Part II

Classical Dynamics

Paper 1, Section I

8D Classical Dynamics

The Lagrangian for a particle of charge q and mass m in an electromagnetic field is

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}) \, ,$$

where $\mathbf{A}(\mathbf{r}, t)$ is the vector potential and $\phi(\mathbf{r}, t)$ is the scalar potential associated with the electromagnetic field.

(a) Determine how L changes under the gauge transformation

$$\phi \mapsto \phi - \frac{\partial f}{\partial t} \, , \qquad \mathbf{A} \mapsto \mathbf{A} + \boldsymbol{\nabla} f \, ,$$

where $f(\mathbf{r}, t)$ is a smooth function. Why does this change in L not affect the Euler-Lagrange equations?

(b) Show that the Euler-Lagrange equations imply the Lorentz force law.

(c) Now suppose that the electric field vanishes and the magnetic field is constant and uniform. Show that the component of the particle's canonical momentum along the direction of the magnetic field is conserved.

Paper 2, Section I

8D Classical Dynamics

A rigid body rotates with angular velocity $\boldsymbol{\omega}(t)$ around its centre of mass. Define what is meant by the *fixed space frame* and the *principal body frame.*

Write down an expression for how the body axes change in time. Hence derive Euler's equations for the torque-free motion of a rigid body.

Consider an axisymmetric body with principal moments of inertia $I_1 = I_2 \neq I_3$. Show that Euler's equations imply the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$ and the body's symmetry axis are always coplanar.

Paper 3, Section I

8D Classical Dynamics

Consider a 3-dimensional system with phase space coordinates $(\mathbf{q}, \mathbf{p})$.

(a) Define the *Poisson bracket* $\{f, g\}$ of two smooth functions on phase space.

(b) Show that $f(\mathbf{q}, \mathbf{p})$ is conserved along a particle's trajectory if and only if $\{f(\mathbf{q}, \mathbf{p}), H\} = 0$, where H is the Hamiltonian.

(c) Derive a constraint satisfied by a function $f(\mathbf{q}, \mathbf{p})$ given that $\{f(\mathbf{q}, \mathbf{p}), \mathbf{q} \cdot \mathbf{p}\} = 0$. Show that any smooth function obeying $f(\lambda \mathbf{q}, \lambda^{-1} \mathbf{p}) = f(\mathbf{q}, \mathbf{p})$, where λ is a real constant, satisfies this constraint.

Paper 4, Section I

8D Classical Dynamics

What is meant by an *adiabatic invariant* of a mechanical system?

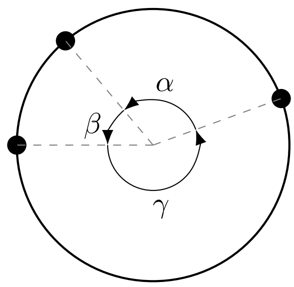
A particle of mass m and energy E moves between two fixed, parallel walls that are a distance L apart. The particle travels freely in a direction perpendicular to the walls except when it collides elastically with a wall at which point its velocity changes instantaneously. Compute the action $I = \oint p\, dq$ and verify that $T = dI/dE$ is the period of oscillation.

Suppose that the distance between the walls is varied very slowly so that $L(t)$ depends on time. How does the energy of the particle depend on time? Give a brief physical explanation for why the particle's energy changes.

Paper 2, Section II

14D Classical Dynamics

Three identical particles, each of mass m, are constrained to move around a fixed circle of radius r that lies in a horizontal plane. You may assume that the particles do not collide. The angles between the locations of the particles are α, β, γ as in the figure, which shows the view from above.



(a) Write down a constraint obeyed by α, β, γ. What degree of freedom is not described by these three angles?

(b) The particles feel the influence of a potential

$$V(\alpha, \beta, \gamma) = V_0(e^{-2\alpha} + e^{-2\beta} + e^{-2\gamma}),$$

where V_0 is a positive constant. Solving your constraint to find $\gamma = \gamma(\alpha, \beta)$, obtain a Lagrangian governing the dynamics of the particles' relative separations as a function of α, β, $\dot{\alpha}$ and $\dot{\beta}$.

(c) Find an equilibrium configuration of the system and show that it is stable. Find three linearly independent normal modes, together with their frequencies, that describe small perturbations about this equilibrium.

(d) The physical system is unchanged by permutations of (α, β, γ). Explain how this is consistent with your answer to part (c).

Paper 4, Section II

15D Classical Dynamics

What does it mean for a phase space coordinate transformation to be *canonical*? Consider a coordinate transformation $(q,p) \mapsto (Q,P)$ on the phase space of a system with one degree of freedom. Show that if this transformation is defined in terms of a generating function $F(q,P)$ via

$$Q = \left.\frac{\partial F}{\partial P}\right|_q \qquad \text{and} \qquad p = \left.\frac{\partial F}{\partial q}\right|_P$$

then it is canonical.

Find the phase space coordinate transformation associated to the generating function

$$F(q,P) = \int_0^q \sqrt{2P - u^2}\, du\,.$$

Obtain Hamilton's equations for Q and P in the case $H(q,p) = \frac{1}{2}(p^2+q^2)$. Hence find $Q(t)$ and $P(t)$ and check that these agree with the usual solution for a simple harmonic oscillator.

A particle of energy E has Hamiltonian $H(q,p) = \frac{1}{2}(p^2+q^2) + \epsilon q^4$, where $2q^2\epsilon \ll 1$ for all q in the range $-\sqrt{2E} \leqslant q \leqslant \sqrt{2E}$. By choosing an appropriately modified generating function $F_\epsilon(q,P)$, show that

$$\frac{q(t)}{p(t)} = \tan(t-t_0) - \epsilon\, I(q_0(t),E)\,(1+\tan^2(t-t_0)) + \epsilon q_0^2(t)\tan^3(t-t_0) + \mathcal{O}(\epsilon^2),$$

where $q_0(t) = \sqrt{2E}\sin(t-t_0)$ and $I(x,y)$ is defined by

$$I(x,y) = \int_0^x \frac{u^4}{(2y-u^2)^{3/2}}\, du\,.$$

Paper 1, Section I
8B Classical Dynamics

(a) Show that the canonical transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P})$ associated with a generating function $F_2(\mathbf{q}, \mathbf{P})$ of type 2 satisfies

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \qquad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}.$$

(b) A physical system with two degrees of freedom is described by the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = H_0(p_1, p_2) + H_1(p_1, p_2)\cos\theta,$$

where

$$\theta = n_1 q_1 + n_2 q_2$$

and n_1 and n_2 are non-zero integers.

Show that a certain linear combination of p_1 and p_2 is conserved, and that there is a (linear) canonical transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P})$ such that $Q_1 = \theta$ and the transformed Hamiltonian does not depend on Q_2.

Explain why the system is integrable.

Paper 2, Section I
8B Classical Dynamics

Show that Hamilton's equations for a system with n degrees of freedom can be written in the form

$$\dot{x}_a = \Omega_{ab}\frac{\partial H}{\partial x_b},$$

where $a, b \in \{1, 2, \ldots, 2n\}$ and Ω is a matrix that you should define.

Using a similar notation, define the Poisson bracket $\{f, g\}$ of two functions $f(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$. Evaluate the Poisson bracket $\{x_a, x_b\}$.

