Part II

Electrodynamics

Paper 1, Section II

37A Electrodynamics

Consider spacetime with coordinates $x^\mu = (ct, \mathbf{x})$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, where $\mu, \nu = 0, 1, 2, 3$ and c is the speed of light. An electromagnetic field described by the vector potential $A_\mu(x)$ fills spacetime, and a particle of mass m and charge q moves through it along the worldline $x^\mu(\lambda)$, where λ is a parameter along the worldline.

(a) Explain using the requirements of Lorentz invariance and gauge invariance why the action

$$S = -mc \int (-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{\frac{1}{2}}\, d\lambda + q \int A_\mu(x)\dot{x}^\mu d\lambda$$

is suitable for describing the relativistic mechanics of the particle, where $\dot{x}^\mu = dx^\mu/d\lambda$.

(b) By varying the action with respect to a worldline with fixed end points, obtain the Euler-Lagrange equations of motion

$$m\frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu$$

where $u^\mu(\tau) = dx^\mu/d\tau$ is the four-velocity, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and τ is the proper time.

(c) Show that the rate of change of the particle energy $\epsilon = \gamma mc^2$ satisfies

$$\frac{d\epsilon}{dt} = q\mathbf{E}\cdot\mathbf{v}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the particle velocity, $\mathbf{E}$ is the electric field and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(d) Hence, or otherwise, derive the following expression for the acceleration of the particle

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m\gamma}[\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}\mathbf{v}(\mathbf{v}\cdot\mathbf{E})],$$

where $\mathbf{B}$ is the magnetic field. Derive the non-relativistic limit of the above expression and comment on its relationship with the Lorentz force law.

Paper 3, Section II

36A Electrodynamics

The retarded four-potential $A^\mu(\mathbf{x},t) = (\phi/c, \mathbf{A})$ due to a charge density $J^\mu(\mathbf{x}',t')$ is

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d^3x',$$

where the integral is over all space.

(a) Explain briefly the physical meaning of the above expression and why causality requires $t' = t_{\rm ret}$, where $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|/c$.

(b) Consider a particle of charge q moving along the worldline $y^\mu = (ct, \mathbf{y}(t))$ and let $\mathbf{R}(t) = \mathbf{x} - \mathbf{y}(t)$ be the vector from the location of the charge at time t to the field point $\mathbf{x}$. Explain why the implicit equation

$$t_{\rm ret} + \frac{R(t_{\rm ret})}{c} = t,$$

determining the retarded potential, can have only one solution.

(c) Hence, or otherwise, obtain the Lienard-Wiechert potentials

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R - \frac{\mathbf{v}}{c}\cdot\mathbf{R}} \quad \text{and} \quad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}}{R - \frac{\mathbf{v}}{c}\cdot\mathbf{R}}$$

for the charge, where $\mathbf{v} = d\mathbf{y}/dt$ is the particle velocity. Clearly specify the time at which the right hand sides are to be evaluated.

(d) For a charge moving without acceleration, show by explicit computation that the resulting potentials satisfy the gauge-fixing condition

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0.$$

Paper 4, Section II

36A Electrodynamics

Consider a dielectric medium whose electromagnetic properties are described by the electric displacement $\mathbf{D}$, the magnetisation $\mathbf{H}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$.

(a) Write down the Maxwell equations for these four fields in the presence of a free charge density ρ and a free current density $\mathbf{J}$.

(b) Hence establish the identity

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J}.$$

(c) Consider a linear dielectric medium with the constitutive relations

$$D_i = \varepsilon_{ij} E_j, \quad B_i = \mu_{ij} H_j$$

where ε_{ij} and μ_{ij} are symmetric matrices, independent of t, representing the anisotropic dielectric response of the material, and the summation convention applies here and below. For a volume V enclosed by the surface S, derive the integral relation

$$\frac{\partial}{\partial t} \int_V \frac{1}{2}(\varepsilon_{ij} E_i E_j + \mu_{ij} H_i H_j) dV + \int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\int_V \mathbf{E} \cdot \mathbf{J}\, dV.$$

In the absence of free currents, interpret the above relation in terms of an energy density ϵ and an energy flux $\mathbf{N}$, clearly identifying each.

(d) Consider a linear dielectric medium with

$$D_i = \varepsilon_{ij} E_j, \quad B_i = \mu \delta_{ij} H_j,$$

where μ and ε_{ij} are independent of space and time, and δ_{ij} is the Kronecker delta. Assuming plane waves

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{e} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t), \quad \mathbf{B}(\mathbf{x}, t) = \mathbf{b} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

in this medium, show that Maxwell's equations in the absence of free charges and currents imply that the wave vector $\mathbf{k}$, the frequency ω and polarisation $\mathbf{e}$ must satisfy

$$[\mathbf{k} \times (\mathbf{k} \times \mathbf{e})]_i + \omega^2 \mu \varepsilon_{ij} e_j = 0.$$

(e) Show that the energy flux $\mathbf{N}$ identified above, applied to the situation in part (d), points in the direction of wave propagation when the polarisation is an eigenvector of the matrix ε_{ij}.

Paper 1, Section II

37B Electrodynamics

Consider a localised electromagnetic field in vacuum with electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively in the absence of charges and currents.

(a) Show that the energy density $\epsilon = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$ obeys a local conservation law

$$\partial_t \epsilon + \boldsymbol{\nabla} \cdot \mathbf{N} = 0\,.$$

Hence obtain an expression for the vector $\mathbf{N}$ and remark on its physical significance. Here ε_0 and μ_0 are the electric and magnetic permeabilities of the vacuum.

(b) Show that the momentum density $\mathbf{g} = \varepsilon_0 \mathbf{E} \times \mathbf{B}$ obeys a local conservation law

$$\partial_t g_j + \nabla_i \sigma_{ij} = 0\,.$$

Hence obtain an expression for the second-rank tensor σ_{ij} and remark on its physical significance.

(c) Defining the tensor

$$T^{\mu\nu} = \begin{bmatrix} \epsilon & cg_j \\ N_i/c & \sigma_{ij} \end{bmatrix}$$

show that the results of (a) and (b) can be expressed as $\partial_\mu T^{\mu\nu} = 0$.

(d) Using the fact that the tensor σ_{ij} is symmetric, show that the integral over all space of the angular momentum density $\mathbf{L} = \mathbf{x} \times \mathbf{g}$ is independent of time. Here $\mathbf{x}$ is the position with respect to the origin of an inertial frame.

(e) Show that the symmetry of σ_{ij} in all inertial frames requires $\mu_0 \epsilon_0 = 1/c^2$.

Paper 3, Section II
36B Electrodynamics

Consider a time-dependent localised electromagnetic field in vacuum with a four-current density J^μ and vector potential A^μ.

(a) Determine the differential equation that relates the four-current density to the vector potential in the gauge choice $\partial_\mu A^\mu = 0$.

(b) Show that the solution to the above differential equation can be expressed as

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{J^\mu(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'$$

where you should specify the form of t'.

(c) Show that the time derivative of the dipole moment $\mathbf{p}$ satisfies

$$\dot{\mathbf{p}} = \int \mathbf{J}(\mathbf{x}, t) \, d^3x$$

where $\mathbf{J}$ is the current density.

(d) A small circular loop of radius r is centred at the origin. The unit vector normal to the plane of the loop is $\mathbf{n}$. A current $I(t) = \sum_{n=0}^{\infty} I_n \sin(n\omega t)$ flows in the loop. Find the three vector potential $\mathbf{A}(\mathbf{x}, t)$ to first order in $r/|\mathbf{x}|$.

Paper 4, Section II
36B Electrodynamics

(a) Explain what is meant by a *dielectric material*.

(b) Define the *polarisation* of, and the *bound charge* in, a dielectric material. Explain the reason for the distinction between the electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ in a dielectric material.

Consider a sphere of a dielectric material of radius R and permittivity ε_1 embedded in another dielectric material of infinite extent and permittivity ε_2. A point charge q is placed at the centre of the sphere. Determine the bound charge on the surface of the sphere.

(c) Define the *magnetisation* of, and the *bound current* in, a dielectric material. Explain the reason for making a distinction between the magnetic flux density $\mathbf{B}$ and the magnetic intensity $\mathbf{H}$ in a dielectric material.

Consider a cylinder of dielectric material of infinite length, radius R and permeability μ_1 embedded in another dielectric material of infinite extent and permeability μ_2. A line current I is placed on the axis of the cylinder. Determine the magnitude and direction of the bound current density on the surface of the cylinder.

Paper 1, Section II

37C Electrodynamics

(a) An electromagnetic field is specified by a four-vector potential

$$A^{\mu}(\mathbf{x},t) = \big(\,\phi(\mathbf{x},t)/c\,,\;\mathbf{A}(\mathbf{x},t)\,\big)\,.$$

Define the corresponding *field-strength tensor* $F^{\mu\nu}$ and state its transformation property under a general Lorentz transformation.

(b) Write down two independent Lorentz scalars that are quadratic in the field strength and express them in terms of the electric and magnetic fields, $\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Show that both these scalars vanish when evaluated on an electromagnetic plane-wave solution of Maxwell's equations of arbitrary wavevector and polarisation.

(c) Find (non-zero) constant, homogeneous background fields $\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0$ and $\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0$ such that both the Lorentz scalars vanish. Show that, for any such background, the field-strength tensor obeys

$$F^{\mu}_{\ \ \rho}\,F^{\rho}_{\ \ \sigma}F^{\sigma}_{\ \ \nu} = 0\,.$$

(d) Hence find the trajectory of a relativistic particle of mass m and charge q in this background. You should work in an inertial frame where the particle is at rest at the origin at $t=0$ and in which $\mathbf{B}_0 = (0,0,B_0)$.

Paper 3, Section II
36C Electrodynamics

(a) Derive the *Larmor formula* for the total power P emitted through a large sphere of radius R by a non-relativistic particle of mass m and charge q with trajectory $\mathbf{x}(t)$. You may assume that the electric and magnetic fields describing radiation due to a source localised near the origin with electric dipole moment $\mathbf{p}(t)$ can be approximated as

$$\begin{aligned}\mathbf{B}_{\text{Rad}}(\mathbf{x},t) &= -\frac{\mu_0}{4\pi rc}\,\widehat{\mathbf{x}}\times\ddot{\mathbf{p}}(t-r/c)\,,\\ \mathbf{E}_{\text{Rad}}(\mathbf{x},t) &= -c\,\widehat{\mathbf{x}}\times\mathbf{B}_{\text{Rad}}(\mathbf{x},t)\,.\end{aligned}$$

Here, the radial distance $r=|\mathbf{x}|$ is assumed to be much larger than the wavelength of emitted radiation which, in turn, is large compared to the spatial extent of the source.

