Part II

Dynamical Systems

Paper 1, Section II

32A Dynamical Systems

(a) State and prove Dulac's theorem. State the Poincaré-Bendixson theorem.

(b) Consider the system

$$\dot{x} = r - x(1+s) + x^2 y \tag{1}$$
$$\dot{y} = sx - x^2 y, \tag{2}$$

where r and s are positive numbers. Show that there is a unique fixed point. Show that for a suitable choice of α to be determined, with $0 < \alpha < r$, trajectories enter the closed region bounded by $x = \alpha$, $y = s/\alpha$, $x + y = r + s/\alpha$ and $y = 0$. Deduce that when $s - 1 > r^2$ the system has a periodic orbit.

Paper 2, Section II

33A Dynamical Systems

(a) Define a Lyapunov function for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ with a fixed point $\mathbf{x}^*$. Explain what it means for a fixed point of the flow to be Lyapunov stable. State and prove Lyapunov's first stability theorem.

(b) Consider the second order differential equation

$$\ddot{x} = F(x) - \mu \dot{x},$$

where $\mu > 0$ and $F(x) = -2x(1-x^2)^2$. Show that there are three fixed points in the $(x, \dot{x})$ plane. Show that one of these is the origin and that it is Lyapunov stable. Show further that the origin is asymptotically stable, and that the ω-limit set of each point in the phase space is one of the three fixed points, justifying your answer carefully.

Paper 3, Section II

31A Dynamical Systems

Consider the dependence of the system

$$\dot{x} = (a^2 - x)(a - y^2) \tag{1}$$

$$\dot{y} = x - y \tag{2}$$

on the parameter a. Find the fixed points and plot their location on the (a, x)-plane. Hence, or deduce, that there are bifurcations at $a = 0$ and $a = a^* > 0$ which is to be determined.

Investigate the bifurcation at $a = 0$ by making the substitutions $X = x - a^2$ and $Y = y - a^2$. Find the extended centre manifold in the form $Y(X, a)$ correct to second order. Find the evolution on the extended centre manifold and hence determine the stability of the fixed points.

Use a plot to show which branches of the fixed points in the (a, x)-plane are stable and which are unstable and state, without calculation, the type of bifurcation at a^*. Hence sketch the structure of the (x, y) phase plane close to the bifurcation at a^* where $|a - a^*| \ll 1$ in the cases i) $a < a^*$ and ii) $a > a^*$.

Paper 4, Section II

32A Dynamical Systems

For the map $x_{n+1} = F(x_n, \lambda) := \lambda x_n(1 - x_n^2)$ with $\lambda > 0$ and $x_n \in [0, 1]$, show the following:

(i) There is an upper limit on λ if points are not to be mapped outside the domain $[0, 1]$. Find this value.

(ii) For $\lambda < 1$ the origin is the only fixed point and is stable.

(iii) If $\lambda > 1$, then the origin is unstable and a new fixed point x^* exists. This new fixed point x^* is stable for $1 < \lambda < 2$ and unstable for $\lambda > 2$.

(iv) For λ close to but larger than 2, and with $X_n = x_n - x^*$ and $0 < \mu = \lambda - 2 \ll 1$, the map can be locally represented as

$$X_{n+1} = -X_n + \alpha\mu X_n + \beta X_n^2 + \gamma X_n^3 + O(\mu^2), \tag{$*$}$$

where α, β and γ are constants that you should evaluate in terms of appropriate derivatives of F. Hence show that the 2-cycle born in the bifurcation at $\lambda = 2$ has points

$$x_{\pm} = x^* \pm \sqrt{\frac{-\alpha\mu}{\gamma + \beta^2}}.$$

[You do not need to substitute the expressions you found for α, β and γ into this formula.]

(v) The 2-cycle is stable for $\lambda > 2$, with $\lambda - 2$ small.

Paper 1, Section II
32B Dynamical Systems

(a) Consider a dynamical system of the form

$$\begin{aligned}\dot{x} &= f(x,y)\,,\\ \dot{y} &= g(x,y)+\epsilon p(x,y)\,,\end{aligned}$$

which is Hamiltonian for $\epsilon = 0$. Explain the *energy balance method.* What does it tell us about periodic orbits of this system for small ϵ?

(b) (i) For $0 < \epsilon \ll 1$, use the energy balance method to seek leading-order approximations to periodic orbits of this system

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= -4x+\epsilon\left[(1-2x^2)ky-(1-3x^2)y^3\right]\,,\end{aligned}$$

where $k > 0$.

[*Hint:* $\int_0^{2\pi}\sin^4\theta d\theta = \frac{3}{4}\pi$ *and* $\int_0^{2\pi}\sin^6\theta d\theta = \frac{5}{8}\pi$.]

(ii) For the cases $0 < k < 6$ and for $k > 6$, deduce the stability of any periodic orbits.

(iii) What can we deduce from this approach about the existence of periodic orbits near $k = 6$?

Paper 2, Section II

33B Dynamical Systems

(a) Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \in \mathbb{R}$. Define what it means for F to have a *horseshoe*.

Define what it means for F to be *chaotic*. [Glendinning's definition should be used throughout this question.]

Prove that if F has a 3-cycle then F^2 has a horseshoe. [You may assume corollaries of the Intermediate Value Theorem.]

(b) Suppose now that F has a 4-cycle, and consider each of these orderings of the points of the 4-cycle:

(i) $x_0 < x_1 < x_2 < x_3$

(ii) $x_0 < x_1 < x_3 < x_2$

(iii) $x_0 < x_2 < x_1 < x_3$

For each of these orderings, construct a suitable directed graph. Based on each of these directed graphs, determine if the corresponding F must be chaotic and also give the minimum number of distinct 3-cycles that F must have.

Give an explicit example of a continuous map $F : [0,1] \to [0,1]$ which has a 4-cycle and is not chaotic. [*Hint: choose a suitable ordering for the points on the 4-cycle, construct a function which is piece-wise linear between these points, and examine the dynamics of this map.*]

Paper 3, Section II
31B Dynamical Systems
Consider the system

$$\begin{aligned}\dot{x} &= -ax + 3y + x(x^2 + y^2)\\ \dot{y} &= -x - ay + y(x^2 + y^2)\,,\end{aligned}$$

where $a > 0$ is a real constant. Throughout this question, you should state carefully any theorems or standard results used.

(a) Show that the origin is asymptotically stable.

(b) Define the term *Lyapunov function.* For the system above, for what values of k is $V(x, y) = x^2 + ky^2$ a valid Lyapunov function in some neighbourhood of the origin? Give your answer in the form $k_1(a) < k < k_2(a)$ where $k_1(a)$ and $k_2(a)$ should be given explicitly.

(c) By considering $V(x, y)$ for $k = 1$, what can be deduced about the domain of stability (for values of a for which $V(x, y)$ is a valid Lyapunov function)?

(d) State the *Poincaré-Bendixson theorem.* Show that the system above has a periodic orbit.

Paper 4, Section II
32B Dynamical Systems
Consider the dynamical system

$$\begin{aligned}\dot{x} &= x(y - k - 3x + x^2)\\ \dot{y} &= y(y - 1 - x)\,,\end{aligned}$$

where k is a constant.

(a) Find all the fixed points of this system. By considering the existence and location of the fixed points, determine the values of k for which bifurcations occur. For each of these, what types of bifurcation are suggested from this approach?

(b) For the fixed points whose positions are independent of k, determine their linear stability. Verify that these results are consistent with the bifurcations suggested above.

(c) Focusing only on the bifurcations which occur for $0 \leqslant k \leqslant \frac{1}{2}$, use centre manifold theory to analyse these bifurcations. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation. [*Hint: There are two bifurcations in this range.*]

Paper 1, Section II
32A Dynamical Systems

(a) State the properties defining a *Lyapunov function* for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. State Lyapunov's first theorem and La Salle's invariance principle.

(b) Consider the system

$$\dot{x} = y\,,$$
$$\dot{y} = -\frac{2x(1-x^2)}{(1+x^2)^3} - ky\,.$$

Show that for $k > 0$ the origin is asymptotically stable, stating clearly any arguments that you use.

$$\left[\textit{Hint: } \frac{d}{dx}\frac{x^2}{(1+x^2)^2} = \frac{2x(1-x^2)}{(1+x^2)^3}\,.\right]$$

(c) Sketch the phase plane, (i) for $k = 0$ and (ii) for $0 < k \ll 1$, giving brief details of any reasoning and identifying the fixed points. Include the domain of stability of the origin in your sketch for case (ii).

(d) For $k > 0$ show that the trajectory $\mathbf{x}(t)$ with $\mathbf{x}(0) = (1, y_0)$, where $y_0 > 0$, satisfies $0 < y(t) < \sqrt{y_0^2 + \frac{1}{2}}$ for $t > 0$. Show also that, for any $\epsilon > 0$, the trajectory cannot remain outside the region $0 < y < \epsilon$.

Paper 2, Section II
33A Dynamical Systems

Consider a modified van der Pol system defined by

$$\dot{x} = y - \mu(\tfrac{1}{3}x^3 - x),$$
$$\dot{y} = -x + F,$$

where $\mu > 0$ and F are constants.

(a) A parallelogram $PQRS$ of width $2L$ is defined by

$$P = \big(L,\, \mu f(L)\big)\,, \qquad Q = \big(L,\, 2L - \mu f(L)\big)\,,$$
$$R = \big(-L,\, -\mu f(L)\big)\,, \quad S = \big(-L,\, \mu f(L) - 2L\big),$$

where $f(L) = \frac{1}{3}L^3 - L$. Show that if L is sufficiently large then trajectories never leave the region inside the parallelogram.

Hence show that if $F^2 < 1$ there must be a periodic orbit. Explain your reasoning carefully.

(b) Use the energy-balance method to analyse the behaviour of the system for $\mu \ll 1$, identifying the difference in behaviours between $F^2 < 1$ and $F^2 > 1$.

(c) Describe the behaviour of the system for $\mu \gg 1$, using sketches of the phase plane to illustrate your arguments for the cases $0 < F < 1$ and $F > 1$.

Paper 3, Section II
31A Dynamical Systems
Consider the system

$$\dot{x} = \mu y + \beta xy + y^2,$$
$$\dot{y} = x - y - x^2,$$

where μ and β are constants with $\beta > 0$.

