (\sqrt{2} \eta+c)}
$$

yields a solution provided the constant $c$ has one of two possible values. Which is the likely physical choice?

## Paper 4, Section II

## 38B Fluid Dynamics II

Consider a two-dimensional horizontal vortex sheet of strength $U$ in a homogeneous inviscid fluid at height $h$ above a horizontal rigid boundary at $y=0$ so that the fluid velocity is

$$
\boldsymbol{u}=\left\{\begin{array}{cr}
U \hat{\boldsymbol{x}}, & 0<y<h \\
\mathbf{0}, & h<y
\end{array}\right.
$$

(i) Investigate the linear instability of the sheet by determining the relevant dispersion relation for small, inviscid, irrotational perturbations. For what wavelengths is the sheet unstable?
(ii) Evaluate the temporal growth rate and the wave propagation speed in the limits of both short and long waves. Using these results, sketch how the growth rate varies with the wavenumber.
(iii) Comment briefly on how the introduction of a stable density difference (fluid in $y>h$ is less dense than that in $0<y<h$ ) and surface tension at the interface would affect the growth rates.

## Paper 4, Section II

## 37A Fluid Dynamics

(a) Show that the Stokes flow around a rigid moving sphere has the minimum viscous dissipation rate of all incompressible flows which satisfy the no-slip boundary conditions on the sphere.
(b) Let $\boldsymbol{u}=\boldsymbol{\nabla}(\boldsymbol{x} \cdot \boldsymbol{\Phi}+\chi)-2 \boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ and $\chi$ are solutions of Laplace's equation, i.e. $\nabla^{2} \boldsymbol{\Phi}=\mathbf{0}$ and $\nabla^{2} \chi=0$.
(i) Show that $\boldsymbol{u}$ is incompressible.
(ii) Show that $\boldsymbol{u}$ satisfies Stokes equation if the pressure $p=2 \mu \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}$.
(c) Consider a rigid sphere moving with velocity $\boldsymbol{U}$. The Stokes flow around the sphere is given by

$$
\mathbf{\Phi}=\alpha \frac{\boldsymbol{U}}{r} \quad \text { and } \quad \chi=\beta \boldsymbol{U} \cdot \nabla\left(\frac{1}{r}\right)
$$

where the origin is chosen to be at the centre of the sphere. Find the values for $\alpha$ and $\beta$ which ensure no-slip conditions are satisfied on the sphere.

## Paper 2, Section II

## 37A Fluid Dynamics

A viscous fluid is contained in a channel between rigid planes $y=-h$ and $y=h$. The fluid in the upper region $\sigma<y<h$ (with $-h<\sigma<h$ ) has dynamic viscosity $\mu_{-}$ while the fluid in the lower region $-h<y<\sigma$ has dynamic viscosity $\mu_{+}>\mu_{-}$. The plane at $y=h$ moves with velocity $U_{-}$and the plane at $y=-h$ moves with velocity $U_{+}$, both in the $x$ direction. You may ignore the effect of gravity.
(a) Find the steady, unidirectional solution of the Navier-Stokes equations in which the interface between the two fluids remains at $y=\sigma$.
(b) Using the solution from part (a):
(i) calculate the stress exerted by the fluids on the two boundaries;
(ii) calculate the total viscous dissipation rate in the fluids;
(iii) demonstrate that the rate of working by boundaries balances the viscous dissipation rate in the fluids.
(c) Consider the situation where $U_{+}+U_{-}=0$. Defining the volume flux in the upper region as $Q_{-}$and the volume flux in the lower region as $Q_{+}$, show that their ratio is independent of $\sigma$ and satisfies

$$
\frac{Q_{-}}{Q_{+}}=-\frac{\mu_{-}}{\mu_{+}} .
$$

## Paper 3, Section II

## 38A Fluid Dynamics

For a fluid with kinematic viscosity $\nu$, the steady axisymmetric boundary-layer equations for flow primarily in the $z$-direction are

$$
\begin{aligned}
u \frac{\partial w}{\partial r}+w \frac{\partial w}{\partial z} & =\frac{\nu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right) \\
\frac{1}{r} \frac{\partial(r u)}{\partial r}+\frac{\partial w}{\partial z} & =0
\end{aligned}
$$

where $u$ is the fluid velocity in the $r$-direction and $w$ is the fluid velocity in the $z$-direction. A thin, steady, axisymmetric jet emerges from a point at the origin and flows along the $z$-axis in a fluid which is at rest far from the $z$-axis.
(a) Show that the momentum flux

$$
M:=\int_{0}^{\infty} r w^{2} d r
$$

is independent of the position $z$ along the jet. Deduce that the thickness $\delta(z)$ of the jet increases linearly with $z$. Determine the scaling dependence on $z$ of the centre-line velocity $W(z)$. Hence show that the jet entrains fluid.
(b) A similarity solution for the streamfunction,

$$
\psi(x, y, z)=\nu z g(\eta) \quad \text { with } \quad \eta:=r / z,
$$

exists if $g$ satisfies the second order differential equation

$$
\eta g^{\prime \prime}-g^{\prime}+g g^{\prime}=0 .
$$

Using appropriate boundary and normalisation conditions (which you should state clearly) to solve this equation, show that

$$
g(\eta)=\frac{12 M \eta^{2}}{32 \nu^{2}+3 M \eta^{2}} .
$$

## Paper 1, Section II

## 38A Fluid Dynamics

A disc of radius $R$ and weight $W$ hovers at a height $h$ on a cushion of air above a horizontal air table - a fine porous plate through which air of density $\rho$ and dynamic viscosity $\mu$ is pumped upward at constant speed $V$. You may assume that the air flow is axisymmetric with no flow in the azimuthal direction, and that the effect of gravity on the air may be ignored.
(a) Write down the relevant components of the Navier-Stokes equations. By estimating the size of the individual terms, simplify these equations when $\varepsilon:=h / R \ll 1$ and $R e:=\rho V h / \mu \ll 1$.
(b) Explain briefly why it is reasonable to expect that the vertical velocity of the air below the disc is a function of distance above the air table alone, and thus find the steady pressure distribution below the disc. Hence show that

$$
W=\frac{3 \pi \mu V R}{2 \varepsilon^{3}} .
$$

## Paper 2, Section II

## 38C Fluid Dynamics II

An initially unperturbed two-dimensional inviscid jet in $-h<y<h$ has uniform speed $U$ in the $x$ direction, while the surrounding fluid is stationary. The unperturbed velocity field $\mathbf{u}=(u, v)$ is therefore given by

$$
\begin{array}{ll}
u=0 & \text { in } \quad y>h, \\
u=U & \text { in } \quad-h<y<h, \\
u=0 & \text { in } \quad y<-h .
\end{array}
$$

Consider separately disturbances in which the layer occupies $-h-\eta<y<h+\eta$ (varicose disturbances) and disturbances in which the layer occupies $-h+\eta<y<h+\eta$ (sinuous disturbances), where $\eta(x, t)=\hat{\eta} e^{i k x+\sigma t}$, and determine the dispersion relation $\sigma(k)$ in each case.

Find asymptotic expressions for the real part $\sigma_{R}$ of $\sigma$ in the limits $k \rightarrow 0$ and $k \rightarrow \infty$ and draw sketches of $\sigma_{R}(k)$ in each case.

Compare the rates of growth of the two types of disturbance.

## Paper 1, Section II

## 38C Fluid Dynamics II

A two-dimensional layer of very viscous fluid of uniform thickness $h(t)$ sits on a stationary, rigid surface $y=0$. It is impacted by a stream of air (which can be assumed inviscid) such that the air pressure at $y=h$ is $p_{0}-\frac{1}{2} \rho_{a} E^{2} x^{2}$, where $p_{0}$ and $E$ are constants, $\rho_{a}$ is the density of the air, and $x$ is the coordinate parallel to the surface.

What boundary conditions apply to the velocity $\mathbf{u}=(u, v)$ and stress tensor $\sigma$ of the viscous fluid at $y=0$ and $y=h$ ?

By assuming the form $\psi=x f(y)$ for the stream function of the flow, or otherwise, solve the Stokes equations for the velocity and pressure fields. Show that the layer thins at a rate

$$
V=-\frac{\mathrm{d} h}{\mathrm{~d} t}=\frac{1}{3} \frac{\rho_{a}}{\mu} E^{2} h^{3} .
$$

## Paper 4, Section II

## 38C Fluid Dynamics II

A cylinder of radius $a$ rotates about its axis with angular velocity $\Omega$ while its axis is fixed parallel to and at a distance $a+h_{0}$ from a rigid plane, where $h_{0} \ll a$. Fluid of kinematic viscosity $\nu$ fills the space between the cylinder and the plane. Determine the gap width $h$ between the cylinder and the plane as a function of a coordinate $x$ parallel to the surface of the wall and orthogonal to the axis of the cylinder. What is the characteristic length scale, in the $x$ direction, for changes in the gap width? Taking an appropriate approximation for $h(x)$, valid in the region where the gap width $h$ is small, use lubrication theory to determine that the volume flux between the wall and the cylinder (per unit length along the axis) has magnitude $\frac{2}{3} a \Omega h_{0}$, and state its direction.

Evaluate the tangential shear stress $\tau$ on the surface of the cylinder. Approximating the torque on the cylinder (per unit length along the axis) in the form of an integral $T=a \int_{-\infty}^{\infty} \tau d x$, find the torque $T$ to leading order in $h_{0} / a \ll 1$.

Explain the restriction $a^{1 / 2} \Omega h_{0}^{3 / 2} / \nu \ll 1$ for the theory to be valid.
[You may use the facts that $\int_{-\infty}^{\infty} \frac{d x}{\left(1+x^{2}\right)^{2}}=\frac{\pi}{2}$ and $\int_{-\infty}^{\infty} \frac{d x}{\left(1+x^{2}\right)^{3}}=\frac{3 \pi}{8}$.]

## Paper 3, Section II

## 39C Fluid Dynamics II

For two Stokes flows $\mathbf{u}^{(1)}(\mathbf{x})$ and $\mathbf{u}^{(2)}(\mathbf{x})$ inside the same volume $V$ with different boundary conditions on its boundary $S$, prove the reciprocal theorem

$$
\int_{S} u_{i}^{(1)} \sigma_{i j}^{(2)} n_{j} d S=\int_{S} u_{i}^{(2)} \sigma_{i j}^{(1)} n_{j} d S,
$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are the stress tensors associated with the flows.
Stating clearly any properties of Stokes flow that you require, use the reciprocal theorem to prove that the drag $\mathbf{F}$ on a body translating with uniform velocity $\mathbf{U}$ is given by

$$
F_{i}=A_{i j} U_{j},
$$

where $\mathbf{A}$ is a symmetric second-rank tensor that depends only on the geometry of the body.

