Part II

Differential Geometry

Paper 1, Section II

26G Differential Geometry

(a) Let X, Y be smooth manifolds and $f : X \to Y$ be a smooth map. What does it mean for y_0 to be a *regular value* of f? Give an example of a smooth map f that has a regular value, together with a regular value of f, justifying your answer. State Sard's theorem.

(b) Let X and Y be compact manifolds of dimension n and $f : X \to Y$ be a smooth map. Define the degree mod 2 of f, quoting carefully (but without proof) the results from the course necessary to make this well defined.

(c) Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Define the exponential map $\exp_p$, explaining carefully its domain. Explain also briefly why the exponential map is smooth. Give an explicit example where the domain of $\exp_p$ is not T_pS, and an example where $\exp_p$ is not surjective, justifying carefully your answers.

(d) Let $S \subset \mathbb{R}^3$ be a compact surface, and let V be a smooth vector field on S. Consider the map $\phi : S \to S$ defined by $\phi(p) = \exp_p(V(p))$. Explain why this map is well-defined and smooth. What is its degree mod 2?

Paper 2, Section II

26G Differential Geometry

(a) For regular curves in $\mathbb{R}^3$, parametrised by arc length s, define *curvature* k and *torsion* τ and derive the Frénet formulas. Indicate carefully all additional assumptions for these to be well defined.

(b) Suppose two regular curves in $\mathbb{R}^3$ both have curvature identically zero and the same arc length. Are they related by a proper Euclidean motion? Justify your answer. Does the answer change if we replace curvature identically zero with curvature identically one?

(c) We say that a quantity $Q(\gamma, s)$, defined for all regular curves γ parametrised by arc length s, is a *pointwise Euclidean invariant of curves* if

$$Q(\gamma, s) = Q(E \circ \gamma, s) \text{ for all proper Euclidean motions } E, \text{and}$$

$$Q(\gamma_{s_0}, s) = Q(\gamma, s - s_0) \text{ for all } s_0 \in \mathbb{R}, \text{ where } \gamma_{s_0}(s) := \gamma(s - s_0).$$

Show that $Q(\gamma, s) := k(s)$ and $Q(\gamma, s) := \tau(s)$, where k and τ refer to the curvature and torsion of the curve γ respectively, are both examples of such pointwise Euclidean invariants of curves.

(d) One can trivially construct other such pointwise Euclidean invariants by applying functions of curvature and torsion, e.g. defining $Q(\gamma, s) := k^2(s)$ or $Q(\gamma, s) := k(s) + \tau(s)$. Are these the only examples, i.e. is it true that if $Q(\gamma, s)$ is any pointwise Euclidean invariant, then $Q(\gamma, s) = f(k(s), \tau(s))$ for some function f (independent of the curve γ)? Justify your answer.

Paper 3, Section II

25G Differential Geometry

(a) Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the Gaussian curvature $K(p)$ and mean curvature $H(p)$ of S at p. Prove that these are Euclidean invariants, i.e. if $E : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper Euclidean motion and $\widetilde{S} = E(S)$ and $\widetilde{K}$, $\widetilde{H}$ denote the Gaussian and mean curvature of $\widetilde{S}$ (with a choice of orientation that you should describe), respectively, then $\widetilde{K}(E(p)) = K(p)$, $\widetilde{H}(E(p)) = H(p)$. Do the Gaussian and mean curvatures depend on the orientation?

(b) Show that there is no Euclidean motion taking a piece of the cylinder to a piece of the plane, and infer that for a general surface S, the property $K = 0$ identically does not imply that there is a Euclidean motion taking S to a piece of the plane. Exhibit similarly two surfaces each with $K = 1$ identically, no respective pieces of which are related by a Euclidean motion, and similarly two surfaces each with $K = -1$ identically.

(c) Let $\mathcal{R} \subset \mathbb{R}^3$ be a compact submanifold of dimension 3 with connected boundary $S = \partial\mathcal{R}$. Note that $S \subset \mathbb{R}^3$ is an orientable surface and can be oriented by the unique normal vector N pointing towards $\mathcal{R}$. Now let $\widetilde{S} \subset \mathcal{R}$ be a surface (without boundary). Suppose $p \in S \cap \widetilde{S}$. Show that $\widetilde{H}(p) \geqslant H(p)$, where H and $\widetilde{H}$ denote the mean curvature of S and $\widetilde{S}$, respectively, where both surfaces are (locally) oriented at p by the N described above. Is it necessarily the case that $\widetilde{K}(p) \geqslant K(p)$? Justify your answer.

Paper 4, Section II

25G Differential Geometry

(a) Given a compact orientable surface with smooth boundary, define the *area element* dA, *Euler characteristic* χ, and *geodesic curvature* k_g of the boundary, explaining briefly why the first two are well defined. State the Gauss–Bonnet theorem for the surface. [You need not consider the case of corners.]

(b) Let S be a compact orientable surface without boundary, and let $\gamma : I \to S$ be a smooth closed curve on S, parametrised by arc length, which separates S into two surfaces with boundary, S_1 and S_2, such that S is the union $S = S_1 \cup S_2$ where $\partial S_1 = \partial S_2 = S_1 \cap S_2 = \gamma(I)$. Suppose there exists an isometry $\phi : S_1 \to S_2$, and moreover, for each $x, y \in \gamma(I)$, an isometry $\phi_{x,y} : S \to S$ such that $\phi_{x,y}(\gamma(I)) = \gamma(I)$ and such that $\phi_{x,y}(x) = y$. Show that γ is a geodesic.

(c) In the above problem, suppose we drop the assumption of the existence of the isometry ϕ. Is γ still necessarily a geodesic?

(d) Alternatively, suppose we drop the assumption of the existence of the isometries $\phi_{x,y}$. Is γ still necessarily a geodesic?

Paper 1, Section II
26I Differential Geometry

Let $S \subset \mathbb{R}^3$ be an oriented surface. Define its *Gauss map* N. For each $p \in S$, show that the derivative of N defines a self-adjoint operator on T_pS, and define the *principal curvatures* of S at a point p. What does it mean for p to be an *umbilical point*? What does it mean for S to be a *minimal surface*?

(a) We say that a smooth map $f : S \to R$ between two surfaces in $\mathbb{R}^3$ is *conformal* if

$$\langle Df_p(u), Df_p(v)\rangle = \lambda(p)\langle u, v\rangle$$

for all $p \in S$ and $u, v \in T_pS$, where $\lambda(p) > 0$.

Show that, if S does not have any umbilical points, then S is a minimal surface if and only if its Gauss map is conformal.

(b) Now drop the assumption about umbilical points. If S is a minimal surface, must its Gauss map be conformal? If the Gauss map is conformal, must S be a minimal surface? Justify your answers.

(c) Suppose S is a connected minimal surface. Can the image of its Gauss map be a great circle in S^2?

Paper 2, Section II
26I Differential Geometry

Define a *k-dimensional smooth manifold*, and a *regular value* of a smooth map between smooth manifolds. State the *inverse function theorem*, and use it to prove the *preimage theorem.*

Suppose X and Y are smooth manifolds and $f : X \to Y$ is a smooth map. If X is compact, show that the set of regular values of f in Y is open.

Consider the space

$$X_a = \{x + y - z^2 - w^2 = a\} \cap \{x^2 + y^2 - z^4/2 = 0\},$$

where x, y, z, w are the standard coordinates on $\mathbb{R}^4$, and $a \in \mathbb{R}$ is a constant. Show that X_a is a 2-dimensional manifold whenever $a \neq 0$. Is X_0 a manifold? Justify your answer.

Paper 3, Section II
25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface. Define the *first fundamental form* of S. If $R \subset \mathbb{R}^3$ is also a surface, we say that a smooth map $\phi : S \to R$ is a *local isometry* if $D\phi$ preserves the first fundamental form at each point.

(a) Let $\alpha : I \to S$ be a curve, and let V be a vector field along α. Define the *covariant derivative* of V. What does it mean for α to be *geodesic*? If $\phi : S \to R$ is a local isometry, show that for an arbitrary geodesic $\alpha : I \to S$, $\phi \circ \alpha$ is also a geodesic. [You may use without proof the fact that Christoffel symbols only depend on the first fundamental form.] Must the converse be true? Give a proof or counterexample.

(b) Define the *Gauss curvature* of S. Suppose $\phi : S \to R$ is a local isometry, and let K_S and K_R denote the Gauss curvatures of S and R respectively. Is it true that $K_R \circ \phi = K_S$? State any theorem you use.

(c) Let R be the surface of revolution defined by the curve $\gamma(u) = (e^u, 0, u)$, with $-\infty < u < \infty$. Let S be the surface of revolution defined by the curve $\delta(s) = (\cosh s, 0, s)$, with $0 < s < \infty$.

(i) Show that there is a diffeomorphism $\phi : S \to R$ such that $K_R \circ \phi = K_S$.

(ii) Does there exist a local isometry $\psi : S \to R$? Justify your answer.

[*Hint: You may use without proof that the surface of revolution defined by the curve* $(f, 0, g)$ *has Gauss curvature given by*

$$\frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}\,.$$

Standard facts about surfaces of revolution may be used without proof if clearly stated.]

Paper 4, Section II
25I Differential Geometry

(a) State *Wirtinger's inequality*. State and prove the *isoperimetric inequality* for domains $\Omega \subset \mathbb{R}^2$ with compact closure and C^1 boundary $\partial\Omega$.

(b) Let $Q \subset \mathbb{R}^2$ be a *cyclic* quadrilateral, meaning that there is a circle through its four vertices. Say its edges have lengths a, b, c and d (in cyclic order). Assume $Q' \subset \mathbb{R}^2$ is another quadrilateral with edges of lengths a, b, c and d (in the same order). Show that $\text{Area}(Q) \geqslant \text{Area}(Q')$. Explain briefly for which Q' equality holds.

Paper 1, Section II

26F Differential Geometry

(a) Let $S \subset \mathbb{R}^3$ be a surface. Give a parametrisation-free definition of the *first fundamental form* of S. Use this definition to derive a description of it in terms of the partial derivatives of a local parametrisation $\phi : U \subset \mathbb{R}^2 \to S$.

(b) Let a be a positive constant. Show that the half-cone

$$\Sigma = \{(x, y, z) \mid z^2 = a(x^2 + y^2),\ z > 0\}$$

is locally isometric to the Euclidean plane. [*Hint: Use polar coordinates on the plane.*]

(c) Define the *second fundamental form* and the *Gaussian curvature* of S. State Gauss' Theorema Egregium. Consider the set

$$V = \{(x, y, z) \mid x^2 + y^2 + z^2 - 2xy - 2yz = 0\} \backslash \{(0, 0, 0)\} \subset \mathbb{R}^3.$$

(i) Show that V is a surface.

(ii) Calculate the Gaussian curvature of V at each point. [*Hint: Complete the square.*]

Paper 2, Section II

26F Differential Geometry

Let U be a domain in $\mathbb{R}^2$, and let $\phi : U \to \mathbb{R}^3$ be a smooth map. Define what it means for ϕ to be an *immersion*. What does it mean for an immersion to be *isothermal*?

Write down a formula for the mean curvature of an immersion in terms of the first and second fundamental forms. What does it mean for an immersed surface to be *minimal*? Assume that $\phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$ is an isothermal immersion. Prove that it is minimal if and only if x, y, z are harmonic functions of u, v.

For $u \in \mathbb{R}, v \in [0, 2\pi]$, and smooth functions $f, g : \mathbb{R} \to \mathbb{R}$, assume that

$$\phi(u, v) = \big(f(u) \cos v, f(u) \sin v, g(u)\big)$$

is an isothermal immersion. Find all possible pairs (f, g) such that this immersion is minimal.

Paper 3, Section II

25F Differential Geometry

Let X and Y be smooth boundaryless manifolds. Suppose $f : X \to Y$ is a smooth map. What does it mean for $y \in Y$ to be a *regular value* of f? State Sard's theorem and the stack-of-records theorem.

Suppose $g : X \to Y$ is another smooth map. What does it mean for f and g to be *smoothly homotopic*? Assume now that X is compact, and has the same dimension as Y. Suppose that $y \in Y$ is a regular value for both X and Y. Prove that

$$\#f^{-1}(y) = \#g^{-1}(y) \pmod 2.$$

Let $U \subset S^n$ be a non-empty open subset of the sphere. Suppose that $h : S^n \to S^n$ is a smooth map such that $\#h^{-1}(y) = 1 \pmod 2$ for all $y \in U$. Show that there must exist a pair of antipodal points on S^n which is mapped to another pair of antipodal points by h.

