Part II

Cosmology

Paper 1, Section I

9B Cosmology

(a) A homogeneous and isotropic fluid has an energy density $\rho(t)$ and pressure $P(t)$. Use the relation $\mathrm{d}E = -P\,\mathrm{d}V$ for the energy E to derive the continuity equation in a universe with scale factor $a(t)$,

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\,(\rho + P) = 0\,,$$

where the overdot indicates differentiation with respect to time t. *[Hint: recall that the physical volume $V(t) = a(t)^3 V_0$, where V_0 is the co-moving volume.]*

(b) Given that the scale factor $a(t)$ satisfies the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P)\,,$$

where G is Newton's constant and c is the speed of light, show that the quantity

$$Q = \frac{8\pi G}{3c^2}\rho\, a^2 - \dot{a}^2\,,$$

is time independent.

(c) The pressure P is related to ρ by the equation of state

$$P = \omega\,\rho, \quad |\omega| < 1\,.$$

Given that $a(t_0) = 1$, find $a(t)$ for an *expanding* universe with $Q = 0$, and hence show that $a(t_\star) = 0$ for some $t_\star < t_0$.

Paper 2, Section I

9B Cosmology

The number density n of photons in thermal equilibrium at temperature T takes the form

$$n = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2 \mathrm{d}\nu}{\exp(h\nu/k_B T) - 1}\,, \qquad (\star)$$

where h is Planck's constant, k_B is the Boltzmann constant and c is the speed of light.

Using $(\star)$, show that the photon number density n and energy density ρ can be expressed in the form

$$n = \alpha\, T^3 \qquad \text{and} \qquad \rho = \xi\, T^4\,,$$

where the constants α and ξ need not be evaluated explicitly.

At time $t = t_{\rm dec}$ and temperature $T = T_{\rm dec}$, photons decouple from thermal equilibrium. By considering how the photon frequency redshifts as a flat universe expands, show that the form of the equilibrium frequency distribution is preserved if the temperature for $t > t_{\rm dec}$ is defined by

$$T = \frac{a(t_{\rm dec})}{a(t)}T_{\rm dec}\,.$$

Paper 3, Section I

9B Cosmology

The equilibrium number density of fermions of mass m at temperature T and chemical potential μ is

$$n = \frac{4\pi g_s}{h^3}\int_0^\infty \frac{p^2 \mathrm{d}p}{\exp\left[\frac{E(p)-\mu}{k_B T}\right]+1},$$

where g_s is the degeneracy factor, $E(p) = c\sqrt{p^2+m^2c^2}$, c is the speed of light, k_B is the Boltzmann constant, p is the magnitude of the particle momentum and h is Planck's constant. For a non-relativistic gas with $pc \ll mc^2$ and $k_B T \ll mc^2 - \mu$, show that the number density becomes

$$n = g_s\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\exp\left[\frac{\mu - m\,c^2}{k_B T}\right]. \qquad (\star)$$

[You may assume that $\int_0^\infty \mathrm{d}x\, x^2\, e^{-x^2/\alpha} = \sqrt{\pi}\alpha^{3/2}/4$ for $\alpha > 0$.]

Before recombination, equilibrium is maintained between neutral hydrogen, free electrons, protons and photons through the interaction

$$p + e^- \leftrightarrow H + \gamma\,.$$

Using the non-relativistic number density $(\star)$, deduce Saha's equation relating the electron and hydrogen number densities,

$$\frac{n_e^2}{n_H} \approx \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2}\exp\left[-\frac{E_{\rm bind}}{k_B T}\right],$$

where $E_{\rm bind} = (m_p + m_e - m_H)c^2$ is the hydrogen binding energy. State clearly any assumptions made.

Paper 4, Section I

9B Cosmology

What is the *flatness problem*? By using the Friedmann and continuity equations, show that a period of accelerated expansion of the scale factor $a(t)$ in the early stages of the universe can solve the flatness problem if $\rho + 3P < 0$, where ρ is the energy density and P is the pressure. *[Hint: it may be useful to compute* $\mathrm{d}(\rho a^2)/\mathrm{d}t$*.]*

In the very early universe one can neglect the spatial curvature and the cosmological constant. Suppose that in addition there is a homogenous scalar field ϕ with potential

$$V(\phi) = m^2\phi^2\,,$$

and the Friedmann equation is

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$

where $H = \dot{a}/a$ is the Hubble parameter. The field ϕ obeys the evolution equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,.$$

During inflation, ϕ evolves slowly after starting from a large initial value ϕ_i at $t = 0$. State what is meant by the *slow-roll approximation*. Show that in this approximation

$$\begin{aligned} \phi(t) &= \phi_i - \frac{2}{\sqrt{3}}mt \\ a(t) &= a_i \exp\left[\frac{m\phi_i}{\sqrt{3}}t - \frac{1}{3}m^2t^2\right] = a_i \exp\left[\frac{\phi_i^2 - \phi(t)^2}{4}\right]\,, \end{aligned}$$

where a_i is the initial value of a.

Paper 1, Section II

15B Cosmology

In a homogeneous and isotropic universe, the scale factor $a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K\,c^2}{a^2} = \frac{8\,\pi\,G}{3c^2}\rho\,,$$

where K is a constant curvature parameter and ρ is the energy density which, together with the pressure P, satisfies the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,. \qquad (\star)$$

(a) Use the equations to show that the rate of change of the Hubble parameter $H = \dot{a}/a$ satisfies

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\rho + 3P)\,.$$

(b) Suppose that an *expanding* universe is filled with radiation (with energy density ρ_r and pressure $P_r = \rho_r/3$) as well as a cosmological constant component (with density ρ_Λ and pressure $P_\Lambda = -\rho_\Lambda$). Both radiation and cosmological constant components satisfy the continuity equation $(\star)$ separately.

Given that the energy densities of these two components are measured today $(t = t_0)$ to be

$$\rho_{r0} = \beta\frac{3c^2H_0^2}{8\pi G} \quad \text{and} \quad \rho_{\Lambda 0} = \frac{3c^2H_0^2}{8\pi G} \quad \text{with constant} \quad \beta > 0 \quad \text{and} \quad a(t_0) = 1\,,$$

show that the curvature parameter must satisfy $Kc^2 = \beta\,H_0^2$. Hence, derive the following relations for the Hubble parameter H and its time derivative:

$$H^2 = \frac{H_0^2}{a^4}(\beta - \beta\,a^2 + a^4)\,,$$
$$\dot{H} = -\beta\frac{H_0^2}{a^4}(2 - a^2)\,.$$

(c) Show qualitatively that universes with $a(0) = 0$ and $\beta > 4$ will recollapse to a Big Crunch in the future. *[Hint: you may find it useful to sketch a^4H^2 and $a^4\dot{H}$ versus a^2 for representative values of β.]*

(d) For $\beta = 4$, find an explicit solution for the scale factor $a(t)$ satisfying $a(0) = 0$. Find the limiting behaviours of this solution for large and small t.

Paper 3, Section II

14B Cosmology

Small density perturbations $\delta_{\mathbf{k}}(t)$ in pressureless matter inside the cosmological horizon obey the following Fourier evolution equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - \frac{4\pi G\bar{\rho}_c}{c^2}\delta_{\mathbf{k}} = 0\,,$$

where the overdot indicates differentiation with respect to time t, $a(t)$ the scale factor of the universe, G is Newton's constant, c the speed of light, $\mathbf{k}$ is the co-moving wavevector and $\bar{\rho}_c$ is the background density of the pressureless gravitating matter.

(a) Let t_{eq} be the time of matter-radiation equality. Show that during the matter-dominated epoch, $\delta_{\mathbf{k}}$ behaves as

$$\delta_{\mathbf{k}}(t) = A(\mathbf{k})\left(\frac{t}{t_{\text{eq}}}\right)^{2/3} + B(\mathbf{k})\left(\frac{t}{t_{\text{eq}}}\right)^{-1}\,,$$

where $A(\mathbf{k})$ and $B(\mathbf{k})$ are functions of $\mathbf{k}$ only.

(b) For a given wavenumber $k \equiv |\mathbf{k}|$, show that the time t_H at which this mode crosses inside the horizon, *i.e.* $c\,t_H \approx 2\pi a(t_H)/k$, is given by

$$\frac{t_H}{t_0} \approx \begin{cases} \left(\frac{k_0}{k}\right)^3\,, & t_H > t_{eq} \\ \dfrac{1}{\sqrt{1+z_{\text{eq}}}}\left(\dfrac{k_0}{k}\right)^2\,, & t_H < t_{eq} \end{cases}$$

where t_0 is the age of this universe, $k_0 \equiv 2\pi/(c\,t_0)$, and the matter-radiation equality redshift is given by $1 + z_{\text{eq}} = (t_0/t_{\text{eq}})^{2/3}$.

(c) Assume that early in the radiation era there is no significant perturbation growth in $\delta_{\mathbf{k}}$ and that primordial perturbations from inflation are scale-invariant with a constant amplitude at the time of horizon crossing given by $\langle\delta_{\mathbf{k}}(t_H)^2\rangle \approx V^{-1}C/k^3$, where C is a constant and V is a volume. Use the results in parts (a) and (b) to project these perturbations forward to $t_0 \gg t_H$, and show that the power spectrum of perturbations today (at $t = t_0$) is given by

$$P(k) \equiv V\langle\delta_{\mathbf{k}}(t_0)^2\rangle = \begin{cases} \frac{Ck}{k_0^4}\,, & k < k_{eq} \\ \frac{Ck_{\text{eq}}}{k_0^4}\left(\frac{k_{\text{eq}}}{k}\right)^3\,, & k > k_{\text{eq}} \end{cases}$$

where k_{eq} is the wavenumber of modes that entered the horizon at matter-radiation equality.

Paper 1, Section I

9A Cosmology

Consider the process where protons and electrons combine to form neutral hydrogen atoms at temperature T. Let n_H be the number density of hydrogen atoms, n_e the number density of electrons, m_e the mass of the electron and E_{bind} the binding energy of hydrogen. Derive *Saha's equation* which relates the ratio n_H/n_e^2 to m_e, E_{bind} and T. Clearly describe the steps required.

[*You may use without proof that at temperature T and chemical potential μ, the number density n of a non-relativistic particle species with mass $m \gg k_B T/c^2$ is given by*

$$n = g\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \exp\left[-\frac{(m\,c^2-\mu)}{k_B\,T}\right],$$

where g is the number of degrees of freedom of this particle species and k_B, $\hbar$ and c are the Boltzmann, Planck and speed of light constants, respectively.]

Paper 2, Section I

9A Cosmology

Consider a ball centered on the origin which is initially of uniform energy density ρ and radius L. The ball expands outwards away from the origin. Additionally, take a particle of mass m at some position $\mathbf{x}$ with $r \equiv |\mathbf{x}| \ll L$. Assume that the particle only experiences gravity through Newton's inverse-square law.

Using the above model of the expanding universe, derive the *Friedmann equation* describing the evolution of the scale factor $a(t)$ appearing in the relation $\mathbf{x}(t) = a(t)\,\mathbf{x}_0$.

Describe the two main flaws in this derivation of the Friedmann equation.

Paper 3, Section I

9A Cosmology

Combining the Friedmann and continuity equations

$$H^2 = \frac{8\pi G}{3c^2}\left(\rho - \frac{k\,c^2}{R^2\,a^2}\right), \qquad \dot\rho + 3\,H\,(\rho + P) = 0\,,$$

derive the *Raychaudhuri equation* (also known as the *acceleration equation*), which expresses $\ddot a/a$ in terms of the energy density ρ and pressure P.

Assume that the strong energy condition $\rho + 3P \geqslant 0$ holds. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H^{-1}\right) \geqslant 1.$$

Deduce that $H \to +\infty$ and $a \to 0$ at a finite time in the past or in the future. What property of H distinguishes the two cases? In one sentence, describe the implications for the evolution of this model universe.

Paper 4, Section I
9A Cosmology

Consider a closed Friedmann-Robertson-Walker universe filled with a fluid endowed with an energy density $\rho \geqslant 0$ and pressure $P \geqslant 0$. For such a universe the Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{c^2}{R^2\, a^2}\,,$$

where $a(t)$ is the scale factor.

What is the meaning of R? Show that a closed universe cannot expand forever.

[*Hint: Use the continuity equation to show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\, a^3) \leqslant 0\,. \qquad]$$

Paper 1, Section II
15A Cosmology

The continuity, Euler and Poisson equations governing how a non-relativistic fluid composed of particles with mass m, number density n, pressure P and velocity $\mathbf{v}$ propagate in an expanding universe take the form

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla}\cdot(\rho\,\mathbf{v}) = 0\,,$$
$$\rho a\left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a}\cdot\boldsymbol{\nabla}\right)\mathbf{u} = -c^2\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi\,,$$
$$\nabla^2\Phi = \frac{4\pi G}{c^2}\rho\, a^2\,,$$

where $\rho = mc^2 n$, $\mathbf{u} = \mathbf{v} + a\,H\,\mathbf{x}$, $H = \dot{a}/a$, Φ is the gravitational potential and $a(t)$ is the scale factor.

Consider small perturbations about a homogeneous and isotropic flow,

$$n = \bar{n}(t) + \epsilon\,\delta n\,, \quad \mathbf{v} = \epsilon\,\delta\mathbf{v}\,, \quad P = \bar{P}(t) + \epsilon\,\delta P \quad \text{and} \quad \Phi = \bar{\Phi}(t,\mathbf{x}) + \epsilon\,\delta\Phi\,,$$

with $\epsilon \ll 1$.

(a) Show that, to first order in ϵ, the continuity equation can be written as

$$\dot{\delta} + \frac{1}{a}\boldsymbol{\nabla}\cdot\delta\mathbf{v} = 0\,, \qquad (\dagger)$$

where $\delta = \delta n/\bar{n}$ is the *density contrast.*

(b) Show that, to first order in ϵ, the Euler equation can be written as

$$m\,\bar{n}\,a\,(\dot{\delta\mathbf{v}} + H\,\delta\mathbf{v}) = -\boldsymbol{\nabla}\delta P - m\,\bar{n}\boldsymbol{\nabla}\delta\Phi\,. \qquad (\dagger\dagger)$$

(c) Now assume that $\delta P = c_s^2\, m\,\delta n$. Using ($\dagger$), ($\dagger\dagger$) and the perturbed Poisson equation, show that the density contrast δ obeys

$$\ddot{\delta} + 2\,H\,\dot{\delta} - c_s^2\left(\frac{1}{a^2}\nabla^2 + k_{\mathrm{J}}^2\right)\delta = 0 \qquad (\star)$$

and express k_{J} as a function of $\bar{n}$, m and c_s^2.

(d) Neglecting the bracketed terms in equation ($\star$), solve it to find the form of the growth of matter perturbations in a radiation-dominated universe.

Paper 3, Section II
14A Cosmology

(a) What are the cosmological *flatness* and *horizon* problems? Explain what forms of time evolution of the cosmological scale factor $a(t)$ must occur during a period of inflationary expansion in a Friedmann-Robertson-Walker universe. How can inflation solve the flatness and horizon problems? [You may assume an equation of state where the pressure P is proportional to the energy density ρ.]

(b) Consider a universe with a Hubble expansion rate $H = \dot{a}/a$ containing a single inflaton field ϕ with a potential $V(\phi) \geqslant 0$. The density and pressure are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$
$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,.$$

Show that the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

demands

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,. \qquad (\dagger)$$

(c) Consider the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho\,, \qquad (\dagger\dagger)$$

and show that

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3c^2}\left[V(\phi) - \dot{\phi}^2\right]\,.$$

Under what conditions does an inflationary phase occur?

(d) What is *slow roll inflation*? Show that in slow roll inflation, the scalar equation ($\dagger$) and Friedmann equation ($\dagger\dagger$) reduce to

$$3H\dot{\phi} \approx -\frac{\mathrm{d}V}{\mathrm{d}\phi} \quad \text{and} \quad H^2 \approx \frac{8\pi G}{3c^2}V(\phi)\,. \qquad (\star)$$

(e) Using the slow roll equations ($\star$), determine $a(\phi)$ and $\phi(t)$ when $V(\phi) = \frac{1}{4}\lambda\phi^4$, with $\lambda > 0$.

Paper 1, Section I

9B Cosmology

The continuity, Euler and Poisson equations governing how non-relativistic fluids with energy density ρ, pressure P and velocity $\mathbf{v}$ propagate in an expanding universe take the form

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0\,,$$

$$\rho\, a\left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a}\cdot\boldsymbol{\nabla}\right)\mathbf{u} = -\frac{1}{c^2}\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi\,,$$

$$\nabla^2\Phi = \frac{4\pi G}{c^2}\,\rho\, a^2\,,$$

where $\mathbf{u} = \mathbf{v} + a\,H\,\mathbf{x}$, $H = \dot{a}/a$ and $a(t)$ is the scale factor.

(a) Show that, for a homogeneous and isotropic flow with $P = \overline{P}(t)$, $\rho = \overline{\rho}(t)$, $\mathbf{v} = \mathbf{0}$ and $\Phi = \overline{\Phi}(t, \mathbf{x})$, consistency of the Euler equation with the Poisson equation implies Raychaudhuri's equation.

(b) Explain why this derivation of Raychaudhuri's equation is an improvement over the derivation of the Friedmann equation using only Newtonian gravity.

(c) Consider small perturbations about a homogeneous and isotropic flow,

$$\rho = \overline{\rho}(t) + \epsilon\,\delta\rho, \quad \mathbf{v} = \epsilon\,\delta\mathbf{v}, \quad P = \overline{P}(t) + \epsilon\,\delta P \quad \text{and} \quad \Phi = \overline{\Phi}(t,\mathbf{x}) + \epsilon\,\delta\Phi\,,$$

with $\epsilon \ll 1$. Show that, to first order in ϵ, the continuity equation can be written as

$$\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\overline{\rho}}\right) = -\frac{1}{a}\boldsymbol{\nabla}\cdot\delta\mathbf{v}\,.$$

Paper 2, Section I

9B Cosmology

(a) The generalised Boltzmann distribution $P(\mathbf{p})$ is given by

$$P(\mathbf{p}) = \frac{e^{-\beta(E_{\mathbf{p}}\, n_{\mathbf{p}} - \mu\, n_{\mathbf{p}})}}{\mathcal{Z}_{\mathbf{p}}}\,,$$

where $\beta = (k_B T)^{-1}$, μ is the chemical potential,

$$\mathcal{Z}_{\mathbf{p}} = \sum_{n_{\mathbf{p}}} e^{-\beta(E_{\mathbf{p}}\, n_{\mathbf{p}} - \mu\, n_{\mathbf{p}})}, \quad E_{\mathbf{p}} = \sqrt{m^2c^4 + p^2c^2} \quad \text{and} \quad p = |\mathbf{p}|\,.$$

Find the average particle number $\langle N(\mathbf{p})\rangle$ with momentum $\mathbf{p}$, assuming that all particles have rest mass m and are either

(i) bosons, or

(ii) fermions.