A slender rod falls slowly through very viscous fluid with its axis inclined to the vertical. Explain why the rod does not rotate, stating any properties of Stokes flow that you use.

When the axis of the rod is inclined at an angle $\theta$ to the vertical, the centre of mass of the rod travels at an angle $\phi$ to the vertical. Given that the rod falls twice as quickly when its axis is vertical as when its axis is horizontal, show that

$$
\tan \phi=\frac{\sin \theta \cos \theta}{1+\cos ^{2} \theta} .
$$

## Paper 2, Section II

## 36B Fluid Dynamics II

A cylinder of radius $a$ falls at speed $U$ without rotating through viscous fluid adjacent to a vertical plane wall, with its axis horizontal and parallel to the wall. The distance between the cylinder and the wall is $h_{0} \ll a$. Use lubrication theory in a frame of reference moving with the cylinder to determine that the two-dimensional volume flux between the cylinder and the wall is

$$
q=\frac{2 h_{0} U}{3}
$$

upwards, relative to the cylinder.
Determine an expression for the viscous shear stress on the cylinder. Use this to calculate the viscous force and hence the torque on the cylinder. If the cylinder is free to rotate, what does your result say about the sense of rotation of the cylinder?
[Hint: You may quote the following integrals:

$$
\int_{-\infty}^{\infty} \frac{d t}{1+t^{2}}=\pi, \quad \int_{-\infty}^{\infty} \frac{d t}{\left(1+t^{2}\right)^{2}}=\frac{\pi}{2}, \quad \int_{-\infty}^{\infty} \frac{d t}{\left(1+t^{2}\right)^{3}}=\frac{3 \pi}{8}
$$

## Paper 1, Section II

## 37B Fluid Dynamics II

Fluid of density $\rho$ and dynamic viscosity $\mu$ occupies the region $y>0$ in Cartesian coordinates $(x, y, z)$. A semi-infinite, dense array of cilia occupy the half plane $y=0$, $x>0$ and apply a stress in the $x$-direction on the adjacent fluid, working at a constant and uniform rate $\rho P$ per unit area, which causes the fluid to move with steady velocity $\mathbf{u}=(u(x, y), v(x, y), 0)$. Give a careful physical explanation of the boundary condition

$$
\left.u \frac{\partial u}{\partial y}\right|_{y=0}=-\frac{P}{\nu} \quad \text { for } \quad x>0
$$

paying particular attention to signs, where $\nu$ is the kinematic viscosity of the fluid. Why would you expect the fluid motion to be confined to a thin region near $y=0$ for sufficiently large values of $x$ ?

Write down the viscous-boundary-layer equations governing the thin region of fluid motion. Show that the flow can be approximated by a stream function

$$
\psi(x, y)=U(x) \delta(x) f(\eta), \quad \text { where } \quad \eta=\frac{y}{\delta(x)}
$$

Determine the functions $U(x)$ and $\delta(x)$. Show that the dimensionless function $f(\eta)$ satisfies

$$
f^{\prime \prime \prime}=\frac{1}{5} f^{\prime 2}-\frac{3}{5} f f^{\prime \prime}
$$

What boundary conditions must be satisfied by $f(\eta)$ ? By considering how the volume flux varies with downstream location $x$, or otherwise, determine (with justification) the sign of the transverse flow $v$.

## Paper 3, Section II

## 37B Fluid Dynamics II

A spherical bubble of radius $a$ moves with velocity $\mathbf{U}$ through a viscous fluid that is at rest far from the bubble. The pressure and velocity fields outside the bubble are given by

$$
p=\mu \frac{a}{r^{3}} \mathbf{U} \cdot \mathbf{x} \quad \text { and } \quad \mathbf{u}=\frac{a}{2 r} \mathbf{U}+\frac{a}{2 r^{3}}(\mathbf{U} \cdot \mathbf{x}) \mathbf{x}
$$

respectively, where $\mu$ is the dynamic viscosity of the fluid, $\mathbf{x}$ is the position vector from the centre of the bubble and $r=|\mathbf{x}|$. Using suffix notation, or otherwise, show that these fields satisfy the Stokes equations.

Obtain an expression for the stress tensor for the fluid outside the bubble and show that the velocity field above also satisfies all the appropriate boundary conditions.

Compute the drag force on the bubble.
[Hint: You may use

$$
\int_{S} n_{i} n_{j} d S=\frac{4}{3} \pi a^{2} \delta_{i j}
$$

where the integral is taken over the surface of a sphere of radius a and $\mathbf{n}$ is the outward unit normal to the surface.]

## Paper 4, Section II

## 37B Fluid Dynamics II

A horizontal layer of inviscid fluid of density $\rho_{1}$ occupying $0<y<h$ flows with velocity $(U, 0)$ above a horizontal layer of inviscid fluid of density $\rho_{2}>\rho_{1}$ occupying $-h<y<0$ and flowing with velocity $(-U, 0)$, in Cartesian coordinates $(x, y)$. There are rigid boundaries at $y= \pm h$. The interface between the two layers is perturbed to position $y=\operatorname{Re}\left(A e^{i k x+\sigma t}\right)$.

Write down the full set of equations and boundary conditions governing this flow. Derive the linearised boundary conditions appropriate in the limit $A \rightarrow 0$. Solve the linearised equations to show that the perturbation to the interface grows exponentially in time if

$$
U^{2}>\frac{\rho_{2}^{2}-\rho_{1}^{2}}{\rho_{1} \rho_{2}} \frac{g}{4 k} \tanh k h
$$

Sketch the right-hand side of this inequality as a function of $k$. Thereby deduce the minimum value of $U$ that makes the system unstable for all wavelengths.

## Paper 4, Section II

## 36B Fluid Dynamics II

A thin layer of fluid of viscosity $\mu$ occupies the gap between a rigid flat plate at $y=0$ and a flexible no-slip boundary at $y=h(x, t)$. The flat plate moves with constant velocity $U \mathbf{e}_{x}$ and the flexible boundary moves with no component of velocity in the $x$-direction.

State the two-dimensional lubrication equations governing the dynamics of the thin layer of fluid. Given a pressure gradient $\mathrm{d} p / \mathrm{d} x$, solve for the velocity profile $u(x, y, t)$ in the fluid and calculate the flux $q(x, t)$. Deduce that the pressure gradient satisfies

$$
\frac{\partial}{\partial x}\left(\frac{h^{3}}{12 \mu} \frac{\mathrm{~d} p}{\mathrm{~d} x}\right)=\frac{\partial h}{\partial t}+\frac{U}{2} \frac{\partial h}{\partial x}
$$

The shape of the flexible boundary is a periodic travelling wave, i.e. $h(x, t)=$ $h(x-c t)$ and $h(\xi+L)=h(\xi)$, where $c$ and $L$ are constants. There is no applied average pressure gradient, so the pressure is also periodic with $p(\xi+L)=p(\xi)$. Show that

$$
\frac{\mathrm{d} p}{\mathrm{~d} x}=6 \mu(U-2 c)\left(\frac{1}{h^{2}}-\frac{\left\langle h^{-2}\right\rangle}{\left\langle h^{-3}\right\rangle} \frac{1}{h^{3}}\right),
$$

where $\langle\ldots\rangle=\frac{1}{L} \int_{0}^{L} \ldots \mathrm{~d} x$ denotes the average over a period. Calculate the shear stress $\sigma_{x y}$ on the plate.

The speed $U$ is such that there is no need to apply an external tangential force to the plate in order to maintain its motion. Show that

$$
U=6 c \frac{\left\langle h^{-2}\right\rangle\left\langle h^{-2}\right\rangle-\left\langle h^{-1}\right\rangle\left\langle h^{-3}\right\rangle}{3\left\langle h^{-2}\right\rangle\left\langle h^{-2}\right\rangle-4\left\langle h^{-1}\right\rangle\left\langle h^{-3}\right\rangle} .
$$

## Paper 3, Section II

## 36B Fluid Dynamics II

A cylindrical pipe of radius $a$ and length $L \gg a$ contains two viscous fluids arranged axisymmetrically with fluid 1 of viscosity $\mu_{1}$ occupying the central region $r<\beta a$, where $0<\beta<1$, and fluid 2 of viscosity $\mu_{2}$ occupying the surrounding annular region $\beta a<r<a$. The flow in each fluid is assumed to be steady and unidirectional, with velocities $u_{1}(r) \mathbf{e}_{z}$ and $u_{2}(r) \mathbf{e}_{z}$ respectively, with respect to cylindrical coordinates $(r, \theta, z)$ aligned with the pipe. A fixed pressure drop $\Delta p$ is applied between the ends of the pipe.

Starting from the Navier-Stokes equations, derive the equations satisfied by $u_{1}(r)$ and $u_{2}(r)$, and state all the boundary conditions. Show that the pressure gradient is constant.

Solve for the velocity profile in each fluid and calculate the corresponding flow rates, $Q_{1}$ and $Q_{2}$.

Derive the relationship between $\beta$ and $\mu_{2} / \mu_{1}$ that yields the same flow rate in each fluid. Comment on the behaviour of $\beta$ in the limits $\mu_{2} / \mu_{1} \gg 1$ and $\mu_{2} / \mu_{1} \ll 1$, illustrating your comment by sketching the flow profiles.
[Hint: In cylindrical coordinates $(r, \theta, z)$,

$$
\left.\nabla^{2} u=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}}+\frac{\partial^{2} u}{\partial z^{2}}, \quad e_{r z}=\frac{1}{2}\left(\frac{\partial u_{r}}{\partial z}+\frac{\partial u_{z}}{\partial r}\right) .\right]
$$

## Paper 2, Section II

## 36B Fluid Dynamics II

For a two-dimensional flow in plane polar coordinates $(r, \theta)$, state the relationship between the streamfunction $\psi(r, \theta)$ and the flow components $u_{r}$ and $u_{\theta}$. Show that the vorticity $\omega$ is given by $\omega=-\nabla^{2} \psi$, and deduce that the streamfunction for a steady two-dimensional Stokes flow satisfies the biharmonic equation

$$
\nabla^{4} \psi=0
$$

A rigid stationary circular disk of radius $a$ occupies the region $r \leqslant a$. The flow far from the disk tends to a steady straining flow $\mathbf{u}_{\infty}=(-E x, E y)$, where $E$ is a constant. Inertial forces may be neglected. Calculate the streamfunction, $\psi_{\infty}(r, \theta)$, for the far-field flow.

By making an appropriate assumption about its dependence on $\theta$, find the streamfunction $\psi$ for the flow around the disk, and deduce the flow components, $u_{r}(r, \theta)$ and $u_{\theta}(r, \theta)$.