[*You may assume results about compact 1-manifolds provided they are accurately stated.*]

Paper 4, Section II

25F Differential Geometry

Let $I \subset \mathbb{R}$ be an interval, and $S \subset \mathbb{R}^3$ be a surface. Assume that $\alpha : I \to S$ is a regular curve parametrised by arc-length. Define the *geodesic curvature* of α. What does it mean for α to be a *geodesic curve*?

State the global Gauss–Bonnet theorem including boundary terms.

Suppose that $S \subset \mathbb{R}^3$ is a surface diffeomorphic to a cylinder. How large can the number of simple closed geodesics on S be in each of the following cases?

(i) S has Gaussian curvature everywhere zero;

(ii) S has Gaussian curvature everywhere positive;

(iii) S has Gaussian curvature everywhere negative.

In cases where there can be two or more simple closed geodesics, must they always be disjoint? Justify your answer.

[*A formula for the Gaussian curvature of a surface of revolution may be used without proof if clearly stated. You may also use the fact that a piecewise smooth curve on a cylinder without self-intersections either bounds a domain homeomorphic to a disc or is homotopic to the waist-curve of the cylinder.*]

Paper 1, Section II

26I Differential Geometry

(a) Let $X \subset \mathbb{R}^N$ be a manifold. Give the definition of the *tangent space* T_pX of X at a point $p \in X$.

(b) Show that $X := \{-x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1\} \cap \{x_0 > 0\}$ defines a submanifold of $\mathbb{R}^4$ and identify explicitly its tangent space $T_{\mathbf{x}}X$ for any $\mathbf{x} \in X$.

(c) Consider the matrix group $O(1,3) \subset \mathbb{R}^{4^2}$ consisting of all 4×4 matrices A satisfying

$$A^t M A = M$$

where M is the diagonal 4×4 matrix $M = \mathrm{diag}(-1, 1, 1, 1)$.

(i) Show that $O(1,3)$ forms a group under matrix multiplication, i.e. it is closed under multiplication and every element in $O(1,3)$ has an inverse in $O(1,3)$.

(ii) Show that $O(1,3)$ defines a 6-dimensional manifold. Identify the tangent space $T_A O(1,3)$ for any $A \in O(1,3)$ as a set $\{AY\}_{Y \in \mathfrak{G}}$ where Y ranges over a linear subspace $\mathfrak{G} \subset \mathbb{R}^{4^2}$ which you should identify explicitly.

(iii) Let X be as defined in (b) above. Show that $O^+(1,3) \subset O(1,3)$ defined as the set of all $A \in O(1,3)$ such that $A\mathbf{x} \in X$ for all $\mathbf{x} \in X$ is both a subgroup and a submanifold of full dimension.

[*You may use without proof standard theorems from the course concerning regular values and transversality.*]

Paper 2, Section II

25I Differential Geometry

(a) State the fundamental theorem for regular curves in $\mathbb{R}^3$.

(b) Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a regular curve, parameterised by arc length, such that its image $\alpha(\mathbb{R}) \subset \mathbb{R}^3$ is a one-dimensional submanifold. Suppose that the set $\alpha(\mathbb{R})$ is preserved by a nontrivial proper Euclidean motion $\phi : \mathbb{R}^3 \to \mathbb{R}^3$.

Show that there exists $\sigma_0 \in \mathbb{R}$ corresponding to ϕ such that $\phi(\alpha(s)) = \alpha(\pm s + \sigma_0)$ for all $s \in \mathbb{R}$, where the choice of $\pm$ sign is independent of s. Show also that the curvature $k(s)$ and torsion $\tau(s)$ of α satisfy

$$k(\pm s + \sigma_0) = k(s) \quad \text{and} \tag{1}$$

$$\tau(\pm s + \sigma_0) = \tau(s), \tag{2}$$

with equation (2) valid only for s such that $k(s) > 0$. In the case where the sign is $+$ and $\sigma_0 = 0$, show that $\alpha(\mathbb{R})$ is a straight line.

(c) Give an explicit example of a curve α satisfying the requirements of (b) such that neither of $k(s)$ and $\tau(s)$ is a constant function, and such that the curve α is closed, i.e. such that $\alpha(s) = \alpha(s + s_0)$ for some $s_0 > 0$ and all s. [Here a drawing would suffice.]

(d) Suppose now that $\alpha : \mathbb{R} \to \mathbb{R}^3$ is an embedded regular curve parameterised by arc length s. Suppose further that $k(s) > 0$ for all s and that $k(s)$ and $\tau(s)$ satisfy (1) and (2) for some σ_0, where the choice $\pm$ is independent of s, and where $\sigma_0 \neq 0$ in the case of $+$ sign. Show that there exists a nontrivial proper Euclidean motion ϕ such that the set $\alpha(\mathbb{R})$ is preserved by ϕ. [You may use the theorem of part (a) without proof.]

Paper 3, Section II

25I Differential Geometry

(a) Show that for a compact regular surface $S \subset \mathbb{R}^3$, there exists a point $p \in S$ such that $K(p) > 0$, where K denotes the Gaussian curvature. Show that if S is contained in a closed ball of radius R in $\mathbb{R}^3$, then there is a point p such that $K(p) \geqslant R^{-2}$.

(b) For a regular surface $S \subset \mathbb{R}^3$, give the definition of a *geodesic polar coordinate system* at a point $p \in S$. Show that in such a coordinate system, $\lim_{r \to 0} G(r, \theta) = 0$, $\lim_{r \to 0} (\sqrt{G})_r(r, \theta) = 1$, $E(r, \theta) = 1$ and $F(r, \theta) = 0$. [You may use without proof standard properties of the exponential map provided you state them clearly.]

(c) Let $S \subset \mathbb{R}^3$ be a regular surface. Show that if $K \leqslant 0$, then any geodesic polar coordinate ball $B(p, \epsilon_0) \subset S$ of radius ϵ_0 around p has area satisfying

$$\text{Area } B(p, \epsilon_0) \geqslant \pi \epsilon_0^2.$$

[You may use without proof the identity $(\sqrt{G})_{rr}(r, \theta) = -\sqrt{G}K$.]

(d) Let $S \subset \mathbb{R}^3$ be a regular surface, and now suppose $-\infty < K \leqslant C$ for some constant $0 < C < \infty$. Given any constant $0 < \gamma < 1$, show that there exists $\epsilon_0 > 0$, depending only on C and γ, so that if $B(p, \epsilon) \subset S$ is any geodesic polar coordinate ball of radius $\epsilon \leqslant \epsilon_0$, then

$$\text{Area } B(p, \epsilon) \geqslant \gamma \pi \epsilon^2.$$

[*Hint: For any fixed* θ_0*, consider the function* $f(r) := \sqrt{G}(r, \theta_0) - \alpha \sin(\sqrt{C} r)$*, for all* $0 < \alpha < \frac{1}{\sqrt{C}}$*. Derive the relation* $f'' \geqslant -Cf$ *and show* $f(r) > 0$ *for an appropriate range of* r*. The following variant of Wirtinger's inequality may be useful and can be assumed without proof: if* g *is a* C^1 *function on* $[0, L]$ *vanishing at* 0*, then* $\int_0^L |g(x)|^2 dx \leqslant \frac{L}{2\pi} \int_0^L |g'(x)|^2 dx$*.*]

Paper 4, Section II

25I Differential Geometry

(a) State the Gauss–Bonnet theorem for compact regular surfaces $S \subset \mathbb{R}^3$ without boundary. Identify all expressions occurring in any formulae.

(b) Let $S \subset \mathbb{R}^3$ be a compact regular surface without boundary and suppose that its Gaussian curvature $K(x) \geqslant 0$ for all $x \in S$. Show that S is diffeomorphic to the sphere.

Let S_n be a sequence of compact regular surfaces in $\mathbb{R}^3$ and let $K_n(x)$ denote the Gaussian curvature of S_n at $x \in S_n$. Suppose that

$$\limsup_{n \to \infty} \inf_{x \in S_n} K_n(x) \geqslant 0. \qquad (\star)$$

(c) Give an example to show that it does not follow that for all sufficiently large n the surface S_n is diffeomorphic to the sphere.

(d) Now assume, in addition to $(\star)$, that all of the following conditions hold:

(1) There exists a constant $R < \infty$ such that for all n, S_n is contained in a ball of radius R around the origin.

(2) There exists a constant $M < \infty$ such that $\text{Area}(S_n) \leqslant M$ for all n.

(3) There exists a constant $\epsilon_0 > 0$ such that for all n, all points $p \in S_n$ admit a geodesic polar coordinate system centred at p of radius at least ϵ_0.

(4) There exists a constant $C < \infty$ such that on all such geodesic polar neighbourhoods, $|\partial_r K_n| \leqslant C$ for all n, where r denotes a geodesic polar coordinate.

(i) Show that for all sufficiently large n, the surface S_n is diffeomorphic to the sphere. [*Hint: It may be useful to identify a geodesic polar ball $B(p_n, \epsilon_0)$ in each S_n for which $\int_{B(p_n,\epsilon_0)} K_n dA$ is bounded below by a positive constant independent of n.*]

(ii) Explain how your example from (c) fails to satisfy one or more of these extra conditions (1)–(4).

[*You may use without proof the standard computations for geodesic polar coordinates: $E = 1$, $F = 0$, $\lim_{r\to 0} G(r,\theta) = 0$, $\lim_{r \to 0}(\sqrt{G})_r(r,\theta) = 1$, and $(\sqrt{G})_{rr} = -K\sqrt{G}$.*]

Paper 4, Section II

25H Differential Geometry

(a) Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular curve without self-intersection given by $\gamma(v) = (f(v), g(v))$ with $f(v) > 0$ for $v \in (a,b)$ and let S be the surface of revolution defined globally by the parametrisation

$$\phi : (0, 2\pi) \times (a,b) \to \mathbb{R}^3,$$

where $\phi(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$, i.e. $S = \phi((0,2\pi)\times(a,b))$. Compute its mean curvature H and its Gaussian curvature K.

(b) Define what it means for a regular surface $S \subset \mathbb{R}^3$ to be *minimal.* Give an example of a minimal surface which is not locally isometric to a cone, cylinder or plane. Justify your answer.

(c) Let S be a regular surface such that $K \equiv 1$. Is it necessarily the case that given any $p \in S$, there exists an open neighbourhood $\mathcal{U} \subset S$ of p such that $\mathcal{U}$ lies on some sphere in $\mathbb{R}^3$? Justify your answer.

Paper 3, Section II
25H Differential Geometry

(a) Let $\alpha : (a,b) \to \mathbb{R}^2$ be a regular curve without self intersection given by $\alpha(v) = (f(v), g(v))$ with $f(v) > 0$ for $v \in (a,b)$.

Consider the local parametrisation given by

$$\phi : (0, 2\pi) \times (a,b) \to \mathbb{R}^3,$$

where $\phi(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$.

(i) Show that the image $\phi((0,2\pi) \times (a,b))$ defines a regular surface S in $\mathbb{R}^3$.

(ii) If $\gamma(s) = \phi(u(s), v(s))$ is a geodesic in S parametrised by arc length, then show that $f(v(s))^2 u'(s)$ is constant in s. If $\theta(s)$ denotes the angle that the geodesic makes with the parallel $S \cap \{z = g(v(s))\}$, then show that $f(v(s))\cos\theta(s)$ is constant in s.

(b) Now assume that $\alpha(v) = (f(v), g(v))$ extends to a smooth curve $\alpha : [a,b] \to \mathbb{R}^2$ such that $f(a) = 0$, $f(b) = 0$, $f'(a) \neq 0$, $f'(b) \neq 0$. Let $\overline{S}$ be the closure of S in $\mathbb{R}^3$.

(i) State a necessary and sufficient condition on $\alpha(v)$ for $\overline{S}$ to be a compact regular surface. Justify your answer.

(ii) If $\overline{S}$ is a compact regular surface, and $\gamma : (-\infty, \infty) \to \overline{S}$ is a geodesic, show that there exists a non-empty open subset $U \subset \overline{S}$ such that $\gamma((-\infty,\infty)) \cap U = \emptyset$.

Paper 2, Section II

25H Differential Geometry

(a) Let $\alpha : (a,b) \to \mathbb{R}^3$ be a smooth regular curve parametrised by arclength. For $s \in (a,b)$, define the *curvature* $k(s)$ and (where defined) the *torsion* $\tau(s)$ of α. What condition must be satisfied in order for the torsion to be defined? Derive the Frenet equations.

(b) If $\tau(s)$ is defined and equal to 0 for all $s \in (a,b)$, show that α lies in a plane.

(c) State the *fundamental theorem for regular curves* in $\mathbb{R}^3$, giving necessary and sufficient conditions for when curves $\alpha(s)$ and $\widetilde{\alpha}(s)$ are related by a proper Euclidean motion.