Show that the transformation $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x})$ preserves the form of Hamilton's equations if and only if the Jacobian matrix

$$J_{ab} = \frac{\partial X_a}{\partial x_b}$$

satisfies

$$J\Omega J^T = \Omega.$$

Deduce that such a canonical transformation leaves the phase-space volume invariant.

Paper 3, Section I
8B Classical Dynamics

The Lagrangian of the Lagrange top can be written as

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mgl \cos \theta \,.$$

Define the *generalized momenta* p_ϕ and p_ψ, and describe how they evolve in time.

Show that the nutation of the top is governed by the equation

$$\frac{1}{2} I_1 \dot{\theta}^2 + V_{\text{eff}}(\theta) = \text{constant} \,,$$

where $V_{\text{eff}}(\theta)$ is an effective potential energy that you should define.

Explain why p_ϕ and p_ψ must be equal in order for the top to reach the vertical position $\theta = 0$. In this case, show that $\theta = 0$ is a stable equilibrium provided that the top spins sufficiently fast.

Paper 4, Section I
8B Classical Dynamics

A particle of mass $m_1 = 3m$ is connected to a fixed point by a massless spring of natural length l and spring constant k. A second particle of mass $m_2 = 2m$ is connected to the first particle by an identical spring. The masses move along a vertical line in a uniform gravitational field g, such that mass m_i is a distance $z_i(t)$ below the fixed point and $z_2 > z_1 > 0$.

[*You may assume that the potential energy of a spring of length $l + x$ is $\frac{1}{2} k x^2$, where k is the spring constant and l is the natural length.*]

Write down the Lagrangian of the system.

Determine the equilibrium values of z_i.

Let q_i be the departure of z_i from its equilibrium value. Show that the Lagrangian can be written as

$$L = \frac{1}{2} T_{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} V_{ij} q_i q_j + \text{constant} \,,$$

and determine the matrices T and V.

Calculate the angular frequencies and eigenvectors of the normal modes of the system.

In what sense are the eigenvectors orthogonal?

Paper 2, Section II
14B Classical Dynamics

(a) A homogeneous, solid ellipsoid of mass M occupies the region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \,,$$

where a, b and c are positive constants. Calculate the inertia tensor of the ellipsoid.

(b) According to Poinsot's construction, the evolution of the angular velocity vector $\boldsymbol{\omega}(t)$ of a rigid body undergoing free rotational motion corresponds to the movement of an inertia ellipsoid on an invariable plane. Derive this construction, explaining why the inertia ellipsoid is tangent to the invariable plane and rolls on it.

(c) Describe qualitatively the general free rotational motion of the body considered in part (a) in an inertial frame of reference, in the special case $a = b < c$.

Paper 4, Section II
15B Classical Dynamics

An isolated three-body system consists of particles with masses m_1, m_2 and m_3 and position vectors $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and $\mathbf{r}_3(t)$. The particles move under the action of their mutual gravitational attraction. Write down the Lagrangian L of the system.

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be defined by

$$\mathbf{a} = \mathbf{r}_1 - \mathbf{r}_2 \,, \qquad \mathbf{b} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_3 \,, \qquad \mathbf{c} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} \,.$$

By expressing $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, or otherwise, show that the total kinetic energy can be written as

$$\frac{1}{2}\alpha|\dot{\mathbf{a}}|^2 + \frac{1}{2}\beta|\dot{\mathbf{b}}|^2 + \frac{1}{2}\gamma|\dot{\mathbf{c}}|^2 \,,$$

and obtain expressions for α, β and γ.

Show that the total potential energy can be expressed as a function of $\mathbf{a}$ and $\mathbf{b}$ only. What does this imply for the evolution of $\mathbf{c}$? Give a physical interpretation of this result.

Show also that the total angular momentum of the system about the origin is

$$\alpha\,\mathbf{a} \times \dot{\mathbf{a}} + \beta\,\mathbf{b} \times \dot{\mathbf{b}} + \gamma\,\mathbf{c} \times \dot{\mathbf{c}} \,.$$

Paper 1, Section I

8D Classical Dynamics

Two equal masses m move along a straight line between two stationary walls. The mass on the left is connected to the wall on its left by a spring of spring constant k_1, and the mass on the right is connected to the wall on its right by a spring of spring constant k_2. The two masses are connected by a third spring of spring constant k_3.

(a) Show that the Lagrangian of the system can be written in the form

$$L = \frac{1}{2}T_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}V_{ij}x_ix_j\,,$$

where $x_i(t)$, for $i = 1, 2$, are the displacements of the two masses from their equilibrium positions, and T_{ij} and V_{ij} are symmetric 2×2 matrices that should be determined.

(b) Let

$$k_1 = k(1+\epsilon\delta)\,, \qquad k_2 = k(1-\epsilon\delta)\,, \qquad k_3 = k\epsilon\,,$$

where $k > 0$, $\epsilon > 0$ and $|\epsilon\delta| < 1$. Using Lagrange's equations of motion, show that the angular frequencies ω of the normal modes of the system are given by

$$\omega^2 = \lambda\,\frac{k}{m}\,,$$

where

$$\lambda = 1 + \epsilon\left(1 \pm \sqrt{1+\delta^2}\right)\,.$$

Paper 2, Section I

8D Classical Dynamics

Show that, in a uniform gravitational field, the net gravitational torque on a system of particles, about its centre of mass, is zero.

Let S be an inertial frame of reference, and let S' be the frame of reference with the same origin and rotating with angular velocity $\boldsymbol{\omega}(t)$ with respect to S. You may assume that the rates of change of a vector $\mathbf{v}$ observed in the two frames are related by

$$\left(\frac{d\mathbf{v}}{dt}\right)_S = \left(\frac{d\mathbf{v}}{dt}\right)_{S'} + \boldsymbol{\omega}\times\mathbf{v}\,.$$

Derive Euler's equations for the torque-free motion of a rigid body.

Show that the general torque-free motion of a symmetric top involves precession of the angular-velocity vector about the symmetry axis of the body. Determine how the direction and rate of precession depend on the moments of inertia of the body and its angular velocity.

Paper 3, Section I

8D Classical Dynamics

The Lagrangian of a particle of mass m and charge q in an electromagnetic field takes the form

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + q\left(-\phi + \dot{\mathbf{r}} \cdot \mathbf{A}\right).$$

Explain the meaning of ϕ and $\mathbf{A}$, and how they are related to the electric and magnetic fields.

Obtain the canonical momentum $\mathbf{p}$ and the Hamiltonian $H(\mathbf{r}, \mathbf{p}, t)$.

Suppose that the electric and magnetic fields have Cartesian components $(E, 0, 0)$ and $(0, 0, B)$, respectively, where E and B are positive constants. Explain why the Hamiltonian of the particle can be taken to be

$$H = \frac{p_x^2}{2m} + \frac{(p_y - qBx)^2}{2m} + \frac{p_z^2}{2m} - qEx.$$

State three independent integrals of motion in this case.

Paper 4, Section I

8D Classical Dynamics

Briefly describe a physical object (a *Lagrange top*) whose Lagrangian is

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta.$$

Explain the meaning of the symbols in this equation.