Calculate the tangential surface stress, $\sigma_{r \theta}$, acting on the boundary of the disk.
[Hints: In plane polar coordinates $(r, \theta)$,

$$
\begin{gathered}
\nabla \cdot \mathbf{u}=\frac{1}{r} \frac{\partial\left(r u_{r}\right)}{\partial r}+\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}, \quad \omega=\frac{1}{r} \frac{\partial\left(r u_{\theta}\right)}{\partial r}-\frac{1}{r} \frac{\partial u_{r}}{\partial \theta} \\
\left.\nabla^{2} V=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} V}{\partial \theta^{2}}, \quad e_{r \theta}=\frac{1}{2}\left(r \frac{\partial}{\partial r}\left(\frac{u_{\theta}}{r}\right)+\frac{1}{r} \frac{\partial u_{r}}{\partial \theta}\right) .\right]
\end{gathered}
$$

## Paper 1, Section II

## 36B Fluid Dynamics II

State the vorticity equation and interpret the meaning of each term.
A planar vortex sheet is diffusing in the presence of a perpendicular straining flow. The flow is everywhere of the form $\mathbf{u}=(U(y, t),-E y, E z)$, where $U \rightarrow \pm U_{0}$ as $y \rightarrow \pm \infty$, and $U_{0}$ and $E>0$ are constants. Show that the vorticity has the form $\boldsymbol{\omega}=\omega(y, t) \mathbf{e}_{z}$, and obtain a scalar equation describing the evolution of $\omega$.

Explain physically why the solution approaches a steady state in which the vorticity is concentrated near $y=0$. Use scaling to estimate the thickness $\delta$ of the steady vorticity layer as a function of $E$ and the kinematic viscosity $\nu$.

Determine the steady vorticity profile, $\omega(y)$, and the steady velocity profile, $U(y)$.
[Hint: $\left.\quad \operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^{2}} \mathrm{~d} u.\right]$
State, with a brief physical justification, why you might expect this steady flow to be unstable to long-wavelength perturbations, defining what you mean by long.

## Paper 4, Section II

## 35E Fluid Dynamics II

A stationary inviscid fluid of thickness $h$ and density $\rho$ is located below a free surface at $y=h$ and above a deep layer of inviscid fluid of the same density in $y<0$ flowing with uniform velocity $U>0$ in the $\mathbf{e}_{x}$ direction. The base velocity profile is thus

$$
u=U, y<0 ; \quad u=0,0<y<h
$$

while the free surface at $y=h$ is maintained flat by gravity.
By considering small perturbations of the vortex sheet at $y=0$ of the form $\eta=\eta_{0} e^{i k x+\sigma t}, k>0$, calculate the dispersion relationship between $k$ and $\sigma$ in the irrotational limit. By explicitly deriving that

$$
\operatorname{Re}(\sigma)= \pm \frac{\sqrt{\tanh (h k)}}{1+\tanh (h k)} U k
$$

deduce that the vortex sheet is unstable at all wavelengths. Show that the growth rates of the unstable modes are approximately $U k / 2$ when $h k \gg 1$ and $U k \sqrt{h k}$ when $h k \ll 1$.

## Paper 2, Section II

## 35E Fluid Dynamics II

Consider an infinite rigid cylinder of radius $a$ parallel to a horizontal rigid stationary surface. Let $\mathbf{e}_{x}$ be the direction along the surface perpendicular to the cylinder axis, $\mathbf{e}_{y}$ the direction normal to the surface (the surface is at $y=0$ ) and $\mathbf{e}_{z}$ the direction along the axis of the cylinder. The cylinder moves with constant velocity $U \mathbf{e}_{x}$. The minimum separation between the cylinder and the surface is denoted by $h_{0} \ll a$.
(i) What are the conditions for the flow in the thin gap between the cylinder and the surface to be described by the lubrication equations? State carefully the relevant length scale in the $\mathbf{e}_{x}$ direction.
(ii) Without doing any calculation, explain carefully why, in the lubrication limit, the net fluid force $\mathbf{F}$ acting on the stationary surface at $y=0$ has no component in the $\mathbf{e}_{y}$ direction.
(iii) Using the lubrication approximation, calculate the $\mathbf{e}_{x}$ component of the velocity field in the gap between the cylinder and the surface, and determine the pressure gradient as a function of the gap thickness $h(x)$.
(iv) Compute the tangential component of the force, $\mathbf{e}_{x} \cdot \mathbf{F}$, acting on the bottom surface per unit length in the $\mathbf{e}_{z}$ direction.
[You may quote the following integrals:

$$
\int_{-\infty}^{\infty} \frac{d u}{\left(1+u^{2}\right)}=\pi, \quad \int_{-\infty}^{\infty} \frac{d u}{\left(1+u^{2}\right)^{2}}=\frac{\pi}{2}, \quad \int_{-\infty}^{\infty} \frac{d u}{\left(1+u^{2}\right)^{3}}=\frac{3 \pi}{8}
$$

## Paper 3, Section II

## 36E Fluid Dynamics II

Consider a three-dimensional high-Reynolds number jet without swirl induced by a force $\mathbf{F}=F \mathbf{e}_{z}$ imposed at the origin in a fluid at rest. The velocity in the jet, described using cylindrical coordinates $(r, \theta, z)$, is assumed to remain steady and axisymmetric, and described by a boundary layer analysis.
(i) Explain briefly why the flow in the jet can be described by the boundary layer equations

$$
u_{r} \frac{\partial u_{z}}{\partial r}+u_{z} \frac{\partial u_{z}}{\partial z}=\nu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_{z}}{\partial r}\right) .
$$

(ii) Show that the momentum flux in the jet, $F=\int_{S} \rho u_{z}^{2} d S$, where $S$ is an infinite surface perpendicular to $\mathbf{e}_{z}$, is not a function of $z$. Combining this result with scalings from the boundary layer equations, derive the scalings for the unknown width $\delta(z)$ and typical velocity $U(z)$ of the jet as functions of $z$ and the other parameters of the problem $(\rho, \nu, F)$.
(iii) Solving for the flow using a self-similar Stokes streamfunction

$$
\psi(r, z)=U(z) \delta^{2}(z) f(\eta), \quad \eta=r / \delta(z)
$$

show that $f(\eta)$ satisfies the differential equation

$$
f f^{\prime}-\eta\left(f^{\prime 2}+f f^{\prime \prime}\right)=f^{\prime}-\eta f^{\prime \prime}+\eta^{2} f^{\prime \prime \prime}
$$

What boundary conditions should be applied to this equation? Give physical reasons for them.
[Hint: In cylindrical coordinates for axisymmetric incompressible flow $\left(u_{r}(r, z), 0, u_{z}(r, z)\right)$ you are given the incompressibility condition as

$$
\frac{1}{r} \frac{\partial}{\partial r}\left(r u_{r}\right)+\frac{\partial u_{z}}{\partial z}=0
$$

the z-component of the Navier-Stokes equation as

$$
\rho\left(\frac{\partial u_{z}}{\partial t}+u_{r} \frac{\partial u_{z}}{\partial r}+u_{z} \frac{\partial u_{z}}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u_{z}}{\partial r}\right)+\frac{\partial^{2} u_{z}}{\partial z^{2}}\right]
$$

and the relationship between the Stokes streamfunction, $\psi(r, z)$, and the velocity components as

$$
u_{r}=-\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad u_{z}=\frac{1}{r} \frac{\partial \psi}{\partial r}
$$

## Paper 1, Section II

## 36E Fluid Dynamics II

(i) In a Newtonian fluid, the deviatoric stress tensor is linearly related to the velocity gradient so that the total stress tensor is

$$
\sigma_{i j}=-p \delta_{i j}+A_{i j k l} \frac{\partial u_{k}}{\partial x_{l}}
$$

Show that for an incompressible isotropic fluid with a symmetric stress tensor we necessarily have

$$
A_{i j k l} \frac{\partial u_{k}}{\partial x_{l}}=2 \mu e_{i j}
$$

where $\mu$ is a constant which we call the dynamic viscosity and $e_{i j}$ is the symmetric part of $\partial u_{i} / \partial x_{j}$.
(ii) Consider Stokes flow due to the translation of a rigid sphere $S_{a}$ of radius $a$ so that the sphere exerts a force $\mathbf{F}$ on the fluid. At distances much larger than the radius of the sphere, the instantaneous velocity and pressure fields are

$$
u_{i}(\mathbf{x})=\frac{1}{8 \mu \pi}\left(\frac{F_{i}}{r}+\frac{F_{m} x_{m} x_{i}}{r^{3}}\right), \quad p(\mathbf{x})=\frac{1}{4 \pi} \frac{F_{m} x_{m}}{r^{3}}
$$

where $\mathbf{x}$ is measured with respect to an origin located at the centre of the sphere, and $r=|\mathbf{x}|$.

Consider a sphere $S_{R}$ of radius $R \gg a$ instantaneously concentric with $S_{a}$. By explicitly computing the tractions and integrating them, show that the force $\mathbf{G}$ exerted by the fluid located in $r>R$ on $S_{R}$ is constant and independent of $R$, and evaluate it.
(iii) Explain why the Stokes equations in the absence of body forces can be written as

$$
\frac{\partial \sigma_{i j}}{\partial x_{j}}=0
$$

Show that by integrating this equation in the fluid volume located instantaneously between $S_{a}$ and $S_{R}$, you can recover the result in (ii) directly.

## Paper 4, Section II

## 37B Fluid Dynamics II

An incompressible fluid of density $\rho$ and kinematic viscosity $\nu$ is confined in a channel with rigid stationary walls at $y= \pm h$. A spatially uniform pressure gradient $-G \cos \omega t$ is applied in the $x$-direction. What is the physical significance of the dimensionless number $S=\omega h^{2} / \nu$ ?

Assuming that the flow is unidirectional and time-harmonic, obtain expressions for the velocity profile and the total flux. [You may leave your answers as the real parts of complex functions.]

In each of the limits $S \rightarrow 0$ and $S \rightarrow \infty$, find and sketch the flow profiles, find leading-order asymptotic expressions for the total flux, and give a physical interpretation.

Suppose now that $G=0$ and that the channel walls oscillate in their own plane with velocity $U \cos \omega t$ in the $x$-direction. Without explicit calculation of the solution, sketch the flow profile in each of the limits $S \rightarrow 0$ and $S \rightarrow \infty$.

## Paper 2, Section II

## 37B Fluid Dynamics II

Air is blown over the surface of a large, deep reservoir of water in such a way as to exert a tangential stress in the $x$-direction of magnitude $K x^{2}$ for $x>0$, with $K>0$. The water is otherwise at rest and occupies the region $y>0$. The surface $y=0$ remains flat.

Find order-of-magnitude estimates for the boundary-layer thickness $\delta(x)$ and tangential surface velocity $U(x)$ in terms of the relevant physical parameters.