(d) Now suppose that $\widetilde{\alpha} : (a,b) \to \mathbb{R}^3$ is another smooth regular curve parametrised by arclength, and that $\widetilde{k}(s)$ and $\widetilde{\tau}(s)$ are its curvature and torsion. Determine whether the following statements are true or false. Justify your answer in each case.

(i) If $\tau(s) = 0$ whenever it is defined, then α lies in a plane.

(ii) If $\tau(s)$ is defined and equal to 0 for all but one value of s in (a,b), then α lies in a plane.

(iii) If $k(s) = \widetilde{k}(s)$ for all s, $\tau(s)$ and $\widetilde{\tau}(s)$ are defined for all $s \neq s_0$, and $\tau(s) = \widetilde{\tau}(s)$ for all $s \neq s_0$, then α and $\widetilde{\alpha}$ are related by a rigid motion.

Paper 1, Section II

26H Differential Geometry

Let $n \geqslant 1$ be an integer.

(a) Show that $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ defines a submanifold of $\mathbb{R}^{n+1}$ and identify explicitly its tangent space $T_x\mathbb{S}^n$ for any $x \in \mathbb{S}^n$.

(b) Show that the matrix group $SO(n) \subset \mathbb{R}^{n^2}$ defines a submanifold. Identify explicitly the tangent space $T_R SO(n)$ for any $R \in SO(n)$.

(c) Given $v \in \mathbb{S}^n$, show that the set $S_v = \{R \in SO(n+1) : Rv = v\}$ defines a submanifold $S_v \subset SO(n+1)$ and compute its dimension. For $v \neq w$, is it ever the case that S_v and S_w are transversal?

[*You may use standard theorems from the course concerning regular values and transversality.*]

Paper 4, Section II

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define what it means for a curve $\gamma : I \to S$ to be a *geodesic*, where $I = (a, b)$ and $-\infty \leqslant a < b \leqslant \infty$.

(b) A geodesic $\gamma : I \to S$ is said to be *maximal* if any geodesic $\widetilde{\gamma} : \widetilde{I} \to S$ with $I \subset \widetilde{I}$ and $\widetilde{\gamma}|_I = \gamma$ satisfies $I = \widetilde{I}$. A surface is said to be *geodesically complete* if all maximal geodesics are defined on $I = (-\infty, \infty)$, otherwise, the surface is said to be *geodesically incomplete*. Give an example, with justification, of a non-compact geodesically complete surface S which is not a plane.

(c) Assume that along any maximal geodesic

$$\gamma : (-T_-, T_+) \to S,$$

the following holds:

$$T_\pm < \infty \implies \limsup_{s \to T_\pm} |K(\gamma(\pm s))| = \infty. \qquad (*)$$

Here K denotes the Gaussian curvature of S.

(i) Show that S is *inextendible*, i.e. if $\widetilde{S} \subset \mathbb{R}^3$ is a connected surface with $S \subset \widetilde{S}$, then $\widetilde{S} = S$.

(ii) Give an example of a surface S which is geodesically incomplete and satisfies $(*)$. Do all geodesically incomplete inextendible surfaces satisfy $(*)$? Justify your answer.

[*You may use facts about geodesics from the course provided they are clearly stated.*]

Paper 3, Section II

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define the *Gaussian curvature* K of S in terms of the coefficients of the first and second fundamental forms, computed with respect to a local parametrization $\phi(u, v)$ of S.

Prove the *Theorema Egregium*, i.e. show that the Gaussian curvature can be expressed entirely in terms of the coefficients of the first fundamental form and their first and second derivatives with respect to u and v.

(b) State the global Gauss–Bonnet theorem for a compact orientable surface S.

(c) Now assume that S is non-compact and diffeomorphic to $\mathbb{S}^2 \setminus \{(1,0,0)\}$ but that there is a point $p \in \mathbb{R}^3$ such that $S \cup \{p\}$ is a compact subset of $\mathbb{R}^3$. Is it necessarily the case that $\int_S K dA = 4/\pi$? Justify your answer.

Paper 2, Section II

25I Differential Geometry

Let $\gamma(t) : [a, b] \to \mathbb{R}^3$ denote a regular curve.

(a) Show that there exists a parametrization of γ by arc length.

(b) Under the assumption that the curvature is non-zero, define the *torsion* of γ. Give an example of two curves γ_1 and γ_2 in $\mathbb{R}^3$ whose curvature (as a function of arc length s) coincides and is non-vanishing, but for which the curves are <u>not</u> related by a rigid motion, i.e. such that $\gamma_1(s)$ is not identically $\rho_{(R,T)}(\gamma_2(s))$ where $R \in SO(3)$, $T \in \mathbb{R}^3$ and

$$\rho_{(R,T)}(v) := T + Rv.$$

(c) Give an example of a simple closed curve γ, other than a circle, which is preserved by a non-trivial rigid motion, i.e. which satisfies

$$\rho_{(R,T)}(v) \in \gamma([a, b]) \text{ for all } v \in \gamma([a, b])$$

for some choice of $R \in SO(3)$, $T \in \mathbb{R}^3$ with $(R, T) \neq (\mathrm{Id}, 0)$. Justify your answer.

(d) Now show that a simple closed curve γ which is preserved by a nontrivial smooth 1-parameter family of rigid motions is necessarily a circle, i.e. show the following:

Let $(R, T) : (-\epsilon, \epsilon) \to SO(3) \times \mathbb{R}^3$ be a regular curve. If for all $\tilde{t} \in (-\epsilon, \epsilon)$,

$$\rho_{(R(\tilde{t}),T(\tilde{t}))}(v) \in \gamma([a, b]) \text{ for all } v \in \gamma([a, b]),$$

then $\gamma([a, b])$ is a circle. [You may use the fact that the set of fixed points of a non-trivial rigid motion is either $\emptyset$ or a line $L \subset \mathbb{R}^3$.]

Paper 1, Section II

26I Differential Geometry

(a) Let $X \subset \mathbb{R}^n$ be a manifold and $p \in X$. Define the *tangent space* T_pX and show that it is a vector subspace of $\mathbb{R}^n$, independent of local parametrization, of dimension equal to $\dim X$.

(b) Now show that T_pX depends continuously on p in the following sense: if p_i is a sequence in X such that $p_i \to p \in X$, and $w_i \in T_{p_i}X$ is a sequence such that $w_i \to w \in \mathbb{R}^n$, then $w \in T_pX$. If $\dim X > 0$, show that all $w \in T_pX$ arise as such limits where p_i is a sequence in $X \setminus p$.

(c) Consider the set $X_a \subset \mathbb{R}^4$ defined by $X_a = \{x_1^2 + 2x_2^2 = a^2\} \cap \{x_3 = ax_4\}$, where $a \in \mathbb{R}$. Show that, for all $a \in \mathbb{R}$, the set X_a is a smooth manifold. Compute its dimension.

(d) For X_a as above, does T_pX_a depend continuously on p <u>and</u> a for all $a \in \mathbb{R}$? In other words, let $a_i \in \mathbb{R}$, $p_i \in X_{a_i}$ be sequences with $a_i \to a \in \mathbb{R}$, $p_i \to p \in X_a$. Suppose that $w_i \in T_{p_i}X_{a_i}$ and $w_i \to w \in \mathbb{R}^4$. Is it necessarily the case that $w \in T_pX_a$? Justify your answer.

Paper 2, Section II

23I Differential Geometry

Let $\alpha : I \to \mathbb{R}^3$ be a regular smooth curve. Define the *curvature* k and *torsion* τ of α and derive the Frenet formulae. Give the assumption which must hold for torsion to be well-defined, and state the Fundamental Theorem for curves in $\mathbb{R}^3$.

Let α be as above and $\tilde{\alpha} : I \to \mathbb{R}^3$ be another regular smooth curve with curvature $\tilde{k}$ and torsion $\tilde{\tau}$. Suppose $\tilde{k}(s) = k(s) \neq 0$ and $\tilde{\tau}(s) = \tau(s)$ for all $s \in I$, and that there exists a non-empty open subinterval $J \subset I$ such that $\tilde{\alpha}|_J = \alpha|_J$. Show that $\tilde{\alpha} = \alpha$.

Now let $S \subset \mathbb{R}^3$ be an oriented surface and let $\alpha : I \to S \subset \mathbb{R}^3$ be a regular smooth curve contained in S. Define *normal curvature* and *geodesic curvature*. When is α a geodesic? Give an example of a surface S and a geodesic α whose normal curvature vanishes identically. Must such a surface S contain a piece of a plane? Can such a geodesic be a simple closed curve? Justify your answers.

Show that if α is a geodesic and the Gaussian curvature of S satisfies $K \geqslant 0$, then we have the inequality $k(s) \leqslant 2|H(\alpha(s))|$, where H denotes the mean curvature of S and k the curvature of α. Give an example where this inequality is sharp.

Paper 3, Section II

23I Differential Geometry

Let $S \subset \mathbb{R}^N$ be a manifold and let $\alpha : [a, b] \to S \subset \mathbb{R}^N$ be a smooth regular curve on S. Define the *total length* $L(\alpha)$ and the *arc length* parameter s. Show that α can be reparametrized by arc length.

Let $S \subset \mathbb{R}^3$ denote a regular surface, let $p, q \in S$ be distinct points and let $\alpha : [a, b] \to S$ be a smooth regular curve such that $\alpha(a) = p$, $\alpha(b) = q$. We say that α is *length minimising* if for all smooth regular curves $\tilde{\alpha} : [a, b] \to S$ with $\tilde{\alpha}(a) = p$, $\tilde{\alpha}(b) = q$, we have $L(\tilde{\alpha}) \geqslant L(\alpha)$. By deriving a formula for the derivative of the energy functional corresponding to a variation of α, show that a length minimising curve is necessarily a geodesic. [You may use the following fact: given a smooth vector field $V(t)$ along α with $V(a) = V(b) = 0$, there exists a variation $\alpha(s, t)$ of α such that $\partial_s \alpha(s, t)|_{s=0} = V(t)$.]

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere and let S denote the surface $\mathbb{S}^2 \setminus (0, 0, 1)$. For which pairs of points $p, q \in S$ does there exist a length minimising smooth regular curve $\alpha : [a, b] \to S$ with $\alpha(a) = p$ and $\alpha(b) = q$? Justify your answer.

Paper 4, Section II

24I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Define the *exponential map* $\exp_p$ and compute its differential $d\exp_p|_0$. Deduce that $\exp_p$ is a local diffeomorphism.

Give an example of a surface S and a point $p \in S$ for which the exponential map $\exp_p$ fails to be defined globally on T_pS. Can this failure be remedied by extending the surface? In other words, for any such S, is there always a surface $S \subset \widehat{S} \subset \mathbb{R}^3$ such that the exponential map $\widehat{\exp}_p$ defined with respect to $\widehat{S}$ is globally defined on $T_pS = T_p\widehat{S}$?

State the version of the Gauss–Bonnet theorem with boundary term for a surface $S \subset \mathbb{R}^3$ and a closed disc $D \subset S$ whose boundary ∂D can be parametrized as a smooth closed curve in S.

Let $S \subset \mathbb{R}^3$ be a flat surface, i.e. $K = 0$. Can there exist a closed disc $D \subset S$, whose boundary ∂D can be parametrized as a smooth closed curve, and a surface $\tilde{S} \subset \mathbb{R}^3$ such that all of the following hold:

(i) $(S \setminus D) \cup \partial D \subset \tilde{S}$;

(ii) letting $\tilde{D}$ be $(\tilde{S} \setminus (S \setminus D)) \cup \partial D$, we have that $\tilde{D}$ is a closed disc in $\tilde{S}$ with boundary $\partial\tilde{D} = \partial D$;

(iii) the Gaussian curvature $\tilde{K}$ of $\tilde{S}$ satisfies $\tilde{K} \geqslant 0$, and there exists a $p \in \tilde{S}$ such that $\tilde{K}(p) > 0$?

Justify your answer.

Paper 1, Section II

25I Differential Geometry

Define what it means for a subset $X \subset \mathbb{R}^N$ to be a *manifold*.

For manifolds X and Y, state what it means for a map $f : X \to Y$ to be *smooth*. For such a smooth map, and $x \in X$, define the *differential map* df_x.

What does it mean for $y \in Y$ to be a *regular value* of f? Give an example of a map $f : X \to Y$ and a $y \in Y$ which is *not* a regular value of f.