(a) Find the fixed points, and classify those on $y = 0$. State how the number of fixed points depends on μ and β. Hence, or otherwise, deduce the values of μ at which stationary bifurcations occur for fixed $\beta > 0$.

(b) Sketch bifurcation diagrams in the (μ, x)-plane for the cases $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$, indicating the stability of the fixed points and the type of the bifurcations in each case. [You are not required to prove that the stabilities or bifurcation types are as you indicate.]

(c) For the case $\beta = 1$, analyse the bifurcation at $\mu = -1$ using extended centre manifold theory and verify that the evolution equation on the centre manifold matches the behaviour you deduced from the bifurcation diagram in part (b).

(d) For $0 < \mu + 1 \ll 1$, sketch the phase plane in the immediate neighbourhood of where the bifurcation of part (c) occurs.

Paper 4, Section II
32A Dynamical Systems
(a) A continuous map F of an interval into itself has a periodic orbit of period 3. Prove that F also has periodic orbits of period n for all positive integers n.

(b) What is the minimum number of distinct orbits of F of periods 2, 4 and 5? Explain your reasoning with a directed graph. [Formal proof is not required.]

(c) Consider the piecewise linear map $F : [0, 1] \to [0, 1]$ defined by linear segments between $F(0) = \frac{1}{2}$, $F(\frac{1}{2}) = 1$ and $F(1) = 0$. Calculate the orbits of periods 2, 4 and 5 that are obtained from the directed graph in part (b).

[*In part (a) you may assume without proof:*

(i) If U and V are non-empty closed bounded intervals such that $V \subseteq F(U)$ then there is a closed bounded interval $K \subseteq U$ such that $F(K) = V$.

(ii) The Intermediate Value Theorem.]

Paper 1, Section II

32E Dynamical Systems

(i) For the dynamical system

$$\dot{x} = -x(x^2 - 2\mu)(x^2 - \mu + a)\,, \qquad (\dagger)$$

sketch the bifurcation diagram in the (μ, x) plane for the three cases $a < 0$, $a = 0$ and $a > 0$. Describe the bifurcation points that occur in each case.

(ii) For the case when $a < 0$ only, confirm the types of bifurcation by finding the system to leading order near each of the bifurcations.

(iii) Explore the structural stability of these bifurcations by adding a small positive constant ϵ to the right-hand side of $(\dagger)$ and by sketching the bifurcation diagrams, for the three cases $a < 0$, $a = 0$ and $a > 0$. Which of the original bifurcations are structurally stable?

Paper 2, Section II

32E Dynamical Systems

(a) State and prove Dulac's criterion. State clearly the Poincaré–Bendixson theorem.

(b) For $(x, y) \in \mathbb{R}^2$ and $k > 0$, consider the dynamical system

$$\begin{aligned} \dot{x} &= kx - 5y - (3x + y)(5x^2 - 6xy + 5y^2)\,, \\ \dot{y} &= 5x + (k - 6)y - (x + 3y)(5x^2 - 6xy + 5y^2). \end{aligned}$$

(i) Use Dulac's criterion to find a range of k for which this system does not have any periodic orbit.

(ii) Find a suitable $f(k) > 0$ such that trajectories enter the disc $x^2 + y^2 \leqslant f(k)$ and do not leave it.

(iii) Given that the system has no fixed points apart from the origin for $k < 10$, give a range of k for which there will exist at least one periodic orbit.

Paper 3, Section II

31E Dynamical Systems

(a) A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a fixed point at the origin. Define the terms *asymptotic stability*, *Lyapunov function* and *domain of stability* of the fixed point $\mathbf{x} = \mathbf{0}$. State and prove Lyapunov's first theorem and state (without proof) La Salle's invariance principle.

(b) Consider the system

$$\begin{aligned} \dot{x} &= -2x + x^3 + \sin(2y)\,, \\ \dot{y} &= -x - y^3. \end{aligned}$$

(i) Show that trajectories cannot leave the square $S = \{(x,y) : |x| < 1, |y| < 1\}$. Show also that there are no fixed points in S other than the origin. Is this enough to deduce that S is in the domain of stability of the origin?

(ii) Construct a Lyapunov function of the form $V = x^2/2 + g(y)$. Deduce that the origin is asymptotically stable.

(iii) Find the largest rectangle of the form $|x| < x_0, |y| < y_0$ on which V is a strict Lyapunov function. Is this enough to deduce that this region is in the domain of stability of the origin?

(iv) Purely from using the Lyapunov function V, what is the most that can be deduced about the domain of stability of the origin?

Paper 4, Section II

32E Dynamical Systems

(a) Let $F : I \to I$ be a continuous map defined on an interval $I \subset \mathbb{R}$. Define what it means (i) for F to have a *horseshoe* and (ii) for F to be *chaotic*. [Glendinning's definition should be used throughout this question.]

(b) Consider the map defined on the interval $[-1, 1]$ by

$$F(x) = 1 - \mu|x|$$

with $0 < \mu \leqslant 2$.

(i) Sketch $F(x)$ and $F^2(x)$ for a case when $0 < \mu < 1$ and a case when $1 < \mu < 2$.

(ii) Describe fully the long term dynamics for $0 < \mu < 1$. What happens for $\mu = 1$?

(iii) When does F have a horseshoe? When does F^2 have a horseshoe?

(iv) For what values of μ is the map F chaotic?

Paper 4, Section II
31E Dynamical Systems
Consider the dynamical system

$$\begin{aligned} \dot{x} &= x + y^2 - a\,, \\ \dot{y} &= y(4x - x^2 - a)\,, \end{aligned}$$

for $(x, y) \in \mathbb{R}^2$, $a \in \mathbb{R}$.

Find all fixed points of this system. Find the three different values of a at which bifurcations appear. For each such value give the location (x, y) of all bifurcations. For each of these, what types of bifurcation are suggested from this analysis?

Use centre manifold theory to analyse these bifurcations. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation.

Paper 3, Section II
31E Dynamical Systems
Consider a dynamical system of the form

$$\begin{aligned} \dot{x} &= x(1 - y + ax)\,, \\ \dot{y} &= ry(-1 + x - by)\,, \end{aligned}$$

on $\Lambda = \{(x, y) : x > 0 \text{ and } y > 0\}$, where a, b and r are real constants and $r > 0$.

(a) For $a = b = 0$, by considering a function of the form $V(x, y) = f(x) + g(y)$, show that all trajectories in Λ are either periodic orbits or a fixed point.

(b) Using the same V, show that no periodic orbits in Λ persist for small a and b if $ab < 0$.

[*Hint: for $a = b = 0$ on the periodic orbits with period T, show that $\int_0^T (1 - x)dt = 0$ and hence that $\int_0^T x(1 - x)dt = \int_0^T \left[-(1 - x)^2 + (1 - x)\right] dt < 0$.*]

(c) By considering Dulac's criterion with $\phi = 1/(xy)$, show that there are no periodic orbits in Λ if $ab < 0$.

(d) Purely by consideration of the existence of fixed points in Λ and their Poincaré indices, determine those (a, b) for which the possibility of periodic orbits can be excluded.

(e) Combining the results above, sketch the a-b plane showing where periodic orbits in Λ might still be possible.

Paper 2, Section II

31E Dynamical Systems

For a map $F : \Lambda \to \Lambda$ give the definitions of chaos according to (i) Devaney (D-chaos) and (ii) Glendinning (G-chaos).

Consider the dynamical system

$$F(x) = ax \pmod 1$$

on $\Lambda = [0, 1)$, for $a > 1$ (note that a is not necessarily an integer). For both definitions of chaos, show that this system is chaotic.

Paper 1, Section II

31E Dynamical Systems

For a dynamical system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, give the definition of the *alpha-limit set* $\alpha(\mathbf{x})$ and the *omega-limit set* $\omega(\mathbf{x})$ of a point $\mathbf{x}$.

Consider the dynamical system

$$\begin{aligned} \dot{x} &= x^2 - 1\,, \\ \dot{y} &= kxy\,, \end{aligned}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and k is a real constant. Answer the following for all values of k, taking care over boundary cases (both in k and in $\mathbf{x}$).

(i) What symmetries does this system have?

(ii) Find and classify the fixed points of this system.

(iii) Does this system have any periodic orbits?

(iv) Give $\alpha(\mathbf{x})$ and $\omega(\mathbf{x})$ (considering all $\mathbf{x} \in \mathbb{R}^2$).

(v) For $\mathbf{x}_0 = (0, y_0)$, give the orbit of $\mathbf{x}_0$ (considering all $y_0 \in \mathbb{R}$). You should give your answer in the form $y = y(x, y_0, k)$, and specify the range of x.

Paper 1, Section II
31E Dynamical Systems

Consider the system

$$\dot{x} = -2ax + 2xy\,, \qquad \dot{y} = 1 - x^2 - y^2\,,$$

where a is a constant.

(a) Find and classify the fixed points of the system. For $a = 0$ show that the linear classification of the non-hyperbolic fixed points is nonlinearly correct. For $a \neq 0$ show that there are no periodic orbits. [Standard results for periodic orbits may be quoted without proof.]

(b) Sketch the phase plane for the cases (i) $a = 0$, (ii) $a = \frac{1}{2}$, and (iii) $a = \frac{3}{2}$, showing any separatrices clearly.

(c) For what values of a do *stationary* bifurcations occur? Consider the bifurcation with $a > 0$. Let y_0, a_0 be the values of y, a at which the bifurcation occurs, and define $Y = y - y_0, \mu = a - a_0$. Assuming that $\mu = O(x^2)$, find the extended centre manifold $Y = Y(x, \mu)$ to leading order. Further, determine the evolution equation on the centre manifold to leading order. Hence identify the type of bifurcation.

Paper 4, Section II
32E Dynamical Systems

Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Define what it means (i) for F to have a *horseshoe* (ii) for F to be *chaotic*. [Glendinning's definition should be used throughout this question.]

Prove that if F has a 3-cycle $x_1 < x_2 < x_3$ then F is chaotic. [You may assume the intermediate value theorem and any corollaries of it.]

State Sharkovsky's theorem.

Use the above results to deduce that if F has an N-cycle, where N is any integer that is not a power of 2, then F is chaotic.

Explain briefly why if F is chaotic then F has N-cycles for many values of N that are not powers of 2. [You may assume that a map with a horseshoe acts on some set Λ like the Bernoulli shift map acts on [0,1).]