(b) A non-relativistic particle of mass m, moving at speed v along the x-axis in the positive direction, encounters a step potential of width L and height $V_0>0$ described by

$$V(x) = \begin{cases} 0\,, & x<0\,,\\ f(x)\,, & 0\leqslant x\leqslant L\,,\\ V_0\,, & x>L\,,\end{cases}$$

where $f(x)$ is a monotonically increasing function with $f(0)=0$ and $f(L)=V_0$. The particle carries charge q and loses energy by emitting electromagnetic radiation. Assume that the total energy loss through emission ΔE_{Rad} is negligible compared with the particle's initial kinetic energy $E=mv^2/2$. For $E>V_0$, show that the total energy lost is

$$\Delta E_{\text{Rad}} = \frac{q^2\mu_0}{6\pi m^2 c}\sqrt{\frac{m}{2}}\int_0^L dx\,\frac{1}{\sqrt{E-f(x)}}\left(\frac{df}{dx}\right)^2 .$$

Find the total energy lost also for the case $E<V_0$.

(c) Take $f(x)=V_0x/L$ and explicitly evaluate the particle energy loss ΔE_{Rad} in each of the cases $E>V_0$ and $E<V_0$. What is the maximum value attained by ΔE_{Rad} as E is varied?

Paper 4, Section II

36C Electrodynamics

(a) Define the *electric displacement* $\mathbf{D}(\mathbf{x},t)$ for a medium which exhibits a linear response with polarisation constant ϵ to an applied electric field $\mathbf{E}(\mathbf{x},t)$ with polarisation constant ϵ. Write down the effective Maxwell equation obeyed by $\mathbf{D}(\mathbf{x})$ in the time-independent case and in the absence of any additional mobile charges in the medium. Describe appropriate boundary conditions for the electric field at an interface between two regions with differing values of the polarisation constant. [You should discuss separately the components of the field normal to and tangential to the interface.]

(b) Consider a sphere of radius a, centred at the origin, composed of dielectric material with polarisation constant ϵ placed in a vacuum and subjected to a constant, asymptotically homogeneous, electric field, $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})$ with $\mathbf{E}(\mathbf{x}) \to \mathbf{E}_0$ as $|\mathbf{x}| \to \infty$. Using the ansatz

$$\mathbf{E}(\mathbf{x}) \;=\; \begin{cases} \alpha\mathbf{E}_0\,, & |\mathbf{x}| < a\,, \\ \mathbf{E}_0 \;+\; \big(\beta(\widehat{\mathbf{x}}\cdot\mathbf{E}_0)\widehat{\mathbf{x}} + \delta\mathbf{E}_0\big)/|\mathbf{x}|^3\,, & |\mathbf{x}| > a\,, \end{cases}$$

with constants α, β and δ to be determined, find a solution to Maxwell's equations with appropriate boundary conditions at $|\mathbf{x}| = a$.

(c) By comparing your solution with the long-range electric field due to a dipole consisting of electric charges $\pm q$ located at displacements $\pm\mathbf{d}/2$ find the induced electric dipole moment of the dielectric sphere.

Paper 1, Section II

37D Electrodynamics

A relativistic particle of rest mass m and electric charge q follows a worldline $x^\mu(\lambda)$ in Minkowski spacetime where $\lambda = \lambda(\tau)$ is an arbitrary parameter which increases monotonically with the proper time τ. We consider the motion of the particle in a background electromagnetic field with four-vector potential $A^\mu(x)$ between initial and final values of the proper time denoted τ_i and τ_f respectively.

(i) Write down an *action* for the particle's motion. Explain what is meant by a *gauge transformation* of the electromagnetic field. How does the action change under a gauge transformation?

(ii) Derive an equation of motion for the particle by considering the variation of the action with respect to the worldline $x^\mu(\lambda)$. Setting $\lambda = \tau$ show that your equation of motion reduces to the Lorentz force law,

$$m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu\,, \qquad (*)$$

where $u^\mu = dx^\mu/d\tau$ is the particle's four-velocity and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the Maxwell field-strength tensor.

(iii) Working in an inertial frame with spacetime coordinates $x^\mu = (ct, x, y, z)$, consider the case of a constant, homogeneous magnetic field of magnitude B, pointing in the z-direction, and vanishing electric field. In a gauge where $A^\mu = (0, 0, Bx, 0)$, show that the equation of motion $(*)$ is solved by circular motion in the x-y plane with proper angular frequency $\omega = qB/m$.

(iv) Let v denote the speed of the particle in this inertial frame with Lorentz factor $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$. Find the radius $R = R(v)$ of the circle as a function of v. Setting $\tau_f = \tau_i + 2\pi/\omega$, evaluate the action $S = S(v)$ for a single period of the particle's motion.

Paper 3, Section II

36D Electrodynamics

The Maxwell stress tensor σ of the electromagnetic fields is a two-index Cartesian tensor with components

$$\sigma_{ij} \;=\; -\epsilon_0\left(E_iE_j-\frac{1}{2}|\mathbf{E}|^2\delta_{ij}\right)-\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}|\mathbf{B}|^2\delta_{ij}\right),$$

where $i,j=1,2,3$, and E_i and B_i denote the Cartesian components of the electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ respectively.

(i) Consider an electromagnetic field sourced by charge and current densities denoted by $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ respectively. Using Maxwell's equations and the Lorentz force law, show that the components of σ obey the equation

$$\sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j}+\frac{\partial g_i}{\partial t} \;=\; -\left(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right)_i\,,$$

where g_i, for $i=1,2,3$, are the components of a vector field $\mathbf{g}(\mathbf{x},t)$ which you should give explicitly in terms of $\mathbf{E}$ and $\mathbf{B}$. Explain the physical interpretation of this equation and of the quantities σ and $\mathbf{g}$.

(ii) A localised source near the origin, $\mathbf{x}=0$, emits electromagnetic radiation. Far from the source, the resulting electric and magnetic fields can be approximated as

$$\mathbf{B}(\mathbf{x},t)\simeq\mathbf{B}_0(\mathbf{x})\,\sin\left(\omega t-\mathbf{k}\cdot\mathbf{x}\right)\,,\qquad \mathbf{E}(\mathbf{x},t)\simeq\mathbf{E}_0(\mathbf{x})\,\sin\left(\omega t-\mathbf{k}\cdot\mathbf{x}\right)\,,$$

where $\mathbf{B}_0(\mathbf{x})=\dfrac{\mu_0\omega^2}{4\pi rc}\,\hat{\mathbf{x}}\times\mathbf{p}_0$ and $\mathbf{E}_0(\mathbf{x})=-c\hat{\mathbf{x}}\times\mathbf{B}_0(\mathbf{x})$ with $r=|\mathbf{x}|$ and $\hat{\mathbf{x}}=\mathbf{x}/r$. Here, $\mathbf{k}=(\omega/c)\hat{\mathbf{x}}$ and $\mathbf{p}_0$ is a constant vector.

Calculate the pressure exerted by these fields on a spherical shell of very large radius R centred on the origin. [You may assume that $\mathbf{E}$ and $\mathbf{B}$ vanish for $r>R$ and that the shell material is absorbant, i.e. no reflected wave is generated.]

Paper 4, Section II

36D Electrodynamics

(a) A dielectric medium exhibits a linear response if the electric displacement $\mathbf{D}(\mathbf{x},t)$ and magnetizing field $\mathbf{H}(\mathbf{x},t)$ are related to the electric and magnetic fields, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, as

$$\mathbf{D} = \epsilon \mathbf{E}\,, \qquad \mathbf{B} = \mu \mathbf{H}\,,$$

where ϵ and μ are constants characterising the electric and magnetic polarisability of the material respectively. Write down the Maxwell equations obeyed by the fields $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{E}$ in this medium in the absence of free charges or currents.

(b) Two such media with constants ϵ_- and ϵ_+ (but the same μ) fill the regions $x < 0$ and $x > 0$ respectively in three-dimensions with Cartesian coordinates (x, y, z).

(i) Starting from Maxwell's equations, derive the appropriate boundary conditions at $x = 0$ for a time-independent electric field $\mathbf{E}(\mathbf{x})$.

(ii) Consider a candidate solution of Maxwell's equations describing the reflection and transmission of an incident electromagnetic wave of wave vector $\mathbf{k}_I$ and angular frequency ω_I off the interface at $x = 0$. The electric field is given as,

$$\mathbf{E}(\mathbf{x},t) \quad = \quad \begin{cases} \sum\limits_{X=I,R} \mathrm{Im}\left[\mathbf{E}_X \, \exp\left(i\mathbf{k}_X \cdot \mathbf{x} \,-\, i\omega_X t\right)\right] \,, & x < 0\,, \\ \mathrm{Im}\left[\mathbf{E}_T \, \exp\left(i\mathbf{k}_T \cdot \mathbf{x} \,-\, i\omega_T t\right)\right] \,, & x > 0\,, \end{cases}$$

where $\mathbf{E}_I$, $\mathbf{E}_R$ and $\mathbf{E}_T$ are constant real vectors and $\mathrm{Im}[z]$ denotes the imaginary part of a complex number z. Give conditions on the parameters $\mathbf{E}_X, \mathbf{k}_X, \omega_X$ for $X = I, R, T$, such that the above expression for the electric field $\mathbf{E}(\mathbf{x},t)$ solves Maxwell's equations for all $x \neq 0$, together with an appropriate magnetic field $\mathbf{B}(\mathbf{x},t)$ which you should determine.

(iii) We now parametrize the incident wave vector as $\mathbf{k}_I = k_I(\cos(\theta_I)\hat{\mathbf{i}}_x + \sin(\theta_I)\hat{\mathbf{i}}_z)$, where $\hat{\mathbf{i}}_x$ and $\hat{\mathbf{i}}_z$ are unit vectors in the x- and z-directions respectively, and choose the incident polarisation vector to satisfy $\mathbf{E}_I \cdot \hat{\mathbf{i}}_x = 0$. By imposing appropriate boundary conditions for $\mathbf{E}(\mathbf{x},t)$ at $x = 0$, which you may assume to be the same as those for the time-independent case considered above, determine the Cartesian components of the wavevector $\mathbf{k}_T$ as functions of k_I, θ_I, ϵ_+ and ϵ_-.

(iv) For $\epsilon_+ < \epsilon_-$ find a critical value θ_I^{cr} of the angle of incidence θ_I above which there is no real solution for the wavevector $\mathbf{k}_T$. Write down a solution for $\mathbf{E}(\mathbf{x},t)$ when $\theta_I > \theta_I^{\mathrm{cr}}$ and comment on its form.

Paper 4, Section II

35E Electrodynamics

Consider a medium in which the electric displacement $\mathbf{D}(t, \mathbf{x})$ and magnetising field $\mathbf{H}(t, \mathbf{x})$ are linearly related to the electric and magnetic fields respectively with corresponding polarisation constants ε and μ;

$$\mathbf{D} = \varepsilon\,\mathbf{E}, \qquad\qquad \mathbf{B} = \mu\,\mathbf{H}.$$

Write down Maxwell's equations for $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ in the absence of free charges and currents.

Consider EM waves of the form,

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &= \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)\,, \\ \mathbf{B}(t, \mathbf{x}) &= \mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)\,. \end{aligned}$$

Find conditions on the electric and magnetic polarisation vectors $\mathbf{E}_0$ and $\mathbf{B}_0$, wave-vector $\mathbf{k}$ and angular frequency ω such that these fields satisfy Maxwell's equations for the medium described above. At what speed do the waves propagate?