(b) The photon total number density n_γ is given by

$$n_\gamma = \frac{2\zeta(3)}{\pi^2\hbar^3c^3}(k_B\, T)^3\,,$$

where $\zeta(3) \approx 1.2$. Consider now the fractional ionisation of hydrogen

$$X_e = \frac{n_e}{n_e + n_H}\,.$$

In our universe $n_e + n_H = n_p + n_H \approx \eta\, n_\gamma$, where $\eta \sim 10^{-9}$ is the baryon-to-photon number density. Find an expression for the ratio

$$\frac{1 - X_e}{X_e^2}$$

in terms of η, $(k_B\, T)$, the electron mass m_e, the speed of light c and the ionisation energy of hydrogen $I \approx 13.6\,\text{eV}$.

One might expect neutral hydrogen to form at a temperature $k_B\, T \sim I$, but instead in our universe it happens at the much lower temperature $k_B\, T \approx 0.3\,\text{eV}$. Briefly explain why this happens.

[*You may use without proof the Saha equation*

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi\hbar^2}{m_e\, k_B\, T}\right)^{3/2} e^{\beta I}\,,$$

for chemical equilibrium in the reaction $e^- + p^+ \leftrightarrow H + \gamma\,.$]

Paper 3, Section I

9B Cosmology

The expansion of the universe during inflation is governed by the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right],$$

and the equation of motion for the inflaton field ϕ,

$$\ddot{\phi} + 3\,\frac{\dot{a}}{a}\,\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,.$$

Consider the potential

$$V = V_0\, e^{-\lambda\,\phi}$$

with $V_0 > 0$ and $\lambda > 0$.

(a) Show that the inflationary equations have the exact solution

$$a(t) = \left(\frac{t}{t_0}\right)^{\gamma} \qquad \text{and} \qquad \phi = \phi_0 + \alpha \log t,$$

for arbitrary t_0 and appropriate choices of α, γ and ϕ_0. Determine the range of λ for which the solution exists. For what values of λ does inflation occur?

(b) Using the inflaton equation of motion and

$$\rho = \frac{1}{2}\dot{\phi}^2 + V\,,$$

together with the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,,$$

determine P.

(c) What is the range of the pressure–energy density ratio $\omega \equiv P/\rho$ for which inflation occurs?

Paper 4, Section I

9B Cosmology

A collection of N particles, with masses m_i and positions $\mathbf{x}_i$, interact through a gravitational potential

$$V = \sum_{i<j} V_{ij} = -\sum_{i<j} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|}\,.$$

Assume that the system is gravitationally bound, and that the positions $\mathbf{x}_i$ and velocities $\dot{\mathbf{x}}_i$ are bounded for all time. Further, define the *time average* of a quantity X by

$$\overline{X} = \lim_{t\to\infty} \frac{1}{t} \int_0^t X(t')\, \mathrm{d}t'\,.$$

(a) Assuming that the time average of the kinetic energy T and potential energy V are well defined, show that

$$\overline{T} = -\frac{1}{2}\overline{V}\,.$$

[You should consider the quantity $I = \dfrac{1}{2}\displaystyle\sum_{i=1}^{N} m_i\, \mathbf{x}_i \cdot \mathbf{x}_i$, with all $\mathbf{x}_i$ measured relative to the centre of mass.]

(b) Explain how part (a) can be used, together with observations, to provide evidence in favour of dark matter. [You may assume that time averaging may be replaced by an average over particles.]

Paper 1, Section II

15B Cosmology

(a) Consider the following action for the inflaton field ϕ

$$S = \int \mathrm{d}^3x\,\mathrm{d}t\,a(t)^3\left[\frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2a(t)^2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi - V(\phi)\right].$$

Use the principle of least action to derive the equation of motion for the inflaton ϕ,

$$\ddot{\phi} + 3H\dot{\phi} - \frac{c^2}{a(t)^2}\nabla^2\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0\,, \qquad (*)$$

where $H = \dot{a}/a$. [In the derivation you may discard boundary terms.]

(b) Consider a regime where $V(\phi)$ is approximately constant so that the universe undergoes a period of exponential expansion during which $a = a_0\,e^{H_{\rm inf}\,t}$. Show that $(*)$ can be written in terms of the spatial Fourier transform $\widehat{\phi}_{\mathbf{k}}(t)$ of $\phi(\mathbf{x},t)$ as

$$\ddot{\widehat{\phi}}_{\mathbf{k}} + 3H_{\rm inf}\dot{\widehat{\phi}}_{\mathbf{k}} + \frac{c^2k^2}{a^2}\widehat{\phi}_{\mathbf{k}} = 0\,. \qquad (**)$$

(c) Define *conformal time* τ and determine the range of τ when $a = a_0\,e^{H_{\rm inf}\,t}$. Show that $(**)$ can be written in terms of the conformal time as

$$\frac{\mathrm{d}^2\widetilde{\phi}_{\mathbf{k}}}{\mathrm{d}\tau^2} + \left(c^2k^2 - \frac{2}{\tau^2}\right)\widetilde{\phi}_{\mathbf{k}} = 0\,, \qquad \text{where} \qquad \widetilde{\phi}_{\mathbf{k}} = -\frac{1}{H_{\rm inf}\tau}\widehat{\phi}_{\mathbf{k}}\,.$$

(d) Let $|\mathrm{BD}\rangle$ denote the state that in the far past was in the ground state of the standard harmonic oscillator with frequency $\omega = c\,k$. Assuming that the quantum variance of $\widehat{\phi}_{\mathbf{k}}$ is given by

$$P_{\mathbf{k}} \equiv \langle\mathrm{BD}|\widehat{\phi}_{\mathbf{k}}\widehat{\phi}^{\dagger}_{\mathbf{k}}|\mathrm{BD}\rangle = \frac{\hbar\,H^2_{\rm inf}}{2c^3k^3}\left(1+\tau^2c^2k^2\right),$$

explain in which sense inflation naturally generates a scale-invariant power spectrum. [You may use that $P_{\mathbf{k}}$ has dimensions of $[\text{length}]^3$.]

Paper 3, Section II
14B Cosmology

(a) Consider a closed universe endowed with cosmological constant $\Lambda > 0$ and filled with radiation with pressure P and energy density ρ. Using the equation of state $P = \frac{1}{3}\rho$ and the continuity equation

$$\dot{\rho} + \frac{3\,\dot{a}}{a}\,(\rho + P) = 0\,,$$

determine how ρ depends on a. Give the physical interpretation of the scaling of ρ with a.

(b) For such a universe the Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{c^2}{R^2 a^2} + \frac{\Lambda}{3}\,.$$

What is the physical meaning of R?

(c) Making the substitution $a(t) = \alpha\,\tilde{a}(t)$, determine α and $\Gamma > 0$ such that the Friedmann equation takes the form

$$\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2 = \frac{\Gamma}{\tilde{a}^4} - \frac{1}{\tilde{a}^2} + \frac{\Lambda}{3}\,.$$

Using the substitution $y(t) = \tilde{a}(t)^2$ and the boundary condition $y(0) = 0$, deduce the boundary condition for $\dot{y}(0)$.

Show that

$$\ddot{y} = \frac{4\Lambda}{3}y - 2\,,$$

and hence that

$$\tilde{a}^2(t) = \frac{3}{2\Lambda}\left[1 - \cosh\left(\sqrt{\frac{4\Lambda}{3}}\,t\right) + \lambda\sinh\left(\sqrt{\frac{4\Lambda}{3}}\,t\right)\right]\,.$$

Express the constant λ in terms of Λ and Γ.

Sketch the graphs of $\tilde{a}(t)$ for the cases $\lambda > 1$, $\lambda < 1$ and $\lambda = 1$.

Paper 1, Section I

9D Cosmology

The Friedmann equation is

$$H^2 = \frac{8\pi G}{3c^2}\left(\rho - \frac{kc^2}{R^2a^2}\right) .$$

Briefly explain the meaning of H, ρ, k and R.

Derive the Raychaudhuri equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) ,$$

where P is the pressure, stating clearly any results that are required.

Assume that the strong energy condition $\rho + 3P \geqslant 0$ holds. Show that there was necessarily a Big Bang singularity at time t_{BB} such that

$$t_0 - t_{BB} \leqslant H_0^{-1} ,$$

where $H_0 = H(t_0)$ and t_0 is the time today.

Paper 2, Section I

9D Cosmology

During inflation, the expansion of the universe is governed by the Friedmann equation,

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) ,$$

and the equation of motion for the inflaton field ϕ,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 .$$

The slow-roll conditions are $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll H\dot{\phi}$. Under these assumptions, solve for $\phi(t)$ and $a(t)$ for the potentials:

(i) $V(\phi) = \frac{1}{2}m^2\phi^2$ and

(ii) $V(\phi) = \frac{1}{4}\lambda\phi^4$, $(\lambda > 0)$.

Paper 3, Section I

9D Cosmology

At temperature T, with $\beta = 1/(k_B T)$, the distribution of ultra-relativistic particles with momentum $\mathbf{p}$ is given by

$$n(\mathbf{p}) = \frac{1}{e^{\beta pc} \mp 1},$$

where the minus sign is for bosons and the plus sign for fermions, and with $p = |\mathbf{p}|$.

Show that the total number of fermions, $n_{\rm f}$, is related to the total number of bosons, $n_{\rm b}$, by $n_{\rm f} = \frac{3}{4} n_{\rm b}$.

Show that the total energy density of fermions, $\rho_{\rm f}$, is related to the total energy density of bosons, $\rho_{\rm b}$, by $\rho_{\rm f} = \frac{7}{8}\rho_{\rm b}$.

Paper 4, Section I

9D Cosmology

At temperature T and chemical potential μ, the number density of a non-relativistic particle species with mass $m \gg k_B T/c^2$ is given by

$$n = g \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-(mc^2 - \mu)/k_B T},$$

where g is the number of degrees of freedom of this particle.

At recombination, electrons and protons combine to form hydrogen. Use the result above to derive the Saha equation

$$n_H \approx n_e^2 \left(\frac{2\pi\hbar^2}{m_e k_B T} \right)^{3/2} e^{E_{\rm bind}/k_B T},$$

where n_H is the number density of hydrogen atoms, n_e the number density of electrons, m_e the mass of the electron and $E_{\rm bind}$ the binding energy of hydrogen. State any assumptions that you use in this derivation.

Paper 1, Section II

15D Cosmology

A fluid with pressure P sits in a volume V. The change in energy due to a change in volume is given by $dE = -PdV$. Use this in a cosmological context to derive the continuity equation,

$$\dot{\rho} = -3H(\rho + P)\,,$$

with ρ the energy density, $H = \dot{a}/a$ the Hubble parameter, and a the scale factor.

In a flat universe, the Friedmann equation is given by

$$H^2 = \frac{8\pi G}{3c^2}\rho\,.$$

Given a universe dominated by a fluid with equation of state $P = w\rho$, where w is a constant, determine how the scale factor $a(t)$ evolves.

Define *conformal time* τ. Assume that the early universe consists of two fluids: radiation with $w = 1/3$ and a network of cosmic strings with $w = -1/3$. Show that the Friedmann equation can be written as

$$\left(\frac{da}{d\tau}\right)^2 = B\rho_{\mathrm{eq}}(a^2 + a_{\mathrm{eq}}^2)\,,$$

where ρ_{eq} is the energy density in radiation, and a_{eq} is the scale factor, both evaluated at radiation-string equality. Here, B is a constant that you should determine. Find the solution $a(\tau)$.

Paper 3, Section II

14D Cosmology

In an expanding spacetime, the density contrast $\delta(\mathbf{x},t)$ satisfies the linearised equation

$$\ddot{\delta} + 2H\dot{\delta} - c_s^2\left(\frac{1}{a^2}\nabla^2 + k_J^2\right)\delta = 0\,, \qquad (*)$$

where a is the scale factor, H is the Hubble parameter, c_s is a constant, and k_J is the Jeans wavenumber, defined by

$$c_s^2 k_J^2 = \frac{4\pi G}{c^2}\bar{\rho}(t)\,,$$

with $\bar{\rho}(t)$ the background, homogeneous energy density.

(i) Solve for $\delta(\mathbf{x},t)$ in a static universe, with $a = 1$ and $H = 0$ and $\bar{\rho}$ constant. Identify two regimes: one in which sound waves propagate, and one in which there is an instability.

(ii) In a matter-dominated universe with $\bar{\rho} \sim 1/a^3$, use the Friedmann equation $H^2 = 8\pi G\bar{\rho}/3c^2$ to find the growing and decaying long-wavelength modes of δ as a function of a.

(iii) Assuming $c_s^2 \approx c_s^2 k_J^2 \approx 0$ in equation $(*)$, find the growth of matter perturbations in a radiation-dominated universe and find the growth of matter perturbations in a curvature-dominated universe.

Paper 3, Section I

9B Cosmology

Consider a spherically symmetric distribution of mass with density $\rho(r)$ at distance r from the centre. Derive the pressure support equation that the pressure $P(r)$ has to satisfy for the system to be in static equilibrium.

Assume now that the mass density obeys $\rho(r) = Ar^2 P(r)$, for some positive constant A. Determine whether or not the system has a stable solution corresponding to a star of finite radius.

Paper 4, Section I

9B Cosmology

Derive the relation between the neutrino temperature T_ν and the photon temperature T_γ at a time long after electrons and positrons have become non-relativistic.

[*In this question you may work in units of the speed of light, so that $c = 1$. You may also use without derivation the following formulae. The energy density ϵ_a and pressure P_a for a single relativistic species a with a number g_a of degenerate states at temperature T are given by*

$$\epsilon_a = \frac{4\pi g_a}{h^3} \int \frac{p^3 dp}{e^{p/(k_B T)} \mp 1}\,, \qquad\qquad P_a = \frac{4\pi g_a}{3h^3} \int \frac{p^3 dp}{e^{p/(k_B T)} \mp 1}\,,$$

where k_B is Boltzmann's constant, h is Planck's constant, and the minus or plus depends on whether the particle is a boson or a fermion respectively. For each species a, the entropy density s_a at temperature T_a is given by,

$$s_a = \frac{\epsilon_a + P_a}{k_B T_a}\,.$$

The effective total number g_* of relativistic species is defined in terms of the numbers of bosonic and fermionic particles in the theory as,

$$g_* = \sum_{bosons} g_{bosons} + \frac{7}{8} \sum_{fermions} g_{fermions}\,,$$

with the specific values $g_\gamma = g_{e^+} = g_{e^-} = 2$ for photons, positrons and electrons.]

Paper 1, Section I

9B Cosmology

[You may work in units of the speed of light, so that $c = 1$.]

By considering a spherical distribution of matter with total mass M and radius R and an infinitesimal mass δm located somewhere on its surface, derive the *Friedmann equation* describing the evolution of the scale factor $a(t)$ appearing in the relation $R(t) = R_0 a(t)/a(t_0)$ for a spatially-flat FLRW spacetime.

Consider now a spatially-flat, *contracting* universe filled by a single component with energy density ρ, which evolves with time as $\rho(t) = \rho_0[a(t)/a(t_0)]^{-4}$. Solve the Friedmann equation for $a(t)$ with $a(t_0) = 1$.

Paper 2, Section I

9B Cosmology

[You may work in units of the speed of light, so that $c = 1$.]

(a) Combining the Friedmann and continuity equations

$$H^2 = \frac{8\pi G}{3}\rho\,, \qquad\qquad \dot{\rho} + 3H(\rho + P) = 0\,,$$

derive the *Raychaudhuri equation* (also known as the *acceleration equation*) which expresses $\ddot{a}/a$ in terms of the energy density ρ and the pressure P.

(b) Assuming an equation of state $P = w\rho$ with constant w, for what w is the expansion of the universe accelerated or decelerated?

(c) Consider an expanding, spatially-flat FLRW universe with both a cosmological constant and non-relativistic matter (also known as dust) with energy densities ρ_{cc} and ρ_{dust} respectively. At some time corresponding to a_{eq}, the energy densities of these two components are equal $\rho_{cc}(a_{eq}) = \rho_{dust}(a_{eq})$. Is the expansion of the universe accelerated or decelerated at this time?

(d) For what numerical value of a/a_{eq} does the universe transition from deceleration to acceleration?

Paper 3, Section II

14B Cosmology

[You may work in units of the speed of light, so that $c = 1$.]

Consider the process where protons and electrons combine to form neutral hydrogen atoms;

$$p^+ + e^- \leftrightarrow H^0 + \gamma.$$

Let n_p, n_e and n_H denote the number densities for protons, electrons and hydrogen atoms respectively. The ionization energy of hydrogen is denoted I. State and derive *Saha's equation* for the ratio $n_e n_p / n_H$, clearly describing the steps required.

[You may use without proof the following formula for the equilibrium number density of a non-relativistic species a with g_a degenerate states of mass m at temperature T such that $k_B T \ll m$,

$$n_a = g_a \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} \exp\left([\mu - m] / k_B T \right) ,$$

where μ is the chemical potential and k_B and h are the Boltzmann and Planck constants respectively.]

The photon number density n_γ is given as

$$n_\gamma = \frac{16\pi}{h^3} \zeta(3) \left(k_B T \right)^3 ,$$

where $\zeta(3) \simeq 1.20$. Consider now the fractional ionization $X_e = n_e/(n_e + n_H)$. In our universe $n_e + n_H = n_p + n_H \simeq \eta n_\gamma$ where η is the baryon-to-photon number ratio. Find an expression for the ratio

$$\frac{(1 - X_e)}{X_e^2}$$

in terms of $k_B T$, η, I and the particle masses. One might expect neutral hydrogen to form at a temperature given by $k_B T \sim I \sim 13$ eV, but instead in our universe it forms at the much lower temperature $k_B T \sim 0.3$ eV. Briefly explain why this happens. Estimate the temperature at which neutral hydrogen would form in a hypothetical universe with $\eta = 1$. Briefly explain your answer.

Paper 1, Section II

15B Cosmology

[You may work in units of the speed of light, so that $c = 1$.]

Consider a spatially-flat FLRW universe with a single, canonical, homogeneous scalar field $\phi(t)$ with a potential $V(\phi)$. Recall the Friedmann equation and the Raychaudhuri equation (also known as the acceleration equation)

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V\right],$$
$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V\right).$$

(a) Assuming $\dot{\phi} \neq 0$, derive the equations of motion for ϕ, i.e.

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0\,.$$

(b) Assuming the special case $V(\phi) = \lambda\phi^4$, find $\phi(t)$, for some initial value $\phi(t_0) = \phi_0$ in the slow-roll approximation, i.e. assuming that $\dot{\phi}^2 \ll 2V$ and $\ddot{\phi} \ll 3H\dot{\phi}$.

(c) The number N of efoldings is defined by $dN = d\ln a$. Using the chain rule, express dN first in terms of dt and then in terms of $d\phi$. Write the resulting relation between dN and $d\phi$ in terms of V and $\partial_\phi V$ only, using the slow-roll approximation.

(d) Compute the number N of efoldings of expansion between some initial value $\phi_i < 0$ and a final value $\phi_f < 0$ (so that $\dot{\phi} > 0$ throughout).

(e) Discuss qualitatively the horizon and flatness problems in the old hot big bang model (i.e. without inflation) and how inflation addresses them.