The logistic map is not chaotic when $\mu < \mu_\infty \approx 3.57$ and it has 3-cycles when $\mu > 1 + \sqrt{8} \approx 3.84$. What can be deduced from these statements about the values of μ for which the logistic map has a 10-cycle?

Paper 3, Section II

32E Dynamical Systems

Consider the system

$$\dot{x} = y\,, \qquad \dot{y} = \mu_1 x + \mu_2 y - (x+y)^3\,,$$

where μ_1 and μ_2 are parameters.

By considering a function of the form $V(x,y) = f(x+y) + \frac{1}{2}y^2$, show that when $\mu_1 = \mu_2 = 0$ the origin is globally asymptotically stable. Sketch the phase plane for this case.

Find the fixed points for the general case. Find the values of μ_1 and μ_2 for which the fixed points have (i) a stationary bifurcation and (ii) oscillatory (Hopf) bifurcations. Sketch these bifurcation values in the (μ_1, μ_2)-plane.

For the case $\mu_2 = -1$, find the leading-order approximation to the extended centre manifold of the bifurcation as μ_1 varies, assuming that $\mu_1 = O(x^2)$. Find also the evolution equation on the extended centre manifold to leading order. Deduce the type of bifurcation, and sketch the bifurcation diagram in the (μ_1, x)-plane.

Paper 2, Section II

32E Dynamical Systems

Consider the system

$$\dot{x} = y, \qquad \dot{y} = x - x^3 + \epsilon(1 - \alpha x^2)y\,,$$

where α and ϵ are real constants, and $0 \leqslant \epsilon \ll 1$. Find and classify the fixed points.

Show that when $\epsilon = 0$ the system is Hamiltonian and find H. Sketch the phase plane for this case.

Suppose now that $0 < \epsilon \ll 1$. Show that the small change in H following a trajectory of the perturbed system around an orbit $H = H_0$ of the unperturbed system is given to leading order by an equation of the form

$$\Delta H = \epsilon \int_{x_1}^{x_2} F(x; \alpha, H_0)\, dx\,,$$

where F should be found explicitly, and where x_1 and x_2 are the minimum and maximum values of x on the unperturbed orbit.

Use the energy-balance method to find the value of α, correct to leading order in ϵ, for which the system has a homoclinic orbit. [*Hint: The substitution $u = 1 - \frac{1}{2}x^2$ may prove useful.*]

Over what range of α would you expect there to be periodic solutions that enclose only one of the fixed points?

Paper 1, Section II

30A Dynamical Systems

Consider the dynamical system

$$\begin{aligned}\dot{x} &= -x + x^3 + \beta x y^2\,,\\ \dot{y} &= -y + \beta x^2 y + y^3\,,\end{aligned}$$

where $\beta > -1$ is a constant.

(a) Find the fixed points of the system, and classify them for $\beta \neq 1$.

Sketch the phase plane for each of the cases (i) $\beta = \frac{1}{2}$ (ii) $\beta = 2$ and (iii) $\beta = 1$.

(b) Given $\beta > 2$, show that the domain of stability of the origin includes the union over $k \in \mathbb{R}$ of the regions

$$x^2 + k^2 y^2 < \frac{4k^2(1+k^2)(\beta-1)}{\beta^2(1+k^2)^2 - 4k^2}\,.$$

By considering $k \gg 1$, or otherwise, show that more information is obtained from the union over k than considering only the case $k = 1$.

$$\left[\textit{Hint: If } B > A, C \textit{ then } \max_{u\in[0,1]} \left\{ Au^2 + 2Bu(1-u) + C(1-u)^2 \right\} = \frac{B^2 - AC}{2B - A - C}\,.\right]$$

Paper 2, Section II
30A Dynamical Systems

(a) State Liapunov's first theorem and La Salle's invariance principle. Use these results to show that the fixed point at the origin of the system

$$\ddot{x} + k\dot{x} + \sin^3 x = 0\,, \quad k > 0,$$

is asymptotically stable.

(b) State the Poincaré–Bendixson theorem. Show that the forced damped pendulum

$$\dot{\theta} = p, \qquad \dot{p} = -kp - \sin\theta + F\,, \quad k > 0, \tag{$*$}$$

with $F > 1$, has a periodic orbit that encircles the cylindrical phase space $(\theta, p) \in \mathbb{R}[\text{mod } 2\pi] \times \mathbb{R}$, and that it is unique.

[You may assume that the Poincaré–Bendixson theorem also holds on a cylinder, and comment, without proof, on the use of any other standard results.]

(c) Now consider $(*)$ for $F, k = O(\epsilon)$, where $\epsilon \ll 1$. Use the energy-balance method to show that there is a homoclinic orbit in $p \geqslant 0$ if $F = F_h(k)$, where $F_h \approx 4k/\pi > 0$.

Explain briefly why there is no homoclinic orbit in $p \leqslant 0$ for $F > 0$.

Paper 3, Section II

30A Dynamical Systems

State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3\,, \\ \dot{y} &= rx - y - yz\,, \\ \dot{z} &= xy - z\,, \end{aligned}$$

where $a \neq 1$ is a constant, is nonhyperbolic at $r = 1$. What are the dimensions of the linear stable and (non-extended) centre subspaces at this point?

Make the substitutions $2u = x + y$, $2v = x - y$ and $\mu = r - 1$ and derive the resultant equations for $\dot{u}, \dot{v}$ and $\dot{z}$.

The extended centre manifold is given by

$$v = V(u, \mu), \qquad z = Z(u, \mu)\,,$$

where V and Z can be expanded as power series about $u = \mu = 0$. What is known about V and Z from the centre manifold theorem? Assuming that $\mu = O(u^2)$, determine Z to $O(u^2)$ and V to $O(u^3)$. Hence obtain the evolution equation on the centre manifold correct to $O(u^3)$, and identify the type of bifurcation distinguishing between the cases $a > 1$ and $a < 1$.

If now $a = 1$, assume that $\mu = O(u^4)$ and extend your calculations of Z to $O(u^4)$ and of the dynamics on the centre manifold to $O(u^5)$. Hence sketch the bifurcation diagram in the neighbourhood of $u = \mu = 0$.

Paper 4, Section II

31A Dynamical Systems

Consider the one-dimensional map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$x_{i+1} = F(x_i; \mu) = x_i(ax_i^2 + bx_i + \mu),$$

where a and b are constants, μ is a parameter and $a \neq 0$.

(a) Find the fixed points of F and determine the linear stability of $x = 0$. Hence show that there are bifurcations at $\mu = 1$, at $\mu = -1$ and, if $b \neq 0$, at $\mu = 1 + b^2/(4a)$.

Sketch the bifurcation diagram for each of the cases:

$$\text{(i) } a > b = 0, \quad \text{(ii) } a, b > 0 \quad \text{and} \quad \text{(iii) } a, b < 0.$$

In each case show the locus and stability of the fixed points in the (μ, x)-plane, and state the type of each bifurcation. [Assume that there are no further bifurcations in the region sketched.]

(b) For the case $F(x) = x(\mu - x^2)$ (i.e. $a = -1$, $b = 0$), you may assume that

$$F^2(x) = x + x(\mu - 1 - x^2)(\mu + 1 - x^2)(1 - \mu x^2 + x^4).$$

Show that there are at most three 2-cycles and determine when they exist. By considering $F'(x_i)F'(x_{i+1})$, or otherwise, show further that one 2-cycle is always unstable when it exists and that the others are unstable when $\mu > \sqrt{5}$. Sketch the bifurcation diagram showing the locus and stability of the fixed points and 2-cycles. State briefly what you would expect to occur for $\mu > \sqrt{5}$.

Paper 3, Section II

29E Dynamical Systems

Consider the dependence of the system

$$\begin{aligned}\dot{x} &= (a - x^2)(a^2 - y)\,,\\ \dot{y} &= x - y\end{aligned}$$

on the parameter a. Find the fixed points and plot their location in the (a, x)-plane. Hence, or otherwise, deduce that there are bifurcations at $a = 0$ and $a = 1$.

Investigate the bifurcation at $a = 1$ by making the substitutions $u = x - 1$, $v = y - 1$ and $\mu = a - 1$. Find the extended centre manifold in the form $v(u, \mu)$ correct to second order. Find the evolution equation on the extended centre manifold to second order, and determine the stability of its fixed points.

Use a plot to show which branches of fixed points in the (a, x)-plane are stable and which are unstable, and state, without calculation, the type of bifurcation at $a = 0$. Hence sketch the structure of the (x, y) phase plane very close to the origin for $|a| \ll 1$ in the cases (i) $a < 0$ and (ii) $a > 0$.

Paper 1, Section II

29E Dynamical Systems

Consider the dynamical system

$$\begin{aligned}\dot{x} &= x(y - a)\,,\\ \dot{y} &= 1 - x - y^2\,,\end{aligned}$$

where $-1 < a < 1$. Find and classify the fixed points of the system.

Use Dulac's criterion with a weighting function of the form $\phi = x^p$ and a suitable choice of p to show that there are no periodic orbits for $a \neq 0$. For the case $a = 0$ use the same weighting function to find a function $V(x, y)$ which is constant on trajectories. [*Hint: $\phi\dot{\mathbf{x}}$ is Hamiltonian.*]

Calculate the stable manifold at $(0, -1)$ correct to quadratic order in x.

Sketch the phase plane for the cases (i) $a = 0$ and (ii) $a = \frac{1}{2}$.

Paper 4, Section II

30E Dynamical Systems

Consider the map defined on $\mathbb{R}$ by

$$F(x) = \begin{cases} 3x & x \leqslant \frac{1}{2} \\ 3(1-x) & x \geqslant \frac{1}{2} \end{cases}$$

and let I be the open interval $(0,1)$. Explain what it means for F to have a *horseshoe* on I by identifying the relevant intervals in the definition.

Let $\Lambda = \{x : F^n(x) \in I, \forall n \geqslant 0\}$. Show that $F(\Lambda) = \Lambda$.

Find the sets $\Lambda_1 = \{x : F(x) \in I\}$ and $\Lambda_2 = \{x : F^2(x) \in I\}$.