Write down three independent integrals of motion for this system, and show that the nutation of the top is governed by the equation

$$\dot{u}^2 = f(u),$$

where $u = \cos\theta$ and $f(u)$ is a certain cubic function that you need not determine.

Paper 2, Section II

14D Classical Dynamics

(a) Show that the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \,,$$

where ω is a positive constant, describes a simple harmonic oscillator with angular frequency ω. Show that the energy E and the action I of the oscillator are related by $E = \omega I$.

(b) Let $0 < \epsilon < 2$ be a constant. Verify that the differential equation

$$\ddot{x} + \frac{x}{(\epsilon t)^2} = 0 \qquad \text{subject to} \qquad x(1) = 0\,, \quad \dot{x}(1) = 1$$

is solved by

$$x(t) = \frac{\sqrt{t}}{k}\sin(k \log t)$$

when $t > 1$, where k is a constant you should determine in terms of ϵ.

(c) Show that the solution in part (b) obeys

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{x^2}{(\epsilon t)^2} = \frac{1 - \cos(2k\log t) + 2k\sin(2k\log t) + 4k^2}{8k^2 t} \,.$$

Hence show that the fractional variation of the action in the limit $\epsilon \ll 1$ is $O(\epsilon)$, but that these variations do not accumulate. Comment on this behaviour in relation to the theory of adiabatic invariance.

Paper 4, Section II

15D Classical Dynamics

(a) Let $(\mathbf{q}, \mathbf{p})$ be a set of canonical phase-space variables for a Hamiltonian system with n degrees of freedom. Define the *Poisson bracket* $\{f, g\}$ of two functions $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$. Write down the canonical commutation relations that imply that a second set $(\mathbf{Q}, \mathbf{P})$ of phase-space variables is also canonical.

(b) Consider the near-identity transformation

$$\mathbf{Q} = \mathbf{q} + \delta\mathbf{q}\,, \qquad \mathbf{P} = \mathbf{p} + \delta\mathbf{p}\,,$$

where $\delta\mathbf{q}(\mathbf{q}, \mathbf{p})$ and $\delta\mathbf{p}(\mathbf{q}, \mathbf{p})$ are small. Determine the approximate forms of the canonical commutation relations, accurate to first order in $\delta\mathbf{q}$ and $\delta\mathbf{p}$. Show that these are satisfied when

$$\delta\mathbf{q} = \epsilon\,\frac{\partial F}{\partial \mathbf{p}}\,, \qquad \delta\mathbf{p} = -\epsilon\,\frac{\partial F}{\partial \mathbf{q}}\,,$$

where ϵ is a small parameter and $F(\mathbf{q}, \mathbf{p})$ is some function of the phase-space variables.

(c) In the limit $\epsilon \to 0$ this near-identity transformation is called the *infinitesimal canonical transformation generated by* F. Let $H(\mathbf{q}, \mathbf{p})$ be an autonomous Hamiltonian. Show that the change in the Hamiltonian induced by the infinitesimal canonical transformation is

$$\delta H = -\epsilon\{F, H\}\,.$$

Explain why F is an integral of motion if and only if the Hamiltonian is invariant under the infinitesimal canonical transformation generated by F.

(d) The Hamiltonian of the gravitational N-body problem in three-dimensional space is

$$H = \frac{1}{2}\sum_{i=1}^{N} \frac{|\mathbf{p}_i|^2}{2m_i} - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}\,,$$

where m_i, $\mathbf{r}_i$ and $\mathbf{p}_i$ are the mass, position and momentum of body i. Determine the form of F and the infinitesimal canonical transformation that correspond to the translational symmetry of the system.

Paper 1, Section I

8B Classical Dynamics

A linear molecule is modelled as four equal masses connected by three equal springs. Using the Cartesian coordinates x_1, x_2, x_3, x_4 of the centres of the four masses, and neglecting any forces other than those due to the springs, write down the Lagrangian of the system describing longitudinal motions of the molecule.

Rewrite and simplify the Lagrangian in terms of the generalized coordinates

$$q_1 = \frac{x_1 + x_4}{2}, \qquad q_2 = \frac{x_2 + x_3}{2}, \qquad q_3 = \frac{x_1 - x_4}{2}, \qquad q_4 = \frac{x_2 - x_3}{2}.$$

Deduce Lagrange's equations for q_1, q_2, q_3, q_4. Hence find the normal modes of the system and their angular frequencies, treating separately the symmetric and antisymmetric modes of oscillation.

Paper 2, Section I

8B Classical Dynamics

A particle of mass m has position vector $\mathbf{r}(t)$ in a frame of reference that rotates with angular velocity $\boldsymbol{\omega}(t)$. The particle moves under the gravitational influence of masses that are fixed in the rotating frame. Explain why the Lagrangian of the particle is of the form

$$L = \frac{1}{2} m(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})^2 - V(\mathbf{r}).$$

Show that Lagrange's equations of motion are equivalent to

$$m\left(\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right) = -\boldsymbol{\nabla} V.$$

Identify the canonical momentum $\mathbf{p}$ conjugate to $\mathbf{r}$. Obtain the Hamiltonian $H(\mathbf{r}, \mathbf{p})$ and Hamilton's equations for this system.

Paper 3, Section I

8B Classical Dynamics

A particle of mass m experiences a repulsive central force of magnitude k/r^2, where $r = |\mathbf{r}|$ is its distance from the origin. Write down the Hamiltonian of the system.

The Laplace–Runge–Lenz vector for this system is defined by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} + mk\,\hat{\mathbf{r}}\,,$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum and $\hat{\mathbf{r}} = \mathbf{r}/r$ is the radial unit vector. Show that

$$\{\mathbf{L}, H\} = \{\mathbf{A}, H\} = \mathbf{0}\,,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. What are the integrals of motion of the system? Show that the polar equation of the orbit can be written as

$$r = \frac{\lambda}{e\cos\theta - 1}\,,$$

where λ and e are non-negative constants.

Paper 4, Section I

8B Classical Dynamics

Derive expressions for the angular momentum and kinetic energy of a rigid body in terms of its mass M, the position $\mathbf{X}(t)$ of its centre of mass, its inertia tensor I (which should be defined) about its centre of mass, and its angular velocity $\boldsymbol{\omega}$.

A spherical planet of mass M and radius R has density proportional to $r^{-1}\sin(\pi r/R)$. Given that $\int_0^\pi x\sin x\,dx = \pi$ and $\int_0^\pi x^3\sin x\,dx = \pi(\pi^2 - 6)$, evaluate the inertia tensor of the planet in terms of M and R.

Paper 2, Section II

14B Classical Dynamics

A symmetric top of mass M rotates about a fixed point that is a distance l from the centre of mass along the axis of symmetry; its principal moments of inertia about the fixed point are $I_1 = I_2$ and I_3. The Lagrangian of the top is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mgl \cos \theta \, .$$

(i) Draw a diagram explaining the meaning of the Euler angles θ, ϕ and ψ.

(ii) Derive expressions for the three integrals of motion E, L_3 and L_z.

(iii) Show that the nutational motion is governed by the equation

$$\frac{1}{2} I_1 \dot{\theta}^2 + V_{\text{eff}}(\theta) = E' \, ,$$

and derive expressions for the effective potential $V_{\text{eff}}(\theta)$ and the modified energy E' in terms of E, L_3 and L_z.