Paper 2, Section I

9B Cosmology

(a) Consider a homogeneous and isotropic universe with a uniform distribution of galaxies. For three galaxies at positions $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_C$, show that spatial homogeneity implies that their non-relativistic velocities $\mathbf{v}(\mathbf{r})$ must satisfy

$$\mathbf{v}(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{v}(\mathbf{r}_B - \mathbf{r}_C) - \mathbf{v}(\mathbf{r}_A - \mathbf{r}_C),$$

and hence that the velocity field coordinates v_i are linearly related to the position coordinates r_j via

$$v_i = H_{ij} r_j \,,$$

where the matrix coefficients H_{ij} are independent of the position. Show why isotropy then implies Hubble's law

$$\mathbf{v} = H\,\mathbf{r}\,, \qquad \text{with } H \text{ independent of } \mathbf{r}\,.$$

Explain how the velocity of a galaxy is determined by the scale factor a and express the Hubble parameter H_0 today in terms of a.

(b) Define the *cosmological horizon* $d_H(t)$. For an Einstein–de Sitter universe with $a(t) \propto t^{2/3}$, calculate $d_H(t_0)$ at $t = t_0$ today in terms of H_0. Briefly describe the horizon problem of the standard cosmology.

Paper 3, Section I

9B Cosmology

The energy density of a particle species is defined by

$$\epsilon = \int_0^\infty E(p) n(p) dp \,,$$

where $E(p) = c\sqrt{p^2 + m^2c^2}$ is the energy, and $n(p)$ the distribution function, of a particle with momentum p. Here c is the speed of light and m is the rest mass of the particle. If the particle species is in thermal equilibrium then the distribution function takes the form

$$n(p) = \frac{4\pi}{h^3} g \frac{p^2}{\exp((E(p) - \mu)/kT) \mp 1},$$

where g is the number of degrees of freedom of the particle, T is the temperature, h and k are constants and $-$ is for bosons and $+$ is for fermions.

(a) Stating any assumptions you require, show that in the very early universe the energy density of a given particle species i is

$$\epsilon_i = \frac{4\pi g_i}{(hc)^3} (kT)^4 \int_0^\infty \frac{y^3}{e^y \mp 1} dy \,.$$

(b) Show that the total energy density in the very early universe is

$$\epsilon = \frac{4\pi^5}{15(hc)^3} g^* (kT)^4 \,,$$

where g^* is defined by

$$g^* \equiv \sum_{\text{Bosons}} g_i + \frac{7}{8} \sum_{\text{Fermions}} g_i .$$

[*Hint: You may use the fact that* $\int_0^\infty y^3 (e^y - 1)^{-1} dy = \pi^4/15$.]

Paper 1, Section I

9B Cosmology

For a homogeneous and isotropic universe filled with pressure-free matter ($P = 0$), the Friedmann and Raychaudhuri equations are, respectively,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \qquad \text{and} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\,,$$

with mass density ρ, curvature k, and where $\dot{a} \equiv da/dt$. Using conformal time τ with $d\tau = dt/a$, show that the relative density parameter can be expressed as

$$\Omega(t) \equiv \frac{\rho}{\rho_{\text{crit}}} = \frac{8\pi G\rho\, a^2}{3\mathcal{H}^2}\,,$$

where $\mathcal{H} = \frac{1}{a}\frac{da}{d\tau}$ and ρ_{crit} is the critical density of a flat $k = 0$ universe (Einstein–de Sitter). Use conformal time τ again to show that the Friedmann and Raychaudhuri equations can be re-expressed as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1 \qquad \text{and} \qquad 2\frac{d\mathcal{H}}{d\tau} + \mathcal{H}^2 + kc^2 = 0\,.$$

From these derive the evolution equation for the density parameter Ω:

$$\frac{d\Omega}{d\tau} = \mathcal{H}\,\Omega\,(\Omega - 1)\,.$$

Plot the qualitative behaviour of Ω as a function of time relative to the expanding Einstein–de Sitter model with $\Omega = 1$ (i.e., include curves initially with $\Omega > 1$ and $\Omega < 1$).

Paper 4, Section I

9B Cosmology

A constant overdensity is created by taking a spherical region of a flat matter-dominated universe with radius $\overline{R}$ and compressing it into a region with radius $R < \overline{R}$. The evolution is governed by the parametric equations

$$R = AR_0(1-\cos\theta)\,, \quad t = B(\theta - \sin\theta),$$

where R_0 is a constant and

$$A = \frac{\Omega_{m,0}}{2(\Omega_{m,0}-1)}\,, \quad B = \frac{\Omega_{m,0}}{2H_0\,(\Omega_{m,0}-1)^{3/2}}\,,$$

where H_0 is the Hubble constant and $\Omega_{m,0}$ is the fractional overdensity at time t_0.

Show that, as $t \to 0^+$,

$$R(t) = R_0\,\Omega_{m,0}^{1/3}\,a(t)\left(1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3} + \dots\right)\,,$$

where the scale factor is given by $a(t) = (3H_0t/2)^{2/3}$.

Show that, at the linear level, the density perturbation δ_{linear} grows as $a(t)$. Show that, when the spherical overdensity has collapsed to zero radius, the linear perturbation has value $\delta_{\text{linear}} = \frac{3}{20}\,(12\pi)^{2/3}$.

Paper 3, Section II

14B Cosmology

The pressure support equation for stars is

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho\,,$$

where ρ is the density, P is the pressure, r is the radial distance, and G is Newton's constant.

(a) What two boundary conditions should we impose on the above equation for it to describe a star?

(b) By assuming a polytropic equation of state,

$$P(r) = K\rho^{1+\frac{1}{n}}(r)\,,$$

where K is a constant, derive the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n,$$

where $\rho = \rho_c\theta^n$, with ρ_c the density at the centre of the star, and $r = a\xi$, for some a that you should determine.

(c) Show that the mass of a polytropic star is

$$M = \frac{1}{2\sqrt{\pi}}\left(\frac{(n+1)\,K}{G}\right)^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\,Y_n,$$

where $Y_n \equiv -\left.\xi_1^2\frac{d\theta}{d\xi}\right|_{\xi=\xi_1}$ and ξ_1 is the value of ξ at the surface of the star.

(d) Derive the following relation between the mass, M, and radius, R, of a polytropic star

$$M = A_n K^{\frac{n}{n-1}} R^{\frac{3-n}{1-n}}\,,$$

where you should determine the constant A_n. What type of star does the $n = 3$ polytrope represent and what is the significance of the mass being constant for this star?

Paper 1, Section II

15B Cosmology

A flat ($k=0$) homogeneous and isotropic universe with scale factor $a(t)$ is filled with a scalar field $\phi(t)$ with potential $V(\phi)$. Its evolution satisfies the Friedmann and scalar field equations,

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + c^2 V(\phi)\right), \qquad \ddot{\phi} + 3H\dot{\phi} + c^2\frac{dV}{d\phi} = 0\,,$$

where $H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter, $M_{\rm Pl}$ is the reduced Planck mass, and dots denote derivatives with respect to cosmic time t, e.g. $\dot{\phi} \equiv d\phi/dt$.

(a) Use these equations to derive the Raychaudhuri equation, expressed in the form:

$$\dot{H} \;=\; -\frac{1}{2M_{\rm Pl}^2}\,\dot{\phi}^2\,.$$

(b) Consider the following ansatz for the scalar field evolution,

$$\phi(t) \;=\; \phi_0 \ln\tanh(\lambda t)\,, \qquad (\dagger)$$

where λ, ϕ_0 are constants. Find the specific cosmological solution,

$$\begin{aligned} H(t) &= \lambda\frac{\phi_0^2}{M_{\rm Pl}^2}\coth(2\lambda t)\,, \\ a(t) &= a_0[\sinh(2\lambda t)]^{\phi_0^2/2M_{\rm Pl}^2}\,, \qquad a_0 \text{ constant.} \end{aligned}$$

(c) Hence, show that the Hubble parameter can be expressed in terms of ϕ as

$$H(\phi) \;=\; \lambda\frac{\phi_0^2}{M_{\rm Pl}^2}\cosh\left(\frac{\phi}{\phi_0}\right),$$

and that the scalar field ansatz solution ($\dagger$) requires the following form for the potential:

$$V(\phi) \;=\; \frac{2\lambda^2\phi_0^2}{c^2}\left[\left(\frac{3\phi_0^2}{2M_{\rm Pl}^2} - 1\right)\cosh^2\left(\frac{\phi}{\phi_0}\right) + 1\right].$$

(d) Assume that the given parameters in $V(\phi)$ are such that $2/3 < \phi_0^2/M_{\rm Pl}^2 < 2$. Show that the asymptotic limit for the cosmological solution as $t \to 0$ exhibits decelerating power law evolution and that there is an accelerating solution as $t \to \infty$, that is,

$$\begin{aligned} &t \to 0\,, \qquad \phi \to -\infty\,, \qquad a(t) \sim t^{\phi_0^2/2M_{\rm Pl}^2}\,, \\ &t \to \infty\,, \qquad \phi \to 0\,, \qquad a(t) \sim \exp(\lambda\phi_0^2\, t/M_{\rm Pl}^2)\,. \end{aligned}$$

Find the time $t_{\rm acc}$ at which the solution transitions from deceleration to acceleration.

Paper 1, Section I

9C Cosmology

In a homogeneous and isotropic universe, describe the relative displacement $\mathbf{r}(t)$ of two galaxies in terms of a scale factor $a(t)$. Show how the relative velocity $\mathbf{v}(t)$ of these galaxies is given by the relation $\mathbf{v}(t) = H(t)\mathbf{r}(t)$, where you should specify $H(t)$ in terms of $a(t)$.

From special relativity, the Doppler shift of light emitted by a particle moving away radially with speed v can be approximated by

$$\frac{\lambda_0}{\lambda_e} = \sqrt{\frac{1+v/c}{1-v/c}} = 1 + \frac{v}{c} + \mathcal{O}\left(\frac{v^2}{c^2}\right),$$

where λ_e is the wavelength of emitted light and λ_0 is the observed wavelength. For the observed light from distant galaxies in a homogeneous and isotropic expanding universe, show that the redshift defined by $1 + z \equiv \lambda_0/\lambda_e$ is given by

$$1 + z = \frac{a(t_0)}{a(t_e)},$$

where t_e is the time of emission and t_0 is the observation time.

Paper 2, Section I

9C Cosmology

In a homogeneous and isotropic universe ($\Lambda = 0$), the acceleration equation for the scale factor $a(t)$ is given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right),$$

where $\rho(t)$ is the mass density and $P(t)$ is the pressure.

If the matter content of the universe obeys the strong energy condition $\rho + 3P/c^2 \geqslant 0$, show that the acceleration equation can be rewritten as $\dot{H} + H^2 \leqslant 0$, with Hubble parameter $H(t) = \dot{a}/a$. Show that

$$H \geqslant \frac{1}{H_0^{-1} + t - t_0},$$

where $H_0 = H(t_0)$ is the measured value today at $t = t_0$. Hence, or otherwise, show that

$$a(t) \leqslant 1 + H_0(t - t_0).$$

Use this inequality to find an upper bound on the age of the universe.

Paper 3, Section I
9C Cosmology

(a) In the early universe electrons, protons and neutral hydrogen are in thermal equilibrium and interact via,

$$e^- + p^+ \leftrightharpoons H + \gamma \,.$$

The non-relativistic number density of particles in thermal equlibrium is

$$n_i = g_i \left(\frac{2\pi m_i kT}{h^2}\right)^{\frac{3}{2}} \exp\left(\frac{\mu_i - m_i c^2}{kT}\right) \,,$$

where, for each species i, g_i is the number of degrees of freedom, m_i is its mass, and μ_i is its chemical potential. [You may assume $g_e = g_p = 2$ and $g_H = 4$.]

Stating any assumptions required, use these expressions to derive the Saha equation which governs the relative abundances of electrons, protons and hydrogen,

$$\frac{n_e n_p}{n_H} = \left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{I}{kT}\right) \,,$$

where I is the binding energy of hydrogen, which should be defined.

(b) Naively, we might expect that the majority of electrons and protons combine to form neutral hydrogen once the temperature drops below the binding energy, i.e. $kT \lesssim I$. In fact recombination does not happen until a much lower temperature, when $kT \approx 0.03I$. Briefly explain why this is.

[*Hint: It may help to consider the relative abundances of particles in the early universe.*]

Paper 4, Section I

9C Cosmology

(a) By considering a spherically symmetric star in hydrostatic equilibrium derive the pressure support equation

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2},$$

where r is the radial distance from the centre of the star, $M(r)$ is the stellar mass contained inside that radius, and $P(r)$ and $\rho(r)$ are the pressure and density at radius r respectively.

(b) Propose, and briefly justify, boundary conditions for this differential equation, both at the centre of the star $r = 0$, and at the stellar surface $r = R$.

Suppose that $P = K\rho^2$ for some $K > 0$. Show that the density satisfies the linear differential equation

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\rho}{\partial x}\right) = -\rho$$

where $x = \alpha r$, for some constant α, is a rescaled radial coordinate. Find α.

Paper 3, Section II
13C Cosmology

(a) The scalar moment of inertia for a system of N particles is given by

$$I = \sum_{i=1}^{N} m_i \, \mathbf{r}_i \cdot \mathbf{r}_i \, ,$$

where m_i is the particle's mass and $\mathbf{r}_i$ is a vector giving the particle's position. Show that, for non-relativistic particles,

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i$$

where K is the total kinetic energy of the system and $\mathbf{F}_i$ is the total force on particle i.

Assume that any two particles i and j interact gravitationally with potential energy

$$V_{ij} = -\frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \, .$$

Show that

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = V \, ,$$

where V is the total potential energy of the system. Use the above to prove the virial theorem.

(b) Consider an approximately spherical overdensity of stationary non-interacting massive particles with initial constant density ρ_i and initial radius R_i. Assuming the system evolves until it reaches a stable virial equilibrium, what will the final ρ and R be in terms of their initial values? Would this virial solution be stable if our particles were baryonic rather than non-interacting? Explain your answer.

Paper 1, Section II

14C Cosmology

The evolution of a flat (k=0) homogeneous and isotropic universe with scale factor $a(t)$, mass density $\rho(t)$ and pressure $P(t)$ obeys the Friedmann and energy conservation equations

$$\begin{aligned} H^2(t) &= \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}\,, \\ \dot{\rho} &= -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right)\,, \end{aligned}$$

where $H(t)$ is the Hubble parameter (observed today $t = t_0$ with value $H_0 = H(t_0)$) and $\Lambda > 0$ is the cosmological constant.

Use these two equations to derive the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right) + \frac{\Lambda c^2}{3}\,.$$

For pressure-free matter ($\rho = \rho_{\mathrm{M}}$ and $P_{\mathrm{M}} = 0$), solve the energy conservation equation to show that the Friedmann and acceleration equations can be re-expressed as

$$\begin{aligned} H &= H_0\sqrt{\frac{\Omega_{\mathrm{M}}}{a^3} + \Omega_\Lambda}\,, \\ \frac{\ddot{a}}{a} &= -\frac{H_0^2}{2}\left[\frac{\Omega_{\mathrm{M}}}{a^3} - 2\Omega_\Lambda\right]\,, \end{aligned}$$

where we have taken $a(t_0) = 1$ and we have defined the relative densities today ($t = t_0$) as

$$\Omega_{\mathrm{M}} = \frac{8\pi G}{3H_0^2}\,\rho_{\mathrm{M}}(t_0) \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}\,.$$

Solve the Friedmann equation and show that the scale factor can be expressed as

$$a(t) = \left(\frac{\Omega_{\mathrm{M}}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\tfrac{3}{2}\sqrt{\Omega_\Lambda}\,H_0\,t\right)\,.$$

Find an expression for the time $\bar{t}$ at which the matter density ρ_{M} and the effective density caused by the cosmological constant Λ are equal. (You need not evaluate this explicitly.)

Paper 1, Section I

9C Cosmology

The expansion scale factor, $a(t)$, for an isotropic and spatially homogeneous universe containing material with pressure p and mass density ρ obeys the equations

$$\dot{\rho} + 3(\rho + p)\,\frac{\dot{a}}{a} = 0\,,$$
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}\,,$$

where the speed of light is set equal to unity, G is Newton's constant, k is a constant equal to 0 or ± 1, and Λ is the cosmological constant. Explain briefly the interpretation of these equations.

Show that these equations imply

$$\frac{\ddot{a}}{a} = -\frac{4\pi G(\rho + 3p)}{3} + \frac{\Lambda}{3}\,.$$

Hence show that a static solution with constant $a = a_{\mathrm{s}}$ exists when $p = 0$ if

$$\Lambda = 4\pi G\rho = \frac{k}{a_{\mathrm{s}}^2}\,.$$

What must the value of k be, if the density ρ is non-zero?

Paper 2, Section I

9C Cosmology

A spherical cloud of mass M has radius $r(t)$ and initial radius $r(0) = R$. It contains material with uniform mass density $\rho(t)$, and zero pressure. Ignoring the cosmological constant, show that if it is initially at rest at $t = 0$ and the subsequent gravitational collapse is governed by Newton's law $\ddot{r} = -GM/r^2$, then

$$\dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right).$$

Suppose r is given parametrically by

$$r = R\cos^2\theta\,,$$

where $\theta = 0$ at $t = 0$. Derive a relation between θ and t and hence show that the cloud collapses to radius $r = 0$ at

$$t = \sqrt{\frac{3\pi}{32G\rho_0}}\,,$$

where ρ_0 is the initial mass density of the cloud.

Paper 3, Section I

9C Cosmology

A universe contains baryonic matter with background density $\rho_B(t)$ and density inhomogeneity $\delta_B(\mathbf{x},t)$, together with non-baryonic dark matter with background density $\rho_D(t)$ and density inhomogeneity $\delta_D(\mathbf{x},t)$. After the epoch of radiation–matter density equality, $t_{\rm eq}$, the background dynamics are governed by

$$H = \frac{2}{3t} \quad \text{and} \quad \rho_D = \frac{1}{6\pi G t^2}\,,$$

where H is the Hubble parameter.

The dark-matter density is much greater than the baryonic density ($\rho_D \gg \rho_B$) and so the time-evolution of the coupled density perturbations, at any point $\mathbf{x}$, is described by the equations

$$\begin{aligned}\ddot{\delta}_B \,+\, 2H\dot{\delta}_B \,&=\, 4\pi G\,\rho_D\,\delta_D\,,\\ \ddot{\delta}_D \,+\, 2H\dot{\delta}_D \,&=\, 4\pi G\,\rho_D\,\delta_D\,.\end{aligned}$$

Show that

$$\delta_D \,=\, \frac{\alpha}{t} \,+\, \beta\, t^{2/3}\,,$$

where α and β are independent of time. Neglecting modes in δ_D and δ_B that decay with increasing time, show that the baryonic density inhomogeneity approaches

$$\delta_B \,=\, \beta\, t^{2/3} \,+\, \gamma\,,$$

where γ is independent of time.

Briefly comment on the significance of your calculation for the growth of baryonic density inhomogeneities in the early universe.