Show that the set $SL_n(\mathbb{R})$ of $n \times n$ real-valued matrices with determinant 1 can naturally be viewed as a manifold $SL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$. What is its dimension? Show that matrix multiplication $f : SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$, defined by $f(A, B) = AB$, is smooth. [Standard theorems may be used without proof if carefully stated.] Describe the tangent space of $SL_n(\mathbb{R})$ at the identity $I \in SL_n(\mathbb{R})$ as a subspace of $\mathbb{R}^{n^2}$.

Show that if $n \geqslant 2$ then the set of real-valued matrices with determinant 0, viewed as a subset of $\mathbb{R}^{n^2}$, is not a manifold.

Paper 3, Section II

22G Differential Geometry

Explain what it means for an embedded surface S in $\mathbf{R}^3$ to be *minimal*. What is meant by an *isothermal* parametrization $\phi : U \to V \subset \mathbf{R}^3$ of an embedded surface $V \subset \mathbf{R}^3$? Prove that if ϕ is isothermal then $\phi(U)$ is minimal if and only if the components of ϕ are harmonic functions on U. [You may assume the formula for the mean curvature of a parametrized embedded surface,

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)},$$

where E, F, G (respectively e, f, g) are the coefficients of the first (respectively second) fundamental forms.]

Let S be an embedded connected minimal surface in $\mathbf{R}^3$ which is closed as a subset of $\mathbf{R}^3$, and let $\Pi \subset \mathbf{R}^3$ be a plane which is disjoint from S. Assuming that local isothermal parametrizations always exist, show that if the Euclidean distance between S and Π is attained at some point $P \in S$, i.e. $d(P, \Pi) = \inf_{Q \in S} d(Q, \Pi)$, then S is a plane parallel to Π.

Paper 4, Section II

23G Differential Geometry

For $S \subset \mathbf{R}^3$ a smooth embedded surface, define what is meant by a *geodesic curve* on S. Show that any geodesic curve $\gamma(t)$ has constant speed $|\dot{\gamma}(t)|$.

For any point $P \in S$, show that there is a parametrization $\phi : U \to V$ of some open neighbourhood V of P in S, with $U \subset \mathbf{R}^2$ having coordinates (u, v), for which the first fundamental form is

$$du^2 + G(u, v)dv^2,$$

for some strictly positive smooth function G on U. State a formula for the Gaussian curvature K of S in V in terms of G. If $K \equiv 0$ on V, show that G is a function of v only, and that we may reparametrize so that the metric is locally of the form $du^2 + dw^2$, for appropriate local coordinates (u, w).

[*You may assume that for any $P \in S$ and nonzero $\xi \in T_P S$, there exists (for some $\epsilon > 0$) a unique geodesic $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = P$ and $\dot{\gamma}(0) = \xi$, and that such geodesics depend smoothly on the initial conditions P and ξ.*]

Paper 2, Section II

23G Differential Geometry

If an embedded surface $S \subset \mathbf{R}^3$ contains a line L, show that the Gaussian curvature is non-positive at each point of L. Give an example where the Gaussian curvature is zero at each point of L.

Consider the helicoid S given as the image of $\mathbf{R}^2$ in $\mathbf{R}^3$ under the map

$$\phi(u, v) = (\sinh v \, \cos u, \, \sinh v \, \sin u, \, u).$$

What is the image of the corresponding Gauss map? Show that the Gaussian curvature at a point $\phi(u, v) \in S$ is given by $-1/\cosh^4 v$, and hence is strictly negative everywhere. Show moreover that there is a line in S passing through any point of S.

[*General results concerning the first and second fundamental forms on an oriented embedded surface $S \subset \mathbf{R}^3$ and the Gauss map may be used without proof in this question.*]

Paper 1, Section II

24G Differential Geometry

Define what is meant by the *regular values* and *critical values* of a smooth map $f : X \to Y$ of manifolds. State the Preimage Theorem and Sard's Theorem.

Suppose now that $\dim X = \dim Y$. If X is compact, prove that the set of regular values is open in Y, but show that this may not be the case if X is non-compact.

Construct an example with $\dim X = \dim Y$ and X compact for which the set of critical values is not a submanifold of Y.

[*Hint: You may find it helpful to consider the case $X = S^1$ and $Y = \mathbf{R}$. Properties of bump functions and the function e^{-1/x^2} may be assumed in this question.*]

Paper 4, Section II

21G Differential Geometry

Let $\mathrm{U}(n)$ denote the set of $n \times n$ unitary complex matrices. Show that $\mathrm{U}(n)$ is a smooth (real) manifold, and find its dimension. [You may use any general results from the course provided they are stated correctly.] For A any matrix in $\mathrm{U}(n)$ and H an $n \times n$ complex matrix, determine when H represents a tangent vector to $\mathrm{U}(n)$ at A.

Consider the tangent spaces to $\mathrm{U}(n)$ equipped with the metric induced from the standard (Euclidean) inner product $\langle \cdot, \cdot \rangle$ on the real vector space of $n \times n$ complex matrices, given by $\langle L, K \rangle = \mathrm{Re}\ \mathrm{trace}\,(LK^*)$, where Re denotes the real part and K^* denotes the conjugate transpose of K. Suppose that H represents a tangent vector to $\mathrm{U}(n)$ at the identity matrix I. Sketch an explicit construction of a *geodesic curve* on $\mathrm{U}(n)$ passing through I and with tangent direction H, giving a brief proof that the acceleration of the curve is always orthogonal to the tangent space to $\mathrm{U}(n)$.

[*Hint: You will find it easier to work directly with* $n \times n$ *complex matrices, rather than the corresponding* $2n \times 2n$ *real matrices.*]

Paper 3, Section II

21G Differential Geometry

Show that the surface S of revolution $x^2 + y^2 = \cosh^2 z$ in $\mathbb{R}^3$ is homeomorphic to a cylinder and has everywhere negative Gaussian curvature. Show moreover the existence of a closed geodesic on S.

Let $S \subset \mathbb{R}^3$ be an arbitrary embedded surface which is homeomorphic to a cylinder and has everywhere negative Gaussian curvature. By using a suitable version of the Gauss–Bonnet theorem, show that S contains at most one closed geodesic. [If required, appropriate forms of the Jordan curve theorem in the plane may also be used without proof.]

Paper 2, Section II
22G Differential Geometry

If U denotes a domain in $\mathbb{R}^2$, what is meant by saying that a smooth map $\phi : U \to \mathbb{R}^3$ is an *immersion*? Define what it means for such an immersion to be *isothermal*. Explain what it means to say that an immersed surface is *minimal*.

Let $\phi(u, v) = (x(u, v), y(u, v), z(u, v))$ be an isothermal immersion. Show that it is minimal if and only if x, y, z are harmonic functions of u, v. [You may use the formula for the mean curvature given in terms of the first and second fundamental forms, namely $H = (eG - 2fF + gE)/(2\{EG - F^2\})$.]

Produce an example of an immersed minimal surface which is not an open subset of a catenoid, helicoid, or a plane. Prove that your example does give an immersed minimal surface in $\mathbb{R}^3$.

Paper 1, Section II
22G Differential Geometry

Let $\Omega \subset \mathbb{R}^2$ be a domain (connected open subset) with boundary $\partial\Omega$ a continuously differentiable simple closed curve. Denoting by $A(\Omega)$ the area of Ω and $l(\partial\Omega)$ the length of the curve $\partial\Omega$, state and prove the isoperimetric inequality relating $A(\Omega)$ and $l(\partial\Omega)$ with optimal constant, including the characterization for equality. [You may appeal to Wirtinger's inequality as long as you state it precisely.]

Does the result continue to hold if the boundary $\partial\Omega$ is allowed finitely many points at which it is not differentiable? Briefly justify your answer by giving either a counterexample or an indication of a proof.

Paper 4, Section II

24G Differential Geometry

Let $I = [0, l]$ be a closed interval, $k(s), \tau(s)$ smooth real valued functions on I with k strictly positive at all points, and $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ a positively oriented orthonormal triad of vectors in $\mathbf{R}^3$. An application of the fundamental theorem on the existence of solutions to ODEs implies that there exists a unique smooth family of triples of vectors $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ for $s \in I$ satisfying the differential equations

$$\mathbf{t}' = k\mathbf{n}, \quad \mathbf{n}' = -k\mathbf{t} - \tau\mathbf{b}, \quad \mathbf{b}' = \tau\mathbf{n},$$

with initial conditions $\mathbf{t}(0) = \mathbf{t}_0$, $\mathbf{n}(0) = \mathbf{n}_0$ and $\mathbf{b}(0) = \mathbf{b}_0$, and that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ forms a positively oriented orthonormal triad for all $s \in I$. Assuming this fact, consider $\alpha : I \to \mathbf{R}^3$ defined by $\alpha(s) = \int_0^s \mathbf{t}(t)dt$; show that α defines a smooth immersed curve parametrized by arc-length, which has curvature and torsion given by $k(s)$ and $\tau(s)$, and that α is uniquely determined by this property up to rigid motions of $\mathbf{R}^3$. Prove that α is a plane curve if and only if τ is identically zero.

If $a > 0$, calculate the curvature and torsion of the smooth curve given by

$$\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c), \quad \text{where } c = \sqrt{a^2 + b^2}.$$

Suppose now that $\alpha : [0, 2\pi] \to \mathbf{R}^3$ is a smooth simple closed curve parametrized by arc-length with curvature everywhere positive. If both k and τ are constant, show that $k = 1$ and $\tau = 0$. If k is constant and τ is not identically zero, show that $k > 1$. Explain what it means for α to be *knotted*; if α is knotted and τ is constant, show that $k(s) > 2$ for some $s \in [0, 2\pi]$. [You may use standard results from the course if you state them precisely.]

Paper 3, Section II
24G Differential Geometry

Let $\alpha : I \to S$ be a parametrized curve on a smooth embedded surface $S \subset \mathbf{R}^3$. Define what is meant by a *vector field* V along α and the concept of such a vector field being *parallel*. If V and W are both parallel vector fields along α, show that the inner product $\langle V(t), W(t) \rangle$ is constant.

Given a local parametrization $\phi : U \to S$, define the *Christoffel symbols* Γ^i_{jk} on U. Given a vector $v_0 \in T_{\alpha(0)}S$, prove that there exists a unique parallel vector field $V(t)$ along α with $V(0) = v_0$ (recall that $V(t)$ is called the *parallel transport* of v_0 along α).

Suppose now that the image of α also lies on another smooth embedded surface $S' \subset \mathbf{R}^3$ and that $T_{\alpha(t)}S = T_{\alpha(t)}S'$ for all $t \in I$. Show that parallel transport of a vector v_0 is the same whether calculated on S or S'. Suppose S is the unit sphere in $\mathbf{R}^3$ with centre at the origin and let $\alpha : [0, 2\pi] \to S$ be the curve on S given by

$$\alpha(t) = (\sin\phi\cos t,\ \sin\phi\sin t,\ \cos\phi)$$

for some fixed angle ϕ. Suppose $v_0 \in T_P S$ is the unit tangent vector to α at $P = \alpha(0) = \alpha(2\pi)$ and let v_1 be its image in $T_P S$ under parallel transport along α. Show that the angle between v_0 and v_1 is $2\pi\cos\phi$.
[*Hint: You may find it useful to consider the circular cone S' which touches the sphere S along the curve α.*]

Paper 2, Section II
25G Differential Geometry

Define the terms *Gaussian curvature* K and *mean curvature* H for a smooth embedded oriented surface $S \subset \mathbf{R}^3$. [You may assume the fact that the derivative of the Gauss map is self-adjoint.] If $K = H^2$ at all points of S, show that both H and K are locally constant. [*Hint: Use the symmetry of second partial derivatives of the field of unit normal vectors.*]

If $K = H^2 = 0$ at all points of S, show that the unit normal vector $\mathbf{N}$ to S is locally constant and that S is locally contained in a plane. If $K = H^2$ is a strictly positive constant on S and $\phi : U \to S$ is a local parametrization (where U is connected) on S with unit normal vector $\mathbf{N}(u, v)$ for $(u, v) \in U$, show that $\phi(u, v) + \mathbf{N}(u, v)/H$ is constant on U. Deduce that S is locally contained in a sphere of radius $1/|H|$.

If S is connected with $K = H^2$ at all points of S, deduce that S is contained in either a plane or a sphere.

Paper 1, Section II

25G Differential Geometry

Define the concepts of (smooth) *manifold* and *manifold with boundary* for subsets of $\mathbf{R}^N$.

Let $X \subset \mathbf{R}^6$ be the subset defined by the equations

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1, \quad x_4^2 - x_5^2 - x_6^2 = -1.$$

Prove that X is a manifold of dimension four.