Using the boundary-layer equations, find the ordinary differential equation governing the dimensionless function $f$ defined in the streamfunction

$$
\psi(x, y)=U(x) \delta(x) f(\eta), \quad \text { where } \eta=y / \delta(x) .
$$

What are the boundary conditions on $f$ ?
Does $f \rightarrow 0$ as $\eta \rightarrow \infty$ ? Why, or why not?
The total horizontal momentum flux $P(X)$ across the vertical line $x=X$ is proportional to $X^{a}$ for $X>0$. Find the exponent $a$. By considering the steadiness of the momentum balance in the region $0<x<X$, explain why the value of $a$ is consistent with the form of the stress exerted on the boundary.

## Paper 3, Section II

## 38B Fluid Dynamics II

A rigid sphere of radius $a$ falls under gravity through an incompressible fluid of density $\rho$ and viscosity $\mu$ towards a rigid horizontal plane. The minimum gap $h_{0}(t)$ between the sphere and the plane satisfies $h_{0} \ll a$. Find an approximation for the gap thickness $h(r, t)$ between the sphere and the plane in the region $r \ll a$, where $r$ is the distance from the axis of symmetry.

For a prescribed value of $\dot{h}_{0}=d h_{0} / d t$, use lubrication theory to find the radial velocity and the fluid pressure in the region $r \ll a$. Explain why the approximations of lubrication theory require $h_{0} \ll a$ and $\rho h_{0} \dot{h}_{0} \ll \mu$.

Calculate the total vertical force due to the motion that is exerted by the fluid on the sphere. Deduce that if the sphere is settling under its own weight (corrected for buoyancy) then $h_{0}(t)$ decreases exponentially. What is the exponential decay rate for a solid sphere of density $\rho_{s}$ in a fluid of density $\rho_{f}$ ?

## Paper 1, Section II

## 38B Fluid Dynamics II

A particle of arbitrary shape and volume $4 \pi a^{3} / 3$ moves at velocity $\mathbf{U}(t)$ through an unbounded incompressible fluid of density $\rho$ and viscosity $\mu$. The Reynolds number of the flow is very small so that the inertia of the fluid can be neglected. Show that the particle experiences a force $\mathbf{F}(t)$ due to the surface stresses given by

$$
F_{i}(t)=-\mu a A_{i j} U_{j}(t),
$$

where $A_{i j}$ is a dimensionless second-rank tensor determined solely by the shape and orientation of the particle. State the reason why $A_{i j}$ must be positive definite.

Show further that, if the particle has the same reflectional symmetries as a cube, then

$$
A_{i j}=\lambda \delta_{i j} .
$$

Let $b$ be the radius of the smallest sphere that contains the particle (still assuming cubic symmetry). By considering the Stokes flow associated with this sphere, suitably extended, and using the minimum dissipation theorem (which should be stated carefully), show that

$$
\lambda \leqslant 6 \pi b / a .
$$

[You may assume the expression for the Stokes drag on a sphere.]

## Paper 4, Section II

## 37A Fluid Dynamics II

Consider the flow of an incompressible fluid of uniform density $\rho$ and dynamic viscosity $\mu$. Show that the rate of viscous dissipation per unit volume is given by

$$
\Phi=2 \mu e_{i j} e_{i j},
$$

where $e_{i j}$ is the strain rate.
Determine expressions for $e_{i j}$ and $\Phi$ when the flow is irrotational with velocity potential $\phi$.

In deep water a linearised wave with a surface displacement $\eta=a \cos (k x-\omega t)$ has a velocity potential $\phi=-(\omega a / k) e^{-k z} \sin (k x-\omega t)$. Hence determine the rate of the viscous dissipation, averaged over a wave period $2 \pi / \omega$, for an irrotational surface wave of wavenumber $k$ and small amplitude $a \ll 1 / k$ in a fluid with very small viscosity $\mu \ll \rho \omega / k^{2}$ and great depth $H \gg 1 / k$.

Calculate the depth-integrated kinetic energy per unit wavelength. Assuming that the average potential energy is equal to the average kinetic energy, show that the total wave energy decreases to leading order as $e^{-\gamma t}$, where $\gamma$ should be found.

## Paper 2, Section II

## 37A Fluid Dynamics II

Write down the boundary-layer equations for steady two-dimensional flow of a viscous incompressible fluid with velocity $U(x)$ outside the boundary layer. Find the boundary layer thickness $\delta(x)$ when $U(x)=U_{0}$, a constant. Show that the boundarylayer equations can be satisfied in this case by a streamfunction $\psi(x, y)=g(x) f(\eta)$ with suitable scaling function $g(x)$ and similarity variable $\eta$. Find the equation satisfied by $f$ and the associated boundary conditions.

Find the drag on a thin two-dimensional flat plate of finite length $L$ placed parallel to a uniform flow. Why does the drag not increase in proportion to the length of the plate? [You may assume that the boundary-layer solution is applicable except in negligibly small regions near the leading and trailing edges. You may also assume that $f^{\prime \prime}(0)=0.33$.]

## Paper 3, Section II

38A Fluid Dynamics II
A disk hovers on a cushion of air above an air-table - a fine porous plate through which a constant flux of air is pumped. Let the disk have a radius $R$ and a weight $M g$ and hover at a low height $h \ll R$ above the air-table. Let the volume flux of air, which has density $\rho$ and viscosity $\mu$, be $w$ per unit surface area. The conditions are such that $\rho w h^{2} / \mu R \ll 1$. Explain the significance of this restriction.

Find the pressure distribution in the air under the disk. Show that this pressure balances the weight of the disk if

$$
h=R\left(\frac{3 \pi \mu R w}{2 M g}\right)^{1 / 3} .
$$

## Paper 1, Section II

## 38A Fluid Dynamics II

The velocity field $\mathbf{u}$ and stress tensor $\sigma$ satisfy the Stokes equations in a volume $V$ bounded by a surface $S$. Let $\hat{\mathbf{u}}$ be another solenoidal velocity field. Show that

$$
\int_{S} \sigma_{i j} n_{j} \hat{u}_{i} d S=\int_{V} 2 \mu e_{i j} \hat{e}_{i j} d V
$$

where $\mathbf{e}$ and $\hat{\mathbf{e}}$ are the strain-rates corresponding to the velocity fields $\mathbf{u}$ and $\hat{\mathbf{u}}$ respectively, and $\mathbf{n}$ is the unit normal vector out of $V$. Hence, or otherwise, prove the minimum dissipation theorem for Stokes flow.

A particle moves at velocity $\mathbf{U}$ through a highly viscous fluid of viscosity $\mu$ contained in a stationary vessel. As the particle moves, the fluid exerts a drag force $\mathbf{F}$ on it. Show that

$$
-\mathbf{F} \cdot \mathbf{U}=\int_{V} 2 \mu e_{i j} e_{i j} d V
$$

Consider now the case when the particle is a small cube, with sides of length $\ell$, moving in a very large vessel. You may assume that

$$
\mathbf{F}=-k \mu \ell \mathbf{U},
$$

for some constant $k$. Use the minimum dissipation theorem, being careful to declare the domain(s) involved, to show that

$$
3 \pi \leqslant k \leqslant 3 \sqrt{3} \pi
$$

[You may assume Stokes' result for the drag on a sphere of radius $a, \mathbf{F}=-6 \pi \mu a \mathbf{U}$.]

## Paper 4, Section II

## 37C Fluid Dynamics II

A steady, two-dimensional flow in the region $y>0$ takes the form $(u, v)=$ $(E x,-E y)$ at large $y$, where $E$ is a positive constant. The boundary at $y=0$ is rigid and no-slip. Consider the velocity field $u=\partial \psi / \partial y, v=-\partial \psi / \partial x$ with stream function $\psi=\operatorname{Ex\delta } \delta(\eta)$, where $\eta=y / \delta$ and $\delta=(\nu / E)^{1 / 2}$ and $\nu$ is the kinematic viscosity. Show that this velocity field satisfies the Navier-Stokes equations provided that $f(\eta)$ satisfies

$$
f^{\prime \prime \prime}+f f^{\prime \prime}-\left(f^{\prime}\right)^{2}=-1 .
$$

What are the conditions on $f$ at $\eta=0$ and as $\eta \rightarrow \infty$ ?

## Paper 2, Section II

## 37C Fluid Dynamics II

An incompressible viscous liquid occupies the long thin region $0 \leqslant y \leqslant h(x)$ for $0 \leqslant x \leqslant \ell$, where $h(x)=d_{1}+\alpha x$ with $h(0)=d_{1}, h(\ell)=d_{2}<d_{1}$ and $d_{1} \ll \ell$. The top boundary at $y=h(x)$ is rigid and stationary. The bottom boundary at $y=0$ is rigid and moving at velocity $(U, 0,0)$. Fluid can move in and out of the ends $x=0$ and $x=\ell$, where the pressure is the same, namely $p_{0}$.

Explaining the approximations of lubrication theory as you use them, find the velocity profile in the long thin region, and show that the volume flux $Q$ (per unit width in the $z$-direction) is

$$
Q=\frac{U d_{1} d_{2}}{d_{1}+d_{2}} .
$$

Find also the value of $h(x)$ (i) where the pressure is maximum, (ii) where the tangential viscous stress on the bottom $y=0$ vanishes, and (iii) where the tangential viscous stress on the top $y=h(x)$ vanishes.

## Paper 3, Section II

## 38C Fluid Dynamics II

For two Stokes flows $\mathbf{u}^{(1)}(\mathbf{x})$ and $\mathbf{u}^{(2)}(\mathbf{x})$ inside the same volume $V$ with different boundary conditions on its boundary $S$, prove the reciprocal theorem

$$
\int_{S} \sigma_{i j}^{(1)} n_{j} u_{i}^{(2)} d S=\int_{S} \sigma_{i j}^{(2)} n_{j} u_{i}^{(1)} d S
$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are the stress fields associated with the flows.
When a rigid sphere of radius $a$ translates with velocity $\mathbf{U}$ through unbounded fluid at rest at infinity, it may be shown that the traction per unit area, $\boldsymbol{\sigma} \cdot \mathbf{n}$, exerted by the sphere on the fluid has the uniform value $3 \mu \mathbf{U} / 2 a$ over the sphere surface. Find the drag on the sphere.

Suppose that the same sphere is now free of external forces and is placed with its centre at the origin in an unbounded Stokes flow given in the absence of the sphere as $\mathbf{u}^{*}(\mathbf{x})$. By applying the reciprocal theorem to the perturbation to the flow generated by the presence of the sphere, and assuming this tends to zero sufficiently rapidly at infinity, show that the instantaneous velocity of the centre of the sphere is

$$
\frac{1}{4 \pi a^{2}} \int \mathbf{u}^{*}(\mathbf{x}) d S
$$

where the integral is taken over the sphere of radius $a$.