Paper 4, Section I

9C Cosmology

The external gravitational potential $\Phi(r)$ due to a thin spherical shell of radius a and mass per unit area σ, centred at $r = 0$, will equal the gravitational potential due to a point mass M at $r = 0$, at any distance $r > a$, provided

$$\frac{Mr\Phi(r)}{2\pi\sigma a} + K(a)r = \int_{r-a}^{r+a} R\Phi(R)\,dR\,, \qquad (*)$$

where $K(a)$ depends on the radius of the shell. For which values of q does this equation have solutions of the form $\Phi(r) = Cr^q$, where C is constant? Evaluate $K(a)$ in each case and find the relation between the mass of the shell and M.

Hence show that the general gravitational force

$$F(r) = \frac{A}{r^2} + Br$$

has a potential satisfying $(*)$. What is the cosmological significance of the constant B?

Paper 3, Section II

13C Cosmology

The early universe is described by equations (with units such that $c = 8\pi G = \hbar = 1$)

$$3H^2 = \rho\,, \qquad \dot{\rho} + 3H(\rho + p) = 0\,, \tag{1}$$

where $H = \dot{a}/a$. The universe contains only a self-interacting scalar field ϕ with interaction potential $V(\phi)$ so that the density and pressure are given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \end{aligned}$$

Show that

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,. \tag{2}$$

Explain the slow-roll approximation and apply it to equations (1) and (2) to show that it leads to

$$\sqrt{3}\int \frac{\sqrt{V}}{V'}d\phi = -t + \text{const.}$$

If $V(\phi) = \frac{1}{4}\lambda\phi^4$ with λ a positive constant and $\phi(0) = \phi_0$, show that

$$\phi(t) = \phi_0 \exp\left[-\sqrt{\frac{4\lambda}{3}}\,t\right]$$

and that, for small t, the scale factor $a(t)$ expands to leading order in t as

$$a(t) \propto \exp\left[\sqrt{\frac{\lambda}{12}}\,\phi_0^2\,t\right]\,.$$

Comment on the relevance of this result for inflationary cosmology.

Paper 1, Section II

14C Cosmology

The distribution function $f(\mathbf{x}, \mathbf{p}, t)$ gives the number of particles in the universe with position in $(\mathbf{x}, \mathbf{x} + \delta\mathbf{x})$ and momentum in $(\mathbf{p}, \mathbf{p} + \delta\mathbf{p})$ at time t. It satisfies the boundary condition that $f \to 0$ as $|\mathbf{x}| \to \infty$ and as $|\mathbf{p}| \to \infty$. Its evolution obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \left[\frac{df}{dt}\right]_{\text{col}},$$

where the collision term $\left[\frac{df}{dt}\right]_{\text{col}}$ describes any particle production and annihilation that occurs.

The universe expands isotropically and homogeneously with expansion scale factor $a(t)$, so the momenta evolve isotropically with magnitude $p \propto a^{-1}$. Show that the Boltzmann equation simplifies to

$$\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}\mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left[\frac{df}{dt}\right]_{\text{col}}. \qquad (*)$$

The number densities n of particles and $\bar{n}$ of antiparticles are defined in terms of their distribution functions f and $\bar{f}$, and momenta p and $\bar{p}$, by

$$n = \int_0^\infty f\, 4\pi p^2\, dp \quad \text{and} \quad \bar{n} = \int_0^\infty \bar{f}\, 4\pi \bar{p}^2\, d\bar{p},$$

and the collision term may be assumed to be of the form

$$\left[\frac{df}{dt}\right]_{\text{col}} = -\langle \sigma v \rangle \int_0^\infty \bar{f} f\, 4\pi \bar{p}^2\, d\bar{p} + R$$

where $\langle \sigma v \rangle$ determines the annihilation cross-section of particles by antiparticles and R is the production rate of particles.

By integrating equation $(*)$ with respect to the momentum $\mathbf{p}$ and assuming that $\langle \sigma v \rangle$ is a constant, show that

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = -\langle \sigma v \rangle n\bar{n} + Q,$$

where $Q = \int_0^\infty R\, 4\pi p^2\, dp$. Assuming the same production rate R for antiparticles, write down the corresponding equation satisfied by $\bar{n}$ and show that

$$(n - \bar{n})a^3 = \text{constant}.$$

Paper 4, Section I

8C Cosmology

Calculate the total effective number of relativistic spin states g_* present in the early universe when the temperature T is $10^{10}\,\mathrm{K}$ if there are three species of low-mass neutrinos and antineutrinos in addition to photons, electrons and positrons. If the weak interaction rate is $\Gamma = (T/10^{10}\,\mathrm{K})^5\,\mathrm{s}^{-1}$ and the expansion rate of the universe is $H = \sqrt{8\pi G\rho/3}$, where ρ is the total density of the universe, calculate the temperature T_* at which the neutrons and protons cease to interact via weak interactions, and show that $T_* \propto g_*^{1/6}$.

State the formula for the equilibrium ratio of neutrons to protons at T_*, and briefly describe the sequence of events as the temperature falls from T_* to the temperature at which the nucleosynthesis of helium and deuterium ends.

What is the effect of an increase or decrease of g_* on the abundance of helium-4 resulting from nucleosynthesis? Why do changes in g_* have a very small effect on the final abundance of deuterium?

Paper 3, Section I

8C Cosmology

What is the *flatness problem*? Show by reference to the Friedmann equation how a period of accelerated expansion of the scale factor $a(t)$ in the early stages of the universe can solve the flatness problem if $\rho + 3P < 0$, where ρ is the mass density and P is the pressure.

In the very early universe, where we can neglect the spatial curvature and the cosmological constant, there is a homogeneous scalar field ϕ with a vacuum potential energy

$$V(\phi) = m^2\phi^2 \,,$$

and the Friedmann energy equation (in units where $8\pi G = 1$) is

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$

where H is the Hubble parameter. The field ϕ obeys the evolution equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,.$$

During inflation, ϕ evolves slowly after starting from a large initial value ϕ_i at $t = 0$. State what is meant by the *slow-roll approximation.* Show that in this approximation,

$$\begin{aligned} \phi(t) &= \phi_i - \frac{2}{\sqrt{3}} mt\,, \\ a(t) &= a_i \exp\left[\frac{m\phi_i}{\sqrt{3}}\,t - \frac{1}{3}m^2t^2\right] = a_i \exp\left[\frac{\phi_i^2 - \phi^2(t)}{4}\right]\,, \end{aligned}$$

where a_i is the initial value of a.

As $\phi(t)$ decreases from its initial value ϕ_i, what is its approximate value when the slow-roll approximation fails?

Paper 2, Section I

8C Cosmology

The mass density perturbation equation for non-relativistic matter ($P \ll \rho c^2$) with wave number k in the late universe ($t > t_{\rm eq}$) is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \left(4\pi G\rho - \frac{c_s^2\, k^2}{a^2}\right)\delta = 0\,. \qquad (*)$$

Suppose that a non-relativistic fluid with the equation of state $P \propto \rho^{4/3}$ dominates the universe when $a(t) = t^{2/3}$, and the curvature and the cosmological constant can be neglected. Show that the sound speed can be written in the form $c_s^2(t) \equiv dP/d\rho = \bar{c}_s^2\, t^{-2/3}$ where $\bar{c}_s$ is a constant.

Find power-law solutions to $(*)$ of the form $\delta \propto t^{\beta}$ and hence show that the general solution is

$$\delta = A_k\, t^{n_+} + B_k\, t^{n_-}$$

where

$$n_\pm = -\frac{1}{6} \pm \left[\left(\frac{5}{6}\right)^2 - \bar{c}_s^2\, k^2\right]^{1/2}.$$

Interpret your solutions in the two regimes $k \ll k_J$ and $k \gg k_J$ where $k_J = \dfrac{5}{6\bar{c}_s}$.

Paper 1, Section I

8C Cosmology

Consider three galaxies O, A and B with position vectors $\mathbf{r}_O$, $\mathbf{r}_A$ and $\mathbf{r}_B$ in a homogeneous universe. Assuming they move with non-relativistic velocities $\mathbf{v}_O = \mathbf{0}$, $\mathbf{v}_A$ and $\mathbf{v}_B$, show that spatial homogeneity implies that the velocity field $\mathbf{v}(\mathbf{r})$ satisfies

$$\mathbf{v}(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{v}(\mathbf{r}_B - \mathbf{r}_O) - \mathbf{v}(\mathbf{r}_A - \mathbf{r}_O)\,,$$

and hence that $\mathbf{v}$ is linearly related to $\mathbf{r}$ by

$$v_i = \sum_{j=1}^{3} H_{ij} r_j\,,$$

where the components of the matrix H_{ij} are independent of $\mathbf{r}$.

Suppose the matrix H_{ij} has the form

$$H_{ij} = \frac{D}{t}\begin{pmatrix} 5 & -1 & -2 \\ 1 & 5 & -1 \\ 2 & 1 & 5 \end{pmatrix}\,,$$

with $D > 0$ constant. Describe the kinematics of the cosmological expansion.

Paper 3, Section II

12C Cosmology

Massive particles and antiparticles each with mass m and respective number densities $n(t)$ and $\bar{n}(t)$ are present at time t in the radiation era of an expanding universe with zero curvature and no cosmological constant. Assuming they interact with cross-section σ at speed v, explain, by identifying the physical significance of each of the terms, why the evolution of $n(t)$ is described by

$$\frac{dn}{dt} = -3\,\frac{\dot{a}}{a}\,n - \langle \sigma v \rangle\, n\bar{n} + P(t)\,,$$

where the expansion scale factor of the universe is $a(t)$, and where the meaning of $P(t)$ should be briefly explained. Show that

$$(n - \bar{n})a^3 = \text{ constant}\,.$$

Assuming initial particle-antiparticle symmetry, show that

$$\frac{d(na^3)}{dt} = \langle \sigma v \rangle\,(n_{\rm eq}^2 - n^2)a^3\,,$$

where $n_{\rm eq}$ is the equilibrium number density at temperature T.

Let $Y = n/T^3$ and $x = m/T$. Show that

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}(Y^2 - Y_{\rm eq}^2)\,,$$

where $\lambda = m^3 \langle \sigma v \rangle / H_m$ and H_m is the Hubble expansion rate when $T = m$.

When $x > x_f \simeq 10$, the number density n can be assumed to be depleted only by annihilations. If λ is constant, show that as $x \to \infty$ at late time, Y approaches a constant value given by

$$Y = \frac{x_f}{\lambda}\,.$$

Why do you expect weakly interacting particles to survive in greater numbers than strongly interacting particles?

Paper 1, Section II

12C Cosmology

A closed universe contains black-body radiation, has a positive cosmological constant Λ, and is governed by the equation

$$\frac{\dot{a}^2}{a^2} = \frac{\Gamma}{a^4} - \frac{1}{a^2} + \frac{\Lambda}{3}\,,$$

where $a(t)$ is the scale factor and Γ is a positive constant. Using the substitution $y = a^2$ and the boundary condition $y(0) = 0$, deduce the boundary condition for $\dot{y}(0)$ and show that

$$\ddot{y} = \frac{4\Lambda}{3}y - 2$$

and hence that

$$a^2(t) = \frac{3}{2\Lambda}\left[1 - \cosh\left(\sqrt{\frac{4\Lambda}{3}}\,t\right) + \lambda \sinh\left(\sqrt{\frac{4\Lambda}{3}}\,t\right)\right].$$

Express the constant λ in terms of Λ and Γ.

Sketch the graphs of $a(t)$ for the cases $\lambda > 1$ and $0 < \lambda < 1$.

Paper 4, Section I

10E Cosmology

A homogeneous and isotropic universe, with cosmological constant Λ, has expansion scale factor $a(t)$ and Hubble expansion rate $H = \dot{a}/a$. The universe contains matter with density ρ and pressure P which satisfy the positive-energy condition $\rho + 3P/c^2 \geqslant 0$. The acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) + \frac{1}{3}\Lambda c^2.$$

If $\Lambda \leqslant 0$, show that

$$\frac{d}{dt}(H^{-1}) \geqslant 1.$$

Deduce that $H \to \infty$ and $a \to 0$ at a finite time in the past or the future. What property of H distinguishes the two cases?

Give a simple counterexample with $\rho = P = 0$ to show that this deduction fails to hold when $\Lambda > 0$.

Paper 3, Section I

10E Cosmology

Consider a finite sphere of zero-pressure material of uniform density $\rho(t)$ which expands with radius $r(t) = a(t)r_0$, where r_0 is an arbitary constant, due to the evolution of the expansion scale factor $a(t)$. The sphere has constant total mass M and its radius satisfies

$$\ddot{r} = -\frac{d\Phi}{dr},$$

where

$$\Phi(r) = -\frac{GM}{r} - \frac{1}{6}\Lambda r^2 c^2,$$

with Λ constant. Show that the scale factor obeys the equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} + \frac{1}{3}\Lambda c^2,$$

where K is a constant. Explain why the sign, but not the magnitude, of K is important. Find exact solutions of this equation for $a(t)$ when

(i) $K = \Lambda = 0$ and $\rho(t) \neq 0$,

(ii) $\rho = K = 0$ and $\Lambda > 0$,

(iii) $\rho = \Lambda = 0$ and $K \neq 0$.

Which two of the solutions (i)–(iii) are relevant for describing the evolution of the universe after the radiation-dominated era?

Paper 2, Section I

10E Cosmology

A self-gravitating fluid with density ρ, pressure $P(\rho)$ and velocity $\mathbf{v}$ in a gravitational potential Φ obeys the equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) &= 0\,, \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{\nabla}\Phi &= \mathbf{0}\,, \\ \nabla^2 \Phi &= 4\pi G \rho\,. \end{aligned}$$

Assume that there exists a static constant solution of these equations with $\mathbf{v} = \mathbf{0}$, $\rho = \rho_0$ and $\Phi = \Phi_0$, for which $\boldsymbol{\nabla}\Phi_0$ can be neglected. This solution is perturbed. Show that, to first order in the perturbed quantities, the density perturbations satisfy

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \nabla^2 \rho_1 + 4\pi G \rho_0 \rho_1\,,$$

where $\rho = \rho_0 + \rho_1(\mathbf{x}, t)$ and $c_s^2 = dP/d\rho$. Show that there are solutions to this equation of the form

$$\rho_1(\mathbf{x}, t) = A \exp[-i\mathbf{k} \cdot \mathbf{x} + i\omega t]\,,$$

where A, ω and $\mathbf{k}$ are constants and

$$\omega^2 = c_s^2\, \mathbf{k} \cdot \mathbf{k} - 4\pi G \rho_0\,.$$

Interpret these solutions physically in the limits of small and large $|\mathbf{k}|$, explaining what happens to density perturbations on large and small scales, and determine the critical wavenumber that divides the two distinct behaviours of the perturbation.

Paper 1, Section I
10E Cosmology

Which particle states are expected to be relativistic and which interacting when the temperature T of the early universe satisfies

(i) $10^{10}\,\mathrm{K} < T < 5 \times 10^{10}\,\mathrm{K}$,

(ii) $5 \times 10^{9}\,\mathrm{K} < T < 10^{10}\,\mathrm{K}$,

(iii) $T < 5 \times 10^{9}\,\mathrm{K}$?

Calculate the total spin weight factor, g_*, of the relativistic particles and the total spin weight factor, g_I, of the interacting particles, in each of the three temperature intervals.

What happens when the temperature falls below $5 \times 10^9\,\mathrm{K}$? Calculate the ratio of the temperatures of neutrinos and photons. Find the effective value of g_* after the universe cools below this temperature. [*Note that the equilibrium entropy density is given by $s = (\rho c^2 + P)/T$, where ρ is the density and P is the pressure.*]

Paper 3, Section II
15E Cosmology

The luminosity distance to an astronomical light source is given by $d_L = \chi/a(t)$, where $a(t)$ is the expansion scale factor and χ is the comoving distance in the universe defined by $dt = a(t)d\chi$. A zero-curvature Friedmann universe containing pressure-free matter and a cosmological constant with density parameters Ω_m and $\Omega_\Lambda \equiv 1 - \Omega_m$, respectively, obeys the Friedmann equation

$$H^2 = H_0^2 \left(\frac{\Omega_{m0}}{a^3} + \Omega_{\Lambda 0} \right),$$

where $H = (da/dt)/a$ is the Hubble expansion rate of the universe and the subscript $_0$ denotes present-day values, with $a_0 \equiv 1$.

If z is the redshift, show that

$$d_L(z) = \frac{1+z}{H_0} \int_0^z \frac{dz'}{\left[(1-\Omega_{\Lambda 0})(1+z')^3 + \Omega_{\Lambda 0}\right]^{1/2}}.$$

Find $d_L(z)$ when $\Omega_{\Lambda 0} = 0$ and when $\Omega_{m0} = 0$. Roughly sketch the form of $d_L(z)$ for these two cases. What is the effect of a cosmological constant Λ on the luminosity distance at a fixed value of z? Briefly describe how the relation between luminosity distance and redshift has been used to establish the acceleration of the expansion of the universe.

Paper 1, Section II

15E Cosmology

What are the cosmological *flatness* and *horizon* problems? Explain what form of time evolution of the cosmological expansion scale factor $a(t)$ must occur during a period of inflationary expansion in a Friedmann universe. How can inflation solve the horizon and flatness problems? [You may assume an equation of state where pressure P is proportional to density ρ.]

The universe has Hubble expansion rate $H = \dot{a}/a$ and contains only a scalar field ϕ with self-interaction potential $V(\phi) > 0$. The density and pressure are given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \\ P &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \end{aligned}$$

in units where $c = \hbar = 1$. Show that the conservation equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

requires

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0\,.$$

If the Friedmann equation has the form

$$3H^2 = 8\pi G\rho$$

and the scalar-field potential has the form

$$V(\phi) = V_0 e^{-\lambda\phi}\,,$$

where V_0 and λ are positive constants, show that there is an exact cosmological solution with

$$\begin{aligned} a(t) &\propto t^{16\pi G/\lambda^2}\,, \\ \phi(t) &= \phi_0 + \frac{2}{\lambda}\ln(t)\,, \end{aligned}$$

where ϕ_0 is a constant. Find the algebraic relation between λ, V_0 and ϕ_0. Show that a solution only exists when $0 < \lambda^2 < 48\pi G$. For what range of values of λ^2 does inflation occur? Comment on what happens when $\lambda \to 0$.

Paper 4, Section I

10D Cosmology

List the relativistic species of bosons and fermions from the standard model of particle physics that are present in the early universe when the temperature falls to $1\,MeV/k_B$.

Which of the particles above will be interacting when the temperature is above $1\,MeV/k_B$ and between $1\,MeV/k_B \gtrsim T \gtrsim 0.51\,MeV/k_B$, respectively?

Explain what happens to the populations of particles present when the temperature falls to $0.51\,MeV/k_B$.

The entropy density of fermion and boson species with temperature T is $s \propto g_s T^3$, where g_s is the number of relativistic spin degrees of freedom, that is,

$$g_s = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i \,.$$

Show that when the temperature of the universe falls below $0.51\,MeV/k_B$ the ratio of the neutrino and photon temperatures will be given by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

Paper 3, Section I

10D Cosmology

The number densities of protons of mass m_p or neutrons of mass m_n in kinetic equilibrium at temperature T, in the absence of any chemical potentials, are each given by (with $i = n$ or p)

$$n_i = g_i \left(\frac{m_i k_B T}{2\pi \hbar^2} \right)^{3/2} \exp\left[-m_i c^2 / k_B T \right] \ ,$$

where k_B is Boltzmann's constant and g_i is the spin degeneracy.