For $a > 0$, let $B(a) \subset \mathbf{R}^6$ denote the spherical ball $x_1^2 + \ldots + x_6^2 \leqslant a$. Prove that $X \cap B(a)$ is empty if $a < 2$, is a manifold diffeomorphic to $S^2 \times S^1$ if $a = 2$, and is a manifold with boundary if $a > 2$, with each component of the boundary diffeomorphic to $S^2 \times S^1$.

[You may quote without proof any general results from lectures that you may need.]

Paper 4, Section II

24H Differential Geometry

Define what is meant by the *geodesic curvature* k_g of a regular curve $\alpha : I \to S$ parametrized by arc length on a smooth oriented surface $S \subset \mathbf{R}^3$. If S is the unit sphere in $\mathbf{R}^3$ and $\alpha : I \to S$ is a parametrized geodesic circle of radius ϕ, with $0 < \phi < \pi/2$, justify the fact that $|k_g| = \cot \phi$.

State the general form of the Gauss–Bonnet theorem with boundary on an oriented surface S, explaining briefly the terms which occur.

Let $S \subset \mathbf{R}^3$ now denote the circular cone given by $z > 0$ and $x^2 + y^2 = z^2 \tan^2 \phi$, for a fixed choice of ϕ with $0 < \phi < \pi/2$, and with a fixed choice of orientation. Let $\alpha : I \to S$ be a simple closed piecewise regular curve on S, with (signed) exterior angles $\theta_1, \ldots, \theta_N$ at the vertices (that is, θ_i is the angle between limits of tangent directions, with sign determined by the orientation). Suppose furthermore that the smooth segments of α are geodesic curves. What possible values can $\theta_1 + \cdots + \theta_N$ take? Justify your answer.

[You may assume that a simple closed curve in $\mathbf{R}^2$ bounds a region which is homeomorphic to a disc. Given another simple closed curve in the interior of this region, you may assume that the two curves bound a region which is homeomorphic to an annulus.]

Paper 3, Section II

24H Differential Geometry

We say that a parametrization $\phi : U \to S \subset \mathbf{R}^3$ of a smooth surface S is *isothermal* if the coefficients of the first fundamental form satisfy $F = 0$ and $E = G = \lambda(u,v)^2$, for some smooth non-vanishing function λ on U. For an isothermal parametrization, prove that

$$\phi_{uu} + \phi_{vv} = 2\lambda^2 H \mathbf{N},$$

where $\mathbf{N}$ denotes the unit normal vector and H the mean curvature, which you may assume is given by the formula

$$H = \frac{g+e}{2\lambda^2},$$

where $g = -\langle \mathbf{N}_u, \phi_u \rangle$ and $e = -\langle \mathbf{N}_v, \phi_v \rangle$ are coefficients in the second fundamental form.

Given a parametrization $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$ of a surface $S \subset \mathbf{R}^3$, we consider the complex valued functions on U:

$$\theta_1 = x_u - ix_v, \quad \theta_2 = y_u - iy_v, \quad \theta_3 = z_u - iz_v. \tag{1}$$

Show that ϕ is isothermal if and only if $\theta_1^2 + \theta_2^2 + \theta_3^2 = 0$. If ϕ is isothermal, show that S is a minimal surface if and only if $\theta_1, \theta_2, \theta_3$ are holomorphic functions of the complex variable $\zeta = u + iv$.

Consider the holomorphic functions on $D := \mathbf{C} \setminus \mathbf{R}_{\geqslant 0}$ (with complex coordinate $\zeta = u + iv$ on $\mathbf{C}$) given by

$$\theta_1 := \frac{1}{2}(1 - \zeta^{-2}), \quad \theta_2 := -\frac{i}{2}(1 + \zeta^{-2}), \quad \theta_3 := -\zeta^{-1}. \tag{2}$$

Find a smooth map $\phi(u,v) = (x(u,v), y(u,v), z(u,v)) : D \to \mathbf{R}^3$ for which $\phi(-1,0) = \mathbf{0}$ and the θ_i defined by (2) satisfy the equations (1). Show furthermore that ϕ extends to a smooth map $\widetilde{\phi} : \mathbf{C}^* \to \mathbf{R}^3$. If $w = x + iy$ is the complex coordinate on $\mathbf{C}$, show that

$$\widetilde{\phi}(\exp(iw)) = (\cosh y \cos x + 1,\ \cosh y \sin x,\ y).$$

Paper 2, Section II

25H Differential Geometry

Let $\alpha : [0, L] \to \mathbf{R}^3$ be a regular curve parametrized by arc length having nowhere-vanishing curvature. State the Frenet relations between the tangent, normal and binormal vectors at a point, and their derivatives.

Let $S \subset \mathbf{R}^3$ be a smooth oriented surface. Define the *Gauss map* $N : S \to S^2$, and show that its derivative at $P \in S$, $dN_P : T_P S \to T_P S$, is self-adjoint. Define the *Gaussian curvature* of S at P.

Now suppose that $\alpha : [0, L] \to \mathbf{R}^3$ has image in S and that its normal curvature is zero for all $s \in [0, L]$. Show that the Gaussian curvature of S at a point $P = \alpha(s)$ of the curve is $K(P) = -\tau(s)^2$, where $\tau(s)$ denotes the torsion of the curve.

If $S \subset \mathbf{R}^3$ is a standard embedded torus, show that there is a curve on S for which the normal curvature vanishes and the Gaussian curvature of S is zero at all points of the curve.

Paper 1, Section II

25H Differential Geometry

For $f : X \to Y$ a smooth map of manifolds, define the concepts of *critical point*, *critical value* and *regular value*.

With the obvious identification of $\mathbf{C}$ with $\mathbf{R}^2$, and hence also of $\mathbf{C}^3$ with $\mathbf{R}^6$, show that the complex-valued polynomial $z_1^3 + z_2^2 + z_3^2$ determines a smooth map $f : \mathbf{R}^6 \to \mathbf{R}^2$ whose only critical point is at the origin. Hence deduce that $V := f^{-1}((0,0)) \setminus \{\mathbf{0}\} \subset \mathbf{R}^6$ is a 4-dimensional manifold, and find the equations of its tangent space at any given point $(z_1, z_2, z_3) \in V$.

Now let $S^5 \subset \mathbf{C}^3 = \mathbf{R}^6$ be the unit 5-sphere, defined by $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Given a point $P = (z_1, z_2, z_3) \in S^5 \cap V$, by considering the vector $(2z_1, 3z_2, 3z_3) \in \mathbf{C}^3 = \mathbf{R}^6$ or otherwise, show that not all tangent vectors to V at P are tangent to S^5. Deduce that $S^5 \cap V \subset \mathbf{R}^6$ is a compact three-dimensional manifold.

[Standard results may be quoted without proof if stated carefully.]

Paper 4, Section II

24I Differential Geometry

For manifolds $X, Y \subset \mathbb{R}^n$, define the terms *tangent space* to X at a point $x \in X$ and *derivative* df_x of a smooth map $f : X \to Y$. State the Inverse Function Theorem for smooth maps between manifolds without boundary.

Now let X be a submanifold of Y and $f : X \to Y$ the inclusion map. By considering the map $f^{-1} : f(X) \to X$, or otherwise, show that df_x is injective for each $x \in X$.

Show further that there exist local coordinates around x and around $y = f(x)$ such that f is given in these coordinates by

$$(x_1, \ldots, x_l) \in \mathbb{R}^l \mapsto (x_1, \ldots, x_l, 0, \ldots, 0) \in \mathbb{R}^k \,,$$

where $l = \dim X$ and $k = \dim Y$. [You may assume that any open ball in $\mathbb{R}^l$ is diffeomorphic to $\mathbb{R}^l$.]

Paper 3, Section II

24I Differential Geometry

For a surface $S \subset \mathbb{R}^3$, define what is meant by the *exponential mapping* $\exp_p$ at $p \in S$, *geodesic polar coordinates* (r, θ) and *geodesic circles.*

Let E, F, G be the coefficients of the first fundamental form in geodesic polar coordinates (r, θ). Prove that $\lim_{r \to 0} \sqrt{G}(r, \theta) = 0$ and $\lim_{r \to 0} (\sqrt{G})_r(r, \theta) = 1$. Give an expression for the Gaussian curvature K in terms of G.

Prove that the Gaussian curvature at a point $p \in S$ satisfies

$$K(p) = \lim_{r \to 0} \frac{12(\pi r^2 - A_p(r))}{\pi r^4} \,,$$

where $A_p(r)$ is the area of the region bounded by the geodesic circle of radius r centred at p.

[You may assume that $E = 1$, $F = 0$ and $d(\exp_p)_0$ is an isometry. Taylor's theorem with any form of the remainder may be assumed if accurately stated.]

Paper 2, Section II
25I Differential Geometry

Define the *Gauss map* N for an oriented surface $S \subset \mathbb{R}^3$. Show that at each $p \in S$ the derivative of the Gauss map

$$dN_p : T_pS \to T_{N(p)}S^2 = T_pS$$

is self-adjoint. Define the *principal curvatures* k_1, k_2 of S.

Now suppose that S is compact (and without boundary). By considering the square of the distance to the origin, or otherwise, prove that S has a point p with $k_1(p)k_2(p) > 0$.

[You may assume that the intersection of S with a plane through the normal direction at $p \in S$ contains a regular curve through p.]

Paper 1, Section II
25I Differential Geometry

Define the *geodesic curvature* k_g of a regular curve in an oriented surface $S \subset \mathbb{R}^3$. When is $k_g = 0$ along a curve?

Explain briefly what is meant by the *Euler characteristic* χ of a compact surface $S \subset \mathbb{R}^3$. State the global Gauss–Bonnet theorem with boundary terms.

Let S be a surface with positive Gaussian curvature that is diffeomorphic to the sphere S^2 and let γ_1,γ_2 be two disjoint simple closed curves in S. Can both γ_1 and γ_2 be geodesics? Can both γ_1 and γ_2 have constant geodesic curvature? Justify your answers.

[You may assume that the complement of a simple closed curve in S^2 consists of two open connected regions.]

Paper 4, Section I

7D Dynamical Systems

Describe the different types of bifurcation from steady states of a one-dimensional map of the form $x_{n+1} = f(x_n)$, and give examples of simple equations exhibiting each type.

Consider the map $x_{n+1} = \alpha x_n^2(1 - x_n)$, $0 < x_n < 1$. What is the maximum value of α for which the interval is mapped into itself?

Show that as α increases from zero to its maximum value there is a saddle-node bifurcation and a period-doubling bifurcation, and determine the values of α for which they occur.

Paper 3, Section I

7D Dynamical Systems

State without proof Lyapunov's first theorem, carefully defining all the terms that you use.

Consider the dynamical system

$$\begin{aligned}\dot{x} &= -2x + y - xy + 3y^2 - xy^2 + x^3\,,\\ \dot{y} &= -2y - x - y^2 - 3xy + 2x^2 y\,.\end{aligned}$$

By choosing a Lyapunov function $V(x, y) = x^2 + y^2$, prove that the origin is asymptotically stable.

By factorising the expression for $\dot{V}$, or otherwise, show that the basin of attraction of the origin includes the set $V < 7/4$.

Paper 2, Section I
7D Dynamical Systems
Consider the dynamical system

$$\dot{x} = \mu x + x^3 - axy, \quad \dot{y} = \mu - x^2 - y\,,$$

where a is a constant.

(a) Show that there is a bifurcation from the fixed point $(0,\mu)$ at $\mu = 0$.

(b) Find the extended centre manifold at leading non-trivial order in x. Hence find the type of bifurcation, paying particular attention to the special values $a = 1$ and $a = -1$. [*Hint. At leading order, the extended centre manifold is of the form* $y = \mu + \alpha x^2 + \beta\mu x^2 + \gamma x^4$, *where* α, β, γ *are constants to be determined.*]

Paper 1, Section I
7D Dynamical Systems
State the Poincaré–Bendixson theorem.

A model of a chemical process obeys the second-order system

$$\dot{x} = 1 - x(1+a) + x^2 y, \quad \dot{y} = ax - x^2 y\,,$$

where $a > 0$. Show that there is a unique fixed point at $(x,y) = (1,a)$ and that it is unstable if $a > 2$. Show that trajectories enter the region bounded by the lines $x = 1/q$, $y = 0$, $y = aq$ and $x + y = 1 + aq$, provided $q > (1+a)$. Deduce that there is a periodic orbit when $a > 2$.

Paper 4, Section II
14D Dynamical Systems
What is meant by the statement that a continuous map of an interval I into itself has a *horseshoe*? State without proof the properties of such a map.

Define the property of *chaos* of such a map according to Glendinning.