(iv) Suppose that

$$L_z = L_3 \left(1 - \frac{\epsilon^2}{2} \right) \, ,$$

where ϵ is a small positive number. By expanding V_{eff} to second order in ϵ and θ, show that there is a stable equilibrium solution with $\theta = O(\epsilon)$, provided that $L_3^2 > 4MglI_1$. Determine the equilibrium value of θ and the precession rate $\dot{\phi}$, to the same level of approximation.

Paper 4, Section II

15B Classical Dynamics

(a) Explain how the Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ of a system can be obtained from its Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. Deduce that the action can be written as

$$S = \int (\mathbf{p} \cdot d\mathbf{q} - H\, dt) \,.$$

Show that Hamilton's equations are obtained if the action, computed between fixed initial and final configurations $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$, is minimized with respect to independent variations of $\mathbf{q}$ and $\mathbf{p}$.

(b) Let $(\mathbf{Q}, \mathbf{P})$ be a new set of coordinates on the same phase space. If the old and new coordinates are related by a type-2 generating function $F_2(\mathbf{q}, \mathbf{P}, t)$ such that

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}} \,, \qquad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}} \,,$$

deduce that the canonical form of Hamilton's equations applies in the new coordinates, but with a new Hamiltonian given by

$$K = H + \frac{\partial F_2}{\partial t} \,.$$

(c) For each of the Hamiltonians

$$\text{(i)} \quad H = H(p) \,, \qquad\qquad \text{(ii)} \quad H = \frac{1}{2}(q^2 + p^2) \,,$$

express the general solution $(q(t), p(t))$ at time t in terms of the initial values given by $(Q, P) = (q(0), p(0))$ at time $t = 0$. In each case, show that the transformation from (q, p) to (Q, P) is canonical for all values of t, and find the corresponding generating function $F_2(q, P, t)$ explicitly.

Paper 4, Section I

8E Classical Dynamics

(a) The angular momentum of a rigid body about its centre of mass is conserved. Derive Euler's equations,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3\,,$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1\,,$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2\,,$$

explaining the meaning of the quantities appearing in the equations.

(b) Show that there are two independent conserved quantities that are quadratic functions of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, and give a physical interpretation of them.

(c) Derive a linear approximation to Euler's equations that applies when $|\omega_1| \ll |\omega_3|$ and $|\omega_2| \ll |\omega_3|$. Use this to determine the stability of rotation about each of the three principal axes of an asymmetric top.

Paper 3, Section I

8E Classical Dynamics

A simple harmonic oscillator of mass m and spring constant k has the equation of motion

$$m\ddot{x} = -kx\,.$$

(a) Describe the orbits of the system in phase space. State how the action I of the oscillator is related to a geometrical property of the orbits in phase space. Derive the action–angle variables (θ, I) and give the form of the Hamiltonian of the oscillator in action–angle variables.

(b) Suppose now that the spring constant k varies in time. Under what conditions does the theory of adiabatic invariance apply? Assuming that these conditions hold, identify an adiabatic invariant and determine how the energy and amplitude of the oscillator vary with k in this approximation.

Paper 2, Section I

8E Classical Dynamics

(a) State *Hamilton's equations* for a system with n degrees of freedom and Hamiltonian $H(\mathbf{q},\mathbf{p},t)$, where $(\mathbf{q},\mathbf{p}) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ are canonical phase-space variables.

(b) Define the *Poisson bracket* $\{f,g\}$ of two functions $f(\mathbf{q},\mathbf{p},t)$ and $g(\mathbf{q},\mathbf{p},t)$.

(c) State the *canonical commutation relations* of the variables $\mathbf{q}$ and $\mathbf{p}$.

(d) Show that the time-evolution of any function $f(\mathbf{q},\mathbf{p},t)$ is given by

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}\,.$$

(e) Show further that the Poisson bracket of any two conserved quantities is also a conserved quantity.

[*You may assume the Jacobi identity,*

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0\,.]$$

Paper 1, Section I

8E Classical Dynamics

(a) A mechanical system with n degrees of freedom has the Lagrangian $L(\mathbf{q},\dot{\mathbf{q}})$, where $\mathbf{q} = (q_1,\ldots,q_n)$ are the generalized coordinates and $\dot{\mathbf{q}} = d\mathbf{q}/dt$.

Suppose that L is invariant under the continuous symmetry transformation $\mathbf{q}(t) \mapsto \mathbf{Q}(s,t)$, where s is a real parameter and $\mathbf{Q}(0,t) = \mathbf{q}(t)$. State and prove *Noether's theorem* for this system.

(b) A particle of mass m moves in a conservative force field with potential energy $V(\mathbf{r})$, where $\mathbf{r}$ is the position vector in three-dimensional space.

Let (r,ϕ,z) be cylindrical polar coordinates. $V(\mathbf{r})$ is said to have *helical symmetry* if it is of the form

$$V(\mathbf{r}) = f(r, \phi - kz)\,,$$

for some constant k. Show that a particle moving in a potential with helical symmetry has a conserved quantity that is a linear combination of angular and linear momenta.

Paper 2, Section II

14E Classical Dynamics

The Lagrangian of a particle of mass m and charge q moving in an electromagnetic field described by scalar and vector potentials $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ is

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + q(-\phi + \dot{\mathbf{r}} \cdot \mathbf{A}) \,,$$

where $\mathbf{r}(t)$ is the position vector of the particle and $\dot{\mathbf{r}} = d\mathbf{r}/dt$.

(a) Show that Lagrange's equations are equivalent to the equation of motion

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \,,$$

where

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

are the electric and magnetic fields.

(b) Show that the related Hamiltonian is

$$H = \frac{|\mathbf{p} - q\mathbf{A}|^2}{2m} + q\phi \,,$$

where $\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A}$. Obtain Hamilton's equations for this system.

(c) Verify that the electric and magnetic fields remain unchanged if the scalar and vector potentials are transformed according to

$$\phi \mapsto \tilde{\phi} = \phi - \frac{\partial f}{\partial t} \,,$$
$$\mathbf{A} \mapsto \tilde{\mathbf{A}} = \mathbf{A} + \boldsymbol{\nabla} f \,,$$

where $f(\mathbf{r}, t)$ is a scalar field. Show that the transformed Lagrangian $\tilde{L}$ differs from L by the total time-derivative of a certain quantity. Why does this leave the form of Lagrange's equations invariant? Show that the transformed Hamiltonian $\tilde{H}$ and phase-space variables $(\mathbf{r}, \tilde{\mathbf{p}})$ are related to H and $(\mathbf{r}, \mathbf{p})$ by a canonical transformation.

[*Hint: In standard notation, the canonical transformation associated with the type-2 generating function* $F_2(\mathbf{q}, \mathbf{P}, t)$ *is given by*

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}} \,, \qquad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}} \,, \qquad K = H + \frac{\partial F_2}{\partial t} \,.]$$

Paper 4, Section II

15E Classical Dynamics

(a) Explain what is meant by a *Lagrange top*. You may assume that such a top has the Lagrangian

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta$$

in terms of the Euler angles (θ, ϕ, ψ). State the meaning of the quantities I_1, I_3, M and l appearing in this expression.