Use this to show, to a very good approximation, that the ratio of the number of neutrons to protons at a temperature $T \simeq 1 MeV/k_B$ is given by

$$\frac{n_n}{n_p} = \exp\left[-(m_n - m_p) c^2 / k_B T \right] \ ,$$

where $(m_n - m_p)c^2 = 1.3 MeV$. Explain any approximations you have used.

The reaction rate for weak interactions between protons and neutrons at energies $5MeV \geqslant k_B T \geqslant 0.8MeV$ is given by $\Gamma = (k_B T / 1MeV)^5 s^{-1}$ and the expansion rate of the universe at these energies is given by $H = (k_B T/1MeV)^2 s^{-1}$. Give an example of a weak interaction that can maintain equilibrium abundances of protons and neutrons at these energies. Show how the final abundance of neutrons relative to protons can be calculated and use it to estimate the mass fraction of the universe in helium-4 after nucleosynthesis.

What would have happened to the helium abundance if the proton and neutron masses had been exactly equal?

Paper 2, Section I

10D Cosmology

The linearised equation for the growth of small inhomogeneous density perturbations $\delta_{\mathbf{k}}$ with comoving wavevector $\mathbf{k}$ in an isotropic and homogeneous universe is

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\rho\right)\delta_{\mathbf{k}} = 0\,,$$

where ρ is the matter density, $c_s = (dP/d\rho)^{1/2}$ is the sound speed, P is the pressure, $a(t)$ is the expansion scale factor of the unperturbed universe, and overdots denote differentiation with respect to time t.

Define the Jeans wavenumber and explain its physical meaning.

Assume the unperturbed Friedmann universe has zero curvature and cosmological constant and it contains only zero-pressure matter, so that $a(t) = a_0 t^{2/3}$. Show that the solution for the growth of density perturbations is given by

$$\delta_{\mathbf{k}} = A(\mathbf{k})t^{2/3} + B(\mathbf{k})t^{-1}\,.$$

Comment briefly on the cosmological significance of this result.

Paper 1, Section I

10D Cosmology

The Friedmann equation and the fluid conservation equation for a closed isotropic and homogeneous cosmology are given by

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{1}{a^2}\,,$$
$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,,$$

where the speed of light is set equal to unity, G is the gravitational constant, $a(t)$ is the expansion scale factor, ρ is the fluid mass density and P is the fluid pressure, and overdots denote differentiation with respect to the time coordinate t.

If the universe contains only blackbody radiation and $a = 0$ defines the zero of time t, show that

$$a^2(t) = t(t_* - t)\,,$$

where t_* is a constant. What is the physical significance of the time t_*? What is the value of the ratio $a(t)/t$ at the time when the scale factor is largest? Sketch the curve of $a(t)$ and identify its geometric shape.

Briefly comment on whether this cosmological model is a good description of the observed universe at any time in its history.

Paper 3, Section II

15D Cosmology

The contents of a spatially homogeneous and isotropic universe are modelled as a finite mass M of pressureless material whose radius $r(t)$ evolves from some constant reference radius r_0 in proportion to the time-dependent scale factor $a(t)$, with

$$r(t) = a(t)r_0 \, .$$

(i) Show that this motion leads to expansion governed by Hubble's Law. If this universe is expanding, explain why there will be a shift in the frequency of radiation between its emission from a distant object and subsequent reception by an observer. Define the redshift z of the observed object in terms of the values of the scale factor $a(t)$ at the times of emission and reception.

(ii) The expanding universal mass M is given a small rotational perturbation, with angular velocity ω, and its angular momentum is subsequently conserved. If deviations from spherical expansion can be neglected, show that its linear rotational velocity will fall as $V \propto a^{-n}$, where you should determine the value of n. Show that this perturbation will become increasingly insignificant compared to the expansion velocity as the universe expands if $a \propto t^{2/3}$.

(iii) A distant cloud of intermingled hydrogen (H) atoms and carbon monoxide (CO) molecules has its redshift determined simultaneously in two ways: by detecting 21 cm radiation from atomic hydrogen and by detecting radiation from rotational transitions in CO molecules. The ratio of the 21 cm atomic transition frequency to the CO rotational transition frequency is proportional to α^2, where α is the fine structure constant. It is suggested that there may be a small difference in the value of the constant α between the times of emission and reception of the radiation from the cloud.

Show that the difference in the redshift values for the cloud, $\Delta z = z_{CO} - z_{21}$, determined separately by observations of the H and CO transitions, is related to $\delta\alpha = \alpha_r - \alpha_e$, the difference in α values at the times of reception and emission, by

$$\Delta z = 2\left(\frac{\delta\alpha}{\alpha_r}\right)(1 + z_{CO}) \, .$$

(iv) The universe today contains 30% of its total density in the form of pressureless matter and 70% in the form of a dark energy with constant redshift-independent density. If these are the only two significant constituents of the universe, show that their densities were equal when the scale factor of the universe was approximately equal to 75% of its present value.

Paper 1, Section II

15D Cosmology

A spherically symmetric star of total mass M_s has pressure $P(r)$ and mass density $\rho(r)$, where r is the radial distance from its centre. These quantities are related by the equations of hydrostatic equilibrium and mass conservation:

$$\begin{aligned}\frac{dP}{dr} &= -\frac{GM(r)\rho}{r^2}\,,\\ \frac{dM}{dr} &= 4\pi\rho r^2\,,\end{aligned}$$

where $M(r)$ is the mass inside radius r.

By integrating from the centre of the star at $r=0$, where $P=P_c$, to the surface of the star at $r=R_s$, where $P=P_s$, show that

$$4\pi R_s^3 P_s = \Omega + 3\int_0^{M_s} \frac{P}{\rho}\, dM\,,$$

where Ω is the total gravitational potential energy. Show that

$$-\Omega > \frac{GM_s^2}{2R_s}\,.$$

If the surface pressure is negligible and the star is a perfect gas of particles of mass m with number density n and $P = nk_BT$ at temperature T, and radiation pressure can be ignored, then show that

$$3\int_0^{M_s} \frac{P}{\rho}\, dM = \frac{3k_B}{m}\bar{T}\,,$$

where $\bar{T}$ is the mean temperature of the star, which you should define.

Hence, show that the mean temperature of the star satisfies the inequality

$$\bar{T} > \frac{GM_s m}{6k_B R_s}\,.$$

Paper 4, Section I

10E Cosmology

The number density of a species $\star$ of non-relativistic particles of mass m, in equilibrium at temperature T and chemical potential μ, is

$$n_\star = g_\star \left(\frac{2\pi m k T}{h^2} \right)^{3/2} e^{(\mu - mc^2)/kT} \,,$$

where $g_\star$ is the spin degeneracy. During primordial nucleosynthesis, deuterium, D, forms through the nuclear reaction

$$p + n \leftrightarrow D \,,$$

where p and n are non-relativistic protons and neutrons. Write down the relationship between the chemical potentials in equilibrium.

Using the fact that $g_D = 4$, and explaining the approximations you make, show that

$$\frac{n_D}{n_n n_p} \approx \left(\frac{h^2}{\pi m_p k T} \right)^{3/2} \exp\left(\frac{B_D}{kT} \right) \,,$$

where B_D is the deuterium binding energy, i.e. $B_D = (m_n + m_p - m_D)c^2$.

Let $X_\star = n_\star / n_B$ where n_B is the baryon number density of the universe. Using the fact that $n_\gamma \propto T^3$, show that

$$\frac{X_D}{X_n X_p} \propto T^{3/2} \eta \exp\left(\frac{B_D}{kT} \right) \,,$$

where η is the baryon asymmetry parameter

$$\eta = \frac{n_B}{n_\gamma} \,.$$

Briefly explain why primordial deuterium does not form until temperatures well below $kT \sim B_D$.

Paper 3, Section I

10E Cosmology

For an ideal Fermi gas in equilibrium at temperature T and chemical potential μ, the average occupation number of the kth energy state, with energy E_k, is

$$\bar{n}_k = \frac{1}{e^{(E_k-\mu)/k_BT}+1}\,.$$

Discuss the limit $T \to 0$. What is the Fermi energy ϵ_F? How is it related to the Fermi momentum p_F? Explain why the density of states with momentum between p and $p+dp$ is proportional to p^2dp and use this fact to deduce that the fermion number density at zero temperature takes the form

$$n \propto p_F^3\,.$$

Consider an ideal Fermi gas that, at zero temperature, is either (i) non-relativistic or (ii) ultra-relativistic. In each case show that the fermion energy density ϵ takes the form

$$\epsilon \propto n^\gamma\,,$$

for some constant γ which you should compute.

Paper 2, Section I

10E Cosmology

The Friedmann equation for the scale factor $a(t)$ of a homogeneous and isotropic universe of mass density ρ is

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}, \qquad \left(H = \frac{\dot{a}}{a}\right)$$

where $\dot{a} = da/dt$ and k is a constant. The mass conservation equation for a fluid of mass density ρ and pressure P is

$$\dot{\rho} = -3\left(\rho + P/c^2\right) H\,.$$

Conformal time τ is defined by $d\tau = a^{-1}dt$. Show that

$$\mathcal{H} = aH\,, \qquad \left(\mathcal{H} = \frac{a'}{a}\right),$$

where $a' = da/d\tau$. Hence show that the acceleration equation can be written as

$$\mathcal{H}' = -\frac{4\pi}{3} G\left(\rho + 3P/c^2\right) a^2\,.$$

Define the density parameter Ω_m and show that in a matter-dominated era, in which $P = 0$, it satisfies the equation

$$\Omega_m' = \mathcal{H}\,\Omega_m(\Omega_m - 1)\,.$$

Use this result to briefly explain the "flatness problem" of cosmology.

Paper 1, Section I

10E Cosmology

The number density of photons in equilibrium at temperature T is given by

$$n = \frac{8\pi}{(hc)^3}\int_0^\infty \frac{\nu^2 d\nu}{e^{\beta h\nu} - 1}\,,$$

where $\beta = 1/(k_B T)$ (k_B is Boltzmann's constant). Show that $n \propto T^3$. Show further that $\epsilon \propto T^4$, where ϵ is the photon energy density.

Write down the Friedmann equation for the scale factor $a(t)$ of a flat homogeneous and isotropic universe. State the relation between a and the mass density ρ for a radiation-dominated universe and hence deduce the time-dependence of a. How does the temperature T depend on time?

Paper 3, Section II
15E Cosmology

In a flat expanding universe with scale factor $a(t)$, average mass density $\bar{\rho}$ and average pressure $\bar{P} \ll \bar{\rho}c^2$, the fractional density perturbations $\delta_k(t)$ at co-moving wavenumber k satisfy the equation

$$\ddot{\delta}_k = -2\left(\frac{\dot{a}}{a}\right)\dot{\delta}_k + 4\pi G\bar{\rho}\delta_k - \frac{c_s^2k^2}{a^2}\delta_k\,. \qquad (*)$$

Discuss briefly the meaning of each term on the right hand side of this equation. What is the Jeans length λ_J, and what is its significance? How is it related to the Jeans mass?

How does the equation $(*)$ simplify at $\lambda \gg \lambda_J$ in a flat universe? Use your result to show that density perturbations can grow. For a growing density perturbation, how does $\dot{\delta}/\delta$ compare to the inverse Hubble time?

Explain qualitatively why structure only forms after decoupling, and why cold dark matter is needed for structure formation.

Paper 1, Section II
15E Cosmology

The Friedmann equation for the scale factor $a(t)$ of a homogeneous and isotropic universe of mass density ρ is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\,,$$

where $\dot{a} = da/dt$. Explain how the value of the constant k affects the late-time ($t \to \infty$) behaviour of a.

Explain briefly why $\rho \propto 1/a^3$ in a matter-dominated (zero-pressure) universe. By considering the scale factor a of a closed universe as a function of conformal time τ, defined by $d\tau = a^{-1}dt$, show that

$$a(\tau) = \frac{\Omega_0}{2(\Omega_0 - 1)}\left[1 - \cos\left(\sqrt{k}c\tau\right)\right]\,,$$

where Ω_0 is the present ($\tau = \tau_0$) density parameter, with $a(\tau_0) = 1$. Use this result to show that

$$t(\tau) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}\left[\sqrt{k}c\tau - \sin\left(\sqrt{k}c\tau\right)\right]\,,$$

where H_0 is the present Hubble parameter. Find the time t_{BC} at which this model universe ends in a "big crunch".

Given that $\sqrt{k}c\tau_0 \ll 1$, obtain an expression for the present age of the universe in terms of H_0 and Ω_0, according to this model. How does it compare with the age of a flat universe?

Paper 1, Section I

10E Cosmology

Light of wavelength λ_e emitted by a distant object is observed by us to have wavelength λ_0. The redshift z of the object is defined by

$$1 + z = \frac{\lambda_0}{\lambda_e}\,.$$

Assuming that the object is at a fixed comoving distance from us in a homogeneous and isotropic universe with scale factor $a(t)$, show that

$$1 + z = \frac{a(t_0)}{a(t_e)}\,,$$

where t_e is the time of emission and t_0 the time of observation (i.e. today).

[You may assume the non-relativistic Doppler shift formula $\Delta\lambda/\lambda = (v/c)\cos\theta$ for the shift $\Delta\lambda$ in the wavelength of light emitted by a nearby object travelling with velocity v at angle θ to the line of sight.]

Given that the object radiates energy L per unit time, explain why the rate at which energy passes through a sphere centred on the object and intersecting the Earth is $L/(1+z)^2$.

Paper 2, Section I

10E Cosmology

A spherically symmetric star in hydrostatic equilibrium has density $\rho(r)$ and pressure $P(r)$, which satisfy the pressure support equation,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad (*)$$

where $m(r)$ is the mass within a radius r. Show that this implies

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\, r^2 \rho\,.$$

Provide a justification for choosing the boundary conditions $dP/dr = 0$ at the centre of the star ($r = 0$) and $P = 0$ at its outer radius ($r = R$).

Use the pressure support equation $(*)$ to derive the virial theorem for a star,

$$\langle P \rangle V = -\frac{1}{3} E_{\mathrm{grav}}\,,$$

where $\langle P \rangle$ is the average pressure, V is the total volume of the star and E_{grav} is its total gravitational potential energy.

Paper 3, Section I

10E Cosmology

For an ideal gas of fermions of mass m in volume V, and at temperature T and chemical potential μ, the number density n and kinetic energy E are given by

$$n = \frac{4\pi g_s}{h^3} \int_0^\infty \bar{n}(p)\, p^2\, dp\,, \qquad E = \frac{4\pi g_s}{h^3} V \int_0^\infty \bar{n}(p)\epsilon(p) p^2\, dp\,,$$

where g_s is the spin-degeneracy factor, h is Planck's constant, $\epsilon(p) = c\sqrt{p^2 + m^2c^2}$ is the single-particle energy as a function of the momentum p, and

$$\bar{n}(p) = \left[\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + 1\right]^{-1}\,,$$

where k is Boltzmann's constant.

(i) Sketch the function $\bar{n}(p)$ at zero temperature, explaining why $\bar{n}(p) = 0$ for $p > p_F$ (the Fermi momentum). Find an expression for n at zero temperature as a function of p_F.

Assuming that a typical fermion is ultra-relativistic ($pc \gg mc^2$) even at zero temperature, obtain an estimate of the energy density E/V as a function of p_F, and hence show that

$$E \;\sim\; hc\, n^{4/3} V \qquad (*)$$

in the ultra-relativistic limit at zero temperature.

(ii) A white dwarf star of radius R has total mass $M = \frac{4\pi}{3} m_p n_p R^3$, where m_p is the proton mass and n_p the average proton number density. On the assumption that the star's degenerate electrons are ultra-relativistic, so that $(*)$ applies with n replaced by the average electron number density n_e, deduce the following estimate for the star's internal kinetic energy:

$$E_{\mathrm{kin}} \;\sim\; hc \left(\frac{M}{m_p}\right)^{4/3} \frac{1}{R}\,.$$

By comparing this with the total gravitational potential energy, briefly discuss the consequences for white dwarf stability.

Paper 4, Section I

10E Cosmology

The equilibrium number density of fermions at temperature T is

$$n = \frac{4\pi g_s}{h^3} \int_0^\infty \frac{p^2 dp}{\exp[(\epsilon(p) - \mu)/kT] + 1},$$

where g_s is the spin degeneracy and $\epsilon(p) = c\sqrt{p^2 + m^2c^2}$. For a non-relativistic gas with $pc \ll mc^2$ and $kT \ll mc^2 - \mu$, show that the number density becomes

$$n = g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp[(\mu - mc^2)/kT]. \qquad (*)$$

[You may assume that $\int_0^\infty dx\, x^2 e^{-x^2/\alpha} = (\sqrt{\pi}/4)\, \alpha^{3/2}$ for $\alpha > 0$.]

Before recombination, equilibrium is maintained between neutral hydrogen, free electrons, protons and photons through the interaction

$$p + e^- \leftrightarrow H + \gamma.$$

Using the non-relativistic number density $(*)$, deduce Saha's equation relating the electron and hydrogen number densities,

$$\frac{n_e^2}{n_H} \approx \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp(-I/kT),$$

where $I = (m_p + m_e - m_H)c^2$ is the ionization energy of hydrogen. State clearly any assumptions you have made.

Paper 1, Section II

15E Cosmology

A homogeneous and isotropic universe, with scale factor a, curvature parameter k, energy density ρ and pressure P, satisfies the Friedmann and energy conservation equations

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,,$$
$$\dot{\rho} + 3H(\rho + P/c^2) = 0\,,$$

where $H = \dot{a}/a$, and the dot indicates a derivative with respect to cosmological time t.

(i) Derive the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2)\,.$$

Given that the strong energy condition $\rho c^2 + 3P \geqslant 0$ is satisfied, show that $(aH)^2$ is a decreasing function of t in an expanding universe. Show also that the density parameter $\Omega = 8\pi G\rho/(3H^2)$ satisfies

$$\Omega - 1 = \frac{kc^2}{a^2H^2}\,.$$

Hence explain, briefly, the flatness problem of standard big bang cosmology.

(ii) A flat ($k = 0$) homogeneous and isotropic universe is filled with a radiation fluid ($w_R = 1/3$) and a dark energy fluid ($w_\Lambda = -1$), each with an equation of state of the form $P_i = w_i\rho_i c^2$ and density parameters today equal to Ω_{R0} and $\Omega_{\Lambda 0}$ respectively. Given that each fluid independently obeys the energy conservation equation, show that the total energy density $(\rho_R + \rho_\Lambda)c^2$ equals ρc^2, where

$$\rho(t) = \frac{3H_0^2}{8\pi G}\frac{\Omega_{R0}}{a^4}\left(1 + \frac{1-\Omega_{R0}}{\Omega_{R0}}\,a^4\right)\,,$$

with H_0 being the value of the Hubble parameter today. Hence solve the Friedmann equation to get

$$a(t) = \alpha(\sinh\beta t)^{1/2}\,,$$

where α and β should be expressed in terms Ω_{R0} and $\Omega_{\Lambda 0}$. Show that this result agrees with the expected asymptotic solutions at both early ($t \to 0$) and late ($t \to \infty$) times.