Consider two media, filling the regions $x < 0$ and $x > 0$ in three dimensional space, and having two different values ε_- and ε_+ of the electric polarisation constant. Suppose an electromagnetic wave is incident from the region $x < 0$ resulting in a transmitted wave in the region $x > 0$ and also a reflected wave for $x < 0$. The angles of incidence, reflection and transmission are denoted θ_I, θ_R and θ_T respectively. By constructing a corresponding solution of Maxwell's equations, derive the *law of reflection* $\theta_I = \theta_R$ and *Snell's law of refraction*, $n_- \sin\theta_I = n_+ \sin\theta_T$ where $n_\pm = c\sqrt{\varepsilon_\pm \mu}$ are the indices of refraction of the two media.

Consider the special case in which the electric polarisation vectors $\mathbf{E}_I$, $\mathbf{E}_R$ and $\mathbf{E}_T$ of the incident, reflected and transmitted waves are all normal to the plane of incidence (i.e. the plane containing the corresponding wave-vectors). By imposing appropriate boundary conditions for $\mathbf{E}$ and $\mathbf{H}$ at $x = 0$, show that,

$$\frac{|\mathbf{E}_R|}{|\mathbf{E}_T|} = \frac{1}{2}\left(1 - \frac{\tan\theta_R}{\tan\theta_T}\right).$$

Paper 3, Section II

36E Electrodynamics

A time-dependent charge distribution $\rho(t, \mathbf{x})$ localised in some region of size a near the origin varies periodically in time with characteristic angular frequency ω. Explain briefly the circumstances under which the *dipole approximation* for the fields sourced by the charge distribution is valid.

Far from the origin, for $r = |\mathbf{x}| \gg a$, the vector potential $\mathbf{A}(t, \mathbf{x})$ sourced by the charge distribution $\rho(t, \mathbf{x})$ is given by the approximate expression

$$\mathbf{A}(t, \mathbf{x}) \quad \simeq \quad \frac{\mu_0}{4\pi r} \int d^3\mathbf{x}' \, \mathbf{J}\left(t - r/c, \mathbf{x}'\right),$$

where $\mathbf{J}(t, \mathbf{x})$ is the corresponding current density. Show that, in the dipole approximation, the large-distance behaviour of the magnetic field is given by,

$$\mathbf{B}(t, \mathbf{x}) \quad \simeq \quad -\frac{\mu_0}{4\pi r c} \hat{\mathbf{x}} \times \ddot{\mathbf{p}}\,(t - r/c)\,,$$

where $\mathbf{p}(t)$ is the electric dipole moment of the charge distribution. Assuming that, in the same approximation, the corresponding electric field is given as $\mathbf{E} = -c\hat{\mathbf{x}} \times \mathbf{B}$, evaluate the flux of energy through the surface element of a large sphere of radius R centred at the origin. Hence show that the total power $P(t)$ radiated by the charge distribution is given by

$$P(t) \quad = \quad \frac{\mu_0}{6\pi c} \left|\ddot{\mathbf{p}}\,(t - R/c)\right|^2.$$

A particle of charge q and mass m undergoes simple harmonic motion in the x-direction with time period $T = 2\pi/\omega$ and amplitude $\mathcal{A}$ such that

$$\mathbf{x}(t) = \mathcal{A} \sin(\omega t)\, \mathbf{i}_x\,. \qquad (\star)$$

Here $\mathbf{i}_x$ is a unit vector in the x-direction. Calculate the total power $P(t)$ radiated through a large sphere centred at the origin in the dipole approximation and determine its time averaged value,

$$\langle P \rangle \quad = \quad \frac{1}{T} \int_0^T P(t)\, dt\,.$$

For what values of the parameters $\mathcal{A}$ and ω is the dipole approximation valid?

Now suppose that the energy of the particle with trajectory $(\star)$ is given by the usual non-relativistic formula for a harmonic oscillator i.e. $E = m|\dot{\mathbf{x}}|^2/2 + m\omega^2|\mathbf{x}|^2/2$, and that the particle loses energy due to the emission of radiation at a rate corresponding to the time-averaged power $\langle P \rangle$. Work out the half-life of this system (i.e. the time $t_{1/2}$ such that $E(t_{1/2}) = E(0)/2$). Explain why the non-relativistic approximation for the motion of the particle is reliable as long as the dipole approximation is valid.

Paper 1, Section II

36E Electrodynamics

A relativistic particle of charge q and mass m moves in a background electromagnetic field. The four-velocity $u^\mu(\tau)$ of the particle at proper time τ is determined by the equation of motion,

$$m\frac{du^\mu}{d\tau} \quad = \quad qF^\mu{}_\nu u^\nu .$$

Here $F^\mu{}_\nu = \eta_{\nu\rho}F^{\mu\rho}$, where $F_{\mu\nu}$ is the electromagnetic field strength tensor and Lorentz indices are raised and lowered with the metric tensor $\eta = \text{diag}\{-1, +1, +1, +1\}$. In the case of a constant, homogeneous field, write down the solution of this equation giving $u^\mu(\tau)$ in terms of its initial value $u^\mu(0)$.

[In the following you may use the relation, given below, between the components of the field strength tensor $F_{\mu\nu}$, for $\mu, \nu = 0, 1, 2, 3$, and those of the electric and magnetic fields $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$,

$$F_{i0} = -F_{0i} = \frac{1}{c}E_i, \qquad\qquad F_{ij} = \varepsilon_{ijk}B_k$$

for $i, j = 1, 2, 3$.]

Suppose that, in some inertial frame with spacetime coordinates $\mathbf{x} = (x, y, z)$ and t, the electric and magnetic fields are parallel to the x-axis with magnitudes E and B respectively. At time $t = \tau = 0$ the 3-velocity $\mathbf{v} = d\mathbf{x}/dt$ of the particle has initial value $\mathbf{v}(0) = (0, v_0, 0)$. Find the subsequent trajectory of the particle in this frame, giving coordinates x, y, z and t as functions of the proper time τ.

Find the motion in the x-direction explicitly, giving x as a function of coordinate time t. Comment on the form of the solution at early and late times. Show that, when projected onto the y-z plane, the particle undergoes circular motion which is periodic in proper time. Find the radius R of the circle and proper time period of the motion $\Delta\tau$ in terms of q, m, E, B and v_0. The resulting trajectory therefore has the form of a helix with varying pitch $P_n := \Delta x_n/R$ where Δx_n is the distance in the x-direction travelled by the particle during the n'th period of its motion in the y-z plane. Show that, for $n \gg 1$,

$$P_n \quad \sim \quad A\,\exp\left(\frac{2\pi E n}{cB}\right),$$

where A is a constant which you should determine.

Paper 1, Section II

36D Electrodynamics

Define the field strength tensor $F^{\mu\nu}(x)$ for an electromagnetic field specified by a 4-vector potential $A^{\mu}(x)$. How do the components of $F^{\mu\nu}$ change under a Lorentz transformation? Write down two independent Lorentz-invariant quantities which are quadratic in the field strength tensor.

[*Hint: The alternating tensor $\varepsilon^{\mu\nu\rho\sigma}$ takes the values $+1$ and -1 when (μ,ν,ρ,σ) is an even or odd permutation of $(0,1,2,3)$ respectively and vanishes otherwise. You may assume this is an invariant tensor of the Lorentz group. In other words, its components are the same in all inertial frames.*]

In an inertial frame with spacetime coordinates $x^{\mu} = (ct, \mathbf{x})$, the 4-vector potential has components $A^{\mu} = (\phi/c, \mathbf{A})$ and the electric and magnetic fields are given as

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned}$$

Evaluate the components of $F^{\mu\nu}$ in terms of the components of $\mathbf{E}$ and $\mathbf{B}$. Show that the quantities

$$S = |\mathbf{B}|^2 - \frac{1}{c^2}|\mathbf{E}|^2 \qquad \text{and} \qquad T = \mathbf{E}\cdot\mathbf{B}$$

are the same in all inertial frames.

A relativistic particle of mass m, charge q and 4-velocity $u^{\mu}(\tau)$ moves according to the Lorentz force law,

$$\frac{du^{\mu}}{d\tau} = \frac{q}{m} F^{\mu}{}_{\nu} u^{\nu}. \qquad (*)$$

Here τ is the proper time. For the case of a constant, uniform field, write down a solution of $(*)$ giving $u^{\mu}(\tau)$ in terms of its initial value $u^{\mu}(0)$ as an infinite series in powers of the field strength.

Suppose further that the fields are such that both S and T defined above are zero. Work in an inertial frame with coordinates $x^{\mu} = (ct, x, y, z)$ where the particle is at rest at the origin at $t = 0$ and the magnetic field points in the positive z-direction with magnitude $|\mathbf{B}| = B$. The electric field obeys $\mathbf{E}\cdot\hat{\mathbf{y}} = 0$. Show that the particle moves on the curve $y^2 = Ax^3$ for some constant A which you should determine.

Paper 4, Section II

36D Electrodynamics

(a) Define the *polarisation* of a dielectric material and explain what is meant by the term *bound charge*.

Consider a sample of material with spatially dependent polarisation $\mathbf{P}(\mathbf{x})$ occupying a region V with surface S. Show that, in the absence of free charge, the resulting scalar potential $\phi(\mathbf{x})$ can be ascribed to bulk and surface densities of bound charge.

Consider a sphere of radius R consisting of a dielectric material with permittivity ϵ surrounded by a region of vacuum. A point-like electric charge q is placed at the centre of the sphere. Determine the density of bound charge on the surface of the sphere.

(b) Define the *magnetization* of a material and explain what is meant by the term *bound current*.

Consider a sample of material with spatially-dependent magnetization $\mathbf{M}(\mathbf{x})$ occupying a region V with surface S. Show that, in the absence of free currents, the resulting vector potential $\mathbf{A}(\mathbf{x})$ can be ascribed to bulk and surface densities of bound current.

Consider an infinite cylinder of radius r consisting of a material with permeability μ surrounded by a region of vacuum. A thin wire carrying current I is placed along the axis of the cylinder. Determine the direction and magnitude of the resulting bound current density on the surface of the cylinder. What is the magnetization $\mathbf{M}(\mathbf{x})$ on the surface of the cylinder?

Paper 3, Section II

37D Electrodynamics

Starting from the covariant form of the Maxwell equations and making a suitable choice of gauge which you should specify, show that the 4-vector potential due to an arbitrary 4-current $J^\mu(x)$ obeys the wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu \quad = \quad -\mu_0 J^\mu,$$

where $x^\mu = (ct, \mathbf{x})$.

Use the method of Green's functions to show that, for a localised current distribution, this equation is solved by

$$A^\mu(t, \mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{J^\mu(t_{\rm ret}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

for some $t_{\rm ret}$ that you should specify.