A continuous map $f : I \to I$ has a periodic orbit of period 5, in which the elements x_j, $j = 1, \ldots, 5$ satisfy $x_j < x_{j+1}$, $j = 1, \ldots, 4$ and the points are visited in the order $x_1 \to x_3 \to x_4 \to x_2 \to x_5 \to x_1$. Show that the map is chaotic. [The Intermediate Value theorem can be used without proof.]

Paper 3, Section II

14D Dynamical Systems

Consider the dynamical system

$$\ddot{x} - (a - bx)\dot{x} + x - x^2 = 0, \quad a, b > 0\,. \tag{1}$$

(a) Show that the fixed point at the origin is an unstable node or focus, and that the fixed point at $x = 1$ is a saddle point.

(b) By considering the phase plane $(x, \dot{x})$, or otherwise, show graphically that the maximum value of x for any periodic orbit is less than one.

(c) By writing the system in terms of the variables x and $z = \dot{x} - (ax - bx^2/2)$, or otherwise, show that for any periodic orbit $\mathcal{C}$

$$\oint_{\mathcal{C}} (x - x^2)(2ax - bx^2)\, dt = 0\,. \tag{2}$$

Deduce that if $a/b > 1/2$ there are no periodic orbits.

(d) If $a = b = 0$ the system (1) is Hamiltonian and has homoclinic orbit

$$X(t) = \frac{1}{2}\left(3\tanh^2\left(\frac{t}{2}\right) - 1\right)\,, \tag{3}$$

which approaches $X = 1$ as $t \to \pm\infty$. Now suppose that a, b are very small and that we seek the value of a/b corresponding to a periodic orbit very close to $X(t)$. By using equation (3) in equation (2), find an approximation to the largest value of a/b for a periodic orbit when a, b are very small.

[*Hint. You may use the fact that* $(1 - X) = \frac{3}{2}\mathrm{sech}^2(\frac{t}{2}) = 3\frac{d}{dt}(\tanh(\frac{t}{2}))$]

Paper 4, Section II

35B Electrodynamics

The charge and current densities are given by $\rho(t, \mathbf{x}) \neq 0$ and $\mathbf{j}(t, \mathbf{x})$ respectively. The electromagnetic scalar and vector potentials are given by $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ respectively. Explain how one can regard $j^\mu = (\rho, \mathbf{j})$ as a four-vector that obeys the current conservation rule $\partial_\mu j^\mu = 0$.

In the Lorenz gauge $\partial_\mu A^\mu = 0$, derive the wave equation that relates $A^\mu = (\phi, \mathbf{A})$ to j^μ and hence show that it is consistent to treat A^μ as a four-vector.

In the Lorenz gauge, with $j^\mu = 0$, a plane wave solution for A^μ is given by

$$A^\mu = \epsilon^\mu \exp(ik_\nu x^\nu)\,,$$

where ϵ^μ, k^μ and x^μ are four-vectors with

$$\epsilon^\mu = (\epsilon^0, \boldsymbol{\epsilon}), \quad k^\mu = (k^0, \mathbf{k}), \quad x^\mu = (t, \mathbf{x})\,.$$

Show that $k_\mu k^\mu = k_\mu \epsilon^\mu = 0$.

Interpret the components of k^μ in terms of the frequency and wavelength of the wave.

Find what residual gauge freedom there is and use it to show that it is possible to set $\epsilon^0 = 0$. What then is the physical meaning of the components of $\boldsymbol{\epsilon}$?

An observer at rest in a frame S measures the angular frequency of a plane wave travelling parallel to the z-axis to be ω. A second observer travelling at velocity v in S parallel to the z-axis measures the radiation to have frequency ω'. Express ω' in terms of ω.

Paper 3, Section II
36B Electrodynamics

The non-relativistic Larmor formula for the power, P, radiated by a particle of charge q and mass m that is being accelerated with an acceleration $\mathbf{a}$ is

$$P = \frac{\mu_0}{6\pi} q^2 |\mathbf{a}|^{\mathbf{2}} .$$

Starting from the Liénard–Wiechert potentials, sketch a derivation of this result. Explain briefly why the relativistic generalization of this formula is

$$P = \frac{\mu_0}{6\pi} \frac{q^2}{m^2} \left(\frac{dp^\mu}{d\tau} \frac{dp^\nu}{d\tau} \eta_{\mu\nu} \right) ,$$

where p^μ is the relativistic momentum of the particle and τ is the proper time along the worldline of the particle.

A particle of mass m and charge q moves in a plane perpendicular to a constant magnetic field B. At time $t = 0$ as seen by an observer $\mathbf{O}$ at rest, the particle has energy $E = \gamma m$. At what rate is electromagnetic energy radiated by this particle?

At time t according to the observer $\mathbf{O}$, the particle has energy $E' = \gamma' m$. Find an expression for γ' in terms of γ and t.

Paper 1, Section II
36B Electrodynamics

A particle of mass m and charge q moves relativistically under the influence of a constant electric field E in the positive z-direction, and a constant magnetic field B also in the positive z-direction.

In some inertial observer's coordinate system, the particle starts at

$$x = R, \quad y = 0, \quad z = 0, \quad t = 0,$$

with velocity given by

$$\dot{x} = 0, \quad \dot{y} = u, \quad \dot{z} = 0,$$

where the dot indicates differentiation with respect to the proper time of the particle. Show that the subsequent motion of the particle, as seen by the inertial observer, is a helix.

a) What is the radius of the helix as seen by the inertial observer?

b) What are the x and y coordinates of the axis of the helix?

c) What is the z coordinate of the particle after a proper time τ has elapsed, as measured by the particle?

Paper 4, Section II
37C Fluid Dynamics II

A steady, two-dimensional flow in the region $y > 0$ takes the form $(u, v) = (Ex, -Ey)$ at large y, where E is a positive constant. The boundary at $y = 0$ is rigid and no-slip. Consider the velocity field $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ with stream function $\psi = Ex\delta f(\eta)$, where $\eta = y/\delta$ and $\delta = (\nu/E)^{1/2}$ and ν is the kinematic viscosity. Show that this velocity field satisfies the Navier–Stokes equations provided that $f(\eta)$ satisfies

$$f''' + ff'' - (f')^2 = -1\,.$$

What are the conditions on f at $\eta = 0$ and as $\eta \to \infty$?

Paper 2, Section II
37C Fluid Dynamics II

An incompressible viscous liquid occupies the long thin region $0 \leqslant y \leqslant h(x)$ for $0 \leqslant x \leqslant \ell$, where $h(x) = d_1 + \alpha x$ with $h(0) = d_1$, $h(\ell) = d_2 < d_1$ and $d_1 \ll \ell$. The top boundary at $y = h(x)$ is rigid and stationary. The bottom boundary at $y = 0$ is rigid and moving at velocity $(U, 0, 0)$. Fluid can move in and out of the ends $x = 0$ and $x = \ell$, where the pressure is the same, namely p_0.

Explaining the approximations of lubrication theory as you use them, find the velocity profile in the long thin region, and show that the volume flux Q (per unit width in the z-direction) is

$$Q = \frac{Ud_1d_2}{d_1 + d_2}\,.$$

Find also the value of $h(x)$ (i) where the pressure is maximum, (ii) where the tangential viscous stress on the bottom $y = 0$ vanishes, and (iii) where the tangential viscous stress on the top $y = h(x)$ vanishes.

Paper 3, Section II

38C Fluid Dynamics II

For two Stokes flows $\mathbf{u}^{(1)}(\mathbf{x})$ and $\mathbf{u}^{(2)}(\mathbf{x})$ inside the same volume V with different boundary conditions on its boundary S, prove the reciprocal theorem

$$\int_S \sigma_{ij}^{(1)} n_j u_i^{(2)} \, dS = \int_S \sigma_{ij}^{(2)} n_j u_i^{(1)} \, dS \,,$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are the stress fields associated with the flows.

When a rigid sphere of radius a translates with velocity $\mathbf{U}$ through unbounded fluid at rest at infinity, it may be shown that the traction per unit area, $\boldsymbol{\sigma} \cdot \mathbf{n}$, exerted by the sphere on the fluid has the uniform value $3\mu\mathbf{U}/2a$ over the sphere surface. Find the drag on the sphere.

Suppose that the same sphere is now free of external forces and is placed with its centre at the origin in an unbounded Stokes flow given in the absence of the sphere as $\mathbf{u}^*(\mathbf{x})$. By applying the reciprocal theorem to the perturbation to the flow generated by the presence of the sphere, and assuming this tends to zero sufficiently rapidly at infinity, show that the instantaneous velocity of the centre of the sphere is

$$\frac{1}{4\pi a^2} \int \mathbf{u}^*(\mathbf{x}) \, dS \,,$$

where the integral is taken over the sphere of radius a.

Paper 1, Section II

38C Fluid Dynamics II

Define the strain-rate tensor e_{ij} in terms of the velocity components u_i. Write down the relation between e_{ij}, the pressure p and the stress σ_{ij} in an incompressible Newtonian fluid of viscosity μ. Show that the local rate of stress-working $\sigma_{ij}\partial u_i/\partial x_j$ is equal to the local rate of dissipation $2\mu e_{ij}e_{ij}$.

An incompressible fluid of density ρ and viscosity μ occupies the semi-infinite region $y > 0$ above a rigid plane boundary $y = 0$ which oscillates with velocity $(V\cos\omega t, 0, 0)$. The fluid is at rest at infinity. Determine the velocity field produced by the boundary motion after any transients have decayed.

Show that the time-averaged rate of dissipation is

$$\tfrac{1}{4}\sqrt{2}V^2 \left(\mu\rho\omega\right)^{1/2}$$

per unit area of the boundary. Verify that this is equal to the time average of the rate of working by the boundary on the fluid per unit area.

Paper 4, Section I

8E Further Complex Methods

Use the Laplace kernel method to write integral representations in the complex t-plane for two linearly independent solutions of the confluent hypergeometric equation

$$z\frac{d^2w(z)}{dz^2} + (c-z)\,\frac{dw(z)}{dz} - aw(z) = 0\,,$$

in the case that $\mathrm{Re}(z) > 0$, $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, a and $c-a$ are not integers.

Paper 3, Section I

8E Further Complex Methods

The Beta function, denoted by $B(z_1, z_2)$, is defined by

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}\,, \quad z_1, z_2 \in \mathbb{C}\,,$$

where $\Gamma(z)$ denotes the Gamma function. It can be shown that

$$B(z_1, z_2) = \int_0^\infty \frac{v^{z_2-1}\,dv}{(1+v)^{z_1+z_2}}\,, \quad \mathrm{Re}\ z_1 > 0\,,\ \mathrm{Re}\ z_2 > 0\,.$$

By computing this integral for the particular case of $z_1+z_2 = 1$, and by employing analytic continuation, deduce that $\Gamma(z)$ satisfies the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,, \quad z \in \mathbb{C}.$$

Paper 2, Section I

8E Further Complex Methods

The hypergeometric function $F(a, b; c; z)$ is defined as the particular solution of the second order linear ODE characterised by the Papperitz symbol

$$\mathrm{P}\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{array}\right\}$$

that is analytic at $z=0$ and satisfies $F(a, b; c; 0)=1$.

Using the fact that a second solution $w(z)$ of the above ODE is of the form

$$w(z)=z^{1-c} u(z)\,,$$

where $u(z)$ is analytic in the neighbourhood of the origin, express $w(z)$ in terms of F.

Paper 1, Section I

8E Further Complex Methods

Recall that if $f(z)$ is analytic in a neighbourhood of $z_0 \neq 0$, then

$$f(z)+\overline{f(z_0)}=2 u\left(\frac{z+\overline{z_0}}{2}, \frac{z-\overline{z_0}}{2 i}\right),$$

where $u(x, y)$ is the real part of $f(z)$. Use this fact to construct the imaginary part of an analytic function whose real part is given by

$$u(x, y)=y \int_{-\infty}^{\infty} \frac{g(t)\, d t}{(t-x)^2+y^2}, \quad x, y \in \mathbb{R},\ y \neq 0\,,$$

where $g(t)$ is real and has sufficient smoothness and decay.