Explain why the quantity

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}}$$

is conserved, and give two other independent integrals of motion.

Show that steady precession, with a constant value of $\theta \in (0, \frac{\pi}{2})$, is possible if

$$p_\psi^2 \geqslant 4MglI_1\cos\theta\,.$$

(b) A rigid body of mass M is of uniform density and its surface is defined by

$$x_1^2 + x_2^2 = x_3^2 - \frac{x_3^3}{h}\,,$$

where h is a positive constant and (x_1, x_2, x_3) are Cartesian coordinates in the body frame.

Calculate the values of I_1, I_3 and l for this symmetric top, when it rotates about the sharp point at the origin of this coordinate system.

Paper 1, Section I

8B Classical Dynamics

Derive Hamilton's equations from an action principle.

Consider a two-dimensional phase space with the Hamiltonian $H = p^2 + q^{-2}$. Show that $F = pq - ctH$ is the first integral for some constant c which should be determined. By considering the surfaces of constant F in the extended phase space, solve Hamilton's equations, and sketch the orbits in the phase space.

Paper 2, Section I

8B Classical Dynamics

Let $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{x}}^2 + y\dot{x}$$

of a particle constrained to move on a sphere $|\mathbf{x}| = 1/c$ of radius $1/c$. Use Lagrange multipliers to show that

$$\ddot{\mathbf{x}} + \dot{y}\mathbf{i} - \dot{x}\mathbf{j} + c^2(|\dot{\mathbf{x}}|^2 + y\dot{x} - x\dot{y})\mathbf{x} = 0. \qquad (*)$$

Now, consider the system $(*)$ with $c = 0$, and find the particle trajectories.

Paper 3, Section I

8B Classical Dynamics

Three particles of unit mass move along a line in a potential

$$V = \frac{1}{2}\Big((x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_3 - x_2)^2 + x_1^2 + x_2^2 + x_3^2\Big),$$

where x_i is the coordinate of the i'th particle, $i = 1, 2, 3$.

Write the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2}T_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}V_{ij}x_ix_j,$$

and specify the matrices T_{ij} and V_{ij}.

Find the normal frequencies and normal modes for this system.

Paper 4, Section I

8B Classical Dynamics

State and prove Noether's theorem in Lagrangian mechanics.

Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2} - V\left(\frac{x}{y}\right)$$

for a particle moving in the upper half-plane $\{(x, y) \in \mathbb{R}^2, y > 0\}$ in a potential V which only depends on x/y. Find two independent first integrals.

Paper 2, Section II

14B Classical Dynamics

Define a body frame $\mathbf{e}_a(t), a = 1, 2, 3$ of a rotating rigid body, and show that there exists a vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ such that

$$\dot{\mathbf{e}}_a = \boldsymbol{\omega} \times \mathbf{e}_a.$$

Let $\mathbf{L} = I_1\omega_1(t)\mathbf{e}_1 + I_2\omega_2(t)\mathbf{e}_2 + I_3\omega_3(t)\mathbf{e}_3$ be the angular momentum of a free rigid body expressed in the body frame. Derive the Euler equations from the conservation of angular momentum.

Verify that the kinetic energy E, and the total angular momentum L^2 are conserved. Hence show that

$$\dot{\omega}_3^2 = f(\omega_3),$$

where $f(\omega_3)$ is a quartic polynomial which should be explicitly determined in terms of L^2 and E.

Paper 4, Section II

15B Classical Dynamics

Given a Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$ with degrees of freedom q_i, define the *Hamiltonian* and show how Hamilton's equations arise from the Lagrange equations and the Legendre transform.

Consider the Lagrangian for a symmetric top moving in constant gravity:

$$\mathcal{L} = \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}B(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta,$$

where A, B, M, g and l are constants. Construct the corresponding Hamiltonian, and find three independent Poisson-commuting first integrals of Hamilton's equations.

Paper 1, Section I

8E Classical Dynamics

Consider a Lagrangian system with Lagrangian $L(x_A, \dot{x}_A, t)$, where $A = 1, \ldots, 3N$, and constraints

$$f_\alpha(x_A, t) = 0, \quad \alpha = 1, \ldots, 3N - n\,.$$

Use the method of Lagrange multipliers to show that this is equivalent to a system with Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t) \equiv L(x_A(q_i, t), \dot{x}_A(q_i, \dot{q}_i, t), t)$, where $i = 1, \ldots, n$, and q_i are coordinates on the surface of constraints.

Consider a bead of unit mass in $\mathbb{R}^2$ constrained to move (with no potential) on a wire given by an equation $y = f(x)$, where (x, y) are Cartesian coordinates. Show that the Euler–Lagrange equations take the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$

for some $\mathcal{L} = \mathcal{L}(x, \dot{x})$ which should be specified. Find one first integral of the Euler–Lagrange equations, and thus show that

$$t = F(x),$$

where $F(x)$ should be given in the form of an integral.

[*Hint: You may assume that the Euler–Lagrange equations hold in all coordinate systems.*]

Paper 2, Section I

8E Classical Dynamics

Derive the Lagrange equations from the principle of stationary action

$$S[q] = \int_{t_0}^{t_1} \mathcal{L}(q_i(t), \dot{q}_i(t), t)dt, \quad \delta S = 0,$$

where the end points $q_i(t_0)$ and $q_i(t_1)$ are fixed.

Let ϕ and $\mathbf{A}$ be a scalar and a vector, respectively, depending on $\mathbf{r} = (x, y, z)$. Consider the Lagrangian

$$\mathcal{L} = \frac{m\dot{\mathbf{r}}^2}{2} - (\phi - \dot{\mathbf{r}} \cdot \mathbf{A}),$$

and show that the resulting Euler–Lagrange equations are invariant under the transformations

$$\phi \to \phi + \alpha\frac{\partial F}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla F,$$

where $F = F(\mathbf{r}, t)$ is an arbitrary function, and α is a constant which should be determined.

Paper 3, Section I

8E Classical Dynamics

Define an *integrable system* with $2n$-dimensional phase space. Define *angle-action variables.*

Consider a two-dimensional phase space with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}q^{2k},$$

where k is a positive integer and the mass $m = m(t)$ changes slowly in time. Use the fact that the action is an adiabatic invariant to show that the energy varies in time as m^c, where c is a constant which should be found.

Paper 4, Section I

8E Classical Dynamics

Consider the Poisson bracket structure on $\mathbb{R}^3$ given by

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y$$

and show that $\{f, \rho^2\} = 0$, where $\rho^2 = x^2 + y^2 + z^2$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is any polynomial function on $\mathbb{R}^3$.

Let $H = (Ax^2 + By^2 + Cz^2)/2$, where A, B, C are positive constants. Find the explicit form of Hamilton's equations

$$\dot{\mathbf{r}} = \{\mathbf{r}, H\}, \quad \text{where} \quad \mathbf{r} = (x, y, z).$$

Find a condition on A, B, C such that the oscillation described by

$$x = 1 + \alpha(t), \quad y = \beta(t), \quad z = \gamma(t)$$

is linearly unstable, where $\alpha(t), \beta(t), \gamma(t)$ are small.