A point particle, of charge q, moving along a worldline $y^\mu(\tau)$ parameterised by proper time τ, produces a 4-vector potential

$$A^\mu(x) = \frac{\mu_0 q c}{4\pi}\,\frac{\dot{y}^\mu(\tau_\star)}{|R^\nu(\tau_\star)\dot{y}_\nu(\tau_\star)|}$$

where $R^\mu(\tau) = x^\mu - y^\mu(\tau)$. Define $\tau_\star(x)$ and draw a spacetime diagram to illustrate its physical significance.

Suppose the particle follows a circular trajectory,

$$\mathbf{y}(t) \quad = \quad (R\cos(\omega\, t), R\sin(\omega\, t), 0)$$

(with $y^0 = ct$), in some inertial frame with coordinates (ct, x, y, z). Evaluate the resulting 4-vector potential at a point on the z-axis as a function of z and t.

Paper 1, Section II

35D Electrodynamics

In some inertial reference frame S, there is a uniform electric field $\mathbf{E}$ directed along the positive y-direction and a uniform magnetic field $\mathbf{B}$ directed along the positive z-direction. The magnitudes of the fields are E and B, respectively, with $E < cB$. Show that it is possible to find a reference frame in which the electric field vanishes, and determine the relative speed βc of the two frames and the magnitude of the magnetic field in the new frame.

[*Hint: You may assume that under a standard Lorentz boost with speed $v = \beta c$ along the x-direction, the electric and magnetic field components transform as*

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma(\beta)(E_y - vB_z) \\ \gamma(\beta)(E_z + vB_y) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma(\beta)(B_y + vE_z/c^2) \\ \gamma(\beta)(B_z - vE_y/c^2) \end{pmatrix},$$

where the Lorentz factor $\gamma(\beta) = (1 - \beta^2)^{-1/2}$.]

A point particle of mass m and charge q moves relativistically under the influence of the fields $\mathbf{E}$ and $\mathbf{B}$. The motion is in the plane $z = 0$. By considering the motion in the reference frame in which the electric field vanishes, or otherwise, show that, with a suitable choice of origin, the worldline of the particle has components in the frame S of the form

$$\begin{aligned} ct(\tau) &= \gamma(u/c)\gamma(\beta)\left[c\tau + \frac{\beta u}{\omega}\sin\omega\tau\right], \\ x(\tau) &= \gamma(u/c)\gamma(\beta)\left[\beta c\tau + \frac{u}{\omega}\sin\omega\tau\right], \\ y(\tau) &= \frac{u\gamma(u/c)}{\omega}\cos\omega\tau. \end{aligned}$$

Here, u is a constant speed with Lorentz factor $\gamma(u/c)$, τ is the particle's proper time, and ω is a frequency that you should determine.

Using dimensionless coordinates,

$$\tilde{x} = \frac{\omega}{u\gamma(u/c)}x \quad \text{and} \quad \tilde{y} = \frac{\omega}{u\gamma(u/c)}y,$$

sketch the trajectory of the particle in the $(\tilde{x}, \tilde{y})$-plane in the limiting cases $2\pi\beta \ll u/c$ and $2\pi\beta \gg u/c$.

Paper 3, Section II

35D Electrodynamics

By considering the force per unit volume $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$ on a charge density ρ and current density $\mathbf{J}$ due to an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, show that

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = -f_i \,,$$

where $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$ and the symmetric tensor σ_{ij} should be specified.

Give the physical interpretation of $\mathbf{g}$ and σ_{ij} and explain how σ_{ij} can be used to calculate the net electromagnetic force exerted on the charges and currents within some region of space in static situations.

The plane $x = 0$ carries a uniform charge σ per unit area and a current K per unit length along the z-direction. The plane $x = d$ carries the opposite charge and current. Show that between these planes

$$\sigma_{ij} = \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\mu_0 K^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad (*)$$

and $\sigma_{ij} = 0$ for $x < 0$ and $x > d$.

Use $(*)$ to find the electromagnetic force per unit area exerted on the charges and currents in the $x = 0$ plane. Show that your result agrees with direct calculation of the force per unit area based on the Lorentz force law.

If the current K is due to the motion of the charge σ with speed v, is it possible for the force between the planes to be repulsive?

Paper 4, Section II

35D Electrodynamics

A dielectric material has a real, frequency-independent relative permittivity ϵ_r with $|\epsilon_r - 1| \ll 1$. In this case, the macroscopic polarization that develops when the dielectric is placed in an external electric field $\mathbf{E}_{\text{ext}}(t, \mathbf{x})$ is $\mathbf{P}(t, \mathbf{x}) \approx \epsilon_0(\epsilon_r - 1)\mathbf{E}_{\text{ext}}(t, \mathbf{x})$. Explain briefly why the associated bound current density is

$$\mathbf{J}_{\text{bound}}(t, \mathbf{x}) \approx \epsilon_0(\epsilon_r - 1)\frac{\partial \mathbf{E}_{\text{ext}}(t, \mathbf{x})}{\partial t} .$$

[You should ignore any magnetic response of the dielectric.]

A sphere of such a dielectric, with radius a, is centred on $\mathbf{x} = 0$. The sphere scatters an incident plane electromagnetic wave with electric field

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} ,$$

where $\omega = c|\mathbf{k}|$ and $\mathbf{E}_0$ is a constant vector. Working in the Lorenz gauge, show that at large distances $r = |\mathbf{x}|$, for which both $r \gg a$ and $ka^2/r \ll 2\pi$, the magnetic vector potential $\mathbf{A}_{\text{scatt}}(t, \mathbf{x})$ of the scattered radiation is

$$\mathbf{A}_{\text{scatt}}(t, \mathbf{x}) \approx -i\omega \mathbf{E}_0 \frac{e^{i(kr - \omega t)}}{r} \frac{(\epsilon_r - 1)}{4\pi c^2} \int_{|\mathbf{x}'| \leqslant a} e^{i\mathbf{q}\cdot\mathbf{x}'} \, d^3\mathbf{x}' ,$$

where $\mathbf{q} = \mathbf{k} - k\hat{\mathbf{x}}$ with $\hat{\mathbf{x}} = \mathbf{x}/r$.

In the far-field, where $kr \gg 1$, the electric and magnetic fields of the scattered radiation are given by

$$\mathbf{E}_{\text{scatt}}(t, \mathbf{x}) \approx -i\omega \hat{\mathbf{x}} \times [\hat{\mathbf{x}} \times \mathbf{A}_{\text{scatt}}(t, \mathbf{x})] ,$$
$$\mathbf{B}_{\text{scatt}}(t, \mathbf{x}) \approx ik\hat{\mathbf{x}} \times \mathbf{A}_{\text{scatt}}(t, \mathbf{x}) .$$

By calculating the Poynting vector of the scattered and incident radiation, show that the ratio of the time-averaged power scattered per unit solid angle to the time-averaged incident power per unit area (i.e. the differential cross-section) is

$$\frac{d\sigma}{d\Omega} = (\epsilon_r - 1)^2 k^4 \left(\frac{\sin(qa) - qa\cos(qa)}{q^3} \right)^2 |\hat{\mathbf{x}} \times \hat{\mathbf{E}}_0|^2 ,$$

where $\hat{\mathbf{E}}_0 = \mathbf{E}_0/|\mathbf{E}_0|$ and $q = |\mathbf{q}|$.

[*You may assume that, in the Lorenz gauge, the retarded potential due to a localised current distribution is*

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' ,$$

where the retarded time $t_{\text{ret}} = t - |\mathbf{x} - \mathbf{x}'|/c$.]

Paper 1, Section II

34E Electrodynamics

A point particle of charge q and mass m moves in an electromagnetic field with 4-vector potential $A_\mu(x)$, where x^μ is position in spacetime. Consider the action

$$S = -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda \; + \; q \int A_\mu \frac{dx^\mu}{d\lambda}\, d\lambda \,, \qquad (*)$$

where λ is an arbitrary parameter along the particle's worldline and $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric.

(a) By varying the action with respect to $x^\mu(\lambda)$, with fixed endpoints, obtain the equation of motion

$$m \frac{du^\mu}{d\tau} = q F^\mu{}_\nu u^\nu \,,$$

where τ is the proper time, $u^\mu = dx^\mu/d\tau$ is the velocity 4-vector, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

(b) This particle moves in the field generated by a second point charge Q that is held at rest at the origin of some inertial frame. By choosing a suitable expression for A_μ and expressing the first particle's spatial position in spherical polar coordinates (r, θ, ϕ), show from the action $(*)$ that

$$\mathcal{E} \equiv \dot{t} - \Gamma/r \,,$$
$$\ell c \equiv r^2 \dot{\phi} \sin^2\theta$$

are constants, where $\Gamma = -qQ/(4\pi\epsilon_0 mc^2)$ and overdots denote differentiation with respect to τ.

(c) Show that when the motion is in the plane $\theta = \pi/2$,

$$\mathcal{E} + \frac{\Gamma}{r} = \sqrt{1 + \frac{\dot{r}^2}{c^2} + \frac{\ell^2}{r^2}} \,.$$

Hence show that the particle's orbit is bounded if $\mathcal{E} < 1$, and that the particle can reach the origin in finite proper time if $\Gamma > |\ell|$.

Paper 3, Section II

34E Electrodynamics

The current density in an antenna lying along the z-axis takes the form

$$\mathbf{J}(t,\mathbf{x}) = \begin{cases} \hat{\mathbf{z}}\, I_0 \sin\left(kd - k|z|\right) e^{-i\omega t}\delta(x)\delta(y) & |z| \leqslant d \\ \mathbf{0} & |z| > d \end{cases},$$

where I_0 is a constant and $\omega = ck$. Show that at distances $r = |\mathbf{x}|$ for which both $r \gg d$ and $r \gg kd^2/(2\pi)$, the retarded vector potential in Lorenz gauge is

$$\mathbf{A}(t,\mathbf{x}) \;\approx\; \hat{\mathbf{z}}\,\frac{\mu_0 I_0}{4\pi r}\, e^{-i\omega(t-r/c)} \int_{-d}^{d} \sin\left(kd - k|z'|\right) e^{-ikz'\cos\theta}\, dz' \,,$$

where $\cos\theta = \hat{\mathbf{r}}\cdot\hat{\mathbf{z}}$ and $\hat{\mathbf{r}} = \mathbf{x}/|\mathbf{x}|$. Evaluate the integral to show that

$$\mathbf{A}(t,\mathbf{x}) \;\approx\; \hat{\mathbf{z}}\,\frac{\mu_0 I_0}{2\pi k r}\left(\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin^2\theta}\right) e^{-i\omega(t-r/c)} \,.$$

In the far-field, where $kr \gg 1$, the electric and magnetic fields are given by

$$\begin{aligned} \mathbf{E}(t,\mathbf{x}) \;&\approx\; -i\omega\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \mathbf{A}(t,\mathbf{x})] \\ \mathbf{B}(t,\mathbf{x}) \;&\approx\; ik\hat{\mathbf{r}} \times \mathbf{A}(t,\mathbf{x}) \,. \end{aligned}$$

By calculating the Poynting vector, show that the time-averaged power radiated per unit solid angle is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{c\mu_0 I_0^2}{8\pi^2}\left(\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right)^2 .$$

[You may assume that in Lorenz gauge, the retarded potential due to a localised current distribution is

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(t_{\text{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}' \,,$$

where the retarded time $t_{\text{ret}} = t - |\mathbf{x}-\mathbf{x}'|/c$.]