## Paper 1, Section II

## 38C Fluid Dynamics II

Define the strain-rate tensor $e_{i j}$ in terms of the velocity components $u_{i}$. Write down the relation between $e_{i j}$, the pressure $p$ and the stress $\sigma_{i j}$ in an incompressible Newtonian fluid of viscosity $\mu$. Show that the local rate of stress-working $\sigma_{i j} \partial u_{i} / \partial x_{j}$ is equal to the local rate of dissipation $2 \mu e_{i j} e_{i j}$.

An incompressible fluid of density $\rho$ and viscosity $\mu$ occupies the semi-infinite region $y>0$ above a rigid plane boundary $y=0$ which oscillates with velocity $(V \cos \omega t, 0,0)$. The fluid is at rest at infinity. Determine the velocity field produced by the boundary motion after any transients have decayed.

Show that the time-averaged rate of dissipation is

$$
\frac{1}{4} \sqrt{2} V^{2}(\mu \rho \omega)^{1 / 2}
$$

per unit area of the boundary. Verify that this is equal to the time average of the rate of working by the boundary on the fluid per unit area.

## Paper 1, Section II

## 38B Fluid Dynamics II

The steady two-dimensional boundary-layer equations for flow primarily in the $x$ direction are

$$
\begin{gathered}
\rho\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right)=-\frac{d P}{d x}+\mu \frac{\partial^{2} u}{\partial y^{2}}, \\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 .
\end{gathered}
$$

A thin, steady, two-dimensional jet emerges from a point at the origin and flows along the $x$-axis in a fluid at rest far from the $x$-axis. Show that the momentum flux

$$
F=\int_{-\infty}^{\infty} \rho u^{2} d y
$$

is independent of position $x$ along the jet. Deduce that the thickness $\delta(x)$ of the jet increases along the jet as $x^{2 / 3}$, while the centre-line velocity $U(x)$ decreases as $x^{-1 / 3}$.

A similarity solution for the jet is sought with a streamfunction $\psi$ of the form

$$
\psi(x, y)=U(x) \delta(x) f(\eta) \quad \text { with } \quad \eta=y / \delta(x) .
$$

Derive the nonlinear third-order non-dimensional differential equation governing $f$, and write down the boundary and normalisation conditions which must be applied.

## Paper 2, Section II

## 37B Fluid Dynamics II

The energy equation for the motion of a viscous, incompressible fluid states that

$$
\frac{d}{d t} \int_{V} \frac{1}{2} \rho u^{2} d V+\int_{S} \frac{1}{2} \rho u^{2} u_{i} n_{i} d S=\int_{S} u_{i} \sigma_{i j} n_{j} d S-2 \mu \int_{V} e_{i j} e_{i j} d V .
$$

Interpret each term in this equation and explain the meaning of the symbols used.
Consider steady rectilinear flow in a (not necessarily circular) pipe having rigid stationary walls. Deduce a relation between the viscous dissipation per unit length of the pipe, the pressure gradient $G$, and the volume flux $Q$.

Starting from the Navier-Stokes equations, calculate the velocity field for steady rectilinear flow in a circular pipe of radius $a$. Using the relationship derived above, or otherwise, find the viscous dissipation per unit length of this flow in terms of $G$.
[Hint: In cylindrical polar coordinates,

$$
\left.\nabla^{2} w(r)=\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right) \cdot\right]
$$

## Paper 3, Section II

## 37B Fluid Dynamics II

If $A_{i}\left(x_{j}\right)$ is harmonic, i.e. if $\nabla^{2} A_{i}=0$, show that

$$
u_{i}=A_{i}-x_{k} \frac{\partial A_{k}}{\partial x_{i}}, \quad \text { with } \quad p=-2 \mu \frac{\partial A_{n}}{\partial x_{n}},
$$

satisfies the incompressibility condition and the Stokes equation. Show that the stress tensor is

$$
\sigma_{i j}=2 \mu\left(\delta_{i j} \frac{\partial A_{n}}{\partial x_{n}}-x_{k} \frac{\partial^{2} A_{k}}{\partial x_{i} \partial x_{j}}\right) .
$$

Consider the Stokes flow corresponding to

$$
A_{i}=V_{i}\left(1-\frac{a}{2 r}\right),
$$

where $V_{i}$ are the components of a constant vector $\mathbf{V}$. Show that on the sphere $r=a$ the normal component of velocity vanishes and the surface traction $\sigma_{i j} x_{j} / a$ is in the normal direction. Hence deduce that the drag force on the sphere is given by

$$
\mathbf{F}=4 \pi \mu a \mathbf{V}
$$

## Paper 4, Section II

## 37B Fluid Dynamics II

A viscous fluid flows along a slowly varying thin channel between no-slip surfaces at $y=0$ and $y=h(x, t)$ under the action of a pressure gradient $d p / d x$. After explaining the approximations and assumptions of lubrication theory, including a comment on the reduced Reynolds number, derive the expression for the volume flux

$$
q=\int_{0}^{h} u d y=-\frac{h^{3}}{12 \mu} \frac{d p}{d x},
$$

as well as the equation

$$
\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0 .
$$

In peristaltic pumping, the surface $h(x, t)$ has a periodic form in space which propagates at a constant speed $c$, i.e. $h(x-c t)$, and no net pressure gradient is applied, i.e. the pressure gradient averaged over a period vanishes. Show that the average flux along the channel is given by

$$
\langle q\rangle=c\left(\langle h\rangle-\frac{\left\langle h^{-2}\right\rangle}{\left\langle h^{-3}\right\rangle}\right),
$$

where $\langle\cdot\rangle$ denotes an average over one period.

## Paper 1, Section II

## 37A Fluid Dynamics II

Write down the Navier-Stokes equation for the velocity $\mathbf{u}(\mathbf{x}, t)$ of an incompressible viscous fluid of density $\rho$ and kinematic viscosity $\nu$. Cast the equation into dimensionless form. Define rectilinear flow, and explain why the spatial form of any steady rectilinear flow is independent of the Reynolds number.
(i) Such a fluid is contained between two infinitely long plates at $y=0, y=a$. The lower plate is at rest while the upper plate moves at constant speed $U$ in the $x$ direction. There is an applied pressure gradient $d p / d x=-G \rho \nu$ in the $x$ direction. Determine the flow field.
(ii) Now there is no applied pressure gradient, but baffles are attached to the lower plate at a distance $L$ from each other $(L \gg a)$, lying between the plates so as to prevent any net volume flux in the $x$ direction. Assuming that far from the baffles the flow is essentially rectilinear, determine the flow field and the pressure gradient in the fluid.

## Paper 2, Section II

## 37A Fluid Dynamics II

What is lubrication theory? Explain the assumptions that go into the theory.
Viscous fluid with dynamic viscosity $\mu$ and density $\rho$ is contained between two flat plates, which approach each other at uniform speed $V$. The first is fixed at $y=0,-L<x<L$. The second has its ends at $\left(-L, h_{0}-\Delta h-V t\right),\left(L, h_{0}+\Delta h-V t\right)$, where $\Delta h \sim h_{0} \ll L$. There is no flow in the $z$ direction, and all variation in $z$ may be neglected. There is no applied pressure gradient in the $x$ direction.

Assuming that $V$ is so small that lubrication theory applies, derive an expression for the horizontal volume flux $Q(x)$ at $t=0$, in terms of the pressure gradient. Show that mass conservation implies that $d Q / d x=V$, so that $Q(L)-Q(-L)=2 V L$. Derive another relation between $Q(L)$ and $Q(-L)$ by setting the pressures at $x= \pm L$ to be equal, and hence show that

$$
Q( \pm L)=V L\left(\frac{\Delta h}{h_{0}} \pm 1\right)
$$

Show that lubrication theory applies if $V \ll \mu / h_{0} \rho$.

## Paper 3, Section II

## 37A Fluid Dynamics II

The equation for the vorticity $\omega(x, y)$ in two-dimensional incompressible flow takes the form

$$
\frac{\partial \omega}{\partial t}+u \frac{\partial \omega}{\partial x}+v \frac{\partial \omega}{\partial y}=\nu\left(\frac{\partial^{2} \omega}{\partial x^{2}}+\frac{\partial^{2} \omega}{\partial y^{2}}\right),
$$

where

$$
u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x} \quad \text { and } \quad \omega=-\left(\frac{\partial^{2} \psi}{\partial x^{2}}+\frac{\partial^{2} \psi}{\partial y^{2}}\right),
$$

and $\psi(x, y)$ is the stream function.
Show that this equation has a time-dependent similarity solution of the form

$$
\psi=C x H(t)^{-1} \phi(\eta), \quad \omega=-C x H(t)^{-3} \phi_{\eta \eta}(\eta) \quad \text { for } \quad \eta=y H(t)^{-1},
$$

if $H(t)=\sqrt{2 C t}$ and $\phi$ satisfies the equation

$$
\begin{equation*}
3 \phi_{\eta \eta}+\eta \phi_{\eta \eta \eta}-\phi_{\eta} \phi_{\eta \eta}+\phi \phi_{\eta \eta \eta}+\frac{1}{R} \phi_{\eta \eta \eta \eta}=0 \tag{*}
\end{equation*}
$$

and $R=C / \nu$ is the effective Reynolds number.
Show that this solution is appropriate for the problem of two-dimensional flow between the rigid planes $y= \pm H(t)$, and determine the boundary conditions on $\phi$ in that case.

Verify that (*) has exact solutions, satisfying the boundary conditions, of the form

$$
\phi=\frac{(-1)^{k}}{k \pi} \sin (k \pi \eta)-\eta, \quad k=1,2, \ldots,
$$

when $R=k^{2} \pi^{2} / 4$. Sketch this solution when $k$ is large, and discuss whether such solutions are likely to be realised in practice.