Consider the ternary (base-3) representation $x = 0 \cdot x_1 x_2 x_3 \dots$ of numbers in I. Show that

$$F(0 \cdot x_1 x_2 x_3 \dots) = \begin{cases} x_1 \cdot x_2 x_3 x_4 \dots & x \leqslant \frac{1}{2} \\ \sigma(x_1) \cdot \sigma(x_2)\sigma(x_3)\sigma(x_4) \dots & x \geqslant \frac{1}{2} \end{cases},$$

where the function $\sigma(x_i)$ of the ternary digits should be identified. What is the ternary representation of the non-zero fixed point? What do the ternary representations of elements of Λ have in common?

Show that F has sensitive dependence on initial conditions on Λ, that F is topologically transitive on Λ, and that periodic points are dense in Λ. [*Hint: You may assume that* $F^n(0 \cdot x_1 \dots x_{n-1} 0 x_{n+1} x_{n+2} \dots) = 0 \cdot x_{n+1} x_{n+2} \dots$ *for* $x \in \Lambda$.]

Briefly state the relevance of this example to the relationship between Glendinning's and Devaney's definitions of chaos.

Paper 2, Section II

30E Dynamical Systems

Consider the nonlinear oscillator

$$\begin{aligned}\dot{x} &= y - \mu x(\tfrac{1}{2}|x| - 1)\,,\\ \dot{y} &= -x\,.\end{aligned}$$

(a) Use the Hamiltonian for $\mu = 0$ to find a constraint on the size of the domain of stability of the origin when $\mu < 0$.

(b) Assume that given $\mu > 0$ there exists an R such that all trajectories eventually remain within the region $|\mathbf{x}| \leqslant R$. Show that there must be a limit cycle, stating carefully any result that you use. [You need not show that there is only one periodic orbit.]

(c) Use the energy-balance method to find the approximate amplitude of the limit cycle for $0 < \mu \ll 1$.

(d) Find the approximate shape of the limit cycle for $\mu \gg 1$, and calculate the leading-order approximation to its period.

Paper 4, Section II
28B Dynamical Systems

Let $f : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Explain what is meant by the statements (i) that f has a *horseshoe* and (ii) that f is *chaotic* (according to Glendinning's definition).

Assume that f has a 3-cycle $\{x_0, x_1, x_2\}$ with $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_0 = f(x_2)$ and, without loss of generality, $x_0 < x_1 < x_2$. Prove that f^2 has a horseshoe. [You may assume the intermediate value theorem.]

Represent the effect of f on the intervals $I_a = [x_0, x_1]$ and $I_b = [x_1, x_2]$ by means of a directed graph, explaining carefully how the graph is constructed. Explain what feature of the graph implies the existence of a 3-cycle.

The map $g : I \to I$ has a 5-cycle $\{x_0, x_1, x_2, x_3, x_4\}$ with $x_{i+1} = g(x_i)$, $0 \leqslant i \leqslant 3$ and $x_0 = g(x_4)$, and $x_0 < x_1 < x_2 < x_3 < x_4$. For which n, $1 \leqslant n \leqslant 4$, is an n-cycle of g guaranteed to exist? Is g guaranteed to be chaotic? Is g guaranteed to have a horseshoe? Justify your answers. [You may use a suitable directed graph as part of your arguments.]

How do your answers to the above change if instead $x_4 < x_2 < x_1 < x_3 < x_0$?

Paper 3, Section II
28B Dynamical Systems

Consider the dynamical system

$$\begin{aligned}\dot{x} &= -\mu + x^2 - y,\\ \dot{y} &= y(a - x),\end{aligned}$$

where a is to be regarded as a fixed real constant and μ as a real parameter.

Find the fixed points of the system and determine the stability of the system linearized about the fixed points. Hence identify the values of μ at given a where bifurcations occur.

Describe informally the concepts of centre manifold theory and apply it to analyse the bifurcations that occur in the above system with $a = 1$. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation.

What can you say, without further detailed calculation, about the case $a = 0$?

Paper 2, Section II

28B Dynamical Systems

(a) An autonomous dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ has a periodic orbit $\mathbf{x} = \mathbf{X}(t)$ with period T. The linearized evolution of a small perturbation $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$ is given by $\eta_i(t) = \Phi_{ij}(t)\eta_j(0)$. Obtain the differential equation and initial condition satisfied by the matrix $\boldsymbol{\Phi}(t)$.

Define the *Floquet multipliers* of the orbit. Explain why one of the multipliers is always unity and give a brief argument to show that the other is given by

$$\exp\left(\int_0^T \nabla \cdot \mathbf{f}(\mathbf{X}(t))\, dt\right).$$

(b) Use the *energy-balance method* for nearly Hamiltonian systems to find leading-order approximations to the two limit cycles of the equation

$$\ddot{x} + \epsilon(2\dot{x}^3 + 2x^3 - 4x^4\dot{x} - \dot{x}) + x = 0,$$

where $0 < \epsilon \ll 1$.

Determine the stability of each limit cycle, giving reasoning where necessary.

[You may assume that $\int_0^{2\pi} \cos^4\theta\, d\theta = 3\pi/4$ and $\int_0^{2\pi} \cos^6\theta\, d\theta = 5\pi/8$.]

Paper 1, Section II

28B Dynamical Systems

(a) What is a Lyapunov function?

Consider the dynamical system for $\mathbf{x}(t) = \big(x(t), y(t)\big)$ given by

$$\begin{aligned}\dot{x} &= -x + y + x(x^2 + y^2)\,,\\ \dot{y} &= -y - 2x + y(x^2 + y^2)\,.\end{aligned}$$

Prove that the origin is asymptotically stable (quoting carefully any standard results that you use).

Show that the domain of attraction of the origin includes the region $x^2 + y^2 < r_1^2$ where the maximum possible value of r_1 is to be determined.

Show also that there is a region $E = \{\mathbf{x} \,|\, x^2 + y^2 > r_2^2\}$ such that $\mathbf{x}(0) \in E$ implies that $|\mathbf{x}(t)|$ increases without bound. Explain your reasoning carefully. Find the smallest possible value of r_2.

(b) Now consider the dynamical system

$$\begin{aligned}\dot{x} &= x - y - x(x^2 + y^2)\,,\\ \dot{y} &= y + 2x - y(x^2 + y^2)\,.\end{aligned}$$

Prove that this system has a periodic solution (again, quoting carefully any standard results that you use).

Demonstrate that this periodic solution is unique.

Paper 4, Section I

7D Dynamical Systems

Consider the map $x_{n+1} = \lambda x_n(1 - x_n^2)$ for $-1 \leqslant x_n \leqslant 1$. What is the maximum value, λ_{max}, for which the interval $[-1, 1]$ is mapped into itself?

Analyse the first two bifurcations that occur as λ increases from 0 towards λ_{max}, including an identification of the values of λ at which the bifurcation occurs and the type of bifurcation.

What type of bifurcation do you expect as the third bifurcation? Briefly give your reasoning.

Paper 3, Section I

7D Dynamical Systems

Define the *Poincaré index* of a closed curve $\mathcal{C}$ for a vector field $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$.

Explain carefully why the index of $\mathcal{C}$ is fully determined by the fixed points of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ that lie within $\mathcal{C}$.

What is the Poincaré index for a closed curve $\mathcal{C}$ if it (a) encloses only a saddle point, (b) encloses only a focus and (c) encloses only a node?

What is the Poincaré index for a closed curve $\mathcal{C}$ that is a periodic trajectory of the dynamical system?

A dynamical system in $\mathbb{R}^2$ has 2 saddle points, 1 focus and 1 node. What is the maximum number of different periodic orbits? [For the purposes of this question, two orbits are said to be different if they enclose different sets of fixed points.]

Paper 2, Section I

7D Dynamical Systems

Consider the system

$$\begin{aligned}\dot{x} &= -x + y + y^2,\\ \dot{y} &= \mu - xy\,.\end{aligned}$$

Show that when $\mu = 0$ the fixed point at the origin has a stationary bifurcation.

Find the centre subspace of the extended system linearised about $(x, y, \mu) = (0, 0, 0)$.

Find an approximation to the centre manifold giving y as a function of x and μ, including terms up to quadratic order.

Hence deduce an expression for $\dot{x}$ on the centre manifold, and identify the type of bifurcation at $\mu = 0$.

Paper 1, Section I

7D Dynamical Systems

Consider the system

$$\begin{aligned}\dot{x} &= y + xy\,,\\ \dot{y} &= x - \tfrac{3}{2}y + x^2.\end{aligned}$$

Show that the origin is a hyperbolic fixed point and find the stable and unstable invariant subspaces of the linearised system.

Calculate the stable and unstable manifolds correct to quadratic order, expressing y as a function of x for each.

Paper 4, Section II

14D Dynamical Systems

A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a fixed point at the origin. Define the terms *Lyapunov stability*, *asymptotic stability* and *Lyapunov function* with respect to this fixed point. State and prove Lyapunov's first theorem and state (without proof) La Salle's invariance principle.

(a) Consider the system

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= -y - x^3 + x^5.\end{aligned}$$

Construct a Lyapunov function of the form $V = f(x) + g(y)$. Deduce that the origin is asymptotically stable, explaining your reasoning carefully. Find the greatest value of y_0 such that use of this Lyapunov function guarantees that the trajectory through $(0, y_0)$ approaches the origin as $t \to \infty$.

(b) Consider the system

$$\begin{aligned}\dot{x} &= x + 4y + x^2 + 2y^2,\\ \dot{y} &= -3x - 3y\,.\end{aligned}$$

Show that the origin is asymptotically stable and that the basin of attraction of the origin includes the region $x^2 + xy + y^2 < \frac{1}{4}$.

Paper 3, Section II

14D Dynamical Systems

Let $f : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Explain what is meant by saying that f has a *horseshoe*.

A map g on the interval $[a, b]$ is a *tent map* if

(i) $g(a) = a$ and $g(b) = a$;

(ii) for some c with $a < c < b$, g is linear and increasing on the interval $[a, c]$, linear and decreasing on the interval $[c, b]$, and continuous at c.

Consider the tent map defined on the interval $[0, 1]$ by

$$f(x) = \begin{cases} \mu x & 0 \leqslant x \leqslant \frac{1}{2} \\ \mu(1-x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

with $1 < \mu \leqslant 2$. Find the corresponding expressions for $f^2(x) = f(f(x))$.