Paper 2, Section II

14E Further Complex Methods

Let the complex function $q(x,t)$ satisfy

$$i\frac{\partial q(x,t)}{\partial t}+\frac{\partial^2 q(x,t)}{\partial x^2}=0\,,\quad 0<x<\infty\,,\ 0<t<T\,,$$

where T is a positive constant. The unified transform method implies that the solution of any well-posed problem for the above equation is given by

$$\begin{aligned}q(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-ik^2t}\hat{q}_0(k)dk\\&-\frac{1}{2\pi}\int_L e^{ikx-ik^2t}\left[k\tilde{g}_0(ik^2,t)-i\tilde{g}_1(ik^2,t)\right]dk\,,\end{aligned}\tag{1}$$

where L is the union of the rays $(i\infty,0)$ and $(0,\infty)$, $\hat{q}_0(k)$ denotes the Fourier transform of the initial condition $q_0(x)$, and $\tilde{g}_0$, $\tilde{g}_1$ denote the t-transforms of the boundary values $q(0,t)$, $q_x(0,t)$:

$$\hat{q}_0(k)=\int_0^{\infty}e^{-ikx}q_0(x)dx,\quad \mathrm{Im}\ k\leqslant 0\,,$$

$$\tilde{g}_0(k,t)=\int_0^t e^{ks}q(0,s)ds\,,\quad \tilde{g}_1(k,t)=\int_0^t e^{ks}q_x(0,s)ds\,,\quad k\in\mathbb{C}\,,\quad 0<t<T\,.$$

Furthermore, $q_0(x)$, $q(0,t)$ and $q_x(0,t)$ are related via the so-called global relation

$$e^{ik^2t}\hat{q}(k,t)=\hat{q}_0(k)+k\tilde{g}_0(ik^2,t)-i\tilde{g}_1(ik^2,t)\,,\quad \mathrm{Im}\ k\leqslant 0\,,\tag{2}$$

where $\hat{q}(k,t)$ denotes the Fourier transform of $q(x,t)$.

(a) Assuming the validity of (1) and (2), use the global relation to eliminate $\tilde{g}_1$ from equation (1).

(b) For the particular case that

$$q_0(x)=e^{-a^2x}\,,\quad 0<x<\infty\,;\quad q(0,t)=\cos bt\,,\quad 0<t<T\,,$$

where a and b are real numbers, use the representation obtained in (a) to express the solution in terms of an integral along the real axis and an integral along L (you should not attempt to evaluate these integrals). Show that it is possible to deform these two integrals to a single integral along a new contour $\tilde{L}$, which you should sketch.

[You may assume the validity of Jordan's lemma.]

Paper 1, Section II

25I Differential Geometry

Let X and Y be manifolds and $f : X \to Y$ a smooth map. Define the notions *critical point*, *critical value*, *regular value* of f. Prove that if y is a regular value of f, then $f^{-1}(y)$ (if non-empty) is a smooth manifold of dimension $\dim X - \dim Y$.

[The Inverse Function Theorem may be assumed without proof if accurately stated.]

Let $M_n(\mathbb{R})$ be the set of all real $n \times n$ matrices and $\mathrm{SO}(n) \subset M_n(\mathbb{R})$ the group of all orthogonal matrices with determinant 1. Show that $\mathrm{SO}(n)$ is a smooth manifold and find its dimension.

Show further that $\mathrm{SO}(n)$ is compact and that its tangent space at $A \in \mathrm{SO}(n)$ is given by all matrices H such that $AH^t + HA^t = 0$.

Paper 2, Section II

25I Differential Geometry

Let $\alpha : I \to \mathbb{R}^3$ be a smooth curve parametrized by arc-length, with $\alpha''(s) \neq 0$ for all $s \in I$. Define what is meant by the *Frenet frame* $t(s), n(s), b(s)$, the *curvature* and *torsion* of α. State and prove the Frenet formulae.

By considering $\langle \alpha, t \times n \rangle$, or otherwise, show that, if for each $s \in I$ the vectors $\alpha(s)$, $t(s)$ and $n(s)$ are linearly dependent, then $\alpha(s)$ is a plane curve.

State and prove the isoperimetric inequality for C^1 regular plane curves.

[You may assume Wirtinger's inequality, provided you state it accurately.]

Paper 3, Section II

24I Differential Geometry

For an oriented surface S in $\mathbb{R}^3$, define the *Gauss map*, the *second fundamental form* and the *normal curvature* in the direction $w \in T_pS$ at a point $p \in S$.

Let $\tilde{k}_1, \ldots, \tilde{k}_m$ be normal curvatures at p in the directions $v_1, \ldots, v_m$, such that the angle between v_i and v_{i+1} is π/m for each $i = 1, \ldots, m-1$ ($m \geqslant 2$). Show that

$$\tilde{k}_1 + \ldots + \tilde{k}_m = mH\,,$$

where H is the mean curvature of S at p.

What is a minimal surface? Show that if S is a minimal surface, then its Gauss map N at each point $p \in S$ satisfies

$$\langle dN_p(w_1), dN_p(w_2)\rangle = \mu(p)\langle w_1, w_2\rangle\,, \quad \text{for all } w_1, w_2 \in T_pS\,, \qquad (*)$$

where $\mu(p) \in \mathbb{R}$ depends only on p. Conversely, if the identity $(*)$ holds at each point in S, must S be minimal? Justify your answer.

Paper 4, Section II

24I Differential Geometry

Define what is meant by a *geodesic*. Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the *geodesic curvature* k_g of a smooth curve $\gamma : I \to S$ parametrized by arc-length.

Explain without detailed proofs what are the *exponential map* $\exp_p$ and the *geodesic polar coordinates* (r, θ) at $p \in S$. Determine the derivative $d(\exp_p)_0$. Prove that the coefficients of the first fundamental form of S in the geodesic polar coordinates satisfy

$$E = 1\,, \quad F = 0\,, \quad G(0,\theta) = 0\,, \quad (\sqrt{G})_r(0,\theta) = 1\,.$$

State the global Gauss–Bonnet formula for compact surfaces with boundary. [You should identify all terms in the formula.]

Suppose that S is homeomorphic to a cylinder $S^1 \times \mathbb{R}$ and has negative Gaussian curvature at each point. Prove that S has at most one simple (i.e. without self-intersections) closed geodesic.

[Basic properties of geodesics may be assumed, if accurately stated.]

Paper 1, Section II
25H Differential Geometry

(i) State the definition of *smooth manifold with boundary* and define the notion of *boundary*. Show that the boundary ∂X is a manifold (without boundary) with $\dim \partial X = \dim X - 1$.

(ii) Let $0 < a < 1$ and let x_1, x_2, x_3, x_4 denote Euclidean coordinates on $\mathbb{R}^4$. Show that the set

$$X = \{x_1^2+x_2^2+x_3^2-x_4^2 \leqslant a\} \cap \{x_1^2+x_2^2+x_3^2+x_4^2 = 1\} \cap \{x_1^2+2\,x_2^2+x_3^2+x_4^2 = 3/2\}$$

is a manifold with boundary and compute its dimension. You may appeal to standard results concerning regular values of smooth functions.

(iii) Determine if the following statements are true or false, giving reasons:

a. If X and Y are manifolds, $f : X \to Y$ smooth and $Z \subset Y$ a submanifold of codimension r such that f is not transversal to Z, then $f^{-1}(Z)$ is not a submanifold of codimension r in X.

b. If X and Y are manifolds and $f : X \to Y$ is smooth, then the set of regular values of f is open in Y.

c. If X and Y are manifolds and $f : X \to Y$ is smooth then the set of critical points is of measure 0 in X.

Paper 2, Section II
25H Differential Geometry

(i) State and prove the isoperimetric inequality for plane curves. You may appeal to Wirtinger's inequality as long as you state it precisely.

(ii) State Fenchel's theorem for curves in space.

(iii) Let $\alpha : I \to \mathbb{R}^2$ be a closed regular plane curve bounding a region K. Suppose $K \supset [p_1, p_1 + d_1] \times [p_2, p_2 + d_2]$, for $d_1 > 0$, $d_2 > 0$, i.e. K contains a rectangle of dimensions d_1, d_2. Let $k(s)$ denote the signed curvature of α with respect to the inward pointing normal, where α is parametrised anticlockwise. Show that there exists an $s_0 \in I$ such that $k(s_0) \leqslant \sqrt{\pi/(d_1 d_2)}$.

Paper 3, Section II
24H Differential Geometry

(i) State and prove the *Theorema Egregium.*

(ii) Define the notions *principal curvatures*, *principal directions* and *umbilical point.*

(iii) Let $S \subset \mathbb{R}^3$ be a connected compact regular surface (without boundary), and let $D \subset S$ be a dense subset of S with the following property. For all $p \in D$, there exists an open neighbourhood $\mathcal{U}_p$ of p in S such that for all $\theta \in [0, 2\pi)$, $\psi_{p,\theta}(\mathcal{U}_p) = \mathcal{U}_p$, where $\psi_{p,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes rotation by θ around the line through p perpendicular to $T_p S$. Show that S is in fact a sphere.

Paper 4, Section II
24H Differential Geometry

(i) Let $S \subset \mathbb{R}^3$ be a regular surface. Define the notions *exponential map*, *geodesic polar coordinates*, *geodesic circles.*

(ii) State and prove *Gauss' lemma.*

(iii) Let S be a regular surface. For fixed $r > 0$, and points p, q in S, let $S_r(p)$, $S_r(q)$ denote the geodesic circles around p, q, respectively, of radius r. Show the following statement: for each $p \in S$, there exists an $r = r(p) > 0$ and a neighborhood $\mathcal{U}_p$ containing p such that for all $q \in \mathcal{U}_p$, the sets $S_r(p)$ and $S_r(q)$ are smooth 1-dimensional manifolds which intersect transversally. What is the cardinality mod 2 of $S_r(p) \cap S_r(q)$?

Paper 1, Section II
25H Differential Geometry

(i) Define *manifold* and *manifold with boundary* for subsets $X \subset \mathbb{R}^N$.

(ii) Let X and Y be manifolds and $f : X \to Y$ a smooth map. Define what it means for $y \in Y$ to be a *regular value* of f.

(iii) Let $n \geqslant 0$ and let $\mathbb{S}^n$ denote the set $\{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1}(x^i)^2 = 1\}$. Let B^{n+1} denote the set $\{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1}(x^i)^2 \leqslant 1\}$. Show that $\mathbb{S}^n$ is an n-dimensional manifold and B^{n+1} is an $(n+1)$-dimensional manifold with boundary, with $\partial B^{n+1} = \mathbb{S}^n$.

[You may use standard theorems involving regular values of smooth functions provided that you state them clearly.]

(iv) For $n \geqslant 0$, consider the map $h : \mathbb{S}^n \to \mathbb{S}^n$ taking $\mathbf{x}$ to $-\mathbf{x}$. Show that h is smooth. Now let f be a smooth map $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $f \circ h = f$. Show that f is not smoothly homotopic to the identity map.

Paper 2, Section II
25H Differential Geometry

(a) Let $\alpha : I \to \mathbb{R}^3$ be a smooth regular curve, parametrized by arc length, such that $\alpha''(s) \neq 0$ for all $s \in I$. Define the *Frenet frame* associated to α and derive the *Frenet formulae*, identifying *curvature* and *torsion*.

(b) Let $\alpha, \tilde{\alpha} : I \to \mathbb{R}^3$ be as above such that $\tilde{k}(s) = k(s)$, $\tilde{\tau}(s) = -\tau(s)$, where k, $\tilde{k}$ denote the curvature of α, $\tilde{\alpha}$, respectively, and τ, $\tilde{\tau}$ denote the torsion. Show that there exists a $T \in \mathrm{O}_3$ and $v \in \mathbb{R}^3$ such that

$$\alpha = T \circ \tilde{\alpha} + v.$$

[You may appeal to standard facts about ordinary differential equations provided that they are clearly stated.]

(c) Let $\alpha : I \to \mathbb{R}^2$ be a closed regular plane curve, bounding a region K. Let $A(K)$ denote the area of K, and let $k(s)$ denote the signed curvature at $\alpha(s)$.

Show that there exists a point $s_0 \in I$ such that

$$k(s_0) \leqslant \sqrt{\pi / A(K)}.$$

[You may appeal to any standard theorem provided that it is clearly stated.]

Paper 3, Section II
24H Differential Geometry

(a) State and prove the *Theorema Egregium.*

(b) Let X be a minimal surface without boundary in $\mathbb{R}^3$ which is closed as a subset of $\mathbb{R}^3$, and assume that X is not contained in a closed ball. Let Π be a plane in $\mathbb{R}^3$ with the property that $D_n \to \infty$ as $n \to \infty$, where for $n = 0, 1, \ldots$,

$$D_n = \inf_{x \in X, d(x,0) \geqslant n} d(x, \Pi).$$

Here $d(x, y)$ denotes the Euclidean distance between x and y and $d(x, \Pi) = \inf_{y \in \Pi} d(x, y)$. Assume moreover that X contains no planar points. Show that X intersects Π.

Paper 4, Section II
24H Differential Geometry

(a) Let X be a compact surface (without boundary) in $\mathbb{R}^3$. State the global Gauss–Bonnet formula for X, identifying all terms in the formula.