## Paper 4, Section II

## 37A Fluid Dynamics II

An axisymmetric incompressible Stokes flow has the Stokes stream function $\Psi(R, \theta)$ in spherical polar coordinates $(R, \theta, \phi)$. Give expressions for the components $u_{R}, u_{\theta}$ of the flow field in terms of $\Psi$. Show that the equation satisfied by $\Psi$ is

$$
\begin{equation*}
\mathcal{D}^{2}\left(\mathcal{D}^{2} \Psi\right)=0, \quad \text { where } \quad \mathcal{D}^{2}=\frac{\partial^{2}}{\partial R^{2}}+\frac{\sin \theta}{R^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\right) . \tag{*}
\end{equation*}
$$

Fluid is contained between the two spheres $R=a, R=b$, with $b \gg a$. The fluid velocity vanishes on the outer sphere, while on the inner sphere $u_{R}=U \cos \theta, u_{\theta}=0$. It is assumed that Stokes flow applies.
(i) Show that the Stokes stream function,

$$
\Psi(R, \theta)=a^{2} U \sin ^{2} \theta\left(A\left(\frac{a}{R}\right)+B\left(\frac{R}{a}\right)+C\left(\frac{R}{a}\right)^{2}+D\left(\frac{R}{a}\right)^{4}\right),
$$

is the general solution of $(*)$ proportional to $\sin ^{2} \theta$ and write down the conditions on $A, B, C, D$ that allow all the boundary conditions to be satisfied.
(ii) Now let $b \rightarrow \infty$, with $|\mathbf{u}| \rightarrow 0$ as $R \rightarrow \infty$. Show that $A=B=1 / 4$ with $C=D=0$.
(iii) Show that when $b / a$ is very large but finite, then the coefficients have the approximate form

$$
C \approx-\frac{3}{8} \frac{a}{b}, \quad D \approx \frac{1}{8} \frac{a^{3}}{b^{3}}, \quad A \approx \frac{1}{4}-\frac{3}{16} \frac{a}{b}, \quad B \approx \frac{1}{4}+\frac{9}{16} \frac{a}{b} .
$$

## Paper 1, Section II

## 37E Fluid Dynamics II

Explain the assumptions of lubrication theory and its use in determining the flow in thin films.

A cylindrical roller of radius $a$ rotates at angular velocity $\Omega$ below the free surface at $y=0$ of a fluid of density $\rho$ and dynamic viscosity $\mu$. The gravitational acceleration is $g$, and the pressure above the free surface is $p_{0}$. The minimum distance of the roller below the fluid surface is $b$, where $b \ll a$. The depth of the roller $d(x)$ below the free surface may be approximated by $d(x) \approx b+x^{2} / 2 a$, where $x$ is the horizontal distance.
(i) State the conditions for lubrication theory to be applicable to this problem. On the further assumption that the free surface may be taken to be flat, find the flow above the roller and calculate the horizontal volume flux $Q$ (per unit length in the third dimension) and the horizontal pressure gradient.
(ii) Use the pressure gradient you have found to estimate the order of magnitude of the departure $h(x)$ of the free surface from $y=0$, and give conditions on the parameters that ensure that $|h| \ll b$, as required for part (i).
[Hint: Integrals of the form

$$
I_{n}=\int_{-\infty}^{\infty}\left(1+t^{2}\right)^{-n} d t
$$

satisfy $I_{1}=\pi$ and

$$
I_{n+1}=\left(\frac{2 n-1}{2 n}\right) I_{n}
$$

for $n \geqslant 1$.]

## Paper 2, Section II

## 37E Fluid Dynamics II

Show that two-dimensional Stokes flow $\mathbf{u}=(u(r, \phi), v(r, \phi), 0)$ in cylindrical polar coordinates $(r, \phi, z)$ has a stream function $\psi(r, \phi)$, with $u=r^{-1} \partial \psi / \partial \phi, v=-\partial \psi / \partial r$, that satisfies the biharmonic equation

$$
\nabla^{4} \psi=0
$$

Give, in terms of $\psi$ and/or its derivatives, the boundary conditions satisfied by $\psi$ on an impermeable plane of constant $\phi$ which is either (a) rigid or (b) stress-free.

A rigid plane passes through the origin and lies along $\phi=-\alpha$. Fluid with viscosity $\mu$ is confined in the region $-\alpha<\phi<0$. A uniform tangential stress $S$ is applied on $\phi=0$. Show that the resulting flow may be described by a stream function $\psi$ of the form $\psi(r, \phi)=S r^{2} f(\phi)$, where $f(\phi)$ is to be found. Hence show that the radial flow $U(r)=u(r, 0)$ on $\phi=0$ is given by

$$
U(r)=\frac{S r}{\mu}\left(\frac{1-\cos 2 \alpha-\alpha \sin 2 \alpha}{\sin 2 \alpha-2 \alpha \cos 2 \alpha}\right) .
$$

By expanding this expression for small $\alpha$ show that $U$ and $S$ have the same sign, provided that $\alpha$ is not too large. Discuss the situation when $\alpha>\alpha_{c}$, where $\tan 2 \alpha_{c}=2 \alpha_{c}$.
[Hint: In plane polar coordinates

$$
\nabla^{2}=\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2}}{\partial \phi^{2}}
$$

and the component $\sigma_{r \phi}$ of the stress tensor takes the form

$$
\left.\sigma_{r \phi}=\mu\left(r \frac{\partial(v / r)}{\partial r}+\frac{1}{r} \frac{\partial u}{\partial \phi}\right) .\right]
$$

## Paper 3, Section II

## 37E Fluid Dynamics II

An axisymmetric incompressible Stokes flow has the Stokes stream function $\Psi(R, \theta)$ in spherical polar coordinates $(R, \theta, \phi)$. Give expressions for the components $u_{R}$ and $u_{\theta}$ of the flow field in terms of $\Psi$, and show that

$$
\nabla \times \mathbf{u}=\left(0,0,-\frac{D^{2} \Psi}{R \sin \theta}\right)
$$

where

$$
D^{2} \Psi=\frac{\partial^{2} \Psi}{\partial R^{2}}+\frac{\sin \theta}{R^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta}\right)
$$

Write down the equation satisfied by $\Psi$.
Verify that the Stokes stream function

$$
\Psi(R, \theta)=\frac{1}{2} U \sin ^{2} \theta\left(R^{2}-\frac{3}{2} a R+\frac{1}{2} \frac{a^{3}}{R}\right)
$$

represents the Stokes flow past a stationary sphere of radius $a$, when the fluid at large distance from the sphere moves at speed $U$ along the axis of symmetry.

A sphere of radius $a$ moves vertically upwards in the $z$ direction at speed $U$ through fluid of density $\rho$ and dynamic viscosity $\mu$, towards a free surface at $z=0$. Its distance $d$ from the surface is much greater than $a$. Assuming that the surface remains flat, show that the conditions of zero vertical velocity and zero tangential stress at $z=0$ can be approximately met for large $d / a$ by combining the Stokes flow for the sphere with that of an image sphere of the same radius located symmetrically above the free surface. Hence determine the leading-order behaviour of the horizontal flow on the free surface as a function of $r$, the horizontal distance from the sphere's centre line.

What is the size of the next correction to your answer as a power of $a / d$ ? [Detailed calculation is not required.]
[Hint: For an axisymmetric vector field $\mathbf{u}$,

$$
\left.\nabla \times \mathbf{u}=\left(\frac{1}{R \sin \theta} \frac{\partial}{\partial \theta}\left(u_{\phi} \sin \theta\right),-\frac{1}{R} \frac{\partial}{\partial R}\left(R u_{\phi}\right), \frac{1}{R} \frac{\partial}{\partial R}\left(R u_{\theta}\right)-\frac{1}{R} \frac{\partial u_{R}}{\partial \theta}\right) .\right]
$$

## Paper 4, Section II

## 37E Fluid Dynamics II

Two regions of inviscid fluid with the same density are separated by a thin membrane at $y=0$. The fluid in $y>0$ has the uniform velocity ( $U, 0,0$ ) in Cartesian coordinates, while the fluid in $y<0$ is at rest.

The membrane is now slightly perturbed to $y=\eta(x, t)$. The dynamical effect of the membrane is to induce a pressure difference across it equal to $\beta \partial^{4} \eta / \partial x^{4}$, where $\beta$ is a constant and the sign is such that the pressure is higher below the interface when $\partial^{4} \eta / \partial x^{4}>0$.

On the assumption that the flow remains irrotational and all perturbations are small, derive the relation between $\sigma$ and $k$ for disturbances of the form $\eta(x, t)=\operatorname{Re}\left(C e^{i k x+\sigma t}\right)$, where $k$ is real but $\sigma$ may be complex. Show that there is instability only for $|k|<k_{\max }$, where $k_{\max }$ is to be determined. Find the maximum growth rate and the value of $|k|$ for which this is obtained.

## 1/II/36A Fluid Dynamics II

Derive the relation between the stress tensor $\sigma_{i j}$ and the rate-of-strain tensor $e_{i j}$ in an incompressible Newtonian fluid, using the result that there is a linear dependence between the components of $\sigma_{i j}$ and those of $e_{i j}$ that is the same in all frames. Write down the boundary conditions that hold at an interface between two viscous fluids.

Viscous fluid is contained in a channel between the rigid planes $y=-a$ and $y=a$. The fluid in $y<0$ has dynamic viscosity $\mu_{-}$, while that in $y>0$ has dynamic viscosity $\mu_{+}$. Gravity may be neglected. The fluids move through the channel in the $x$-direction under the influence of a pressure gradient applied at the ends of the channel. It may be assumed that the velocity has no $z$-components, and all quantities are independent of $z$.

Find a steady solution of the Navier-Stokes equation in which the interface between the two fluids remains at $y=0$, the fluid velocity is everywhere independent of $x$, and the pressure gradient is uniform. Use it to calculate the following:
(a) the viscous tangential stress at $y=-a$ and at $y=a$; and
(b) the ratio of the volume fluxes of the two different fluids.

Comment on the limits of each of the results in (a) and (b) as $\mu_{+} / \mu_{-} \rightarrow 1$, and as $\mu_{+} / \mu_{-} \rightarrow \infty$.

## 2/II/36A Fluid Dynamics II

Viscous fluid with dynamic viscosity $\mu$ flows with velocity $\left(u_{x}, u_{y}, u_{z}\right) \equiv\left(\mathbf{u}_{H}, u_{z}\right)$ (in cartesian coordinates $x, y, z$ ) in a shallow container with a free surface at $z=0$. The base of the container is rigid, and is at $z=-h(x, y)$. A horizontal stress $\mathbf{S}(x, y)$ is applied at the free surface. Gravity may be neglected.

Using lubrication theory (conditions for the validity of which should be clearly stated), show that the horizontal volume flux $\mathbf{q}(x, y) \equiv \int_{-h}^{0} \mathbf{u}_{H} d z$ satisfies the equations

$$
\nabla \cdot \mathbf{q}=0, \quad \mu \mathbf{q}=-\frac{1}{3} h^{3} \nabla p+\frac{1}{2} h^{2} \mathbf{S}
$$

where $p(x, y)$ is the pressure. Find also an expression for the surface velocity $\mathbf{u}_{0}(x, y) \equiv$ $\mathbf{u}_{H}(x, y, 0)$ in terms of $\mathbf{S}, \mathbf{q}$ and $h$.

Now suppose that the container is cylindrical with boundary at $x^{2}+y^{2}=a^{2}$, where $a \gg h$, and that the surface stress is uniform and in the $x$-direction, so $\mathbf{S}=\left(S_{0}, 0\right)$ with $S_{0}$ constant. It can be assumed that the correct boundary condition to apply at $x^{2}+y^{2}=a^{2}$ is $\mathbf{q} \cdot \mathbf{n}=0$, where $\mathbf{n}$ is the unit normal.