Find the non-zero fixed point x_0 and the points $x_{-1} < \frac{1}{2} < x_{-2}$ that satisfy

$$f^2(x_{-2}) = f(x_{-1}) = x_0 = f(x_0)\,.$$

Sketch graphs of f and f^2 showing the points corresponding to x_{-2}, x_{-1} and x_0. Indicate the values of f and f^2 at their maxima and minima and also the gradients of each piece of their graphs.

Identify a subinterval of $[0, 1]$ on which f^2 is a tent map. Hence demonstrate that f^2 has a horseshoe if $\mu \geqslant 2^{1/2}$.

Explain briefly why f^4 has a horseshoe when $\mu \geqslant 2^{1/4}$.

Why are there periodic points of f arbitrarily close to x_0 for $\mu \geqslant 2^{1/2}$, but no such points for $2^{1/4} \leqslant \mu < 2^{1/2}$? Explain carefully any results or terms that you use.

Paper 4, Section I

7C Dynamical Systems

Consider the system

$$\begin{aligned} \dot{x} &= y + ax + bx^3 \,, \\ \dot{y} &= -x \,. \end{aligned}$$

What is the Poincaré index of the single fixed point? If there is a closed orbit, why must it enclose the origin?

By writing $\dot{x} = \partial H/\partial y + g(x)$ and $\dot{y} = -\partial H/\partial x$ for suitable functions $H(x, y)$ and $g(x)$, show that if there is a closed orbit $\mathcal{C}$ then

$$\oint_{\mathcal{C}} (ax + bx^3)x \, dt = 0 \,.$$

Deduce that there is no closed orbit when $ab > 0$.

If $ab < 0$ and a and b are both $O(\epsilon)$, where ϵ is a small parameter, then there is a single closed orbit that is to within $O(\epsilon)$ a circle of radius R centred on the origin. Deduce a relation between a, b and R.

Paper 3, Section I

7C Dynamical Systems

A one-dimensional map is defined by

$$x_{n+1} = F(x_n, \mu) \,,$$

where μ is a parameter. What is the condition for a bifurcation of a fixed point x_* of F?

Let $F(x, \mu) = x(x^2 - 2x + \mu)$. Find the fixed points and show that bifurcations occur when $\mu = -1$, $\mu = 1$ and $\mu = 2$. Sketch the bifurcation diagram, showing the locus and stability of the fixed points in the (x, μ) plane and indicating the type of each bifurcation.

Paper 2, Section I

7C Dynamical Systems

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a two-dimensional dynamical system with a fixed point at $\mathbf{x} = \mathbf{0}$. Define a Lyapunov function $V(\mathbf{x})$ and explain what it means for $\mathbf{x} = \mathbf{0}$ to be Lyapunov stable.

For the system

$$\begin{aligned} \dot{x} &= -x - 2y + x^3 \,, \\ \dot{y} &= -y + x + \tfrac{1}{2}y^3 + x^2 y \,, \end{aligned}$$

determine the values of C for which $V = x^2 + Cy^2$ is a Lyapunov function in a sufficiently small neighbourhood of the origin.

For the case $C = 2$, find V_1 and V_2 such that $V(\mathbf{x}) < V_1$ at $t = 0$ implies that $V \to 0$ as $t \to \infty$ and $V(\mathbf{x}) > V_2$ at $t = 0$ implies that $V \to \infty$ as $t \to \infty$.

Paper 1, Section I

7C Dynamical Systems

Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$ which has a hyperbolic fixed point at the origin.

Define the stable and unstable invariant subspaces of the system linearised about the origin. Give a constraint on the dimensions of these two subspaces.

Define the local stable and unstable manifolds of the origin for the system. How are these related to the invariant subspaces of the linearised system?

For the system

$$\begin{aligned} \dot{x} &= -x + x^2 + y^2 \,, \\ \dot{y} &= y + y^2 - x^2 \,, \end{aligned}$$

calculate the stable and unstable manifolds of the origin, each correct up to and including cubic order.

Paper 3, Section II

14C Dynamical Systems

Let $f : I \to I$ be a continuous map of an interval $I \subset \mathbb{R}$. Explain what is meant by the statements (a) f has a *horseshoe* and (b) f is *chaotic* according to Glendinning's definition of chaos.

Assume that f has a 3-cycle $\{x_0, x_1, x_2\}$ with $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_0 = f(x_2)$, $x_0 < x_1 < x_2$. Prove that f^2 has a horseshoe. [You may assume the Intermediate Value Theorem.]

Represent the effect of f on the intervals $I_a = [x_0, x_1]$ and $I_b = [x_1, x_2]$ by means of a directed graph. Explain how the existence of the 3-cycle corresponds to this graph.

The map $g : I \to I$ has a 4-cycle $\{x_0, x_1, x_2, x_3\}$ with $x_1 = g(x_0)$, $x_2 = g(x_1)$, $x_3 = g(x_2)$ and $x_0 = g(x_3)$. If $x_0 < x_3 < x_2 < x_1$ is g necessarily chaotic? [You may use a suitable directed graph as part of your argument.]

How does your answer change if $x_0 < x_2 < x_1 < x_3$?

Paper 4, Section II

14C Dynamical Systems

Consider the dynamical system

$$\begin{aligned}\dot{x} &= (x+y+a)(x-y+a)\ ,\\ \dot{y} &= y-x^2-b\ ,\end{aligned}$$

where $a>0$.

Find the fixed points of the dynamical system. Show that for any fixed value of a there exist three values $b_1 > b_2 \geqslant b_3$ of b where a bifurcation occurs. Show that $b_2 = b_3$ when $a = 1/2$.

In the remainder of this question set $a = 1/2$.

(i) Being careful to explain your reasoning, show that the extended centre manifold for the bifurcation at $b = b_1$ can be written in the form $X = \alpha Y + \beta\mu + p(Y,\mu)$, where X and Y denote the departures from the values of x and y at the fixed point, $b = b_1 + \mu$, α and β are suitable constants (to be determined) and p is quadratic to leading order. Derive a suitable approximate form for p, and deduce the nature of the bifurcation and the stability of the different branches of the steady state solution near the bifurcation.

(ii) Repeat the calculations of part (i) for the bifurcation at $b = b_2$.

(iii) Sketch the x values of the fixed points as functions of b, indicating the nature of the bifurcations and where each branch is stable.

Paper 4, Section I

7D Dynamical Systems

Describe the different types of bifurcation from steady states of a one-dimensional map of the form $x_{n+1} = f(x_n)$, and give examples of simple equations exhibiting each type.

Consider the map $x_{n+1} = \alpha x_n^2(1 - x_n)$, $0 < x_n < 1$. What is the maximum value of α for which the interval is mapped into itself?

Show that as α increases from zero to its maximum value there is a saddle-node bifurcation and a period-doubling bifurcation, and determine the values of α for which they occur.

Paper 3, Section I

7D Dynamical Systems

State without proof Lyapunov's first theorem, carefully defining all the terms that you use.

Consider the dynamical system

$$\begin{aligned} \dot{x} &= -2x + y - xy + 3y^2 - xy^2 + x^3 \,, \\ \dot{y} &= -2y - x - y^2 - 3xy + 2x^2 y \,. \end{aligned}$$

By choosing a Lyapunov function $V(x, y) = x^2 + y^2$, prove that the origin is asymptotically stable.

By factorising the expression for $\dot{V}$, or otherwise, show that the basin of attraction of the origin includes the set $V < 7/4$.

Paper 2, Section I

7D Dynamical Systems

Consider the dynamical system

$$\dot{x} = \mu x + x^3 - axy, \quad \dot{y} = \mu - x^2 - y\,,$$

where a is a constant.

(a) Show that there is a bifurcation from the fixed point $(0, \mu)$ at $\mu = 0$.

(b) Find the extended centre manifold at leading non-trivial order in x. Hence find the type of bifurcation, paying particular attention to the special values $a = 1$ and $a = -1$. [*Hint. At leading order, the extended centre manifold is of the form* $y = \mu + \alpha x^2 + \beta \mu x^2 + \gamma x^4$, *where* α, β, γ *are constants to be determined.*]

Paper 1, Section I

7D Dynamical Systems

State the Poincaré–Bendixson theorem.

A model of a chemical process obeys the second-order system

$$\dot{x} = 1 - x(1 + a) + x^2 y, \quad \dot{y} = ax - x^2 y\,,$$

where $a > 0$. Show that there is a unique fixed point at $(x, y) = (1, a)$ and that it is unstable if $a > 2$. Show that trajectories enter the region bounded by the lines $x = 1/q$, $y = 0$, $y = aq$ and $x + y = 1 + aq$, provided $q > (1 + a)$. Deduce that there is a periodic orbit when $a > 2$.

Paper 4, Section II

14D Dynamical Systems

What is meant by the statement that a continuous map of an interval I into itself has a *horseshoe*? State without proof the properties of such a map.

Define the property of *chaos* of such a map according to Glendinning.

A continuous map $f : I \to I$ has a periodic orbit of period 5, in which the elements x_j, $j = 1, \ldots, 5$ satisfy $x_j < x_{j+1}$, $j = 1, \ldots, 4$ and the points are visited in the order $x_1 \to x_3 \to x_4 \to x_2 \to x_5 \to x_1$. Show that the map is chaotic. [The Intermediate Value theorem can be used without proof.]

Paper 3, Section II

14D Dynamical Systems

Consider the dynamical system

$$\ddot{x} - (a - bx)\dot{x} + x - x^2 = 0, \quad a, b > 0\,. \tag{1}$$

(a) Show that the fixed point at the origin is an unstable node or focus, and that the fixed point at $x = 1$ is a saddle point.

(b) By considering the phase plane $(x, \dot{x})$, or otherwise, show graphically that the maximum value of x for any periodic orbit is less than one.

(c) By writing the system in terms of the variables x and $z = \dot{x} - (ax - bx^2/2)$, or otherwise, show that for any periodic orbit $\mathcal{C}$

$$\oint_{\mathcal{C}} (x - x^2)(2ax - bx^2)\, dt = 0\,. \tag{2}$$

Deduce that if $a/b > 1/2$ there are no periodic orbits.

(d) If $a = b = 0$ the system (1) is Hamiltonian and has homoclinic orbit

$$X(t) = \frac{1}{2}\left(3\tanh^2\left(\frac{t}{2}\right) - 1\right)\,, \tag{3}$$

which approaches $X = 1$ as $t \to \pm\infty$. Now suppose that a, b are very small and that we seek the value of a/b corresponding to a periodic orbit very close to $X(t)$. By using equation (3) in equation (2), find an approximation to the largest value of a/b for a periodic orbit when a, b are very small.