Paper 4, Section II

34E Electrodynamics

(a) A uniform, isotropic dielectric medium occupies the half-space $z > 0$. The region $z < 0$ is in vacuum. State the boundary conditions that should be imposed on $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{H}$ at $z = 0$.

(b) A linearly polarized electromagnetic plane wave, with magnetic field in the (x, y)-plane, is incident on the dielectric from $z < 0$. The wavevector $\mathbf{k}$ makes an acute angle θ_I to the normal $\hat{\mathbf{z}}$. If the dielectric has frequency-independent relative permittivity ϵ_r, show that the fraction of the incident power that is reflected is

$$\mathcal{R} = \left(\frac{n\cos\theta_I - \cos\theta_T}{n\cos\theta_I + \cos\theta_T}\right)^2 ,$$

where $n = \sqrt{\epsilon_r}$, and the angle θ_T should be specified. [You should ignore any magnetic response of the dielectric.]

(c) Now suppose that the dielectric moves at speed βc along the x-axis, the incident angle $\theta_I = 0$, and the magnetic field of the incident radiation is along the y-direction. Show that the reflected radiation propagates normal to the surface $z = 0$, has the same frequency as the incident radiation, and has magnetic field also along the y-direction. [*Hint: You may assume that under a standard Lorentz boost with speed $v = \beta c$ along the x-direction, the electric and magnetic field components transform as*

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma(B_y + vE_z/c^2) \\ \gamma(B_z - vE_y/c^2) \end{pmatrix} ,$$

where $\gamma = (1 - \beta^2)^{-1/2}$.]

(d) Show that the fraction of the incident power reflected from the moving dielectric is

$$\mathcal{R}_\beta = \left(\frac{n/\gamma - \sqrt{1 - \beta^2/n^2}}{n/\gamma + \sqrt{1 - \beta^2/n^2}}\right)^2 .$$

Paper 4, Section II

33A Electrodynamics

A point particle of charge q has trajectory $y^\mu(\tau)$ in Minkowski space, where τ is its proper time. The resulting electromagnetic field is given by the Liénard–Wiechert 4-potential

$$A^\mu(x) \;=\; -\frac{q\,\mu_0\,c}{4\pi}\,\frac{u^\mu(\tau_*)}{R^\nu(\tau_*)\,u_\nu(\tau_*)}\,, \quad \text{where} \quad R^\nu = x^\nu - y^\nu(\tau) \quad \text{and} \quad u^\mu = dy^\mu/d\tau\,.$$

Write down the condition that determines the point $y^\mu(\tau_*)$ on the trajectory of the particle for a given value of x^μ. Express this condition in terms of components, setting $x^\mu = (ct, \mathbf{x})$ and $y^\mu = (ct', \mathbf{y})$, and define the retarded time t_r.

Suppose that the 3-velocity of the particle $\mathbf{v}(t') = \dot{\mathbf{y}}(t') = d\mathbf{y}/dt'$ is small in size compared to c, and suppose also that $r = |\mathbf{x}| \gg |\mathbf{y}|$. Working to leading order in $1/r$ and to first order in $\mathbf{v}$, show that

$$\phi(x) \;=\; \frac{q\,\mu_0\,c}{4\pi r}\,(\,c + \hat{\mathbf{r}}\cdot\mathbf{v}(t_r)\,)\;, \quad \mathbf{A}(x) \;=\; \frac{q\,\mu_0}{4\pi r}\,\mathbf{v}(t_r)\,, \quad \text{where} \quad \hat{\mathbf{r}} = \mathbf{x}/r\;.$$

Now assume that t_r can be replaced by $t_- = t - (r/c)$ in the expressions for ϕ and $\mathbf{A}$ above. Calculate the electric and magnetic fields to leading order in $1/r$ and hence show that the Poynting vector is (in this approximation)

$$\mathbf{N}(x) \;=\; \frac{q^2\mu_0}{(4\pi)^2\,c}\,\frac{\hat{\mathbf{r}}}{r^2}\,\Big|\,\hat{\mathbf{r}}\times\dot{\mathbf{v}}(t_-)\,\Big|^2\,.$$

If the charge q is performing simple harmonic motion $\mathbf{y}(t') = A\mathbf{n}\cos\omega t'$, where $\mathbf{n}$ is a unit vector and $A\omega \ll c$, find the total energy radiated during one period of oscillation.

Paper 3, Section II

34A Electrodynamics

(i) Consider the action

$$S = -\frac{1}{4\mu_0 c}\int \left(F_{\mu\nu}F^{\mu\nu} + 2\lambda^2 A_\mu A^\mu\right)\, d^4x + \frac{1}{c}\int A_\mu J^\mu \, d^4x\,,$$

where $A_\mu(x)$ is a 4-vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $J^\mu(x)$ is a conserved current, and $\lambda \geqslant 0$ is a constant. Derive the field equation

$$\partial_\mu F^{\mu\nu} - \lambda^2 A^\nu = -\mu_0 J^\nu\,.$$

For $\lambda = 0$ the action S describes standard electromagnetism. Show that in this case the theory is invariant under gauge transformations of the field $A_\mu(x)$, which you should define. Is the theory invariant under these same gauge transformations when $\lambda > 0$?

Show that when $\lambda > 0$ the field equation above implies

$$\partial_\mu \partial^\mu A^\nu - \lambda^2 A^\nu = -\mu_0 J^\nu\,. \qquad (*)$$

Under what circumstances does $(*)$ hold in the case $\lambda = 0$?

(ii) Now suppose that $A_\mu(x)$ and $J_\mu(x)$ obeying $(*)$ reduce to static 3-vectors $\mathbf{A}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in some inertial frame. Show that there is a solution

$$\mathbf{A}(\mathbf{x}) = -\mu_0 \int G(\,|\mathbf{x}-\mathbf{x}'|\,)\, \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$$

for a suitable Green's function $G(R)$ with $G(R) \to 0$ as $R \to \infty$. Determine $G(R)$ for any $\lambda \geqslant 0$. [*Hint: You may find it helpful to consider first the case* $\lambda = 0$ *and then the case* $\lambda > 0$, *using the result* $\nabla^2\Big(\dfrac{1}{R}f(R)\Big) = \nabla^2\Big(\dfrac{1}{R}\Big)f(R) + \dfrac{1}{R}f''(R)$, *where* $R = |\,\mathbf{x}-\mathbf{x}'\,|$.]

If $\mathbf{J}(\mathbf{x})$ is zero outside some bounded region, comment on the effect of the value of λ on the behaviour of $\mathbf{A}(\mathbf{x})$ when $|\mathbf{x}|$ is large. [No further detailed calculations are required.]

Paper 1, Section II

34A Electrodynamics

Briefly explain how to interpret the components of the relativistic stress–energy tensor in terms of the density and flux of energy and momentum in some inertial frame.

(i) The stress–energy tensor of the electromagnetic field is

$$T^{\mu\nu}_{\text{em}} = \frac{1}{\mu_0}\Big(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\Big)\,,$$

where $F_{\mu\nu}$ is the field strength, $\eta_{\mu\nu}$ is the Minkowski metric, and μ_0 is the permeability of free space. Show that $\partial_\mu T^{\mu\nu}_{\text{em}} = -F^{\nu}{}_{\mu}J^{\mu}$, where J^μ is the current 4-vector.

[Maxwell's equations are $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ and $\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$.]

(ii) A fluid consists of point particles of rest mass m and charge q. The fluid can be regarded as a continuum, with 4-velocity $u^\mu(x)$ depending on the position x in spacetime. For each x there is an inertial frame S_x in which the fluid particles at x are at rest. By considering components in S_x, show that the fluid has a current 4-vector field

$$J^\mu \;=\; q\,n_0 u^\mu\,,$$

and a stress–energy tensor

$$T^{\mu\nu}_{\text{fluid}} \;=\; m\,n_0 u^\mu u^\nu\,,$$

where $n_0(x)$ is the proper number density of particles (the number of particles per unit spatial volume in S_x in a small region around x). Write down the Lorentz 4-force on a fluid particle at x. By considering the resulting 4-acceleration of the fluid, show that the total stress–energy tensor is conserved, i.e.

$$\partial_\mu\left(T^{\mu\nu}_{\text{em}} + T^{\mu\nu}_{\text{fluid}}\right) \;=\; 0\,.$$

Paper 4, Section II

35C Electrodynamics

(i) The action S for a point particle of rest mass m and charge q moving along a trajectory $x^\mu(\lambda)$ in the presence of an electromagnetic 4-vector potential A^μ is

$$S = -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda + q \int A_\mu \frac{dx^\mu}{d\lambda}\, d\lambda \,,$$

where λ is an arbitrary parametrization of the path and $\eta_{\mu\nu}$ is the Minkowski metric. By varying the action with respect to $x^\mu(\lambda)$, derive the equation of motion $m\ddot{x}^\mu = qF^\mu{}_\nu \dot{x}^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and overdots denote differentiation with respect to proper time for the particle.

(ii) The particle moves in constant electric and magnetic fields with non-zero Cartesian components $E_z = E$ and $B_y = B$, with $B > E/c > 0$ in some inertial frame. Verify that a suitable 4-vector potential has components

$$A^\mu = (0, 0, 0, -Bx - Et)$$

in that frame.

Find the equations of motion for x, y, z and t in terms of proper time τ. For the case of a particle that starts at rest at the spacetime origin at $\tau = 0$, show that

$$\ddot{z} + \frac{q^2}{m^2} \left(B^2 - \frac{E^2}{c^2} \right) z = \frac{qE}{m} \,.$$

Find the trajectory $x^\mu(\tau)$ and sketch its projection onto the (x, z) plane.

Paper 3, Section II

36C Electrodynamics

The 4-vector potential $A^\mu(t,\mathbf{x})$ (in the Lorenz gauge $\partial_\mu A^\mu = 0$) due to a localised source with conserved 4-vector current J^μ is

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(t_{\text{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}' ,$$

where $t_{\text{ret}} = t - |\mathbf{x}-\mathbf{x}'|/c$. For a source that varies slowly in time, show that the spatial components of A^μ at a distance $r = |\mathbf{x}|$ that is large compared to the spatial extent of the source are

$$\mathbf{A}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi r}\left.\frac{d\mathbf{P}}{dt}\right|_{t-r/c} ,$$

where $\mathbf{P}$ is the electric dipole moment of the source, which you should define. Explain what is meant by the *far-field* region, and calculate the leading-order part of the magnetic field there.