Paper 2, Section II

13E Classical Dynamics

Show that an object's inertia tensor about a point displaced from the centre of mass by a vector $\mathbf{c}$ is given by

$$(I_{\mathbf{c}})_{ab} = (I_0)_{ab} + M(|\mathbf{c}|^2 \delta_{ab} - c_a c_b),$$

where M is the total mass of the object, and $(I_0)_{ab}$ is the inertia tensor about the centre of mass.

Find the inertia tensor of a cube of uniform density, with edge of length L, about one of its vertices.

Paper 4, Section II

14E Classical Dynamics

Explain how geodesics of a Riemannian metric

$$g = g_{ab}(x^c) dx^a dx^b$$

arise from the kinetic Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{ab}(x^c) \dot{x}^a \dot{x}^b,$$

where $a, b = 1, \ldots, n$.

Find geodesics of the metric on the upper half plane

$$\Sigma = \{(x, y) \in \mathbb{R}^2, y > 0\}$$

with the metric

$$g = \frac{dx^2 + dy^2}{y^2}$$

and sketch the geodesic containing the points $(2, 3)$ and $(10, 3)$.

[*Hint: Consider* dy/dx.]

Paper 4, Section I

8E Classical Dynamics

Using conservation of angular momentum $\mathbf{L} = L_a \mathbf{e}_a$ in the body frame, derive the Euler equations for a rigid body:

$$I_1\,\dot\omega_1 + (I_3{-}I_2)\,\omega_2\,\omega_3 = 0, \quad I_2\,\dot\omega_2 + (I_1{-}I_3)\,\omega_3\,\omega_1 = 0, \quad I_3\,\dot\omega_3 + (I_2{-}I_1)\,\omega_1\,\omega_2 = 0.$$

[You may use the formula $\dot{\mathbf{e}}_a = \boldsymbol{\omega} \wedge \mathbf{e}_a$ without proof.]

Assume that the principal moments of inertia satisfy $I_1 < I_2 < I_3$. Determine whether a rotation about the principal 3-axis leads to stable or unstable perturbations.

Paper 1, Section I

8E Classical Dynamics

Consider a one-parameter family of transformations $q_i(t) \mapsto Q_i(s,t)$ such that $Q_i(0,t) = q_i(t)$ for all time t, and

$$\frac{\partial}{\partial s} L(Q_i, \dot Q_i, t) = 0\,,$$

where L is a Lagrangian and a dot denotes differentiation with respect to t. State and prove Noether's theorem.

Consider the Lagrangian

$$L \;=\; \frac{1}{2}(\,\dot x^2 + \dot y^2 + \dot z^2\,) \;-\; V(\,x{+}y,\; y{+}z\,)\,,$$

where the potential V is a function of two variables. Find a continuous symmetry of this Lagrangian and construct the corresponding conserved quantity. Use the Euler–Lagrange equations to explicitly verify that the function you have constructed is independent of t.

Paper 2, Section I

8E Classical Dynamics

Consider the Lagrangian

$$L = A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + B(\dot{\psi} + \dot{\phi}\cos\theta)^2 - C(\cos\theta)^k ,$$

where A, B, C are positive constants and k is a positive integer. Find three conserved quantities and show that $u = \cos\theta$ satisfies

$$\dot{u}^2 = f(u) ,$$

where $f(u)$ is a polynomial of degree $k+2$ which should be determined.

Paper 3, Section I

8E Classical Dynamics

Consider a six-dimensional phase space with coordinates (q_i, p_i) for $i = 1, 2, 3$. Compute the Poisson brackets $\{L_i, L_j\}$, where $L_i = \epsilon_{ijk}\, q_j\, p_k$.

Consider the Hamiltonian

$$H = \frac{1}{2}|\mathbf{p}|^2 + V(|\mathbf{q}|)$$

and show that the resulting Hamiltonian system admits three Poisson-commuting independent first integrals.

Paper 2, Section II

13E Classical Dynamics

Define what it means for the transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$(q_i, p_i) \mapsto \big(Q_i(q_j, p_j),\, P_i(q_j, p_j)\big) , \quad i, j = 1, \ldots, n$$

to be *canonical*. Show that a transformation is canonical if and only if

$$\{Q_i, Q_j\} = 0 , \quad \{P_i, P_j\} = 0 , \quad \{Q_i, P_j\} = \delta_{ij} .$$

Show that the transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$Q = q\cos\epsilon - p\sin\epsilon , \quad P = q\sin\epsilon + p\cos\epsilon$$

is canonical for any real constant ϵ. Find the corresponding generating function.

Paper 4, Section II

14E Classical Dynamics

A particle of unit mass is attached to one end of a light, stiff rod of length ℓ. The other end of the rod is held at a fixed position, such that the rod is free to swing in any direction. Write down the Lagrangian for the system giving a clear definition of any angular variables you introduce. [You should assume the acceleration g is constant.]

Find two independent constants of the motion.

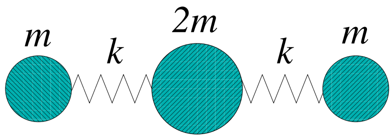
The particle is projected horizontally with speed v from a point where the rod lies at an angle α to the downward vertical, with $0 < \alpha < \pi/2$. In terms of ℓ, g and α, find the critical speed v_c such that the particle always remains at its initial height.

The particle is now projected horizontally with speed v_c but from a point at angle $\alpha + \delta\alpha$ to the vertical, where $\delta\alpha/\alpha \ll 1$. Show that the height of the particle oscillates, and find the period of oscillation in terms of ℓ, g and α.

Paper 4, Section I

7D Classical Dynamics

A triatomic molecule is modelled by three masses moving in a line while connected to each other by two identical springs of force constant k as shown in the figure.



(a) Write down the Lagrangian and derive the equations describing the motion of the atoms.

(b) Find the normal modes and their frequencies. What motion does the lowest frequency represent?

Paper 3, Section I

7D Classical Dynamics

(a) Consider a particle of mass m that undergoes periodic motion in a one-dimensional potential $V(q)$. Write down the Hamiltonian $H(p, q)$ for the system. Explain what is meant by the *angle–action variables* (θ, I) of the system and write down the integral expression for the action variable I.

(b) For $V(q) = \frac{1}{2}m\omega^2 q^2$ and fixed total energy E, describe the shape of the trajectories in phase-space. By using the expression for the area enclosed by the trajectory, or otherwise, find the action variable I in terms of ω and E. Hence describe how E changes with ω if ω varies slowly with time. Justify your answer.

Paper 2, Section I

7D Classical Dynamics

The Lagrangian for a heavy symmetric top of mass M, pinned at a point that is a distance l from the centre of mass, is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta \,.$$

(a) Find all conserved quantities. In particular, show that ω_3, the spin of the top, is constant.

(b) Show that θ obeys the equation of motion

$$I_1 \ddot{\theta} = -\frac{dV_{\text{eff}}}{d\theta} \,,$$

where the explicit form of V_{eff} should be determined.

(c) Determine the condition for uniform precession with no nutation, that is $\dot{\theta} = 0$ and $\dot{\phi} = \text{const}$. For what values of ω_3 does such uniform precession occur?