(b) Let $X \subset \mathbb{R}^3$ be a surface. Define what it means for a curve $\gamma : I \to X$ to be a *geodesic*. State a theorem concerning the existence of geodesics and define the *exponential map*.

(c) Let $\psi : X \to Y$ be an isometry and let γ be a geodesic. Show that $\psi \circ \gamma$ is a geodesic. If K_X denotes the Gaussian curvature of X, and K_Y denotes the Gaussian curvature of Y, show that $K_Y \circ \psi = K_X$.

Now suppose $\psi : X \to Y$ is a smooth map such that $\psi \circ \gamma$ is a geodesic for all γ a geodesic. Is ψ necessarily an isometry? Give a proof or counterexample.

Similarly, suppose $\psi : X \to Y$ is a smooth map such that $K_Y \circ \psi = K_X$. Is ψ necessarily an isometry? Give a proof or counterexample.

1/II/24H Differential Geometry

Let $n \geqslant 1$ be an integer, and let $M(n)$ denote the set of $n \times n$ real-valued matrices. We make $M(n)$ into an n^2-dimensional smooth manifold via the obvious identification $M(n) = \mathbb{R}^{n^2}$.

(a) Let $GL(n)$ denote the subset

$$GL(n) = \{A \in M(n) : A^{-1} \text{ exists}\}.$$

Show that $GL(n)$ is a submanifold of $M(n)$. What is $\dim GL(n)$?

(b) Now let $SL(n) \subset GL(n)$ denote the subset

$$SL(n) = \{A \in GL(n) : \det A = 1\}.$$

Show that for $A \in GL(n)$,

$$(d \det)_A B = \operatorname{tr}(A^{-1}B) \det A.$$

Show that $SL(n)$ is a submanifold of $GL(n)$. What is the dimension of $SL(n)$?

(c) Now consider the set $X = M(n) \setminus GL(n)$. For what values of $n \geqslant 1$ is X a submanifold of $M(n)$?

2/II/24H Differential Geometry

(a) For a regular curve in $\mathbb{R}^3$, define *curvature* and *torsion* and state the *Frenet formulas*.

(b) State and prove the isoperimetric inequality for domains $\Omega \subset \mathbb{R}^2$ with compact closure and C^1 boundary $\partial\Omega$.

[You may assume Wirtinger's inequality.]

(c) Let $\gamma : I \to \mathbb{R}^2$ be a *closed* plane regular curve such that γ is contained in a disc of radius r. Show that there exists $s \in I$ such that $|k(s)| \geqslant r^{-1}$, where $k(s)$ denotes the signed curvature. Show by explicit example that the assumption of closedness is necessary.

3/II/23H **Differential Geometry**

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define the *Gauss Map*, *principal curvatures* k_i, *Gaussian curvature* K and *mean curvature* H. State the *Theorema Egregium*.

(b) Define what is meant for S to be *minimal*. Prove that if S is minimal, then $K \leqslant 0$. Give an example of a minimal surface whose Gaussian curvature is not identically 0, justifying your answer.

(c) Does there exist a *compact* minimal surface $S \subset \mathbb{R}^3$? Justify your answer.

4/II/24H **Differential Geometry**

Let $S \subset \mathbb{R}^3$ be a surface.

(a) In the case where S is compact, define the *Euler characteristic* χ and *genus* g of S.

(b) Define the notion of *geodesic curvature* k_g for regular curves $\gamma : I \to S$. When is $k_g = 0$? State the *Global Gauss–Bonnet Theorem* (including boundary term).

(c) Let $S = \mathbb{S}^2$ (the standard 2-sphere), and suppose $\gamma \subset \mathbb{S}^2$ is a simple closed regular curve such that $\mathbb{S}^2 \setminus \gamma$ is the union of two distinct connected components with equal areas. Can γ have everywhere strictly positive or everywhere strictly negative geodesic curvature?

(d) Prove or disprove the following statement: if S is connected with Gaussian curvature $K = 1$ identically, then S is a subset of a sphere of radius 1.

1/II/24H Differential Geometry

Let $f : X \to Y$ be a smooth map between manifolds without boundary. Recall that f is a *submersion* if $df_x : T_xX \to T_{f(x)}Y$ is surjective for all $x \in X$. The *canonical submersion* is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for $k \geqslant l$, given by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_l).$$

(i) Let f be a submersion, $x \in X$ and $y = f(x)$. Show that there exist local coordinates around x and y such that f, in these coordinates, is the canonical submersion. [*You may assume the inverse function theorem.*]

(ii) Show that submersions map open sets to open sets.

(iii) If X is compact and Y connected, show that every submersion is surjective. Are there submersions of compact manifolds into Euclidean spaces $\mathbb{R}^k$ with $k \geqslant 1$?

2/II/24H Differential Geometry

(i) What is a minimal surface? Explain why minimal surfaces always have non-positive Gaussian curvature.

(ii) A smooth map $f : S_1 \to S_2$ between two surfaces in 3-space is said to be *conformal* if

$$\langle df_p(v_1), df_p(v_2) \rangle = \lambda(p) \langle v_1, v_2 \rangle$$

for all $p \in S_1$ and all $v_1, v_2 \in T_pS_1$, where $\lambda(p) \neq 0$ is a number which depends only on p.

Let S be a surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \to S^2$ is conformal.

(iii) Show that isothermal coordinates exist around a non-planar point in a minimal surface.

3/II/23H Differential Geometry

(i) Let $f : X \to Y$ be a smooth map between manifolds without boundary. Define critical point, critical value and regular value. State Sard's theorem.

(ii) Explain how to define the degree modulo 2 of a smooth map f, indicating clearly the hypotheses on X and Y. Show that a smooth map with non-zero degree modulo 2 must be surjective.

(iii) Let S be the torus of revolution obtained by rotating the circle $(y-2)^2 + z^2 = 1$ in the yz-plane around the z-axis. Describe the critical points and the critical values of the Gauss map N of S. Find the degree modulo 2 of N. Justify your answer by means of a sketch or otherwise.

4/II/24H **Differential Geometry**

(i) What is a geodesic? Show that geodesics are critical points of the energy functional.

(ii) Let S be a surface which admits a parametrization $\phi(u, v)$ defined on an open subset W of $\mathbb{R}^2$ such that $E = G = U + V$ and $F = 0$, where $U = U(u)$ is a function of u alone and $V = V(v)$ is a function of v alone. Let $\gamma : I \to \phi(W)$ be a geodesic and write $\gamma(t) = \phi(u(t), v(t))$. Show that

$$\big[U(u(t)) + V(v(t))\big]\big[V(v(t))\dot{u}^2 - U(u(t))\dot{v}^2\big]$$

is independent of t.

1/II/24H **Differential Geometry**

(a) State and prove the inverse function theorem for a smooth map $f : X \to Y$ between manifolds without boundary.

[*You may assume the inverse function theorem for functions in Euclidean space.*]

(b) Let p be a real polynomial in k variables such that for some integer $m \geqslant 1$,

$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k)$$

for all real $t > 0$ and all $y = (x_1, \ldots, x_k) \in \mathbb{R}^k$. Prove that the set X_a of points y where $p(y) = a$ is a $(k-1)$-dimensional submanifold of $\mathbb{R}^k$, provided it is not empty and $a \neq 0$.
[*You may use the pre-image theorem provided that it is clearly stated.*]

(c) Show that the manifolds X_a with $a > 0$ are all diffeomorphic. Is X_a with $a > 0$ necessarily diffeomorphic to X_b with $b < 0$?

2/II/24H **Differential Geometry**

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define the exponential map $\exp_p$ at a point $p \in S$. Assuming that $\exp_p$ is smooth, show that $\exp_p$ is a diffeomorphism in a neighbourhood of the origin in T_pS.

(b) Given a parametrization around $p \in S$, define the Christoffel symbols and show that they only depend on the coefficients of the first fundamental form.

(c) Consider a system of normal co-ordinates centred at p, that is, Cartesian co-ordinates (x, y) in T_pS and parametrization given by $(x, y) \mapsto \exp_p(xe_1 + ye_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of T_pS. Show that all of the Christoffel symbols are zero at p.

3/II/23H **Differential Geometry**

Let $S \subset \mathbb{R}^3$ be a connected oriented surface.

(a) Define the Gauss map $N : S \to S^2$ of S. Given $p \in S$, show that the derivative of N,

$$dN_p : T_pS \to T_{N(p)}S^2 = T_pS$$

is self-adjoint.

(b) Show that if N is a diffeomorphism, then the Gaussian curvature is positive everywhere. Is the converse true?

4/II/24H **Differential Geometry**

(a) Let $S \subset \mathbb{R}^3$ be an oriented surface and let λ be a real number. Given a point $p \in S$ and a vector $v \in T_pS$ with unit norm, show that there exist $\varepsilon > 0$ and a unique curve $\gamma : (-\varepsilon, \varepsilon) \to S$ parametrized by arc-length and with constant geodesic curvature λ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

[*You may use the theorem on existence and uniqueness of solutions of ordinary differential equations.*]

(b) Let S be an oriented surface with positive Gaussian curvature and diffeomorphic to S^2. Show that two simple closed geodesics in S must intersect. Is it true that two smooth simple closed curves in S with constant geodesic curvature $\lambda \neq 0$ must intersect?

1/II/24H **Differential Geometry**

Let $f : X \to Y$ be a smooth map between manifolds without boundary.

(i) Define what is meant by a *critical point*, *critical value* and *regular value* of f.

(ii) Show that if y is a regular value of f and $\dim X \geqslant \dim Y$, then the set $f^{-1}(y)$ is a submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$.

[*You may assume the inverse function theorem.*]

(iii) Let $SL(n, \mathbb{R})$ be the group of all $n \times n$ real matrices with determinant 1. Prove that $SL(n, \mathbb{R})$ is a submanifold of the set of all $n \times n$ real matrices. Find the tangent space to $SL(n, \mathbb{R})$ at the identity matrix.

2/II/24H **Differential Geometry**

State the isoperimetric inequality in the plane.

Let $S \subset \mathbb{R}^3$ be a surface. Let $p \in S$ and let $S_r(p)$ be a geodesic circle of centre p and radius r (r small). Let L be the length of $S_r(p)$ and A be the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 r^4 K(p) + \varepsilon(r),$$

where $K(p)$ is the Gaussian curvature of S at p and

$$\lim_{r \to 0} \frac{\varepsilon(r)}{r^4} = 0.$$

When $K(p) > 0$ and r is small, compare this briefly with the isoperimetric inequality in the plane.

3/II/23H **Differential Geometry**

(i) Define geodesic curvature and state the Gauss–Bonnet theorem.

(ii) Let $\alpha : I \to \mathbb{R}^3$ be a closed regular curve parametrized by arc-length, and assume that α has non-zero curvature everywhere. Let $n : I \to S^2 \subset \mathbb{R}^3$ be the curve given by the normal vector $n(s)$ to $\alpha(s)$. Let $\bar{s}$ be the arc-length of the curve n on S^2. Show that the geodesic curvature k_g of n is given by

$$k_g = -\frac{d}{ds} \tan^{-1}(\tau/k) \frac{ds}{d\bar{s}},$$

where k and τ are the curvature and torsion of α.

(iii) Suppose now that $n(s)$ is a simple curve (i.e. it has no self-intersections). Show that $n(I)$ divides S^2 into two regions of equal area.

4/II/24H **Differential Geometry**

(i) Define what is meant by an *isothermal parametrization*. Let $\phi : U \to \mathbb{R}^3$ be an isothermal parametrization. Prove that

$$\phi_{uu} + \phi_{vv} = 2\,\lambda^2\,\mathbf{H},$$

where $\mathbf{H}$ is the mean curvature vector and $\lambda^2 = \langle \phi_u, \phi_u \rangle$.

Define what it means for ϕ to be *minimal*, and deduce that ϕ is minimal if and only if $\Delta\phi = 0$.

[*You may assume that the mean curvature H can be written as*

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}.\ ]$$

(ii) Write $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$. Consider the complex valued functions

$$\varphi_1 = x_u - ix_v, \quad \varphi_2 = y_u - iy_v, \quad \varphi_3 = z_u - iz_v.$$

Show that ϕ is isothermal if and only if $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0$.

Suppose now that ϕ is isothermal. Prove that ϕ is minimal if and only if φ_1, φ_2 and φ_3 are holomorphic functions.

(iii) Consider the immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\phi(u,v) = (u - u^3/3 + uv^2,\ -v + v^3/3 - u^2 v,\ u^2 - v^2).$$

Find φ_1, φ_2 and φ_3. Show that ϕ is an isothermal parametrization of a minimal surface.