[*Hint. You may use the fact that* $(1 - X) = \frac{3}{2}\operatorname{sech}^2(\frac{t}{2}) = 3\frac{d}{dt}(\tanh(\frac{t}{2}))$]

Paper 1, Section I

7C Dynamical Systems

Find the fixed points of the dynamical system (with $\mu \neq 0$)

$$\dot{x} = \mu^2 x - xy\,,$$
$$\dot{y} = -y + x^2\,,$$

and determine their type as a function of μ.

Find the stable and unstable manifolds of the origin correct to order 4.

Paper 2, Section I

7C Dynamical Systems

State the Poincaré–Bendixson theorem for two-dimensional dynamical systems.

A dynamical system can be written in polar coordinates (r, θ) as

$$\dot{r} = r - r^3(1 + \alpha \cos\theta)\,,$$
$$\dot{\theta} = 1 - r^2 \beta \cos\theta\,,$$

where α and β are constants with $0 < \alpha < 1$.

Show that trajectories enter the annulus $(1+\alpha)^{-1/2} < r < (1-\alpha)^{-1/2}$.

Show that if there is a fixed point (r_0, θ_0) inside the annulus then $r_0^2 = (\beta - \alpha)/\beta$ and $\cos\theta_0 = 1/(\beta - \alpha)$.

Use the Poincaré–Bendixson theorem to derive conditions on β that guarantee the existence of a periodic orbit.

Paper 3, Section I

7C Dynamical Systems

For the map $x_{n+1} = \lambda x_n(1 - x_n^2)$, with $\lambda > 0$, show the following:

(i) If $\lambda < 1$, then the origin is the only fixed point and is stable.

(ii) If $\lambda > 1$, then the origin is unstable. There are two further fixed points which are stable for $1 < \lambda < 2$ and unstable for $\lambda > 2$.

(iii) If $\lambda < 3\sqrt{3}/2$, then x_n has the same sign as the starting value x_0 if $|x_0| < 1$.

(iv) If $\lambda < 3$, then $|x_{n+1}| < 2\sqrt{3}/3$ when $|x_n| < 2\sqrt{3}/3$. Deduce that iterates starting sufficiently close to the origin remain bounded, though they may change sign.

[*Hint: For (iii) and (iv) a graphical representation may be helpful.*]

Paper 4, Section I

7C Dynamical Systems

(i) Explain the use of the energy balance method for describing approximately the behaviour of nearly Hamiltonian systems.

(ii) Consider the nearly Hamiltonian dynamical system

$$\ddot{x} + \epsilon\dot{x}(-1 + \alpha x^2 - \beta x^4) + x = 0\,, \qquad 0 < \epsilon \ll 1\,,$$

where α and β are positive constants. Show that, for sufficiently small ϵ, the system has periodic orbits if $\alpha^2 > 8\beta$, and no periodic orbits if $\alpha^2 < 8\beta$. Show that in the first case there are two periodic orbits, and determine their approximate size and their stability.

What can you say about the existence of periodic orbits when $\alpha^2 = 8\beta$?

[You may assume that

$$\int_0^{2\pi} \sin^2 t\, dt = \pi\,, \quad \int_0^{2\pi} \sin^2 t \cos^2 t\, dt = \frac{\pi}{4}\,, \quad \int_0^{2\pi} \sin^2 t \cos^4 t\, dt = \frac{\pi}{8}\,.\Big]$$

Paper 3, Section II
14C Dynamical Systems

Explain what is meant by a *steady-state bifurcation* of a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ in $\mathbb{R}^n$, where μ is a real parameter.

Consider the system in $x \geqslant 0$, $y \geqslant 0$, with $\mu > 0$,

$$\begin{aligned} \dot{x} &= x(1 - y^2 - x^2)\,, \\ \dot{y} &= y(\mu - y - x^2)\,. \end{aligned}$$

(i) Show that both the fixed point $(0, \mu)$ and the fixed point $(1, 0)$ have a steady-state bifurcation when $\mu = 1$.

(ii) By finding the first approximation to the extended centre manifold, construct the normal form near the bifurcation point $(1, 0)$ when μ is close to unity, and show that there is a transcritical bifurcation there. Explain why the symmetries of the equations mean that the bifurcation at $(0, 1)$ must be of pitchfork type.

(iii) Show that two fixed points with $x, y > 0$ exist in the range $1 < \mu < 5/4$. Show that the solution with $y < 1/2$ is stable. Identify the bifurcation that occurs at $\mu = 5/4$.

(iv) Draw a sketch of the values of y at the fixed points as functions of μ, indicating the bifurcation points and the regions where each branch is stable. [Detailed calculations are not required.]

Paper 4, Section II
14C Dynamical Systems

(i) State and prove Lyapunov's First Theorem, and state (without proof) La Salle's Invariance Principle. Show by example how the latter result can be used to prove asymptotic stability of a fixed point even when a strict Lyapunov function does not exist.

(ii) Consider the system

$$\begin{aligned} \dot{x} &= -x + 2y + x^3 + 2x^2y + 2xy^2 + 2y^3\,, \\ \dot{y} &= -y - x - 2x^3 + \frac{1}{2}x^2y - 3xy^2 + y^3\,. \end{aligned}$$

Show that the origin is asymptotically stable and that the basin of attraction of the origin includes the region $x^2 + 2y^2 < 2/3$.

Paper 1, Section I

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = -\mu x + y\,, \qquad \dot{y} = \frac{x^2}{1+x^2} - \nu y\,,$$

where $x(t)$ and $y(t)$ are non-negative, the parameters μ and ν are strictly positive and $\mu \neq \nu$. Sketch the nullclines in the x, y plane. Deduce that for $\mu < \mu_c$ (where μ_c is to be determined) there are three fixed points. Find them and determine their type.

Sketch the phase portrait for $\mu < \mu_c$ and identify, qualitatively on your sketch, the stable and unstable manifolds of the saddle point. What is the final outcome of this system?

Paper 2, Section I

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = \mu\left(\frac{1}{3}x^3 - x\right) + y\,, \qquad \dot{y} = -x\,,$$

where the parameter $\mu > 0$. Using Lyapunov's approach, discuss the stability of the fixed point and its domain of attraction. Relevant definitions or theorems that you use should be stated carefully, but proofs are not required.

Paper 3, Section I

7D Dynamical Systems

Let $I = [0,1)$. The sawtooth (Bernoulli shift) map $F : I \to I$ is defined by

$$F(x) = 2\,x \ [\bmod 1]\,.$$

Describe the effect of F using binary notation. Show that F is continuous on I except at $x = \frac{1}{2}$. Show also that F has N-periodic points for all $N \geqslant 2$. Are they stable?

Explain why F is chaotic, using Glendinning's definition.

Paper 4, Section I

7D Dynamical Systems

Consider the 2-dimensional flow

$$\dot{x} = y + \frac{1}{4}x\left(1 - 2x^2 - 2y^2\right), \qquad \dot{y} = -x + \frac{1}{2}y\left(1 - x^2 - y^2\right).$$

Use the Poincaré–Bendixson theorem, which should be stated carefully, to obtain a domain $\mathcal{D}$ in the xy-plane, within which there is at least one periodic orbit.

Paper 3, Section II

14D Dynamical Systems

Describe informally the concepts of extended stable manifold theory. Illustrate your discussion by considering the 2-dimensional flow

$$\dot{x} = \mu x + xy - x^3, \qquad \dot{y} = -y + y^2 - x^2,$$

where μ is a parameter with $|\mu| \ll 1$, in a neighbourhood of the origin. Determine the nature of the bifurcation.

Paper 4, Section II

14D Dynamical Systems

Let $I = [0,1]$ and consider continuous maps $F : I \to I$. Give an informal outline description of the two different bifurcations of fixed points of F that can occur.

Illustrate your discussion by considering in detail the logistic map

$$F(x) = \mu x(1-x),$$

for $\mu \in (0, 1+\sqrt{6}]$.

Describe qualitatively what happens for $\mu \in (1+\sqrt{6}, 4]$.

[You may assume without proof that

$$x - F^2(x) = x(\mu x - \mu + 1)(\mu^2 x^2 - \mu(\mu+1)x + \mu + 1). \qquad]$$

Paper 1, Section I

7E Dynamical Systems

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a two-dimensional dynamical system with a fixed point at $\mathbf{x} = \mathbf{0}$. Define a *Lyapunov function* $V(\mathbf{x})$ and explain what it means for $\mathbf{x} = \mathbf{0}$ to be Lyapunov stable.

Determine the values of β for which $V = x^2 + \beta y^2$ is a Lyapunov function in a sufficiently small neighbourhood of the origin for the system

$$\begin{aligned} \dot{x} &= -x + 2y + 2xy - x^2 - 4y^2, \\ \dot{y} &= -y + xy. \end{aligned}$$

What can be deduced about the basin of attraction of the origin using V when $\beta = 2$?

Paper 2, Section I

7E Dynamical Systems

For each of the one-dimensional systems

(i) $\dot{x} = \mu^2 - a^2 + 2ax^2 - x^4$,

(ii) $\dot{x} = x(\mu^2 - a^2 + 2ax^2 - x^4)$,

determine the location and stability of all the fixed points. For each system sketch bifurcation diagrams in the (μ, x) plane in each of the two cases $a > 0$ and $a < 0$. Identify and carefully describe all the bifurcation points that occur.

[Detailed calculations are not required, but bifurcation diagrams must be clearly labelled, and the locations of bifurcation points should be given.]

Paper 3, Section I

7E Dynamical Systems

Consider the one-dimensional real map $x_{n+1} = F(x_n) = rx_n^2(1 - x_n)$, where $r > 0$. Locate the fixed points and explain for what ranges of the parameter r each fixed point exists. For what range of r does F map the open interval $(0, 1)$ into itself?

Determine the location and type of all the bifurcations from the fixed points which occur. Sketch the location of the fixed points in the (r, x) plane, indicating stability.