Paper 1, Section I

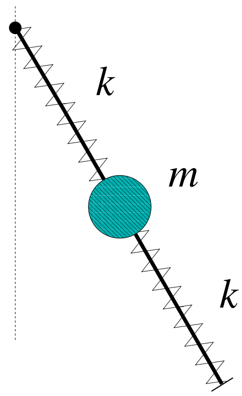
7D Classical Dynamics

(a) The action for a one-dimensional dynamical system with a generalized coordinate q and Lagrangian L is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt\,.$$

State the principle of least action and derive the Euler–Lagrange equation.

(b) A planar spring-pendulum consists of a light rod of length l and a bead of mass m, which is able to slide along the rod without friction and is attached to the ends of the rod by two identical springs of force constant k as shown in the figure. The rod is pivoted at one end and is free to swing in a vertical plane under the influence of gravity.



(i) Identify suitable generalized coordinates and write down the Lagrangian of the system.

(ii) Derive the equations of motion.

Paper 4, Section II

12C Classical Dynamics

Consider a rigid body with angular velocity $\boldsymbol{\omega}$, angular momentum $\mathbf{L}$ and position vector $\mathbf{r}$, in its body frame.

(a) Use the expression for the kinetic energy of the body,

$$\frac{1}{2}\int d^3\mathbf{r}\,\rho(\mathbf{r})\,\dot{\mathbf{r}}^2\,,$$

to derive an expression for the tensor of inertia of the body, $\mathbf{I}$. Write down the relationship between $\mathbf{L}$, $\mathbf{I}$ and $\boldsymbol{\omega}$.

(b) Euler's equations of torque-free motion of a rigid body are

$$\begin{aligned} I_1\,\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,, \\ I_2\,\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,, \\ I_3\,\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,. \end{aligned}$$

Working in the frame of the principal axes of inertia, use Euler's equations to show that the energy E and the squared angular momentum $\mathbf{L}^2$ are conserved.

(c) Consider a cuboid with sides a, b and c, and with mass M distributed uniformly.

(i) Use the expression for the tensor of inertia derived in (a) to calculate the principal moments of inertia of the body.

(ii) Assume $b = 2a$ and $c = 4a$, and suppose that the initial conditions are such that

$$\mathbf{L}^2 = 2I_2E$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axis $\mathbf{e}_2$. Derive the first order differential equation for ω_2 in terms of E, M and a and hence determine the long-term behaviour of $\boldsymbol{\omega}$.

Paper 2, Section II

12C Classical Dynamics

(a) Consider a Lagrangian dynamical system with one degree of freedom. Write down the expression for the Hamiltonian of the system in terms of the generalized velocity $\dot{q}$, momentum p, and the Lagrangian $L(q, \dot{q}, t)$. By considering the differential of the Hamiltonian, or otherwise, derive Hamilton's equations.

Show that if q is ignorable (cyclic) with respect to the Lagrangian, i.e. $\partial L/\partial q = 0$, then it is also ignorable with respect to the Hamiltonian.

(b) A particle of charge q and mass m moves in the presence of electric and magnetic fields such that the scalar and vector potentials are $\phi = yE$ and $\mathbf{A} = (0, xB, 0)$, where (x, y, z) are Cartesian coordinates and E, B are constants. The Lagrangian of the particle is

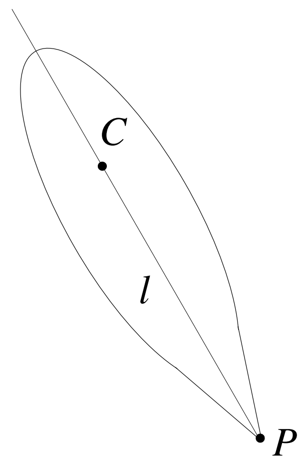
$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 - q\phi + q\dot{\mathbf{r}} \cdot \mathbf{A}\,.$$

Starting with the Lagrangian, derive an explicit expression for the Hamiltonian and use Hamilton's equations to determine the motion of the particle.

Paper 4, Section I

9A Classical Dynamics

Consider a heavy symmetric top of mass M with principal moments of inertia I_1, I_2 and I_3, where $I_1 = I_2 \neq I_3$. The top is pinned at point P, which is at a distance l from the centre of mass, C, as shown in the figure.



Its angular velocity in a body frame $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is given by

$$\boldsymbol{\omega} = [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\,\mathbf{e}_1 + [\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]\,\mathbf{e}_2 + [\dot{\psi} + \dot{\phi}\cos\theta]\,\mathbf{e}_3\,,$$

where ϕ, θ and ψ are the Euler angles.

(a) Assuming that $\{\mathbf{e}_a\}$, $a = 1, 2, 3$, are chosen to be the principal axes, write down the Lagrangian of the top in terms of ω_a and the principal moments of inertia. Hence find the Lagrangian in terms of the Euler angles.

(b) Find all conserved quantities. Show that ω_3, the spin of the top, is constant.

(c) By eliminating $\dot{\phi}$ and $\dot{\psi}$, derive a second-order differential equation for θ.

Paper 3, Section I

9A Classical Dynamics

(a) The action for a one-dimensional dynamical system with a generalized coordinate q and Lagrangian L is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt\,.$$

State the principle of least action. Write the expression for the Hamiltonian in terms of the generalized velocity $\dot{q}$, the generalized momentum p and the Lagrangian L. Use it to derive Hamilton's equations from the principle of least action.

(b) The motion of a particle of charge q and mass m in an electromagnetic field with scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ is characterized by the Lagrangian

$$L = \frac{m\dot{\mathbf{r}}^2}{2} - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A})\,.$$

(i) Write down the Hamiltonian of the particle.

(ii) Consider a particle which moves in three dimensions in a magnetic field with $\mathbf{A} = (0, Bx, 0)$, where B is a constant. There is no electric field. Obtain Hamilton's equations for the particle.

Paper 2, Section I

9A Classical Dynamics

The components of the angular velocity $\boldsymbol{\omega}$ of a rigid body and of the position vector $\mathbf{r}$ are given in a body frame.

(a) The kinetic energy of the rigid body is defined as

$$T = \frac{1}{2}\int d^3\mathbf{r}\,\rho(\mathbf{r})\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,,$$

Given that the centre of mass is at rest, show that T can be written in the form

$$T = \frac{1}{2}I_{ab}\omega_a\omega_b\,,$$

where the explicit form of the tensor I_{ab} should be determined.

(b) Explain what is meant by the *principal moments of inertia.*

(c) Consider a rigid body with principal moments of inertia I_1, I_2 and I_3, which are all unequal. Derive Euler's equations of torque-free motion

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,. \end{aligned}$$

(d) The body rotates about the principal axis with moment of inertia I_1. Derive the condition for stable rotation.

Paper 1, Section I

9A Classical Dynamics

Consider a one-dimensional dynamical system with generalized coordinate and momentum (q, p).

(a) Define the *Poisson bracket* $\{f, g\}$ of two functions $f(q, p, t)$ and $g(q, p, t)$.

(b) Verify the Leibniz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

(c) Explain what is meant by a *canonical transformation* $(q, p) \to (Q, P)$.