Write $\mathbf{q}=\nabla \psi(x, y) \times \hat{\mathbf{z}}$, and show that $\psi$ satisfies the equation

$$
\nabla \cdot\left(\frac{1}{h^{3}} \nabla \psi\right)=-\frac{S_{0}}{2 \mu h^{2}} \frac{\partial h}{\partial y}
$$

Deduce that if $h=h_{0}$ (constant) then $\mathbf{q}=\mathbf{0}$. Find $\mathbf{u}_{\mathbf{0}}$ in this case.
Now suppose that $h=h_{0}(1+\epsilon y / a)$, where $\epsilon \ll 1$. Verify that to leading order in $\epsilon, \psi=\epsilon C\left(x^{2}+y^{2}-a^{2}\right)$ for some constant $C$ to be determined. Hence determine $\mathbf{u}_{0}$ up to and including terms of order $\epsilon$.
$[$ Hint: $\nabla \times(\mathbf{A} \times \hat{\mathbf{z}})=\hat{\mathbf{z}} \cdot \nabla \mathbf{A}-\hat{\mathbf{z}} \nabla \cdot \mathbf{A}$ for any vector field $\mathbf{A}$.

## 3/II/36A Fluid Dynamics II

Show that, in cylindrical polar co-ordinates, the streamfunction $\psi(r, \phi)$ for the velocity $\mathbf{u}=\left(u_{r}(r, \phi), u_{\phi}(r, \phi), 0\right)$ and vorticity $(0,0, \omega(r, \phi))$ of two-dimensional Stokes flow of incompressible fluid satisfies the equations

$$
\mathbf{u}=\left(\frac{1}{r} \frac{\partial \psi}{\partial \phi},-\frac{\partial \psi}{\partial r}, 0\right), \quad \nabla^{2} \omega=-\nabla^{4} \psi=0
$$

Show also that the pressure $p(r, \phi)$ satisfies $\nabla^{2} p=0$.
A stationary rigid circular cylinder of radius $a$ occupies the region $r \leqslant a$. The flow around the cylinder tends at large distances to a simple shear flow, with velocity given in cartesian coordinates $(x, y, z)$ by $\mathbf{u}=(\Gamma y, 0,0)$. Inertial forces may be neglected.

By solving the equation for $\psi$ in cylindrical polars, determine the flow field everywhere. Determine the torque on the cylinder per unit length in $z$.
[Hint: in cylindrical polars

$$
\nabla^{2} V=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} V}{\partial \phi^{2}}
$$

The off-diagonal component of the rate-of-strain tensor is given by

$$
\left.e_{r \phi}=\frac{1}{2}\left(\frac{1}{r} \frac{\partial u_{r}}{\partial \phi}+r \frac{\partial}{\partial r}\left(\frac{u_{\phi}}{r}\right)\right) .\right]
$$

## 4/II/37A Fluid Dynamics II

Viscous incompressible fluid of uniform density is extruded axisymmetrically from a thin circular slit of small radius centred at the origin and lying in the plane $z=0$ in cylindrical polar coordinates $r, \theta, z$. There is no external radial pressure gradient. It is assumed that the fluid forms a thin boundary layer, close to and symmetric about the plane $z=0$. The layer has thickness $\delta(r) \ll r$. The $r$-component of the steady Navier-Stokes equations may be approximated by

$$
u_{r} \frac{\partial u_{r}}{\partial r}+u_{z} \frac{\partial u_{r}}{\partial z}=\nu \frac{\partial^{2} u_{r}}{\partial z^{2}}
$$

(i) Prove that the quantity (proportional to the flux of radial momentum)

$$
\mathcal{F}=\int_{-\infty}^{\infty} u_{r}^{2} r d z
$$

is independent of $r$.
(ii) Show, by balancing terms in the momentum equation and assuming constancy of $\mathcal{F}$, that there is a similarity solution of the form

$$
u_{r}=-\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad u_{z}=\frac{1}{r} \frac{\partial \Psi}{\partial r}, \quad \Psi=-A \delta(r) f(\eta), \quad \eta=\frac{z}{\delta(r)}, \quad \delta(r)=C r
$$

where $A, C$ are constants. Show that for suitable choices of $A$ and $C$ the equation for $f$ takes the form

$$
\begin{gathered}
-f^{\prime 2}-f f^{\prime \prime}=f^{\prime \prime \prime} ; \\
f=f^{\prime \prime}=0 \text { at } \eta=0 ; \quad f^{\prime} \rightarrow 0 \text { as } \eta \rightarrow \infty ; \\
\int_{-\infty}^{\infty} f_{\eta}^{2} d \eta=1 .
\end{gathered}
$$

(iii) Give an inequality connecting $\mathcal{F}$ and $\nu$ that ensures that the boundary layer approximation $(\delta \ll r)$ is valid. Solve the equation to give a complete solution to the problem for $u_{r}$ when this inequality holds.
[Hint: $\left.\int_{-\infty}^{\infty} \operatorname{sech}^{4} x d x=4 / 3.\right]$

## 1/II/36B Fluid Dynamics II

Discuss how the methods of lubrication theory may be used to find viscous fluid flows in thin layers or narrow gaps, explaining carefully what inequalities need to hold in order that the theory may apply.

Viscous fluid of kinematic viscosity $\nu$ flows under the influence of gravity $g$, down an inclined plane making an angle $\alpha \ll 1$ with the horizontal. The fluid layer lies between $y=0$ and $y=h(x, t)$, where $x, y$ are distances measured down the plane and perpendicular to it, and $|\partial h / \partial x|$ is of the same order as $\alpha$. Give conditions involving $h, \alpha, \nu$ and $g$ that ensure that lubrication theory can be used, and solve the lubrication equations, together with the equation of mass conservation, to obtain an equation for $h$ in the form

$$
\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(-A h^{3}+B h^{3} \frac{\partial h}{\partial x}\right)
$$

where $A, B$ are constants to be determined. Show that there is a steady solution with $\partial h / \partial x=k=$ constant, and interpret this physically. Show also that a solution of this equation exists in the form of a front, with $h(x, t)=F(\xi)$, where $\xi=x-c t, F(0)=0$, and $F(\xi) \rightarrow h_{0}$ as $\xi \rightarrow-\infty$. Determine $c$ in terms of $h_{0}$, find the shape of the front implicitly in the form $\xi=G(h)$, and show that $h \propto(-\xi)^{1 / 3}$ as $\xi \rightarrow 0$ from below.

## 2/II/36B Fluid Dynamics II

Viscous fluid is extracted through a small hole in the tip of the cone given by $\theta=\alpha$ in spherical polar coordinates $(R, \theta, \phi)$. The total volume flux through the hole takes the constant value $Q$. It is given that there is a steady solution of the Navier-Stokes equations for the fluid velocity $\mathbf{u}$. For small enough $R$, the velocity $\mathbf{u}$ is well approximated by $\mathbf{u} \sim\left(-A / R^{2}, 0,0\right)$, where $A=Q /[2 \pi(1-\cos \alpha)]$ except in thin boundary layers near $\theta=\alpha$.
(i) Verify that the volume flux through the hole is approximately $Q$.
(ii) Construct a Reynolds number (depending on $R$ ) in terms of $Q$ and the kinematic viscosity $\nu$, and thus give an estimate of the value of $R$ below which solutions of this type will appear.
(iii) Assuming that there is a boundary layer near $\theta=\alpha$, write down the boundary layer equations in the usual form, using local Cartesian coordinates $x$ and $y$ parallel and perpendicular to the boundary. Show that the boundary layer thickness $\delta(x)$ is proportional to $x^{\frac{3}{2}}$, and show that the $x$ component of the velocity $u_{x}$ may be written in the form

$$
u_{x}=-\frac{A}{x^{2}} F^{\prime}(\eta), \quad \text { where } \quad \eta=\frac{y}{\delta(x)}
$$

Derive the equation and boundary conditions satisfied by $F$. Give an expression, in terms of $F$, for the volume flux through the boundary layer, and use this to derive the $R$ dependence of the first correction to the flow outside the boundary layer.

## 3/II/36B Fluid Dynamics II



Viscous fluid of kinematic viscosity $\nu$ and density $\rho$ flows in a curved pipe of constant rectangular cross section and constant curvature. The cross-section has height $2 a$ and width $2 b$ (in the radial direction) with $b \gg a$, and the radius of curvature of the inner wall is $R$, with $R \gg b$. A uniform pressure gradient $-G$ is applied along the pipe.
(i) Assume to a first approximation that the pipe is straight, and ignore variation in the $x$-direction, where $(x, y, z)$ are Cartesian coordinates referred to an origin at the centre of the section, with $x$ increasing radially and $z$ measured along the pipe. Find the flow field along the pipe in the form $\mathbf{u}=(0,0, w(y))$.
(ii) It is given that the largest component of the inertial acceleration $\mathbf{u} \cdot \nabla \mathbf{u}$ due to the curvature of the pipe is $-w^{2} / R$ in the $x$ direction. Consider the secondary flow $\mathbf{u}_{s}$ induced in the $x, y$ plane, again ignoring variations in $x$ and any end effects (except for the requirement that there be zero total mass flux in the $x$ direction). Show that $\mathbf{u}_{s}$ takes the form $\mathbf{u}_{s}=(u(y), 0,0)$, where

$$
u(y)=\frac{G^{2}}{120 \rho^{2} \nu^{3} R}\left(5 a^{2} y^{4}-y^{6}\right)+\frac{C}{2} y^{2}+D
$$

and write down two equations determining the constants $C$ and $D$. [It is not necessary to solve these equations.]
Give conditions on the parameters that ensure that $|u| \ll|w|$.