Paper 4, Section I

7E Dynamical Systems

Consider the two-dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ given in polar coordinates by

$$\begin{aligned} \dot{r} &= (r - r^2)(r - g(\theta)), \\ \dot{\theta} &= r, \end{aligned} \qquad (*)$$

where $g(\theta)$ is continuously differentiable and 2π-periodic. Find a periodic orbit γ for $(*)$ and, using the hint or otherwise, compute the Floquet multipliers of γ in terms of $g(\theta)$. Explain why one of the Floquet multipliers is independent of $g(\theta)$. Give a sufficient condition for γ to be asymptotically stable.

Investigate the stability of γ and the dynamics of $(*)$ in the case $g(\theta) = 2\sin\theta$.

[*Hint: The determinant of the fundamental matrix* $\Phi(t)$ *satisfies*

$$\frac{d}{dt}\det\Phi \quad = \quad (\nabla \cdot \mathbf{f})\, \det\Phi.\ \Big]$$

Paper 3, Section II

14E Dynamical Systems

Consider the dynamical system

$$\begin{aligned} \dot{x} &= -ax - 2xy, \\ \dot{y} &= x^2 + y^2 - b, \end{aligned}$$

where $a \geqslant 0$ and $b > 0$.

(i) Find and classify the fixed points. Show that a bifurcation occurs when $4b = a^2 > 0$.

(ii) After shifting coordinates to move the relevant fixed point to the origin, and setting $a = 2\sqrt{b} - \mu$, carry out an extended centre manifold calculation to reduce the two-dimensional system to one of the canonical forms, and hence determine the type of bifurcation that occurs when $4b = a^2 > 0$. Sketch phase portraits in the cases $0 < a^2 < 4b$ and $0 < 4b < a^2$.

(iii) Sketch the phase portrait in the case $a = 0$. Prove that periodic orbits exist if and only if $a = 0$.

Paper 4, Section II

14E Dynamical Systems

Let I, J be closed bounded intervals in $\mathbb{R}$, and let $F : \mathbb{R} \to \mathbb{R}$ be a continuous map.

Explain what is meant by the statement that 'I F-covers J' (written $I \to J$). For a collection of intervals $I_0, \ldots, I_k$ define the associated directed graph Γ and transition matrix A. Derive an expression for the number of (not necessarily least) period-n points of F in terms of A.

Let F have a 5-cycle

$$x_0 < x_1 < x_2 < x_3 < x_4$$

such that $x_{i+1} = F(x_i)$ for $i = 0, \ldots, 4$ where indices are taken modulo 5. Write down the directed graph Γ and transition matrix A for the F-covering relations between the intervals $[x_i, x_{i+1}]$. Compute the number of n-cycles which are guaranteed to exist for F, for each integer $1 \leqslant n \leqslant 4$, and the intervals the points move between.

Explain carefully whether or not F is guaranteed to have a horseshoe. Must F be chaotic? Could F be a unimodal map? Justify your answers.

Similarly, a continuous map $G : \mathbb{R} \to \mathbb{R}$ has a 5-cycle

$$x_3 < x_1 < x_0 < x_2 < x_4.$$

For what integer values of n, $1 \leqslant n \leqslant 4$, is G guaranteed to have an n-cycle?

Is G guaranteed to have a horseshoe? Must G be chaotic? Justify your answers.

1/I/7A **Dynamical Systems**

Sketch the phase plane of the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + x^2 - ky,\end{aligned}$$

(i) for $k = 0$ and (ii) for $k = 1/10$. Include in your sketches any trajectories that are the separatrices of a saddle point. In case (ii) shade the domain of stability of the origin.

3/II/14A **Dynamical Systems**

Define the Poincaré index of a simple closed curve, not necessarily a trajectory, and the Poincaré index of an isolated fixed point $\mathbf{x_0}$ for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. State the Poincaré index of a periodic orbit.

Consider the system

$$\begin{aligned}\dot{x} &= y + ax - bx^3,\\ \dot{y} &= x^3 - x,\end{aligned}$$

where a and b are constants and $a \neq 0$.

(a) Find and classify the fixed points, and state their Poincaré indices.

(b) By considering a suitable function $H(x, y)$, show that any periodic orbit Γ satisfies

$$\oint_\Gamma (x - x^3)(ax - bx^3)dt = 0,$$

where $x(t)$ is evaluated along the orbit.

(c) Deduce that if $b/a < 1$ then the second-order differential equation

$$\ddot{x} - (a - 3bx^2)\dot{x} + x - x^3 = 0$$

has no periodic solutions.

2/I/7A **Dynamical Systems**

Explain the difference between a *stationary bifurcation* and an *oscillatory bifurcation* for a fixed point $\mathbf{x}_0$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x};\mu)$ in $\mathbb{R}^n$ with a real parameter μ.

The normal form of a Hopf bifurcation in polar coordinates is

$$\dot{r} = \mu r - ar^3 + O(r^5),$$
$$\dot{\theta} = \omega + c\mu - br^2 + O(r^4),$$

where a, b, c and ω are constants, $a \neq 0$, and $\omega > 0$. Sketch the phase plane near the bifurcation for each of the cases (i) $\mu < 0 < a$, (ii) $0 < \mu, a$, (iii) $\mu, a < 0$ and (iv) $a < 0 < \mu$.

Let R be the radius and T the period of the limit cycle when one exists. Sketch how R varies with μ for the case when the limit cycle is subcritical. Find the leading-order approximation to $dT/d\mu$ for $|\mu| \ll 1$.

4/II/14A **Dynamical Systems**

Explain the difference between a *hyperbolic* and a *nonhyperbolic* fixed point $\mathbf{x}_0$ for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$.

Consider the system in $\mathbb{R}^2$, where μ is a real parameter,

$$\dot{x} = x(\mu - x + y^2),$$
$$\dot{y} = y(1 - x - y^2).$$

Show that the fixed point $(\mu, 0)$ has a bifurcation when $\mu = 1$, while the fixed points $(0, \pm 1)$ have a bifurcation when $\mu = -1$.

[The fixed point at $(0, -1)$ should not be considered further.]

Analyse each of the bifurcations at $(\mu, 0)$ and $(0, 1)$ in turn as follows. Make a change of variable of the form $\mathbf{X} = \mathbf{x} - \mathbf{x}_0(\mu)$, $\nu = \mu - \mu_0$. Identify the (non-extended) stable and centre linear subspaces at the bifurcation in terms of X and Y. By finding the leading-order approximation to the extended centre manifold, construct the evolution equation on the extended centre manifold, and determine the type of bifurcation. Sketch the local bifurcation diagram, showing which fixed points are stable.

[*Hint: the leading-order approximation to the extended centre manifold of the bifurcation at* $(0, 1)$ *is* $Y = aX$ *for some coefficient* a.]

Show that there is another fixed point in $y > 0$ for $\mu < 1$, and that this fixed point connects the two bifurcations.

3/I/7A **Dynamical Systems**

State the normal-form equations for (i) a saddle-node bifurcation, (ii) a transcritical bifurcation and (iii) a pitchfork bifurcation, for a one-dimensional map $x_{n+1} = F(x_n; \mu)$.

Consider a period-doubling bifurcation of the form

$$x_{n+1} = -x_n + \alpha + \beta x_n + \gamma x_n^2 + \delta x_n^3 + O(x_n^4),$$

where $x_n = O(\mu^{1/2})$, $\alpha, \beta = O(\mu)$, and $\gamma, \delta = O(1)$ as $\mu \to 0$. Show that

$$X_{n+2} = X_n + \hat{\mu} X_n - A X_n^3 + O(X_n^4),$$

where $X_n = x_n - \frac{1}{2}\alpha$, and the parameters $\hat{\mu}$ and A are to be identified in terms of α, β, γ and δ. Deduce the condition for the bifurcation to be supercritical.

4/I/7A **Dynamical Systems**

Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. State when F is chaotic according to Glendinning's definition.

Prove that if F has a 3-cycle then F^2 has a horseshoe.

[You may assume the Intermediate Value Theorem.]

1/I/7E **Dynamical Systems**

Given a non-autonomous kth-order differential equation

$$\frac{d^k y}{dt^k} = g\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{k-1}y}{dt^{k-1}}\right)$$

with $y \in \mathbb{R}$, explain how it may be written in the autonomous first-order form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ for suitably chosen vectors $\mathbf{x}$ and $\mathbf{f}$.

Given an autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$, define the corresponding *flow* $\boldsymbol{\phi}_t(\mathbf{x})$. What is $\boldsymbol{\phi}_s(\boldsymbol{\phi}_t(\mathbf{x}))$ equal to? Define the *orbit* $\mathcal{O}(\mathbf{x})$ through $\mathbf{x}$ and the *limit set* $\omega(\mathbf{x})$ of $\mathbf{x}$. Define a *homoclinic orbit*.

3/II/14E **Dynamical Systems**

The Lorenz equations are

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

where r, σ and b are positive constants and $(x, y, z) \in \mathbb{R}^3$.

(i) Show that the origin is globally asymptotically stable for $0 < r < 1$ by considering a function $V(x, y, z) = \frac{1}{2}(x^2 + Ay^2 + Bz^2)$ with a suitable choice of constants A and B.

(ii) State, without proof, the Centre Manifold Theorem.

Show that the fixed point at the origin is nonhyperbolic at $r = 1$. What are the dimensions of the linear stable and (non-extended) centre subspaces at this point?

(iii) Let $\sigma = 1$ from now on. Make the substitutions $u = x + y$, $v = x - y$ and $\mu = r - 1$ and derive the resulting equations for $\dot{u}, \dot{v}$ and $\dot{z}$.

The extended centre manifold is given by

$$v = V(u, \mu), \qquad z = Z(u, \mu)$$

where V and Z can be expanded as power series about $u = \mu = 0$. What is known about V and Z from the Centre Manifold Theorem? Assuming that $\mu = O(u^2)$, determine Z correct to $O(u^2)$ and V to $O(u^3)$. Hence obtain the evolution equation on the extended centre manifold correct to $O(u^3)$, and identify the type of bifurcation.

2/I/7E **Dynamical Systems**

Find and classify the fixed points of the system

$$\dot{x} = (1 - x^2)y \ ,$$
$$\dot{y} = x(1 - y^2) \ .$$

What are the values of their Poincaré indices? Prove that there are no periodic orbits. Sketch the phase plane.

4/II/14E **Dynamical Systems**

Consider the one-dimensional map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$x_{i+1} = F(x_i) = x_i(ax_i^2 + bx_i + \mu),$$

where a and b are constants, μ is a parameter and $a \neq 0$.