[*Hint:* $\int dx/\sqrt{x^2+1} = \operatorname{arcsinh} x$.]

Paper 3, Section II

15E Cosmology

An expanding universe with scale factor $a(t)$ is filled with (pressure-free) cold dark matter (CDM) of average mass density $\bar{\rho}(t)$. In the Zel'dovich approximation to gravitational clumping, the perturbed position $\mathbf{r}(\mathbf{q},t)$ of a CDM particle with unperturbed comoving position $\mathbf{q}$ is given by

$$\mathbf{r}(\mathbf{q},t) = a(t)[\mathbf{q} + \boldsymbol{\psi}(\mathbf{q},t)]\,, \tag{1}$$

where $\boldsymbol{\psi}$ is the comoving displacement.

(i) Explain why the conservation of CDM particles implies that

$$\rho(\mathbf{r},t)\, d^3r = a^3\bar{\rho}(t)\, d^3q\,,$$

where $\rho(\mathbf{r},t)$ is the CDM mass density. Use (1) to verify that $d^3q = a^{-3}[1 - \boldsymbol{\nabla}_{\mathbf{q}} \cdot \boldsymbol{\psi}]d^3r$, and hence deduce that the fractional density perturbation is, to first order,

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} = -\boldsymbol{\nabla}_{\mathbf{q}} \cdot \boldsymbol{\psi}\,.$$

Use this result to integrate the Poisson equation $\nabla^2\Phi = 4\pi G\bar{\rho}$ for the gravitational potential Φ. Then use the particle equation of motion $\ddot{\mathbf{r}} = -\boldsymbol{\nabla}\Phi$ to deduce a second-order differential equation for $\boldsymbol{\psi}$, and hence that

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G\bar{\rho}\,\delta = 0\,. \tag{2}$$

[You may assume that $\nabla^2\Phi = 4\pi G\bar{\rho}$ implies $\boldsymbol{\nabla}\Phi = (4\pi G/3)\bar{\rho}\,\mathbf{r}$ and that the pressure-free acceleration equation is $\ddot{a} = -(4\pi G/3)\bar{\rho}a$.]

(ii) A flat matter-dominated universe with background density $\bar{\rho} = (6\pi G t^2)^{-1}$ has scale factor $a(t) = (t/t_0)^{2/3}$. The universe is filled with a pressure-free homogeneous (non-clumping) fluid of mass density $\rho_H(t)$, as well as cold dark matter of mass density $\rho_C(\mathbf{r},t)$.

Assuming that the Zel'dovich perturbation equation in this case is as in (2) but with $\bar{\rho}$ replaced by $\bar{\rho}_C$, i.e. that

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G\bar{\rho}_C\delta = 0\,,$$

seek power-law solutions $\delta \propto t^\alpha$ to find growing and decaying modes with

$$\alpha = \frac{1}{6}\left(-1 \pm \sqrt{25 - 24\,\Omega_H}\right)\,,$$

where $\Omega_H = \rho_H/\bar{\rho}$.

Given that matter domination starts ($t = t_{\rm eq}$) at a redshift $z \approx 10^5$, and given an initial perturbation $\delta(t_{\rm eq}) \approx 10^{-5}$, show that $\Omega_H = 2/3$ yields a model that is not compatible with the large-scale structure observed today.

Paper 1, Section I

10D Cosmology

What is meant by the expression 'Hubble time'?

For $a(t)$ the scale factor of the universe and assuming $a(0) = 0$ and $a(t_0) = 1$, where t_0 is the time now, obtain a formula for the size of the particle horizon R_0 of the universe.

Taking

$$a(t) = (t/t_0)^{\alpha},$$

show that R_0 is finite for certain values of α. What might be the physically relevant values of α? Show that the age of the universe is less than the Hubble time for these values of α.

Paper 2, Section I

10D Cosmology

The number density $n = N/V$ for a photon gas in equilibrium is given by

$$n = \frac{8\pi}{c^3} \int_0^{\infty} \frac{\nu^2}{e^{h\nu/kT} - 1}\, d\nu,$$

where ν is the photon frequency. By letting $x = h\nu/kT$, show that

$$n = \alpha T^3,$$

where α is a constant which need not be evaluated.

The photon entropy density is given by

$$s = \beta T^3,$$

where β is a constant. By considering the entropy, explain why a photon gas cools as the universe expands.

Paper 3, Section I

10D Cosmology

Consider a homogenous and isotropic universe with mass density $\rho(t)$, pressure $P(t)$ and scale factor $a(t)$. As the universe expands its energy changes according to the relation $dE = -PdV$. Use this to derive the fluid equation

$$\dot{\rho} = -3\,\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right).$$

Use conservation of energy applied to a test particle at the boundary of a spherical fluid element to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi}{3}\,G\rho - \frac{k}{a^2}\,c^2\,,$$

where k is a constant. State any assumption you have made. Briefly state the significance of k.

Paper 4, Section I

10D Cosmology

The linearised equation for the growth of density perturbations, $\delta_{\mathbf{k}}$, in an isotropic and homogenous universe is

$$\ddot{\delta}_{\mathbf{k}} + 2\,\frac{\dot{a}}{a}\,\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^{\,2}\,\mathbf{k}^2}{a^{\,2}} - 4\pi\,G\rho\right)\delta_{\mathbf{k}} = 0\,,$$

where ρ is the density of matter, c_s the sound speed, $c_s^{\,2} = dP/d\rho$, and $\mathbf{k}$ is the comoving wavevector and $a(t)$ is the scale factor of the universe.

What is the Jean's length? Discuss its significance for the growth of perturbations.

Consider a universe filled with pressure-free matter with $a(t) = (t/t_0)^{2/3}$. Compute the resulting equation for the growth of density perturbations. Show that your equation has growing and decaying modes and comment briefly on the significance of this fact.

Paper 1, Section II

15D Cosmology

A star has pressure $P(r)$ and mass density $\rho(r)$, where r is the distance from the centre of the star. These quantities are related by the pressure support equation

$$P' = -\frac{Gm\rho}{r^2},$$

where $P' = dP/dr$ and $m(r)$ is the mass within radius r. Use this to derive the virial theorem

$$E_{\text{grav}} = -3\langle P\rangle V,$$

where E_{grav} is the total gravitational potential energy and $\langle P\rangle$ the average pressure.

The total kinetic energy of a spherically symmetric star is related to $\langle P\rangle$ by

$$E_{\text{kin}} = \alpha\langle P\rangle V,$$

where α is a constant. Use the virial theorem to determine the condition on α for gravitational binding. By considering the relation between pressure and 'internal energy' U for an ideal gas, determine α for the cases of a) an ideal gas of non-relativistic particles, b) an ideal gas of ultra-relativistic particles.

Why does your result imply a maximum mass for any star? Briefly explain what is meant by the Chandrasekhar limit.

A white dwarf is in orbit with a companion star. It slowly accretes matter from the other star until its mass exceeds the Chandrasekhar limit. Briefly explain its subsequent evolution.

Paper 3, Section II

15D Cosmology

The number density for particles in thermal equilibrium, neglecting quantum effects, is

$$n = g_s \frac{4\pi}{h^3} \int p^2\, dp\, \exp(-(E(p) - \mu)/kT)\,,$$

where g_s is the number of degrees of freedom for the particle with energy $E(p)$ and μ is its chemical potential. Evaluate n for a non-relativistic particle.

Thermal equilibrium between two species of non-relativistic particles is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta\,,$$

where α and β are massless particles. Evaluate the ratio of number densities n_a/n_b given that their respective masses are m_a and m_b and chemical potentials are μ_a and μ_b.

Explain how a reaction like the one above is relevant to the determination of the neutron to proton ratio in the early universe. Why does this ratio not fall rapidly to zero as the universe cools?

Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei. Letting

$$Y_{He} = \rho_{He}/\rho$$

be the fraction of the universe's helium, compute Y_{He} as a function of the ratio $r = n_n/n_p$ at the time of nucleosynthesis.

Paper 1, Section I

10D Cosmology

Prior to a time $t \sim 100,000$ years, the Universe was filled with a gas of photons and non-relativistic free electrons and protons maintained in equilibrium by Thomson scattering. At around $t \sim 400,000$ years, the protons and electrons began combining to form neutral hydrogen,

$$p + e^- \leftrightarrow H + \gamma. \qquad (*)$$

[You may assume that the equilibrium number density of a non-relativistic species ($kT \ll mc^2$) is given by

$$n = g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left((\mu - mc^2)/kT\right)$$

while the photon number density is

$$n_\gamma = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3, \qquad (\zeta(3) \approx 1.20\ldots).$$]

Deduce Saha's equation for the recombination process $(*)$ stating clearly your assumptions and the steps made in the calculation,

$$\frac{n_e^2}{n_H} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp(-I/kT),$$

where I is the ionization energy of hydrogen.

Consider now the fractional ionization $X_e = n_e/n_B$ where $n_B = n_p + n_H = \eta n_\gamma$ is the baryon number of the Universe and η is the baryon to photon ratio. Find an expression for the ratio

$$(1 - X_e)/X_e^2$$

in terms only of kT and constants such as η and I.

Suggest a reason why neutral hydrogen forms at a temperature $kT \approx 0.3\text{eV}$ which is much lower than the hydrogen ionization temperature $kT = I \approx 13\text{eV}$.

Paper 2, Section I

10D Cosmology

(a) The equilibrium distribution for the energy density of a massless neutrino takes the form

$$\epsilon = \frac{4\pi c}{h^3} \int_0^\infty \frac{p^3 dp}{\exp(pc/kT) + 1} \, .$$

Show that this can be expressed in the form $\epsilon = \alpha T^4$, where the constant α need not be evaluated explicitly.

(b) In the early universe, the entropy density s at a temperature T is $s = (8\sigma/3c)\mathcal{N}_S T^3$ where $\mathcal{N}_S$ is the total effective spin degrees of freedom. Briefly explain why $\mathcal{N}_S = \mathcal{N}_* + \mathcal{N}_{SD}$, each term of which consists of two separate components as follows: the contribution from each massless species in equilibrium ($T_i = T$) is

$$\mathcal{N}_* = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i \, ,$$

and a similar sum for massless species which have decoupled,

$$\mathcal{N}_{SD} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 \, ,$$

where in each case g_i is the degeneracy and T_i is the temperature of the species i.

The three species of neutrinos and antineutrinos decouple from equilibrium at a temperature $T \approx 1\text{MeV}$, after which positrons and electrons annihilate at $T \approx 0.5\text{MeV}$, leaving photons in equilibrium with a small excess population of electrons. Using entropy considerations, explain why the ratio of the neutrino and photon temperatures today is given by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \, .$$

Paper 3, Section I

10D Cosmology

(a) Write down an expression for the total gravitational potential energy E_{grav} of a spherically symmetric star of outer radius R in terms of its mass density $\rho(r)$ and the total mass $m(r)$ inside a radius r, satisfying the relation $dm/dr = 4\pi r^2 \rho(r)$.

An isotropic mass distribution obeys the pressure-support equation,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,,$$

where $P(r)$ is the pressure. Multiply this expression by $4\pi r^3$ and integrate with respect to r to derive the virial theorem relating the kinetic and gravitational energy of the star

$$E_{\text{kin}} = -\tfrac{1}{2}E_{\text{grav}}\,,$$

where you may assume for a non-relativistic ideal gas that $E_{\text{kin}} = \frac{3}{2}\langle P\rangle V$, with $\langle P\rangle$ the average pressure.

(b) Consider a white dwarf supported by electron Fermi degeneracy pressure $P \approx h^2 n^{5/3}/m_{\text{e}}$, where m_{e} is the electron mass and n is the number density. Assume a uniform density $\rho(r) = m_{\text{p}} n(r) \approx m_{\text{p}}\langle n\rangle$, so the total mass of the star is given by $M = (4\pi/3)\langle n\rangle m_{\text{p}} R^3$ where m_{p} is the proton mass. Show that the total energy of the white dwarf can be written in the form

$$E_{\text{total}} \;=\; E_{\text{kin}} + E_{\text{grav}} \;=\; \frac{\alpha}{R^2} \;-\; \frac{\beta}{R}\,,$$

where α, β are positive constants which you should specify. Deduce that the white dwarf has a stable radius R_{WD} at which the energy is minimized, that is,

$$R_{\text{WD}} \;\sim\; \frac{h^2 M^{-1/3}}{G m_{\text{e}} m_{\text{p}}^{5/3}}\,.$$

Paper 4, Section I

10D Cosmology

(a) Consider the motion of three galaxies O, A, B at positions $\mathbf{r}_O$, $\mathbf{r}_A$, $\mathbf{r}_B$ in an isotropic and homogeneous universe. Assuming non-relativistic velocities $\mathbf{v}(\mathbf{r})$, show that spatial homogeneity implies

$$\mathbf{v}(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{v}(\mathbf{r}_B - \mathbf{r}_O) - \mathbf{v}(\mathbf{r}_A - \mathbf{r}_O)\,,$$

that is, that the velocity field $\mathbf{v}$ is linearly related to $\mathbf{r}$ by

$$v_i = \sum_j H_{ij} r_j\,,$$

where the matrix coefficients H_{ij} are independent of $\mathbf{r}$. Further show that isotropy implies Hubble's law,

$$\mathbf{v} = H\mathbf{r}\,,$$

where the Hubble parameter H is independent of $\mathbf{r}$. Presuming H to be a function of time t, show that Hubble's law can be integrated to obtain the solution

$$\mathbf{r}(t) = a(t)\mathbf{x}\,,$$

where $\mathbf{x}$ is a constant (comoving) position and the scalefactor $a(t)$ satisfies $H = \dot{a}/a$.

(b) Define the cosmological horizon $d_H(t)$. For models with $a(t) = t^\alpha$ where $0 < \alpha < 1$, show that the cosmological horizon $d_H(t) = ct/(1-\alpha)$ is finite. Briefly explain the horizon problem.

Paper 1, Section II

15D Cosmology

(i) In a homogeneous and isotropic universe, the scalefactor $a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho,$$

where $\rho(t)$ is the matter density which, together with the pressure $P(t)$, satisfies

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right).$$

Use these two equations to derive the Raychaudhuri equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right).$$

(ii) Conformal time τ is defined by taking $dt/d\tau = a$, so that $\dot{a} = a'/a \equiv \mathcal{H}$ where primes denote derivatives with respect to τ. For matter obeying the equation of state $P = w\rho c^2$, show that the Friedmann and energy conservation equations imply

$$\mathcal{H}^2 + kc^2 = \frac{8\pi G}{3}\rho_0 a^{-(1+3w)},$$

where $\rho_0 = \rho(t_0)$ and we take $a(t_0) = 1$ today. Use the Raychaudhuri equation to derive the expression

$$\mathcal{H}' + \tfrac{1}{2}(1+3w)[\mathcal{H}^2 + kc^2] = 0.$$

For a $kc^2 = 1$ closed universe, by solving first for $\mathcal{H}$ (or otherwise), show that the scale factor satisfies

$$a = \alpha(\sin\beta\tau)^{2/(1+3w)}$$

where α, β are constants. [*Hint: You may assume that* $\int dx/(1+x^2) = -\cot^{-1}x + const.$]

For a closed universe dominated by pressure-free matter ($P = 0$), find the complete parametric solution

$$a = \tfrac{1}{2}\alpha(1-\cos 2\beta\tau), \qquad t = \frac{\alpha}{4\beta}(2\beta\tau - \sin 2\beta\tau).$$

Paper 3, Section II

15D Cosmology

In the Zel'dovich approximation, particle trajectories in a flat expanding universe are described by $\mathbf{r}(\mathbf{q},t) = a(t)[\mathbf{q}+\mathbf{\Psi}(\mathbf{q},t)]$, where $a(t)$ is the scale factor of the universe, $\mathbf{q}$ is the unperturbed comoving trajectory and $\mathbf{\Psi}$ is the comoving displacement. The particle equation of motion is

$$\ddot{\mathbf{r}} = -\nabla\Phi - \frac{1}{\rho}\nabla P\,,$$

where ρ is the mass density, P is the pressure ($P \ll \rho c^2$) and Φ is the Newtonian potential which satisfies the Poisson equation $\nabla^2\Phi = 4\pi G\rho$.

(i) Show that the fractional density perturbation and the pressure gradient are given by

$$\delta \equiv \frac{\rho-\bar\rho}{\bar\rho} \approx -\nabla_{\mathbf{q}}\cdot\mathbf{\Psi}\,, \qquad \nabla P \approx -\bar\rho\,\frac{c_s^2}{a}\,\nabla_{\mathbf{q}}^2\mathbf{\Psi}\,,$$

where $\nabla_{\mathbf{q}}$ has components $\partial/\partial q_i$, $\bar\rho = \bar\rho(t)$ is the homogeneous background density and $c_s^2 \equiv \partial P/\partial\rho$ is the sound speed. [You may assume that the Jacobian $|\partial r_i/\partial q_j|^{-1} = |a\,\delta_{ij} + a\,\partial\psi_i/\partial q_j|^{-1} \approx a^{-3}(1-\nabla_{\mathbf{q}}\cdot\mathbf{\Psi})$ for $|\mathbf{\Psi}| \ll |\mathbf{q}|$.]

Use this result to integrate the Poisson equation once and obtain then the evolution equation for the comoving displacement:

$$\ddot{\mathbf{\Psi}} + 2\,\frac{\dot a}{a}\,\dot{\mathbf{\Psi}} - 4\pi G\bar\rho\,\mathbf{\Psi} - \frac{c_s^2}{a^2}\,\nabla_{\mathbf{q}}^2\mathbf{\Psi} = 0\,,$$

[You may assume that the integral of $\nabla^2\Phi = 4\pi G\,\bar\rho$ is $\nabla\Phi = 4\pi G\bar\rho\mathbf{r}/3$, that $\mathbf{\Psi}$ is irrotational and that the Raychaudhuri equation is $\ddot a/a \approx -4\pi G\bar\rho/3$ for $P \ll \rho c^2$.]