4/II/37B Fluid Dynamics II
(i) Assuming that axisymmetric incompressible flow $\mathbf{u}=\left(u_{R}, u_{\theta}, 0\right)$, with vorticity $(0,0, \omega)$ in spherical polar coordinates $(R, \theta, \phi)$ satisfies the equations

$$
\mathbf{u}=\nabla \times\left(0,0, \frac{\Psi}{R \sin \theta}\right), \quad \omega=-\frac{1}{R \sin \theta} D^{2} \Psi
$$

where

$$
D^{2} \equiv \frac{\partial^{2}}{\partial R^{2}}+\frac{\sin \theta}{R^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\right)
$$

show that for Stokes flow $\Psi$ satisfies the equation

$$
\begin{equation*}
D^{4} \Psi=0 \tag{*}
\end{equation*}
$$

(ii) A rigid sphere of radius $a$ moves at velocity $U \hat{\mathbf{z}}$ through viscous fluid of density $\rho$ and dynamic viscosity $\mu$ which is at rest at infinity. Assuming Stokes flow and by applying the boundary conditions at $R=a$ and as $R \rightarrow \infty$, verify that $\Psi=(A R+B / R) \sin ^{2} \theta$ is the appropriate solution to $(*)$ for this flow, where $A$ and $B$ are to be determined.
(iii) Hence find the velocity field outside the sphere. Without direct calculation, explain why the drag is in the $z$ direction and has magnitude proportional to $U$.
(iv) A second identical sphere is introduced into the flow, at a distance $b \gg a$ from the first, and moving at the same velocity. Justify the assertion that, when the two spheres are at the same height, or when one is vertically above the other, the drag on each sphere is the same. Calculate the leading correction to the drag in each case, to leading order in $a / b$.
[You may quote without proof the fact that, for an axisymmetric function $F(R, \theta)$,

$$
\nabla \times(0,0, F)=\left(\frac{1}{R \sin \theta} \frac{\partial}{\partial \theta}(\sin \theta F),-\frac{1}{R} \frac{\partial}{\partial R}(R F), 0\right)
$$

in spherical polar coordinates $(R, \theta, \phi)$.]

## 1/II/36B Fluid Dynamics II

Write down the boundary conditions that are satisfied at the interface between two viscous fluids in motion. Briefly discuss the physical meaning of these boundary conditions.

A layer of incompressible fluid of density $\rho$ and viscosity $\mu$ flows steadily down a plane inclined at an angle $\theta$ to the horizontal. The layer is of uniform thickness $h$ measured perpendicular to the plane and the viscosity of the overlying air can be neglected. Using co-ordinates parallel and perpendicular to the plane, write down the equations of motion, and the boundary conditions on the plane and on the free top surface. Determine the pressure and velocity fields. Show that the volume flux down the plane is $\frac{1}{3} \rho g h^{3} \sin \theta / \mu$ per unit cross-slope width.

Consider now the case where a second layer of fluid, of uniform thickness $\alpha h$, viscosity $\beta \mu$, and density $\rho$ flows steadily on top of the first layer. Determine the pressure and velocity fields in each layer. Why does the velocity profile in the bottom layer depend on $\alpha$ but not on $\beta$ ?

## 2/II/36B Fluid Dynamics II

A very long cylinder of radius $a$ translates steadily at speed $V$ in a direction perpendicular to its axis and parallel to a plane boundary. The centre of the cylinder remains a distance $a+b$ above the plane, where $b \ll a$, and the motion takes place through an incompressible fluid of viscosity $\mu$.

Consider the force $F$ per unit length parallel to the plane that must be applied to the cylinder to maintain the motion. Explain why $F$ scales according to $F \propto \mu V(a / b)^{1 / 2}$.

Approximating the lower cylindrical surface by a parabola, or otherwise, determine the velocity and pressure gradient fields in the space between the cylinder and the plane. Hence, by considering the shear stress on the plane, or otherwise, calculate $F$ explicitly.
[You may use

$$
\left.\int_{-\infty}^{\infty}\left(1+x^{2}\right)^{-1} d x=\pi, \quad \int_{-\infty}^{\infty}\left(1+x^{2}\right)^{-2} d x=\frac{1}{2} \pi \quad \text { and } \quad \int_{-\infty}^{\infty}\left(1+x^{2}\right)^{-3} d x=\frac{3}{8} \pi .\right]
$$

## 3/II/36B Fluid Dynamics II

Define the rate of strain tensor $e_{i j}$ in terms of the velocity components $u_{i}$.
Write down the relation between $e_{i j}$, the pressure $p$ and the stress tensor $\sigma_{i j}$ in an incompressible Newtonian fluid of viscosity $\mu$.

Prove that $2 \mu e_{i j} e_{i j}$ is the local rate of dissipation per unit volume in the fluid.
Incompressible fluid of density $\rho$ and viscosity $\mu$ occupies the semi-infinite domain $y>0$ above a rigid plane boundary $y=0$ that oscillates with velocity $(V \cos \omega t, 0,0)$, where $V$ and $\omega$ are constants. The fluid is at rest at $y=\infty$. Determine the velocity field produced by the boundary motion after any transients have decayed.

Evaluate the time-averaged rate of dissipation in the fluid, per unit area of boundary.

## 4/II/37B Fluid Dynamics II

A line force of magnitude $F$ is applied in the positive $x$-direction to an unbounded fluid, generating a thin two-dimensional jet along the positive $x$-axis. The fluid is at rest at $y= \pm \infty$ and there is negligible motion in $x<0$. Write down the pressure gradient within the boundary layer. Deduce that the function $M(x)$ defined by

$$
M(x)=\int_{-\infty}^{\infty} \rho u^{2}(x, y) d y
$$

is independent of $x$ for $x>0$. Interpret this result, and explain why $M=F$. Use scaling arguments to deduce that there is a similarity solution having stream function

$$
\psi=(F \nu x / \rho)^{1 / 3} f(\eta) \quad \text { where } \quad \eta=y\left(F / \rho \nu^{2} x^{2}\right)^{1 / 3}
$$

Hence show that $f$ satisfies

$$
\begin{equation*}
3 f^{\prime \prime \prime}+f^{\prime 2}+f f^{\prime \prime}=0 . \tag{*}
\end{equation*}
$$

Show that a solution of $(*)$ is

$$
f(\eta)=A \tanh (A \eta / 6)
$$

where $A$ is a constant to be determined by requiring that $M$ is independent of $x$. Find the volume flux, $Q(x)$, in the jet. Briefly indicate why $Q(x)$ increases as $x$ increases.
[Hint: You may use $\int_{-\infty}^{\infty} \operatorname{sech}^{4}(x) d x=4 / 3$.]

## 1/II/36E Fluid Dynamics II

Consider a unidirectional flow with dynamic viscosity $\mu$ along a straight rigid-walled channel of uniform cross-sectional shape $\mathcal{D}$ driven by a uniform applied pressure gradient $G$. Write down the differential equation and boundary conditions governing the velocity $w$ along the channel.

Consider the situation when the boundary includes a sharp corner of angle $2 \alpha$. Explain why one might expect that, sufficiently close to the corner, the solution should be of the form

$$
w=(G / \mu) r^{2} f(\theta)
$$

where $r$ and $\theta$ are polar co-ordinates with origin at the vertex and $\theta= \pm \alpha$ describing the two planes emanating from the corner. Determine $f(\theta)$.

If $\mathcal{D}$ is the sector bounded by the lines $\theta= \pm \alpha$ and the circular arc $r=a$, show that the flow is given by

$$
w=(G / \mu) r^{2} f(\theta)+\sum_{n=0}^{\infty} A_{n} r^{\lambda_{n}} \cos \lambda_{n} \theta
$$

where $\lambda_{n}$ and $A_{n}$ are to be determined.
[Note that $\left.\int \cos (a x) \cos (b x) d x=\{a \sin (a x) \cos (b x)-b \sin (b x) \cos (a x)\} /\left(a^{2}-b^{2}\right).\right]$
Considering the values of $\lambda_{0}$ and $\lambda_{1}$, comment briefly on the cases: (i) $2 \alpha<\frac{1}{2} \pi$; (ii) $\frac{1}{2} \pi<2 \alpha<\frac{3}{2} \pi$; and (iii) $\frac{3}{2} \pi<2 \alpha<2 \pi$.

## 2/II/36E Fluid Dynamics II

A volume $V$ of very viscous fluid of density $\rho$ and dynamic viscosity $\mu$ is released at the origin on a rigid horizontal boundary at time $t=0$. Using lubrication theory, determine the velocity profile in the gravity current once it has spread sufficiently that the axisymmetric thickness $h(r, t)$ of the current is much less than the radius $R(t)$ of the front.

Derive the differential equation

$$
\frac{\partial h}{\partial t}=\frac{\beta}{r} \frac{\partial}{\partial r}\left(r h^{3} \frac{\partial h}{\partial r}\right),
$$

where $\beta$ is to be determined.
Write down the other equations that are needed to determine the appropriate similarity solution for this problem.

Determine the similarity solution and calculate $R(t)$.

## 3/II/36E Fluid Dynamics II

Write down the Navier-Stokes equations for an incompressible fluid.
Explain the concepts of the Euler and Prandtl limits applied to the Navier-Stokes equations near a rigid boundary.

A steady two-dimensional flow given by $(U, 0)$ far upstream flows past a semi-infinite flat plate, held at $y=0, x>0$. Derive the boundary layer equation

$$
\frac{\partial \psi}{\partial y} \frac{\partial^{2} \psi}{\partial x \partial y}-\frac{\partial \psi}{\partial x} \frac{\partial^{2} \psi}{\partial y^{2}}=\nu \frac{\partial^{3} \psi}{\partial y^{3}}
$$

for the stream-function $\psi(x, y)$ near the plate, explaining any approximations made.
Show that the appropriate solution must be of the form

$$
\psi(x, y)=(\nu U x)^{1 / 2} f(\eta)
$$

and determine the dimensionless variable $\eta$.
Derive the equation and boundary conditions satisfied by $f(\eta)$. [You need not solve them.]

Suppose now that the plate has a finite length $L$ in the direction of the flow. Show that the force $F$ on the plate (per unit width perpendicular to the flow) is given by

$$
F=\frac{4 \rho U^{2} L}{(U L / \nu)^{1 / 2}} \frac{f^{\prime \prime}(0)}{\left[f^{\prime}(\infty)\right]^{2}} .
$$

## 4/II/37E Fluid Dynamics II

Consider flow of an incompressible fluid of uniform density $\rho$ and dynamic viscosity $\mu$. Show that the rate of viscous dissipation per unit volume is given by

$$
\Phi=2 \mu e_{i j} e_{i j},
$$

where $e_{i j}$ is the strain rate.
Determine expressions for $e_{i j}$ and $\Phi$ when the flow is irrotational with velocity potential $\phi$. Hence determine the rate of viscous dissipation, averaged over a wave period $2 \pi / \omega$, for an irrotational two-dimensional surface wave of wavenumber $k$ and small amplitude $a \ll k^{-1}$ in a fluid of very small viscosity $\mu \ll \rho \omega / k^{2}$ and great depth $H \gg 1 / k$.
[You may use without derivation that in deep water a linearised wave with surface displacement $\eta=a \cos (k x-w t)$ has velocity potential $\phi=-(\omega a / k) e^{-k z} \sin (k x-\omega t)$.]

Calculate the depth-integrated time-averaged kinetic energy per wavelength. Assuming that the average potential energy is equal to the average kinetic energy, show that the total wave energy decreases to leading order like $e^{-\gamma t}$, where

$$
\gamma=4 \mu k^{2} / \rho
$$