(i) Find the fixed points of F and determine the linear stability of $x = 0$. Hence show that there are bifurcations at $\mu = 1$, at $\mu = -1$ and, if $b \neq 0$, at $\mu = 1 + b^2/4a$.

Sketch the bifurcation diagram for each of the cases:

$$(1)\ a > b = 0, \quad (2)\ a, b > 0 \quad \text{and} \quad (3)\ a, b < 0.$$

In each case show the locus and stability of the fixed points in the (μ, x)-plane, and state the type of each bifurcation. [*Assume that there are no further bifurcations in the region sketched.*]

(ii) For the case $F(x) = x(\mu - x^2)$ (i.e. $a = -1$, $b = 0$), you may assume that

$$F^2(x) = x + x(\mu - 1 - x^2)(\mu + 1 - x^2)(1 - \mu x^2 + x^4) \ .$$

Show that there are at most three 2-cycles and determine when they exist. By considering $F'(x_i)F'(x_{i+1})$, or otherwise, show further that one 2-cycle is always unstable when it exists and that the others are unstable when $\mu > \sqrt{5}$. Sketch the bifurcation diagram showing the locus and stability of the fixed points and 2-cycles. State briefly what you would expect to occur in the region $\mu > \sqrt{5}$.

3/I/7E **Dynamical Systems**

State the Poincaré–Bendixson Theorem for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$.

Prove that if $k^2 < 4$ then the system

$$\begin{aligned} \dot{x} &= x - y - x^3 - xy^2 - k^2xy^2 \\ \dot{y} &= y + x - x^2y - y^3 - k^2x^2y \end{aligned}$$

has a periodic orbit in the region $2/(2+k^2) \leqslant x^2 + y^2 \leqslant 1$.

4/I/7E **Dynamical Systems**

By considering the binary representation of the sawtooth map, $F(x) = 2x$ [mod 1] for $x \in [0,1)$, show that:

(i) F has sensitive dependence on initial conditions on $[0,1)$.

(ii) F has topological transitivity on $[0,1)$.

(iii) Periodic points are dense in $[0,1)$.

Find all the 4-cycles of F and express them as fractions.

1/I/7E **Dynamical Systems**

Find the fixed points of the system

$$\dot{x} = x(x + 2y - 3) \; ,$$
$$\dot{y} = y(3 - 2x - y) \; .$$

Local linearization shows that all the fixed points with $xy = 0$ are saddle points. Why can you be certain that this remains true when nonlinear terms are taken into account? Classify the fixed point with $xy \neq 0$ by its local linearization. Show that the equation has Hamiltonian form, and thus that your classification is correct even when the nonlinear effects are included.

Sketch the phase plane.

1/II/14E **Dynamical Systems**

(a) An autonomous dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ has a periodic orbit $\mathbf{x} = \mathbf{X}(t)$ with period T. The linearized evolution of a small perturbation $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$ is given by $\eta_i(t) = \Phi_{ij}(t)\eta_j(0)$. Obtain the differential equation and initial condition satisfied by the matrix $\mathbf{\Phi}(t)$.

Define the *Floquet multipliers* of the orbit. Explain why one of the multipliers is always unity and show that the other is given by

$$\exp\left(\int_0^T \boldsymbol{\nabla} \cdot \mathbf{f}\big(\mathbf{X}(t)\big) \; dt \right) \; .$$

(b) Use the 'energy-balance' method for nearly Hamiltonian systems to find a leading-order approximation to the amplitude of the limit cycle of the equation

$$\ddot{x} + \epsilon(\alpha x^2 + \beta \dot{x}^2 - \gamma)\dot{x} + x = 0 \; ,$$

where $0 < \epsilon \ll 1$ and $(\alpha + 3\beta)\gamma > 0$.

Compute a leading-order approximation to the nontrivial Floquet multiplier of the limit cycle and hence determine its stability.

[*You may assume that* $\int_0^{2\pi} \sin^2\theta \cos^2\theta \; d\theta = \pi/4$ *and* $\int_0^{2\pi} \cos^4\theta \; d\theta = 3\pi/4$.]

2/I/7E **Dynamical Systems**

Explain what is meant by a *strict Lyapunov function* on a domain $\mathcal{D}$ containing the origin for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$. Define the *domain of stability* of a fixed point $\mathbf{x}_0$.

By considering the function $V = \frac{1}{2}(x^2+y^2)$ show that the origin is an asymptotically stable fixed point of

$$\begin{aligned} \dot{x} &= -2x + y + x^3 - xy^2 \ , \\ \dot{y} &= -x - 2y + 6x^2 y + 4y^3 \ . \end{aligned}$$

Show also that its domain of stability includes $x^2+y^2 < \frac{1}{2}$ and is contained in $x^2+y^2 \leqslant 2$.

2/II/14E **Dynamical Systems**

Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Explain what is meant by saying (a) that F has a *horseshoe*, (b) that F is *chaotic* (Glendinning's definition).

Consider the tent map defined on the interval $[0,1]$ by

$$F(x) = \begin{cases} \mu x & 0 \leqslant x < \frac{1}{2} \\ \mu(1-x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

with $1 < \mu \leqslant 2$.

Find the non-zero fixed point x_0 and the points $x_{-1} < \frac{1}{2} < x_{-2}$ that satisfy

$$F^2(x_{-2}) = F(x_{-1}) = x_0 \ .$$

Sketch a graph of F and F^2 showing the points corresponding to x_{-2}, x_{-1} and x_0. Hence show that F^2 has a horseshoe if $\mu \geqslant 2^{1/2}$.

Explain briefly why F^4 has a horseshoe when $2^{1/4} \leqslant \mu < 2^{1/2}$ and why there are periodic points arbitrarily close to x_0 for $\mu \geqslant 2^{1/2}$, but no such points for $2^{1/4} \leqslant \mu < 2^{1/2}$.

3/I/7E Dynamical Systems

State the normal-form equations for (a) a saddle-node bifurcation, (b) a transcritical bifurcation, and (c) a pitchfork bifurcation, for a dynamical system $\dot{x} = f(x, \mu)$.

Consider the system

$$\begin{aligned}\dot{x} &= \mu + y - x^2 + 2xy + 3y^2 \\ \dot{y} &= -y + 2x^2 + 3xy \ .\end{aligned}$$

Compute the extended centre manifold near $x = y = \mu = 0$, and the evolution equation on the centre manifold, both correct to second order in x and μ. Deduce the type of bifurcation and show that the equation can be put in normal form, to the same order, by a change of variables of the form $T = \alpha t$, $X = x - \beta\mu$, $\tilde{\mu} = \gamma(\mu)$ for suitably chosen α, β and $\gamma(\mu)$.

4/I/7E Dynamical Systems

Consider the logistic map $F(x) = \mu x(1-x)$ for $0 \leqslant x \leqslant 1$, $0 \leqslant \mu \leqslant 4$. Show that there is a period-doubling bifurcation of the nontrivial fixed point at $\mu = 3$. Show further that the bifurcating 2-cycle (x_1, x_2) is given by the roots of

$$\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0 \ .$$

Show that there is a second period-doubling bifurcation at $\mu = 1 + \sqrt{6}$.

1/I/7B **Dynamical Systems**

State Dulac's Criterion and the Poincaré–Bendixson Theorem regarding the existence of periodic solutions to the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Hence show that

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(\mu - 2x^2 - y^2) \end{aligned}$$

has no periodic solutions if $\mu < 0$ and at least one periodic solution if $\mu > 0$.

1/II/14B **Dynamical Systems**

Consider the equations

$$\begin{aligned} \dot{x} &= (a - x^2)(a^2 - y) \\ \dot{y} &= x - y \end{aligned}$$

as a function of the parameter a. Find the fixed points and plot their location in the (a, x) plane. Hence, or otherwise, deduce that there are bifurcations at $a = 0$ and $a = 1$.

Investigate the bifurcation at $a = 1$ by making the substitutions $u = x - 1$, $v = y - x$ and $\mu = a - 1$. Find the equation of the extended centre manifold to second order. Find the evolution equation on the centre manifold to second order, and determine the stability of its fixed points.

Show which branches of fixed points in the (a, x) plane are stable and which are unstable, and state, without calculation, the type of bifurcation at $a = 0$. Hence sketch the structure of the (x, y) phase plane very near the origin for $|a| \ll 1$ in the cases (i) $a < 0$ and (ii) $a > 0$.

The system is perturbed to $\dot{x} = (a - x^2)(a^2 - y) + \epsilon$, where $0 < \epsilon \ll 1$, with $\dot{y} = x - y$ still. Sketch the possible changes to the bifurcation diagram near $a = 0$ and $a = 1$. [*Calculation is not required.*]

2/I/7B **Dynamical Systems**

Define Lyapunov stability and quasi-asymptotic stability of a fixed point $\mathbf{x}_0$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

By considering a Lyapunov function of the form $V = g(x)+y^2$, show that the origin is an asymptotically stable fixed point of

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x^5 \, . \end{aligned}$$

[*Lyapunov's Second Theorem may be used without proof, provided you show that its conditions apply.*]

2/II/14B **Dynamical Systems**

Prove that if a continuous map F of an interval into itself has a periodic orbit of period three then it also has periodic orbits of least period n for all positive integers n.

Explain briefly why there must be at least two periodic orbits of least period 5.

[*You may assume without proof:*

(i) If U and V are non-empty closed bounded intervals such that $V \subseteq F(U)$ then there is a closed bounded interval $K \subseteq U$ such that $F(K) = V$.

(ii) The Intermediate Value Theorem.]

3/I/7B **Dynamical Systems**

Define the stable and unstable invariant subspaces of the linearisation of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ at a saddle point located at the origin in $\mathbb{R}^n$. How, according to the Stable Manifold Theorem, are the stable and unstable manifolds related to the invariant subspaces?

Calculate the stable and unstable manifolds, correct to cubic order, for the system

$$\begin{aligned} \dot{x} &= x + x^2 + 2xy + 3y^2 \\ \dot{y} &= -y + 3x^2 \, . \end{aligned}$$

4/I/7B **Dynamical Systems**

Find and classify the fixed points of the system

$$\begin{aligned}\dot{x} &= x(1-y)\\ \dot{y} &= -y + x^2\,.\end{aligned}$$

Sketch the phase plane.

What is the ω-limit for the point $(2,-1)$? Which points have $(0,0)$ as their ω-limit?