Consider the Fourier expansion $\delta(\mathbf{x},t) = \sum_{\mathbf{k}} \delta_{\mathbf{k}} \exp(i\,\mathbf{k}\cdot\mathbf{x})$ of the density perturbation using the comoving wavenumber $\mathbf{k}$ ($k = |\mathbf{k}|$) and obtain the evolution equation for the mode $\delta_{\mathbf{k}}$:

$$\ddot\delta_{\mathbf{k}} + 2\,\frac{\dot a}{a}\,\dot\delta_{\mathbf{k}} - (4\pi G\bar\rho - c_s^2\,k^2/a^2)\,\delta_{\mathbf{k}} = 0\,. \qquad (*)$$

(ii) Consider a flat matter-dominated universe with $a(t) = (t/t_0)^{2/3}$ (background density $\bar\rho = 1/(6\pi Gt^2)$) and with an equation of state $P = \beta\rho^{4/3}$ to show that $(*)$ becomes

$$\ddot\delta_{\mathbf{k}} + \frac{4}{3t}\,\dot\delta_{\mathbf{k}} - \frac{1}{t^2}\,(\tfrac{2}{3} - \bar v_s^2\,k^2)\,\delta_{\mathbf{k}} = 0\,,$$

where the constant $\bar v_s^2 \equiv (4\beta/3)(6\pi G)^{-1/3}\,t_0^{4/3}$. Seek power law solutions of the form $\delta_{\mathbf{k}} \propto t^\alpha$ to find the growing and decaying modes

$$\delta_{\mathbf{k}} = A_{\mathbf{k}}\,t^{n+} + B_{\mathbf{k}}\,t^{n-} \quad \text{where} \quad n_\pm = -\tfrac{1}{6} \pm \left[(\tfrac{5}{6})^2 - \bar v_s^2\,k^2\right]^{1/2}.$$

1/I/10E **Cosmology**

The number density of particles of mass m at equilibrium in the early universe is given by the integral

$$n = \frac{4\pi g_s}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp[(E(p)-\mu)/kT]\mp 1}, \qquad \begin{cases} - & \text{bosons}\,, \\ + & \text{fermions}, \end{cases}$$

where $E(p) = c\sqrt{p^2+m^2c^2}$, μ is the chemical potential, and g_s is the spin degeneracy. Assuming that the particles remain in equilibrium when they become non-relativistic ($kT, \mu \ll mc^2$), show that the number density can be expressed as

$$n = g_s\left(\frac{2\pi m kT}{h^2}\right)^{3/2} e^{(\mu - mc^2)/kT}.$$

[*Hint: Recall that* $\int_0^\infty dx\, e^{-\sigma^2 x^2} = \sqrt{\pi}/(2\sigma), \quad (\sigma > 0)$.]

At around $t = 100$ seconds, deuterium D forms through the nuclear fusion of nonrelativistic protons p and neutrons n via the interaction $p + n \leftrightarrow D$. In equilibrium, what is the relationship between the chemical potentials of the three species? Show that the ratio of their number densities can be expressed as

$$\frac{n_D}{n_n n_p} \approx \left(\frac{\pi m_p kT}{h^2}\right)^{-3/2} e^{B_D/kT},$$

where the deuterium binding energy is $B_D = (m_n + m_p - m_D)\,c^2$ and you may take $g_D = 4$. Now consider the fractional densities $X_a = n_a/n_B$, where n_B is the baryon density of the universe, to re-express the ratio above in the form $X_D/(X_n X_p)$, which incorporates the baryon-to-photon ratio η of the universe.

[You may assume that the photon density is $n_\gamma = (16\pi\zeta(3)/(hc)^3)(kT)^3$.]

Why does deuterium form only at temperatures much lower than that given by $kT \approx B_D$?

2/I/10E **Cosmology**

A spherically-symmetric star obeys the pressure-support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2},$$

where $P(r)$ is the pressure at a distance r from the centre, $\rho(r)$ is the density, and $m(r)$ is the mass within a sphere of radius r. Show that this implies

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho.$$

Propose and justify appropriate boundary conditions for the pressure $P(r)$ at the centre of the star ($r = 0$) and at its outer edge $r = R$.

Show that the function

$$F(r) = P(r) + \frac{Gm^2}{8\pi r^4}$$

is a decreasing function of r. Deduce that the central pressure $P_{\rm c} \equiv P(0)$ satisfies

$$P_{\rm c} \ > \ \frac{GM^2}{8\pi R^4},$$

where $M \equiv m(R)$ is the mass of the star.

1/II/15E **Cosmology**

(i) A homogeneous and isotropic universe has mass density $\rho(t)$ and scale factor $a(t)$. Show how the conservation of total energy (kinetic plus gravitational potential) when applied to a test particle on the edge of a spherical region in this universe can be used to obtain the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$

where k is a constant. State clearly any assumptions you have made.

(ii) Assume that the universe is flat ($k = 0$) and filled with two major components: pressure-free matter ($P_{\mathrm{M}} = 0$) and dark energy with equation of state $P_\Lambda = -\rho_\Lambda c^2$ where their mass densities today ($t = t_0$) are given respectively by $\rho_{\mathrm{M}0}$ and $\rho_{\Lambda 0}$. Assuming that each component independently satisfies the fluid conservation equation, $\dot{\rho} = -3H(\rho + P/c^2)$, show that the total mass density can be expressed as

$$\rho(t) = \frac{\rho_{\mathrm{M}0}}{a^3} + \rho_{\Lambda 0},$$

where we have set $a(t_0) = 1$.

Hence, solve the Friedmann equation and show that the scale factor can be expressed in the form

$$a(t) = \alpha(\sinh \beta t)^{2/3},$$

where α and β are constants which you should specify in terms of $\rho_{\mathrm{M}0}$, $\rho_{\Lambda 0}$ and t_0.

[*Hint: try the substitution* $b = a^{3/2}$.]

Show that the scale factor $a(t)$ has the expected behaviour for a matter-dominated universe at early times ($t \to 0$) and that the universe accelerates at late times ($t \to \infty$).

3/I/10E **Cosmology**

The energy density ϵ and pressure P of photons in the early universe is given by

$$\epsilon = \frac{4\sigma}{c}T^4, \qquad P = \frac{1}{3}\epsilon,$$

where σ is the Stefan–Boltzmann constant. By using the first law of thermodynamics $dE = TdS - PdV + \mu dN$, deduce that the entropy differential dS can be expressed in the form

$$dS = \frac{16\sigma}{3c}d(T^3V).$$

With the third law, show that the entropy density is given by $s = (16\sigma/3c)T^3$.

While particle interaction rates Γ remain much greater than the Hubble parameter H, justify why entropy will be conserved during the expansion of the universe. Hence, in the early universe (radiation domination) show that the temperature $T \propto a^{-1}$ where $a(t)$ is the scale factor of the universe, and show that the Hubble parameter $H \propto T^2$.

4/I/10E **Cosmology**

The Friedmann and Raychaudhuri equations are respectively

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \qquad \text{and} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right),$$

where ρ is the mass density, P is the pressure, k is the curvature and $\dot{a} \equiv da/dt$ with t the cosmic time. Using conformal time τ (defined by $d\tau = dt/a$) and the equation of state $P = w\rho c^2$, show that these can be rewritten as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1 \qquad \text{and} \qquad 2\frac{d\mathcal{H}}{d\tau} = -(3w+1)\left(\mathcal{H}^2 + kc^2\right),$$

where $\mathcal{H} = a^{-1}da/d\tau$ and the relative density is $\Omega \equiv \rho/\rho_{\text{crit}} = 8\pi G\rho a^2/(3\mathcal{H}^2)$.

Use these relations to derive the following evolution equation for Ω

$$\frac{d\Omega}{d\tau} = (3w+1)\mathcal{H}\Omega(\Omega - 1).$$

For both $w = 0$ and $w = -1$, plot the qualitative evolution of Ω as a function of τ in an expanding universe $\mathcal{H} > 0$ (i.e. include curves initially with $\Omega > 1$ and $\Omega < 1$).

Hence, or otherwise, briefly describe the flatness problem of the standard cosmology and how it can be solved by inflation.

3/II/15E **Cosmology**

Small density perturbations $\delta_{\mathbf{k}}(t)$ in pressureless matter inside the cosmological horizon obey the following Fourier evolution equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho}_{\rm c}\delta_{\mathbf{k}} = 0,$$

where $\bar{\rho}_{\rm c}$ is the average background density of the pressureless gravitating matter and $\mathbf{k}$ is the comoving wavevector.

(i) Seek power law solutions $\delta_{\mathbf{k}} \propto t^{\beta}$ (β constant) during the matter-dominated epoch ($t_{\rm eq} < t < t_0$) to find the approximate solution

$$\delta_{\mathbf{k}}(t) = A(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{2/3} + B(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{-1}, \qquad t \gg t_{\rm eq}$$

where A, B are functions of $\mathbf{k}$ only and $t_{\rm eq}$ is the time of equal matter-radiation.

By considering the behaviour of the scalefactor a and the relative density $\bar{\rho}_{\rm c}/\bar{\rho}_{\rm total}$, show that early in the radiation era ($t \ll t_{\rm eq}$) there is effectively no significant perturbation growth in $\delta_{\mathbf{k}}$ on sub-horizon scales.

(ii) For a given wavenumber $k = |\mathbf{k}|$, show that the time $t_{\rm H}$ at which this mode crosses inside the horizon, i.e., $ct_{\rm H} \approx 2\pi a(t_{\rm H})/k$, is given by

$$\frac{t_{\rm H}}{t_0} \approx \begin{cases} \left(\frac{k_0}{k}\right)^3, & t_{\rm H} \gg t_{\rm eq}, \\ (1+z_{\rm eq})^{-1/2}\left(\frac{k_0}{k}\right)^2, & t_{\rm H} \ll t_{\rm eq}, \end{cases}$$

where $k_0 \equiv 2\pi/(ct_0)$, and the equal matter-radiation redshift is given by $1 + z_{\rm eq} = (t_0/t_{\rm eq})^{2/3}$.

Assume that primordial perturbations from inflation are scale-invariant with a constant amplitude as they cross the Hubble radius given by $\langle|\delta_{\mathbf{k}}(t_{\rm H})|^2\rangle \approx V^{-1}A/k^3$, where A is a constant and V is a large volume. Use the results of (i) to project these perturbations forward to t_0, and show that the power spectrum for perturbations today will be given approximately by

$$P(k) \;\equiv\; V\langle|\delta_{\mathbf{k}}(t_0)|^2\rangle \;\approx\; \frac{A}{k_0^4} \times \begin{cases} k, & k < k_{\rm eq}, \\ k_{\rm eq}\left(\frac{k_{\rm eq}}{k}\right)^3, & k > k_{\rm eq}. \end{cases}$$

1/I/10A **Cosmology**

Describe the motion of light rays in an expanding universe with scale factor $a(t)$, and derive the redshift formula

$$1 + z = \frac{a(t_0)}{a(t_{\rm e})}\,,$$

where the light is emitted at time $t_{\rm e}$ and observed at time t_0.

A galaxy at comoving position $\mathbf{x}$ is observed to have a redshift z. Given that the galaxy emits an amount of energy L per unit time, show that the total energy per unit time crossing a sphere centred on the galaxy and intercepting the earth is $L/(1+z)^2$. Hence, show that the energy per unit time per unit area passing the earth is

$$\frac{L}{(1+z)^2}\,\frac{1}{4\pi|\mathbf{x}|^2 a^2(t_0)}\,.$$

2/I/10A **Cosmology**

The number density of photons in thermal equilibrium at temperature T takes the form

$$n = \frac{8\pi}{c^3}\int \frac{\nu^2 d\nu}{\exp(h\nu/kT) - 1}\,.$$

At time $t = t_{\rm dec}$ and temperature $T = T_{\rm dec}$, photons decouple from thermal equilibrium. By considering how the photon frequency redshifts as the universe expands, show that the form of the equilibrium frequency distribution is preserved, with the temperature for $t > t_{\rm dec}$ defined by

$$T \equiv \frac{a(t_{\rm dec})}{a(t)} T_{\rm dec}\,.$$

Show that the photon number density n and energy density ϵ can be expressed in the form

$$n = \alpha T^3\,, \qquad \epsilon = \xi T^4\,,$$

where the constants α and ξ need not be evaluated explicitly.

1/II/15A **Cosmology**

In a homogeneous and isotropic universe, the scale factor $a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,,$$

where ρ is the matter density, which, together with the pressure P, satisfies

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right)\,.$$

Here, k is a constant curvature parameter. Use these equations to show that the rate of change of the Hubble parameter $H = \dot{a}/a$ satisfies

$$\dot{H} + H^2 = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right)\,.$$

Suppose that an *expanding* Friedmann universe is filled with radiation (density ρ_R and pressure $P_R = \rho_R c^2/3$) as well as a "dark energy" component (density ρ_Λ and pressure $P_\Lambda = -\rho_\Lambda c^2$). Given that the energy densities of these two components are measured today ($t = t_0$) to be

$$\rho_{R0} = \beta\,\frac{3H_0^2}{8\pi G} \quad \text{and} \quad \rho_{\Lambda 0} = \frac{3H_0^2}{8\pi G} \quad \text{with constant } \beta > 0 \quad \text{and} \quad a(t_0) = 1\,,$$

show that the curvature parameter must satisfy $kc^2 = \beta H_0^2$. Hence derive the following relations for the Hubble parameter and its time derivative:

$$H^2 = \frac{H_0^2}{a^4}\left(\beta - \beta a^2 + a^4\right)\,,$$
$$\dot{H} = -\beta\,\frac{H_0^2}{a^4}\left(2 - a^2\right)\,.$$

Show qualitatively that universes with $\beta > 4$ will recollapse to a Big Crunch in the future. [*Hint: Sketch* a^4H^2 *and* $a^4\dot{H}$ *versus* a^2 *for representative values of* β.]

For $\beta = 4$, find an explicit solution for the scale factor $a(t)$ satisfying $a(0) = 0$. Find the limiting behaviours of this solution for large and small t. Comment briefly on their significance.

3/I/10A **Cosmology**

The number density of a non-relativistic species in thermal equilibrium is given by

$$n = g_s \left(\frac{2\pi m kT}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/kT\right] .$$

Suppose that thermal and chemical equilibrium is maintained between protons p (mass $m_{\rm p}$, degeneracy $g_s = 2$), neutrons n (mass $m_{\rm n} \approx m_{\rm p}$, degeneracy $g_s = 2$) and helium-4 nuclei ^{4}He (mass $m_{\rm He} \approx 4m_{\rm p}$, degeneracy $g_s = 1$) via the interaction

$$2{\rm p} + 2{\rm n} \leftrightarrow {}^4{\rm He} + \gamma\,,$$

where you may assume the photons γ have zero chemical potential $\mu_\gamma = 0$. Given that the binding energy of helium-4 obeys $B_{\rm He}/c^2 \equiv 2m_{\rm p} + 2n_{\rm n} - m_{\rm He} \ll m_{\rm He}$, show that the ratio of the number densities can be written as

$$\frac{n_{\rm p}^2\, n_{\rm n}^2}{n_{\rm He}} = 2\left(\frac{2\pi m_{\rm p} kT}{h^2}\right)^{9/2} \exp(-B_{\rm He}/kT)\,. \qquad (\dagger)$$

Explain briefly why the baryon-to-photon ratio $\eta \equiv n_B/n_\gamma$ remains constant during the expansion of the universe, where $n_B \approx n_{\rm p} + n_{\rm n} + 4n_{\rm He}$ and $n_\gamma \approx (16\pi/(hc)^3)(kT)^3$.

By considering the fractional densities $X_i \equiv n_i/n_B$ of the species i, re-express the ratio (†) in the form

$$\frac{X_{\rm p}^2 X_{\rm n}^2}{X_{\rm He}} = \eta^{-3}\frac{1}{32}\left(\frac{\pi}{2}\right)^{3/2}\left(\frac{m_{\rm p}c^2}{kT}\right)^{9/2} \exp\left(-B_{\rm He}/kT\right) .$$

Given that $B_{\rm He} \approx 30$MeV, verify (very approximately) that this ratio approaches unity when $kT \approx 0.3$MeV. In reality, helium-4 is not formed until after deuterium production at a considerably lower temperature. Explain briefly the reason for this delay.

4/I/10A **Cosmology**

The equation governing density perturbation modes $\delta_{\mathbf{k}}(t)$ in a matter-dominated universe (with $a(t) = (t/t_0)^{2/3}$) is

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \delta_{\mathbf{k}} = 0\,,$$

where $\mathbf{k}$ is the comoving wavevector. Find the general solution for the perturbation, showing that there is a growing mode such that

$$\delta_{\mathbf{k}}(t) \approx \frac{a(t)}{a(t_i)}\delta_{\mathbf{k}}(t_i) \quad (t \gg t_i)\,.$$

Show that the physical wavelength corresponding to the comoving wavenumber $k = |\mathbf{k}|$ crosses the Hubble radius cH^{-1} at a time t_k given by

$$\frac{t_k}{t_0} = \left(\frac{k_0}{k}\right)^3, \qquad \text{where } \; k_0 = \frac{2\pi}{cH_0^{-1}}\,.$$

According to inflationary theory, the amplitude of the variance at horizon-crossing is constant, that is, $\langle|\delta_{\mathbf{k}}(t_k)|^2\rangle = AV^{-1}/k^3$ where A and V (the volume) are constants. Given this amplitude and the results obtained above, deduce that the power spectrum today takes the form

$$P(k) \equiv V\langle|\delta_{\mathbf{k}}(t_0)|^2\rangle = \frac{A}{k_0^4}\,k\,.$$

3/II/15A **Cosmology**

A spherically symmetric star with outer radius R has mass density $\rho(r)$ and pressure $P(r)$, where r is the distance from the centre of the star. Show that hydrostatic equilibrium implies the pressure support equation,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad (\dagger)$$

where $m(r)$ is the mass inside radius r. State without proof any results you may need.

Write down an integral expression for the total gravitational potential energy $E_{\rm grav}$ of the star. Hence use ($\dagger$) to deduce the virial theorem

$$E_{\rm grav} = -3\langle P\rangle V\,, \qquad (*)$$

where $\langle P\rangle$ is the average pressure and V is the volume of the star.

Given that a non-relativistic ideal gas obeys $P = 2E_{\rm kin}/3V$ and that an ultra-relativistic gas obeys $P = E_{\rm kin}/3V$, where $E_{\rm kin}$ is the kinetic energy, discuss briefly the gravitational stability of a star in these two limits.

At zero temperature, the number density of particles obeying the Pauli exclusion principle is given by

$$n = \frac{4\pi g_s}{h^3}\int_0^{p_{\rm F}} p^2 dp = \frac{4\pi g_s}{3}\left(\frac{p_{\rm F}}{h}\right)^3\,,$$

where $p_{\rm F}$ is the Fermi momentum, g_s is the degeneracy and h is Planck's constant. Deduce that the non-relativistic internal energy $E_{\rm kin}$ of these particles is

$$E_{\rm kin} = \frac{4\pi g_s V h^2}{10 m_p}\left(\frac{p_{\rm F}}{h}\right)^5\,,$$

where m_p is the mass of a particle. Hence show that the non-relativistic Fermi degeneracy pressure satisfies

$$P \;\sim\; \frac{h^2}{m_p}\, n^{5/3}\,.$$

Use the virial theorem ($*$) to estimate that the radius R of a star supported by Fermi degeneracy pressure is approximately

$$R \;\sim\; \frac{h^2 M^{-1/3}}{G m_p^{8/3}}\,,$$

where M is the total mass of the star.

[*Hint: Assume* $\rho(r) = m_p n(r) \sim m_p\langle n\rangle$ *and note that* $M \approx (4\pi R^3/3)\, m_p\langle n\rangle$.]

1/I/10D **Cosmology**

(a) Introduce the concept of comoving co-ordinates in a homogeneous and isotropic universe and explain how the velocity of a galaxy is determined by the scale factor a. Express the Hubble parameter H_0 today in terms of the scale factor.

(b) The Raychaudhuri equation states that the acceleration of the universe is determined by the mass density ρ and the pressure P as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right) .$$

Now assume that the matter constituents of the universe satisfy $\rho + 3P/c^2 \geqslant 0$. In this case explain clearly why the Hubble time H_0^{-1} sets an upper limit on the age of the universe; equivalently, that the scale factor must vanish ($a(t_i) = 0$) at some time $t_i < t_0$ with $t_0 - t_i \leqslant H_0^{-1}$.