A point charge q moves non-relativistically in a circle of radius a in the (x,y) plane with angular frequency ω (such that $a\omega \ll c$). Show that the magnetic field in the far-field at the point $\mathbf{x}$ with spherical polar coordinates r, θ and ϕ has components along the θ and ϕ directions given by

$$\begin{aligned} B_\theta &\approx -\frac{\mu_0\omega^2 qa}{4\pi rc}\sin[\omega(t-r/c)-\phi] , \\ B_\phi &\approx \frac{\mu_0\omega^2 qa}{4\pi rc}\cos[\omega(t-r/c)-\phi]\cos\theta . \end{aligned}$$

Calculate the total power radiated by the charge.

Paper 1, Section II

36C Electrodynamics

(i) Starting from the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A^\mu = (\phi/c, \mathbf{A})$ is the 4-vector potential with components such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,,$$

derive the transformation laws for the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ under the standard Lorentz boost $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) Two point charges, each with electric charge q, are at rest and separated by a distance d in some inertial frame S. By transforming the fields from the rest frame S, calculate the magnitude and direction of the force between the two charges in an inertial frame in which the charges are moving with speed βc in a direction perpendicular to their separation.

(iii) The 4-force for a particle with 4-momentum p^μ is $F^\mu = dp^\mu/d\tau$, where τ is proper time. Show that the components of F^μ in an inertial frame in which the particle has 3-velocity $\mathbf{v}$ are

$$F^\mu = \gamma\left(\mathbf{F}\cdot\mathbf{v}/c, \mathbf{F}\right)\,,$$

where $\gamma = (1 - \mathbf{v}\cdot\mathbf{v}/c^2)^{-1/2}$ and $\mathbf{F}$ is the 3-force acting on the particle. Hence verify that your result in (ii) above is consistent with Lorentz transforming the electromagnetic 3-force from the rest frame S.

Paper 4, Section II

35B Electrodynamics

(i) For a time-dependent source, confined within a domain D, show that the time derivative $\dot{\mathbf{d}}$ of the dipole moment $\mathbf{d}$ satisfies

$$\dot{\mathbf{d}} = \int_D d^3y \, \mathbf{J}(\mathbf{y}) \,,$$

where $\mathbf{J}$ is the current density.

(ii) The vector potential $\mathbf{A}(\mathbf{x}, t)$ due to a time-dependent source is given by

$$\mathbf{A} = \frac{1}{r} f\,(t - r/c)\,\mathbf{k} \,,$$

where $r = |\mathbf{x}| \neq 0$, and $\mathbf{k}$ is the unit vector in the z direction. Calculate the resulting magnetic field $\mathbf{B}(\mathbf{x}, t)$. By considering the magnetic field for small r show that the dipole moment of the effective source satisfies

$$\frac{\mu_0}{4\pi}\dot{\mathbf{d}} = f(t)\mathbf{k} \,.$$

Calculate the asymptotic form of the magnetic field $\mathbf{B}$ at very large r.

(iii) Using the equation

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} \,,$$

calculate $\mathbf{E}$ at very large r. Show that $\mathbf{E}, \mathbf{B}$ and $\hat{\mathbf{r}} = \mathbf{x}/|\mathbf{x}|$ form a right-handed triad, and moreover $|\mathbf{E}| = c|\mathbf{B}|$. How do $|\mathbf{E}|$ and $|\mathbf{B}|$ depend on r? What is the significance of this?

(iv) Calculate the power $P(\theta, \phi)$ emitted per unit solid angle and sketch its dependence on θ. Show that the emitted radiation is polarised and describe how the plane of polarisation (that is, the plane in which $\mathbf{E}$ and $\hat{\mathbf{r}}$ lie) depends on the direction of the dipole. Suppose the dipole moment has constant amplitude and constant frequency and so the radiation is monochromatic with wavelength λ. How does the emitted power depend on λ?

Paper 3, Section II
36B Electrodynamics

(i) Obtain Maxwell's equations in empty space from the action functional

$$S[A_\mu] = -\frac{1}{\mu_0}\int d^4x \frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(ii) A modification of Maxwell's equations has the action functional

$$\tilde{S}[A_\mu] = -\frac{1}{\mu_0}\int d^4x \Big\{\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\lambda^2} A_\mu A^\mu\Big\}\,,$$

where again $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and λ is a constant. Obtain the equations of motion of this theory and show that they imply $\partial_\mu A^\mu = 0$.

(iii) Show that the equations of motion derived from $\tilde{S}$ admit solutions of the form

$$A^\mu = A_0^\mu e^{ik_\nu x^\nu}\,,$$

where A_0^μ is a constant 4-vector, and the 4-vector k_μ satisfies $A_0^\mu k_\mu = 0$ and $k_\mu k^\mu = -1/\lambda^2$.

(iv) Show further that the tensor

$$T_{\mu\nu} = \frac{1}{\mu_0}\Big\{F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2\lambda^2}\big(\eta_{\mu\nu}A_\alpha A^\alpha - 2A_\mu A_\nu\big)\Big\}$$

is conserved, that is $\partial^\mu T_{\mu\nu} = 0$.

Paper 1, Section II

36B Electrodynamics

(i) Starting from

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

and performing a Lorentz transformation with $\gamma = 1/\sqrt{1-u^2/c^2}$, using

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & -\gamma u/c & 0 & 0 \\ -\gamma u/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

show how $\mathbf{E}$ and $\mathbf{B}$ transform under a Lorentz transformation.

(ii) By taking the limit $c \to \infty$, obtain the behaviour of $\mathbf{E}$ and $\mathbf{B}$ under a Galilei transfomation and verify the invariance under Galilei transformations of the non-relativistic equation

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,.$$

(iii) Show that Maxwell's equations admit solutions of the form

$$\mathbf{E} = \mathbf{E}_0\, f(t - \mathbf{n}\cdot\mathbf{x}/c)\,, \qquad \mathbf{B} = \mathbf{B}_0\, f(t - \mathbf{n}\cdot\mathbf{x}/c)\,, \qquad (\star)$$

where f is an arbitrary function, $\mathbf{n}$ is a unit vector, and the constant vectors $\mathbf{E}_0$ and $\mathbf{B}_0$ are subject to restrictions which should be stated.

(iv) Perform a Galilei transformation of a solution ($\star$), with $\mathbf{n} = (1,0,0)$. Show that, by a particular choice of u, the solution may brought to the form

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 g(\tilde{x})\,, \qquad \tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0 g(\tilde{x})\,, \qquad (\dagger)$$

where g is an arbitrary function and $\tilde{x}$ is a spatial coordinate in the rest frame. By showing that ($\dagger$) is not a solution of Maxwell's equations in the boosted frame, conclude that Maxwell's equations are not invariant under Galilei transformations.

Paper 4, Section II

35B Electrodynamics

The charge and current densities are given by $\rho(t, \mathbf{x}) \neq 0$ and $\mathbf{j}(t, \mathbf{x})$ respectively. The electromagnetic scalar and vector potentials are given by $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ respectively. Explain how one can regard $j^\mu = (\rho, \mathbf{j})$ as a four-vector that obeys the current conservation rule $\partial_\mu j^\mu = 0$.

In the Lorenz gauge $\partial_\mu A^\mu = 0$, derive the wave equation that relates $A^\mu = (\phi, \mathbf{A})$ to j^μ and hence show that it is consistent to treat A^μ as a four-vector.

In the Lorenz gauge, with $j^\mu = 0$, a plane wave solution for A^μ is given by

$$A^\mu = \epsilon^\mu \exp(i k_\nu x^\nu)\,,$$

where ϵ^μ, k^μ and x^μ are four-vectors with

$$\epsilon^\mu = (\epsilon^0, \boldsymbol{\epsilon}), \quad k^\mu = (k^0, \mathbf{k}), \quad x^\mu = (t, \mathbf{x})\,.$$

Show that $k_\mu k^\mu = k_\mu \epsilon^\mu = 0$.

Interpret the components of k^μ in terms of the frequency and wavelength of the wave.

Find what residual gauge freedom there is and use it to show that it is possible to set $\epsilon^0 = 0$. What then is the physical meaning of the components of $\boldsymbol{\epsilon}$?

An observer at rest in a frame S measures the angular frequency of a plane wave travelling parallel to the z-axis to be ω. A second observer travelling at velocity v in S parallel to the z-axis measures the radiation to have frequency ω'. Express ω' in terms of ω.

Paper 3, Section II

36B Electrodynamics

The non-relativistic Larmor formula for the power, P, radiated by a particle of charge q and mass m that is being accelerated with an acceleration $\mathbf{a}$ is

$$P = \frac{\mu_0}{6\pi} q^2 |\mathbf{a}|^{\mathbf{2}} .$$

Starting from the Liénard–Wiechert potentials, sketch a derivation of this result. Explain briefly why the relativistic generalization of this formula is

$$P = \frac{\mu_0}{6\pi} \frac{q^2}{m^2} \left(\frac{dp^\mu}{d\tau} \frac{dp^\nu}{d\tau} \eta_{\mu\nu} \right) ,$$

where p^μ is the relativistic momentum of the particle and τ is the proper time along the worldline of the particle.

A particle of mass m and charge q moves in a plane perpendicular to a constant magnetic field B. At time $t = 0$ as seen by an observer $\mathbf{O}$ at rest, the particle has energy $E = \gamma m$. At what rate is electromagnetic energy radiated by this particle?

At time t according to the observer $\mathbf{O}$, the particle has energy $E' = \gamma' m$. Find an expression for γ' in terms of γ and t.

Paper 1, Section II

36B Electrodynamics

A particle of mass m and charge q moves relativistically under the influence of a constant electric field E in the positive z-direction, and a constant magnetic field B also in the positive z-direction.

In some inertial observer's coordinate system, the particle starts at

$$x = R, \quad y = 0, \quad z = 0, \quad t = 0,$$

with velocity given by

$$\dot{x} = 0, \quad \dot{y} = u, \quad \dot{z} = 0,$$

where the dot indicates differentiation with respect to the proper time of the particle. Show that the subsequent motion of the particle, as seen by the inertial observer, is a helix.

a) What is the radius of the helix as seen by the inertial observer?

b) What are the x and y coordinates of the axis of the helix?

c) What is the z coordinate of the particle after a proper time τ has elapsed, as measured by the particle?

Paper 1, Section II

36C Electrodynamics

In the Landau–Ginzburg model of superconductivity, the energy of the system is given, for constants α and β, by

$$E = \int \left\{ \frac{1}{2\mu_0}\mathbf{B}^2 + \frac{1}{2m}[(i\hbar\nabla - q\mathbf{A})\psi^* \cdot (-i\hbar\nabla - q\mathbf{A})\psi] + \alpha\psi^*\psi + \beta(\psi^*\psi)^2 \right\} d^3\mathbf{x}\,,$$

where $\mathbf{B}$ is the time-independent magnetic field derived from the vector potential $\mathbf{A}$, and ψ is the wavefunction of the charge carriers, which have mass m and charge q.

Describe the physical meaning of each of the terms in the integral.

Explain why in a superconductor one must choose $\alpha < 0$ and $\beta > 0$. Find an expression for the number density n of the charge carriers in terms of α and β.

Show that the energy is invariant under the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda\,, \qquad \psi \to \psi\, e^{iq\Lambda/\hbar}\,.$$

Assuming that the number density n is uniform, show that, if E is a minimum under variations of $\mathbf{A}$, then

$$\operatorname{curl}\mathbf{B} = -\frac{\mu_0 q^2 n}{m}(\mathbf{A} - \frac{\hbar}{q}\nabla\phi)\,,$$

where $\phi = \arg\psi$.

Find a formula for $\nabla^2\mathbf{B}$ and use it to explain why there cannot be a magnetic field inside the bulk of a superconductor.

Paper 3, Section II

36C Electrodynamics

Explain how time-dependent distributions of electric charge $\rho(\mathbf{x},t)$ and current $\mathbf{j}(\mathbf{x},t)$ can be combined into a four-vector $j^a(x)$ that obeys $\partial_a j^a = 0$.

This current generates a four-vector potential $A^a(x)$. Explain how to find A^a in the gauge $\partial_a A^a = 0$.

A small circular loop of wire of radius r is centred at the origin. The unit vector normal to the plane of the loop is $\mathbf{n}$. A current $I_o \sin\omega t$ flows in the loop. Find the three-vector potential $\mathbf{A}(\mathbf{x},t)$ to leading order in $r/|\mathbf{x}|$.

Paper 4, Section II

35C Electrodynamics

Suppose that there is a distribution of electric charge given by the charge density $\rho(\mathbf{x})$. Develop the multipole expansion, up to quadrupole terms, for the electrostatic potential ϕ and define the dipole and quadrupole moments of the charge distribution.

A tetrahedron has a vertex at $(1,1,1)$ where there is a point charge of strength $3q$. At each of the other vertices located at $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$ there is a point charge of strength $-q$.

What is the dipole moment of this charge distribution?

What is the quadrupole moment?

Paper 1, Section II

35B Electrodynamics

The vector potential A^μ is determined by a current density distribution j^μ in the gauge $\partial_\mu A^\mu = 0$ by

$$\Box A^\mu = -\mu_0 j^\mu, \qquad \Box = -\frac{\partial^2}{\partial t^2} + \nabla^2,$$

in units where $c = 1$.

Describe how to justify the result

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{j^\mu(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}, \qquad t' = t - |\mathbf{x} - \mathbf{x}'|.$$

A plane square loop of thin wire, edge lengths l, has its centre at the origin and lies in the (x, y) plane. For $t < 0$, no current is flowing in the loop, but at $t = 0$ a constant current I is turned on.

Find the vector potential at the point $(0, 0, z)$ as a function of time due to a single edge of the loop.

What is the electric field due to the entire loop at $(0, 0, z)$ as a function of time? Give a careful justification of your answer.

Paper 3, Section II

36B Electrodynamics

A particle of rest-mass m, electric charge q, is moving relativistically along the path $x^\mu(s)$ where s parametrises the path.

Write down an action for which the extremum determines the particle's equation of motion in an electromagnetic field given by the potential $A^\mu(x)$.

Use your action to derive the particle's equation of motion in a form where s is the proper time.

Suppose that the electric and magnetic fields are given by

$$\begin{aligned}\mathbf{E} &= (0,0,E)\,,\\ \mathbf{B} &= (0,B,0)\,.\end{aligned}$$

where E and B are constants and $B > E > 0$.

Find $x^\mu(s)$ given that the particle starts at rest at the origin when $s = 0$.

Describe qualitatively the motion of the particle.

Paper 4, Section II

35B Electrodynamics

In a superconductor the number density of charge carriers of charge q is n_s . Suppose that there is a time-independent magnetic field described by the three-vector potential $\mathbf{A}$.

Derive an expression for the superconducting current.

Explain how your answer is gauge invariant.

Suppose that for $z < 0$ there is a constant magnetic field $\mathbf{B}_0$ in a vacuum and, for $z > 0$, there is a uniform superconductor. Derive the magnetic field for $z > 0$.

Paper 1, Section II

35C Electrodynamics

The action for a modified version of electrodynamics is given by

$$I = \int d^4x \left(-\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} m^2 A_a A^a + \mu_0 J^a A_a \right),$$

where m is an arbitrary constant, $F_{ab} = \partial_a A_b - \partial_b A_a$ and J^a is a conserved current.

(i) By varying A_a, derive the field equations analogous to Maxwell's equations by demanding that $\delta I = 0$ for an arbitrary variation δA_a.

(ii) Show that $\partial_a A^a = 0$.

(iii) Suppose that the current $J^a(x)$ is a function of position only. Show that

$$A^a(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{J^a(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} e^{-m|\mathbf{x}-\mathbf{x}'|}.$$

Paper 3, Section II

36C Electrodynamics

A particle of charge of q moves along a trajectory $y^a(s)$ in spacetime where s is the proper time on the particle's world-line.

Explain briefly why, in the gauge $\partial_a A^a = 0$, the potential at the spacetime point x is given by

$$A^a(x) = \frac{\mu_0 q}{2\pi} \int ds \; \frac{dy^a}{ds} \; \theta\left(x^0 - y^0(s)\right) \; \delta\left((x^c - y^c(s))(x^d - y^d(s))\eta_{cd}\right).$$

Evaluate this integral for a point charge moving relativistically along the z-axis, $x = y = 0$, at constant velocity v so that $z = vt$.

Check your result by starting from the potential of a point charge at rest

$$\mathbf{A} = 0,$$
$$\phi = \frac{\mu_0 q}{4\pi(x^2 + y^2 + z^2)^{1/2}},$$

and making an appropriate Lorentz transformation.

Paper 4, Section II

35C Electrodynamics

In a superconductor, the charge carriers have a charge q, mass m and number density n. Describe how to construct the superconducting current in terms of the vector potential $\mathbf{A}$ and the wavefunction of the charge carriers.

Show that the current is gauge invariant.

Derive the Helmholtz equation

$$\nabla^2 \mathbf{B} = \mathbf{B}/\ell^2$$

for a time-independent magnetic field and evaluate the length scale ℓ in terms of n, q and m.

Why does this imply that magnetic flux cannot exist in a superconductor?

1/II/34D **Electrodynamics**

Frame $\mathcal{S}'$ is moving with uniform speed v in the x-direction relative to a laboratory frame $\mathcal{S}$. The components of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the two frames are related by the Lorentz transformation

$$E'_x = E_x, \quad E'_y = \gamma(E_y - vB_z), \quad E'_z = \gamma(E_z + vB_y),$$

$$B'_x = B_x, \quad B'_y = \gamma(B_y + vE_z), \quad B'_z = \gamma(B_z - vE_y),$$

where $\gamma = 1/\sqrt{1-v^2}$ and units are chosen so that $c = 1$. How do the components of the spatial vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ (where $i = \sqrt{-1}$) transform?

Show that $\mathbf{F}'$ is obtained from $\mathbf{F}$ by a rotation through θ about a spatial axis $\mathbf{n}$, where $\mathbf{n}$ and θ should be determined. Hence, or otherwise, show that there are precisely two independent scalars associated with $\mathbf{F}$ which are preserved by the Lorentz transformation, and obtain them.

[*Hint: since* $|v| < 1$ *there exists a unique real* ψ *such that* $v = \tanh\psi$.]

3/II/35D **Electrodynamics**

The retarded scalar potential $\varphi(t, \mathbf{x})$ produced by a charge distribution $\rho(t, \mathbf{x})$ is given by

$$\varphi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_\Omega d^3x' \, \frac{\rho(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

where Ω denotes all 3-space. Describe briefly and qualitatively the physics underlying this formula.

Write the integrand in the formula above as a 1-dimensional integral over a new time coordinate τ. Next consider a special source, a point charge q moving along a trajectory $\mathbf{x} = \mathbf{x}_0(t)$ so that

$$\rho(t, \mathbf{x}) = q\delta^{(3)}(\mathbf{x} - \mathbf{x}_0(t)),$$

where $\delta^{(3)}(\mathbf{x})$ denotes the 3-dimensional delta function. By reversing the order of integration, or otherwise, obtain the Liénard–Wiechert potential

$$\varphi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{q}{R - \mathbf{v} \cdot \mathbf{R}},$$

where $\mathbf{v}$ and $\mathbf{R}$ are to be determined.

Write down the corresponding formula for the vector potential $\mathbf{A}(t, \mathbf{x})$.

4/II/35D **Electrodynamics**

The Maxwell field tensor is given by

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

A general 4-velocity is written as $U^a = \gamma(1, \mathbf{v})$, where $\gamma = (1 - |\mathbf{v}|^2)^{-1/2}$, and $c = 1$. A general 4-current density is written as $J^a = (\rho, \mathbf{j})$, where ρ is the charge density and $\mathbf{j}$ is the 3-current density. Show that

$$F^{ab}U_b = \gamma(\mathbf{E} \cdot \mathbf{v},\ \mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

In the rest frame of a conducting medium, Ohm's law states that $\mathbf{j} = \sigma \mathbf{E}$ where σ is the conductivity. Show that the relativistic generalization to a frame in which the medium moves with uniform velocity $\mathbf{v}$ is

$$J^a - (J^b U_b) U^a = \sigma F^{ab} U_b.$$

Show that this implies

$$\mathbf{j} = \rho \mathbf{v} + \sigma\gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B} - (\mathbf{v} \cdot \mathbf{E})\mathbf{v}).$$

Simplify this formula, given that the charge density vanishes in the rest frame of the medium.

1/II/34E **Electrodynamics**

Frame $\mathcal{S}'$ is moving with uniform speed v in the z-direction relative to a laboratory frame $\mathcal{S}$. Using Cartesian coordinates and units such that $c = 1$, the relevant Lorentz transformation is

$$t' = \gamma(t - vz), \quad x' = x, \quad y' = y, \quad z' = \gamma(z - vt)\,,$$

where $\gamma = 1/\sqrt{1-v^2}$. A straight thin wire of infinite extent lies along the z-axis and carries charge and current line densities σ and J per unit length, as measured in $\mathcal{S}$. Stating carefully your assumptions show that the corresponding quantities in $\mathcal{S}'$ are given by

$$\sigma' = \gamma(\sigma - vJ), \qquad J' = \gamma(J - v\sigma)\,.$$

Using cylindrical polar coordinates, and the integral forms of the Maxwell equations $\nabla \cdot \mathbf{E} = \mu_0 \rho$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, derive the electric and magnetic fields outside the wire in both frames.

In a standard notation the Lorentz transformation for the electric and magnetic fields is

$$\mathbf{E}_{\|}{}' = \mathbf{E}_{\|}, \qquad \mathbf{B}_{\|}{}' = \mathbf{B}_{\|}, \qquad \mathbf{E}_{\perp}{}' = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}), \qquad \mathbf{B}_{\perp}{}' = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}_{\perp}).$$

Is your result consistent with this?