The observed Hubble time is $H_0^{-1} = 1 \times 10^{10}$ years. Discuss two reasons why the above upper limit does not seem to apply to our universe.

2/I/10D **Cosmology**

The total energy of a gas can be expressed in terms of a momentum integral

$$E = \int_0^\infty \mathcal{E}(p)\,\bar{n}(p)\; dp \ ,$$

where p is the particle momentum, $\mathcal{E}(p) = c\sqrt{p^2 + m^2c^2}$ is the particle energy and $\bar{n}(p)\; dp$ is the average number of particles in the momentum range $p \to p + dp$. Consider particles in a cubic box of side L with $p \propto L^{-1}$. Explain why the momentum varies as

$$\frac{dp}{dV} = -\frac{p}{3V} \ .$$

Consider the overall change in energy dE due to the volume change dV. Given that the volume varies slowly, use the thermodynamic result $dE = -P\; dV$ (at fixed particle number N and entropy S) to find the pressure

$$P = \frac{1}{3V}\int_0^\infty p\,\mathcal{E}'(p)\,\bar{n}(p)\; dp\,.$$

Use this expression to derive the equation of state for an ultrarelativistic gas.

During the radiation-dominated era, photons remain in equilibrium with energy density $\epsilon_\gamma \propto T^4$ and number density $n_\gamma \propto T^3$. Briefly explain why the photon temperature falls inversely with the scale factor, $T \propto a^{-1}$. Discuss the implications for photon number and entropy conservation.

2/II/15D **Cosmology**

(a) Consider a homogeneous and isotropic universe filled with relativistic matter of mass density $\rho(t)$ and scale factor $a(t)$. Consider the energy $E(t) \equiv \rho(t)c^2V(t)$ of a small fluid element in a comoving volume V_0 where $V(t) = a^3(t)V_0$. Show that for slow (adiabatic) changes in volume, the density will satisfy the fluid conservation equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right) ,$$

where P is the pressure.

(b) Suppose that a flat ($k = 0$) universe is filled with two matter components:

(i) radiation with an equation of state $P_{\mathrm{r}} = \frac{1}{3}\rho_{\mathrm{r}}c^2$.

(ii) a gas of cosmic strings with an equation of state $P_{\mathrm{s}} = -\frac{1}{3}\rho_{\mathrm{s}}c^2$.

Use the fluid conservation equation to show that the total relativistic mass density behaves as

$$\rho \ = \ \frac{\rho_{\mathrm{r}0}}{a^4} + \frac{\rho_{\mathrm{s}0}}{a^2} \, ,$$

where $\rho_{\mathrm{r}0}$ and $\rho_{\mathrm{s}0}$ are respectively the radiation and string densities today (that is, at $t = t_0$ when $a(t_0) = 1$). Assuming that both the Hubble parameter today H_0 and the ratio $\beta \equiv \rho_{\mathrm{r}0}/\rho_{\mathrm{s}0}$ are known, show that the Friedmann equation can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^4}\left(\frac{a^2+\beta}{1+\beta}\right) .$$

Solve this equation to find the following solution for the scale factor

$$a(t) = \frac{(H_0t)^{1/2}}{(1+\beta)^{1/2}}\left[H_0t + 2\beta^{1/2}(1+\beta)^{1/2}\right]^{1/2} .$$

Show that the scale factor has the expected asymptotic behaviour at early times $t \to 0$.

Hence show that the age of this universe today is

$$t_0 \ = \ H_0^{-1}(1+\beta)^{1/2}\left[(1+\beta)^{1/2} - \beta^{1/2}\right] ,$$

and that the time t_{eq} of equal radiation and string densities ($\rho_{\mathrm{r}} = \rho_{\mathrm{s}}$) is

$$t_{\mathrm{eq}} \ = \ H_0^{-1}\left(\sqrt{2}-1\right)\beta^{1/2}(1+\beta)^{1/2} .$$

3/I/10D **Cosmology**

(a) Consider a spherically symmetric star with outer radius R, density $\rho(r)$ and pressure $P(r)$. By balancing the gravitational force on a shell at radius r against the force due to the pressure gradient, derive the pressure support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,,$$

where $m(r) = \int_0^r \rho(r')\,4\pi r'^2\,dr'$. Show that this implies

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho\,.$$

Suggest appropriate boundary conditions at $r = 0$ and $r = R$, together with a brief justification.

(b) Describe qualitatively the endpoint of stellar evolution for our sun when all its nuclear fuel is spent. Your discussion should briefly cover electron degeneracy pressure and the relevance of stability against inverse beta-decay.

[*Note that* $m_n - m_p \approx 2.6\,m_e$, *where* m_n, m_p, m_e *are the masses of the neutron, proton and electron respectively.*]

4/I/10D **Cosmology**

The number density of fermions of mass m at equilibrium in the early universe with temperature T, is given by the integral

$$n = \frac{4\pi}{h^3} \int_0^\infty \frac{p^2\, dp}{\exp[(\mathcal{E}(p) - \mu)/kT] + 1}$$

where $\mathcal{E}(p) = c\sqrt{p^2 + m^2c^2}$, and μ is the chemical potential. Assuming that the fermions remain in equilibrium when they become non-relativistic (kT, $\mu \ll mc^2$), show that the number density can be expressed as

$$n = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/kT\right] \ .$$

[*Hint: You may assume* $\int_0^\infty dx\, e^{-\sigma^2 x^2} = \sqrt{\pi}/(2\sigma)\,, \quad (\sigma > 0)$.]

Suppose that the fermions decouple at a temperature given by $kT = mc^2/\alpha$ where $\alpha \gg 1$. Assume also that $\mu = 0$. By comparing with the photon number density at $n_\gamma = 16\pi\zeta(3)(kT/hc)^3$, where $\zeta(3) = \sum_{n=1}^\infty n^{-3} = 1.202\ldots$, show that the ratio of number densities at decoupling is given by

$$\frac{n}{n_\gamma} = \frac{\sqrt{2\pi}}{8\zeta(3)}\, \alpha^{3/2}\, e^{-\alpha} \ .$$

Now assume that $\alpha \approx 20$, (which implies $n/n_\gamma \approx 5 \times 10^{-8}$), and that the fermion mass $m = m_p/20$, where m_p is the proton mass. Explain clearly why this new fermion would be a good candidate for solving the dark matter problem of the standard cosmology.

4/II/15D **Cosmology**

The perturbed motion of cold dark matter particles (pressure-free, $P = 0$) in an expanding universe can be parametrized by the trajectories

$$\mathbf{r}(\mathbf{q}, t) = a(t)\,[\mathbf{q} + \boldsymbol{\psi}(\mathbf{q}, t)] \ ,$$

where $a(t)$ is the scale factor of the universe, $\mathbf{q}$ is the unperturbed comoving trajectory and $\boldsymbol{\psi}$ is the comoving displacement. The particle equation of motion is $\ddot{\mathbf{r}} = -\nabla\Phi$, where the Newtonian potential satisfies the Poisson equation $\nabla^2\Phi = 4\pi G\rho$ with mass density $\rho(\mathbf{r}, t)$.

(a) Discuss how matter conservation in a small volume $d^3\mathbf{r}$ ensures that the perturbed density $\rho(\mathbf{r}, t)$ and the unperturbed background density $\bar{\rho}(t)$ are related by

$$\rho(\mathbf{r}, t) d^3\mathbf{r} = \bar{\rho}(t) a^3(t) d^3\mathbf{q} \ .$$

By changing co-ordinates with the Jacobian

$$|\partial r_i/\partial q_j|^{-1} = |a\delta_{ij} + a\,\partial\psi_i/\partial q_j|^{-1} \approx a^{-3}(1 - \nabla_q \cdot \boldsymbol{\psi}) \ ,$$

show that the fractional density perturbation $\delta(\mathbf{q}, t)$ can be written to leading order as

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} = -\nabla_q \cdot \boldsymbol{\psi} \ ,$$

where $\nabla_q \cdot \boldsymbol{\psi} = \sum_i \partial\psi_i/\partial q_i$.

Use this result to integrate the Poisson equation once. Hence, express the particle equation of motion in terms of the comoving displacement as

$$\ddot{\boldsymbol{\psi}} + 2\frac{\dot{a}}{a}\dot{\boldsymbol{\psi}} - 4\pi G\bar{\rho}\boldsymbol{\psi} = 0 \ .$$

Infer that the density perturbation evolution equation is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \ . \qquad (*)$$

[*Hint: You may assume that the integral of* $\nabla^2\Phi = 4\pi G\bar{\rho}$ *is* $\nabla\Phi = -4\pi G\bar{\rho}\mathbf{r}/3$. *Note also that the Raychaudhuri equation (for* $P = 0$*) is* $\ddot{a}/a = -4\pi G\bar{\rho}/3$.]

(b) Find the general solution of equation $(*)$ in a flat ($k = 0$) universe dominated by cold dark matter ($P = 0$). Discuss the effect of late-time Λ or dark energy domination on the growth of density perturbations.

1/I/10D **Cosmology**

(a) Around $t \approx 1\,\mathrm{s}$ after the big bang ($kT \approx 1\,\mathrm{MeV}$), neutrons and protons are kept in equilibrium by weak interactions such as

$$n + \nu_e \leftrightarrow p + e^- \,. \qquad (*)$$

Show that, in equilibrium, the neutron-to-proton ratio is given by

$$\frac{n_n}{n_p} \approx e^{-Q/kT} \,,$$

where $Q = (m_n - m_p)c^2 = 1.29\,\mathrm{MeV}$ corresponds to the mass difference between the neutron and the proton. Explain briefly why we can neglect the difference $\mu_n - \mu_p$ in the chemical potentials.

(b) The ratio of the weak interaction rate $\Gamma_W \propto T^5$ which maintains $(*)$ to the Hubble expansion rate $H \propto T^2$ is given by

$$\frac{\Gamma_W}{H} \approx \left(\frac{kT}{0.8\,\mathrm{MeV}}\right)^3 . \qquad (\dagger)$$

Explain why the neutron-to-proton ratio effectively "freezes out" once $kT < 0.8\,\mathrm{MeV}$, except for some slow neutron decay. Also explain why almost all neutrons are subsequently captured in ^4He; estimate the value of the relative mass density $Y_{^4He} = \rho_{^4He}/\rho_{\mathrm{B}}$ (with $\rho_{\mathrm{B}} = \rho_n + \rho_p$) given a final ratio $n_n/n_p \approx 1/8$.

(c) Suppose instead that the weak interaction rate were very much weaker than that described by equation $(\dagger)$. Describe the effect on the relative helium density $Y_{^4He}$. Briefly discuss the wider implications of this primordial helium-to-hydrogen ratio on stellar lifetimes and life on earth.

2/I/10D **Cosmology**

(a) A spherically symmetric star obeys the pressure-support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad (*)$$

where $P(r)$ is the pressure at a distance r from the centre, $\rho(r)$ is the density, and the mass $m(r)$ is defined through the relation $dm/dr = 4\pi r^2 \rho(r)$. Multiply $(*)$ by $4\pi r^3$ and integrate over the total volume V of the star to derive the virial theorem

$$\langle P \rangle V = -\tfrac{1}{3} E_{\mathrm{grav}}\,,$$

where $\langle P \rangle$ is the average pressure and E_{grav} is the total gravitational potential energy.

(b) Consider a white dwarf supported by electron Fermi degeneracy pressure $P \approx h^2 n^{5/3}/m_{\mathrm{e}}$, where m_{e} is the electron mass and n is the number density. Assume a uniform density $\rho(r) = m_{\mathrm{p}} n(r) \approx m_{\mathrm{p}} \langle n \rangle$, so the total mass of the star is given by $M = (4\pi/3)\langle n \rangle m_{\mathrm{p}} R^3$ where R is the star radius and m_{p} is the proton mass. Show that the total energy of the white dwarf can be written in the form

$$E_{\mathrm{total}} \;=\; E_{\mathrm{kin}} + E_{\mathrm{grav}} \;=\; \frac{\alpha}{R^2} \;-\; \frac{\beta}{R}\,,$$

where α, β are positive constants which you should determine. [*You may assume that for an ideal gas* $E_{\mathrm{kin}} = \frac{3}{2}\langle P \rangle V$.] Use this expression to explain briefly why a white dwarf is stable.

2/II/15D **Cosmology**

(a) Consider a homogeneous and isotropic universe with scale factor $a(t)$ and filled with mass density $\rho(t)$. Show how the conservation of kinetic energy plus gravitational potential energy for a test particle on the edge of a spherical region in this universe can be used to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,, \tag{$*$}$$

where k is a constant. State clearly any assumptions you have made.

(b) Now suppose that the universe was filled throughout its history with radiation with equation of state $P = \rho c^2/3$. Using the fluid conservation equation and the definition of the relative density Ω, show that the density of this radiation can be expressed as

$$\rho \;=\; \frac{3H_0^2}{8\pi G}\,\frac{\Omega_0}{a^4}\,,$$

where H_0 is the Hubble parameter today and Ω_0 is the relative density today ($t = t_0$) and $a_0 \equiv a(t_0) = 1$ is assumed. Show also that $kc^2 = H_0^2(\Omega_0 - 1)$ and hence rewrite the Friedmann equation ($*$) as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_0\left(\frac{1}{a^4} - \frac{\beta}{a^2}\right)\,, \tag{$\dagger$}$$

where $\beta \equiv (\Omega_0 - 1)/\Omega_0$.

(c) Now consider a closed model with $k > 0$ (or $\Omega > 1$). Rewrite ($\dagger$) using the new time variable τ defined by

$$\frac{dt}{d\tau} = a\,.$$

Hence, or otherwise, solve ($\dagger$) to find the parametric solution

$$a(\tau) = \frac{1}{\sqrt{\beta}}(\sin\alpha\tau)\,, \qquad t(\tau) = \frac{1}{\alpha\sqrt{\beta}}(1 - \cos\alpha\tau)\,,$$

where $\alpha \equiv H_0\sqrt{(\Omega_0 - 1)}$. [*Recall that* $\int dx/\sqrt{1-x^2} = \sin^{-1} x$.]

Using the solution for $a(\tau)$, find the value of the new time variable $\tau = \tau_0$ today and hence deduce that the age of the universe in this model is

$$t_0 = H_0^{-1}\frac{\sqrt{\Omega_0} - 1}{\Omega_0 - 1}\,.$$

3/I/10D **Cosmology**

(a) Define and discuss the concept of the *cosmological horizon* and the *Hubble radius* for a homogeneous isotropic universe. Illustrate your discussion with the specific examples of the Einstein–de Sitter universe ($a \propto t^{2/3}$ for $t > 0$) and a de Sitter universe ($a \propto e^{Ht}$ with H constant, $t > -\infty$).

(b) Explain the *horizon problem* for a decelerating universe in which $a(t) \propto t^{\alpha}$ with $\alpha < 1$. How can inflation cure the horizon problem?

(c) Consider a Tolman (radiation-filled) universe ($a(t) \propto t^{1/2}$) beginning at $t_{\mathrm{r}} \sim 10^{-35}$s and lasting until today at $t_0 \approx 10^{17}$s. Estimate the horizon size today $d_H(t_0)$ and project this lengthscale backwards in time to show that it had a physical size of about 1 metre at $t \approx t_{\mathrm{r}}$.

Prior to $t \approx t_{\mathrm{r}}$, assume an inflationary (de Sitter) epoch with constant Hubble parameter H given by its value at $t \approx t_{\mathrm{r}}$ for the Tolman universe. How much expansion during inflation is required for the observable universe today to have begun inside one Hubble radius?

4/I/10D **Cosmology**

The linearised equation for the growth of a density fluctuation δ_k in a homogeneous and isotropic universe is

$$\frac{d^2\delta_k}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta_k}{dt} - \left(4\pi G\rho_{\mathrm{m}} - \frac{v_s^2 k^2}{a^2}\right)\delta_k = 0\,, \qquad (*)$$

where ρ_{m} is the non-relativistic matter density, k is the comoving wavenumber and v_s is the sound speed ($v_s^2 \equiv dP/d\rho$).

(a) Define the Jeans length λ_{J} and discuss its significance for perturbation growth.

(b) Consider an Einstein–de Sitter universe with $a(t) = (t/t_0)^{2/3}$ filled with pressure-free matter ($P = 0$). Show that the perturbation equation $(*)$ can be re-expressed as

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0\,.$$

By seeking power law solutions, find the growing and decaying modes of this equation.

(c) Qualitatively describe the evolution of non-relativistic matter perturbations ($k > aH$) in the radiation era, $a(t) \propto t^{1/2}$, when $\rho_{\mathrm{r}} \gg \rho_{\mathrm{m}}$. What feature in the power spectrum is associated with the matter–radiation transition?

4/II/15D **Cosmology**

For an ideal gas of *bosons*, the average occupation number can be expressed as

$$\bar{n}_k = \frac{g_k}{e^{(E_k-\mu)/kT} - 1}\,, \qquad (*)$$

where g_k has been included to account for the degeneracy of the energy level E_k. In the approximation in which a discrete set of energies E_k is replaced with a continuous set with momentum p, the density of one-particle states with momentum in the range p to $p + dp$ is $g(p)dp$. Explain briefly why

$$g(p) \;\propto\; p^2 V\,,$$

where V is the volume of the gas. Using this formula with equation $(*)$, obtain an expression for the total energy density $\epsilon = E/V$ of an ultra-relativistic gas of bosons at zero chemical potential as an integral over p. Hence show that

$$\epsilon \;\propto\; T^\alpha\,,$$

where α is a number you should find. Why does this formula apply to photons?

Prior to a time $t \sim 100,000$ years, the universe was filled with a gas of photons and non-relativistic free electrons and protons. Subsequently, at around $t \sim 400,000$ years, the protons and electrons began combining to form neutral hydrogen,

$$p + e^- \leftrightarrow H + \gamma\,.$$

Deduce Saha's equation for this recombination process stating clearly the steps required:

$$\frac{n_{\rm e}^2}{n_{\rm H}} = \left(\frac{2\pi m_{\rm e} kT}{h^2}\right)^{3/2} \exp(-I/kT)\,,$$

where I is the ionization energy of hydrogen. [*Note that the equilibrium number density of a non-relativistic species* ($kT \ll mc^2$) *is given by* $n = g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/kT\right]$, *while the photon number density is* $n_\gamma = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3$, where $\zeta(3) \approx 1.20....$]

Consider now the fractional ionization $X_{\rm e} = n_{\rm e}/n_{\rm B}$, where $n_B = n_{\rm p} + n_{\rm H} = \eta n_\gamma$ is the baryon number of the universe and η is the baryon-to-photon ratio. Find an expression for the ratio

$$(1 - X_{\rm e})/X_{\rm e}^2$$

in terms only of kT and constants such as η and I. One might expect neutral hydrogen to form at a temperature given by $kT \approx I \approx 13\,\text{eV}$, but instead in our universe it forms at the much lower temperature $kT \approx 0.3\,\text{eV}$. Briefly explain why.