3/II/35E **Electrodynamics**

Consider a particle of charge q moving with 3-velocity $\mathbf{v}$. If the particle is moving slowly then Larmor's formula asserts that the instantaneous radiated power is

$$\mathcal{P} = \frac{\mu_0}{6\pi} q^2 \left| \frac{d\mathbf{v}}{dt} \right|^2 .$$

Suppose, however, that the particle is moving relativistically. Give reasons why one should conclude that $\mathcal{P}$ is a Lorentz invariant. Writing the 4-velocity as $U^a = (\gamma, \gamma\mathbf{v})$ where $\gamma = 1/\sqrt{1-|\mathbf{v}|^2}$ and $c = 1$, show that

$$\dot{U}^a = (\gamma^3\alpha, \gamma^3\alpha\mathbf{v} + \gamma\dot{\mathbf{v}})$$

where $\alpha = \mathbf{v}\cdot\dot{\mathbf{v}}$ and $\dot{f} = df/ds$ where s is the particle's proper time. Show also that

$$\dot{U}^a\dot{U}_a = -\gamma^4\alpha^2 - \gamma^2|\dot{\mathbf{v}}|^2.$$

Deduce the relativistic version of Larmor's formula.

Suppose the particle moves in a circular orbit perpendicular to a uniform magnetic field $\mathbf{B}$. Show that

$$\mathcal{P} = \frac{\mu_0}{6\pi}\frac{q^4}{m^2}(\gamma^2 - 1)|\mathbf{B}|^2,$$

where m is the mass of the particle, and comment briefly on the slow motion limit.

4/II/35E **Electrodynamics**

An action

$$S[\varphi] = \int d^4x\, L(\varphi, \varphi_{,a})$$

is given, where $\varphi(x)$ is a scalar field. Explain heuristically how to compute the functional derivative $\delta S/\delta\varphi$.

Consider the action for electromagnetism,

$$S[A_a] = -\int d^4x \left\{ \frac{1}{4\mu_0} F^{ab}F_{ab} + J^a A_a \right\} .$$

Here J^a is the 4-current density, A_a is the 4-potential and $F_{ab} = A_{b,a} - A_{a,b}$ is the Maxwell field tensor. Obtain Maxwell's equations in 4-vector form.

Another action that is sometimes suggested is

$$\widehat{S}[A_a] = -\int d^4x \left\{ \frac{1}{2\mu_0} A^{a,b}A_{a,b} + J^a A_a \right\} .$$

Under which additional assumption can Maxwell's equations be obtained using this action?

Using this additional assumption establish the relationship between the actions S and $\widehat{S}$.

1/II/34E **Electrodynamics**

$\mathcal{S}$ and $\mathcal{S}'$ are two reference frames with $\mathcal{S}'$ moving with constant speed v in the x-direction relative to $\mathcal{S}$. The co-ordinates x^a and x'^a are related by $dx'^a = L^a{}_b\, dx^b$ where

$$L^a{}_b = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\gamma = (1 - v^2)^{-1/2}$. What is the transformation rule for the scalar potential φ and vector potential $\mathbf{A}$ between the two frames?

As seen in $\mathcal{S}'$ there is an infinite uniform stationary distribution of charge along the x-axis with uniform line density σ. Determine the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ both in $\mathcal{S}'$ and $\mathcal{S}$. Check your answer by verifying explicitly the invariance of the two quadratic Lorentz invariants.

Comment briefly on the limit $|v| \ll 1$.

3/II/35E **Electrodynamics**

A particle of rest mass m and charge q is moving along a trajectory $x^a(s)$, where s is the particle's proper time, in a given external electromagnetic field with 4-potential $A^a(x^c)$. Consider the action principle $\delta S = 0$ where the action is $S = \int L\, ds$ and

$$L(s, x^a, \dot{x}^a) = -m\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b} - qA_a(x^c)\dot{x}^a,$$

and variations are taken with fixed endpoints.

Show first that the action is invariant both under reparametrizations $s \to \alpha s + \beta$ where α and β are constants and also under a change of electromagnetic gauge. Next define the generalized momentum $P_a = \partial L/\partial \dot{x}^a$, and obtain the equation of motion

$$m\ddot{x}^a = qF^a{}_b\dot{x}^b, \qquad (*)$$

where the tensor $F^a{}_b$ should be defined and you may assume that $d/ds\ (\eta_{ab}\dot{x}^a\dot{x}^b) = 0$. Then verify from $(*)$ that indeed $d/ds\ (\eta_{ab}\dot{x}^a\dot{x}^b) = 0$.

How does P_a differ from the momentum p_a of an uncharged particle? Comment briefly on the principle of minimal coupling.

4/II/35E **Electrodynamics**

The retarded scalar potential produced by a charge distribution $\rho(t', \mathbf{x}')$ is

$$\varphi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \; \frac{\rho(t - R, \mathbf{x}')}{R},$$

where $R = |\mathbf{R}|$ and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. By use of an appropriate delta function rewrite the integral as an integral over both d^3x' and dt' involving $\rho(t', \mathbf{x}')$.

Now specialize to a point charge q moving on a path $\mathbf{x}' = \mathbf{x}_0(t')$ so that we may set

$$\rho(t', \mathbf{x}') = q\, \delta^{(3)}(\mathbf{x}' - \mathbf{x}_0(t')).$$

By performing the volume integral first obtain the Liénard–Wiechert potential

$$\varphi(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{1}{(R^* - \mathbf{v} \cdot \mathbf{R}^*)} \; ,$$

where $\mathbf{R}^*$ and $\mathbf{v}$ should be specified.

Obtain the corresponding result for the magnetic potential.

1/II/34B **Electrodynamics**

In a frame $\mathcal{F}$ the electromagnetic fields $(\mathbf{E}, \mathbf{B})$ are encoded into the Maxwell field 4-tensor F^{ab} and its dual ${}^*F^{ab}$, where

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and

$${}^*F^{ab} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} .$$

[*Here the signature is* $(+ \; - \; - \; -)$ *and units are chosen so that* $c = 1$.] Obtain two independent Lorentz scalars of the electromagnetic field in terms of $\mathbf{E}$ and $\mathbf{B}$.

Suppose that $\mathbf{E} \cdot \mathbf{B} > 0$ in the frame $\mathcal{F}$. Given that there exists a frame $\mathcal{F}'$ in which $\mathbf{E}' \times \mathbf{B}' = \mathbf{0}$, show that

$$E' = \left[(\mathbf{E} \cdot \mathbf{B}) \big(K + \sqrt{1 + K^2} \big) \right]^{1/2}, \qquad B' = \left[\frac{\mathbf{E} \cdot \mathbf{B}}{K + \sqrt{1 + K^2}} \right]^{1/2},$$

where (E', B') are the magnitudes of $(\mathbf{E}', \mathbf{B}')$, and

$$K = \frac{1}{2} \big(|\mathbf{E}|^2 - |\mathbf{B}|^2 \big) / \big(\mathbf{E} \cdot \mathbf{B} \big) .$$

[*Hint: there is no need to consider the Lorentz transformations for* $\mathbf{E}'$ *and* $\mathbf{B}'$.]

3/II/35B **Electrodynamics**

A non-relativistic particle of rest mass m and charge q is moving slowly with velocity $\mathbf{v}(t)$. The power $dP/d\Omega$ radiated per unit solid angle in the direction of a unit vector $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2}\,|\mathbf{n} \times q\dot{\mathbf{v}}|^2\,.$$

Obtain Larmor's formula

$$P = \frac{\mu_0\, q^2}{6\pi}\,|\dot{\mathbf{v}}|^2\,.$$

The particle has energy $\mathcal{E}$ and, starting from afar, makes a head-on collision with a fixed central force described by a potential $V(r)$, where $V(r) > \mathcal{E}$ for $r < r_0$ and $V(r) < \mathcal{E}$ for $r > r_0$. Let W be the total energy radiated by the particle. Given that $W \ll \mathcal{E}$, show that

$$W \approx \frac{\mu_0\, q^2}{3\pi m^2}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\left(\frac{dV}{dr}\right)^2 \frac{dr}{\sqrt{V(r_0) - V(r)}}\,.$$

4/II/35B **Electrodynamics**

In Ginzburg–Landau theory, superconductivity is due to "supercarriers" of mass m_s and charge q_s, which are described by a macroscopic wavefunction ψ with "Mexican hat" potential

$$V = \alpha(T)|\psi|^2 + \frac{1}{2}\beta|\psi|^4 .$$

Here, $\beta > 0$ is constant and $\alpha(T)$ is a function of temperature T such that $\alpha(T) > 0$ for $T > T_c$ but $\alpha(T) < 0$ for $T < T_c$, where T_c is a critical temperature. In the presence of a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, the total energy of the superconducting system is

$$E[\psi, \psi^*, \mathbf{A}] = \int d^3x \left[\frac{1}{2\mu_0} A_{k,j} \big(A_{k,j} - A_{j,k} \big) + \frac{\hbar^2}{2m_s} \Big| \psi_{,k} + i\frac{q_s}{\hbar} A_k \psi \Big|^2 + V \right] .$$

Use this to derive the equations

$$-\frac{\hbar^2}{2m_s} \big(\boldsymbol{\nabla} - i\frac{q_s}{\hbar}\mathbf{A} \big)^2 \psi + \big(\alpha + \beta|\psi|^2 \big)\psi = 0 \qquad (*)$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} \equiv \boldsymbol{\nabla}\big(\boldsymbol{\nabla} \cdot \mathbf{A}\big) - \boldsymbol{\nabla}^2 \mathbf{A} = \mu_0 \, \mathbf{j} , \qquad (\dagger)$$

where

$$\begin{aligned} \mathbf{j} &= -\frac{iq_s\hbar}{2m_s} \big(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^* \big) - \frac{q_s^2}{m_s}|\psi|^2 \mathbf{A} \\ &= \frac{q_s}{2m_s} \left[\psi^* \left(-i\hbar\boldsymbol{\nabla} - q_s\mathbf{A} \right) \psi + \psi \left(i\hbar\boldsymbol{\nabla} - q_s\mathbf{A} \right) \psi^* \right] . \end{aligned}$$

Suppose that we write the wavefunction as

$$\psi = \sqrt{n_s}\, e^{i\theta} ,$$

where n_s is the (real positive) supercarrier density and θ is a real phase function. Given that

$$\left(\boldsymbol{\nabla} - \frac{iq_s}{\hbar}\,\mathbf{A} \right) \psi = 0 ,$$

show that n_s is constant and that $\hbar\boldsymbol{\nabla}\theta = q_s\mathbf{A}$. Given also that $T < T_c$, deduce that $(*)$ allows such solutions for a certain choice of n_s, which should be determined. Verify that your results imply $\mathbf{j} = \mathbf{0}$. Show also that $\mathbf{B} = \mathbf{0}$ and hence that $(\dagger)$ is solved.

Let $\mathcal{S}$ be a surface within the superconductor with closed boundary $\mathcal{C}$. Show that the magnetic flux through $\mathcal{S}$ is

$$\Phi \equiv \int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{dS} = \frac{\hbar}{q_s} \big[\theta\big]_{\mathcal{C}} .$$

Discuss, briefly, flux quantization.