(d) State the condition for a transformation $(q, p) \to (Q, P)$ to be canonical in terms of the Poisson bracket $\{Q, P\}$. Use this to determine whether or not the following transformations are canonical:

(i) $Q = \dfrac{q^2}{2}\,,\; P = \dfrac{p}{q}\,,$

(ii) $Q = \tan q\,,\; P = p\cos q\,,$

(iii) $Q = \sqrt{2q}\,e^{t}\cos p\,,\; P = \sqrt{2q}\,e^{-t}\sin p\,.$

Paper 4, Section II
15A Classical Dynamics

(a) Consider a system with one degree of freedom, which undergoes periodic motion in the potential $V(q)$. The system's Hamiltonian is

$$H(p,q) = \frac{p^2}{2m} + V(q)\,.$$

(i) Explain what is meant by the *angle* and *action variables*, θ and I, of the system and write down the integral expression for the action variable I. Is I conserved? Is θ conserved?

(ii) Consider $V(q) = \lambda q^6$, where λ is a positive constant. Find I in terms of λ, the total energy E, the mass M, and a dimensionless constant factor (which you need not compute explicitly).

(iii) Hence describe how E changes with λ if λ varies slowly with time. Justify your answer.

(b) Consider now a particle which moves in a plane subject to a central force-field $\mathbf{F} = -kr^{-2}\hat{\mathbf{r}}$.

(i) Working in plane polar coordinates (r,ϕ), write down the Hamiltonian of the system. Hence deduce two conserved quantities. Prove that the system is integrable and state the number of action variables.

(ii) For a particle which moves on an elliptic orbit find the action variables associated with radial and tangential motions. Can the relationship between the frequencies of the two motions be deduced from this result? Justify your answer.

(iii) Describe how E changes with m and k if one or both of them vary slowly with time.

[You may use

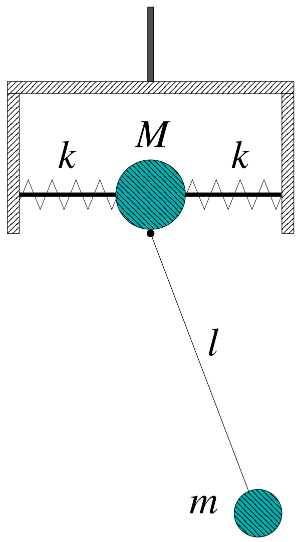
$$\int_{r_1}^{r_2} \left\{ \left(1 - \frac{r_1}{r}\right)\left(\frac{r_2}{r} - 1\right)\right\}^{\frac{1}{2}} dr \;=\; \frac{\pi}{2}\,(r_1 + r_2) - \pi\,\sqrt{r_1\, r_2}\ ,$$

where $0 < r_1 < r_2$.]

Paper 2, Section II

15A Classical Dynamics

A planar pendulum consists of a mass m at the end of a light rod of length l. The pivot of the pendulum is attached to a bead of mass M, which slides along a horizontal rod without friction. The bead is connected to the ends of the horizontal rod by two identical springs of force constant k. The pivot constrains the pendulum to swing in the vertical plane through the horizontal rod. The horizontal rod is mounted on a bracket, so the system could rotate about the vertical axis which goes through its centre as shown in the figure.



(a) Initially, the system is not allowed to rotate about the vertical axis.

- (i) Identify suitable generalized coordinates and write down the Lagrangian of the system.
- (ii) Write down expression(s) for any conserved quantities. Justify your answer.
- (iii) Derive the equations of motion.
- (iv) For $M = m/2$ and $gm/kl = 3$, find the frequencies of small oscillations around the stable equilibrium and the corresponding normal modes. Describe the respective motions of the system.

(b) Assume now that the system is free to rotate about the vertical axis without friction. Write down the Lagrangian of the system. Identify and calculate the additional conserved quantity.

Paper 4, Section I

9B Classical Dynamics

The Lagrangian for a heavy symmetric top of mass M, pinned at point O which is a distance l from the centre of mass, is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta .$$

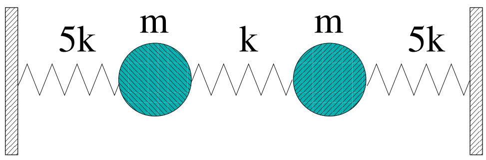
(i) Starting with the fixed space frame $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ and choosing O at its origin, sketch the top with embedded body frame axis $\mathbf{e}_3$ being the symmetry axis. Clearly identify the Euler angles (θ, ϕ, ψ).

(ii) Obtain the momenta p_θ, p_ϕ and p_ψ and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations. Identify the three conserved quantities.

Paper 3, Section I

9B Classical Dynamics

Two equal masses m are connected to each other and to fixed points by three springs of force constant $5k$, k and $5k$ as shown in the figure.



(i) Write down the Lagrangian and derive the equations describing the motion of the system in the direction parallel to the springs.

(ii) Find the normal modes and their frequencies. Comment on your results.

Paper 2, Section I

9B Classical Dynamics

(i) Consider a rigid body with principal moments of inertia I_1, I_2, I_3. Derive Euler's equations of torque-free motion,

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2, \end{aligned}$$

with components of the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ given in the body frame.

(ii) Use Euler's equations to show that the energy E and the square of the total angular momentum $\mathbf{L}^2$ of the body are conserved.

(iii) Consider a torque-free motion of a symmetric top with $I_1 = I_2 = \frac{1}{2}I_3$. Show that in the body frame the vector of angular velocity $\boldsymbol{\omega}$ precesses about the body-fixed $\mathbf{e}_3$ axis with constant angular frequency equal to ω_3.

Paper 1, Section I

9B Classical Dynamics

Consider an n-dimensional dynamical system with generalized coordinates and momenta (q_i, p_i), $i = 1, 2, ..., n$.

(a) Define the Poisson bracket $\{f, g\}$ of two functions $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$.

(b) Assuming Hamilton's equations of motion, prove that if a function $G(q_i, p_i)$ Poisson commutes with the Hamiltonian, that is $\{G, H\} = 0$, then G is a constant of the motion.

(c) Assume that q_j is an ignorable coordinate, that is the Hamiltonian does not depend on it explicitly. Using the formalism of Poisson brackets prove that the conjugate momentum p_j is conserved.

Paper 4, Section II

15B Classical Dynamics

The motion of a particle of charge q and mass m in an electromagnetic field with scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ is characterized by the Lagrangian

$$L = \frac{m\dot{\mathbf{r}}^2}{2} - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}) \ .$$

(i) Write down the Hamiltonian of the particle.

(ii) Write down Hamilton's equations of motion for the particle.

(iii) Show that Hamilton's equations are invariant under the gauge transformation

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla \Lambda,$$

for an arbitrary function $\Lambda(\mathbf{r}, t)$.

(iv) The particle moves in the presence of a field such that $\phi = 0$ and $\mathbf{A} = (-\frac{1}{2}yB, \frac{1}{2}xB, 0)$, where (x, y, z) are Cartesian coordinates and B is a constant.

(a) Find a gauge transformation such that only one component of $\mathbf{A}(x, y, z)$ remains non-zero.

(b) Determine the motion of the particle.

(v) Now assume that B varies very slowly with time on a time-scale much longer than $(qB/m)^{-1}$. Find the quantity which remains approximately constant throughout the motion.
[You may use the expression for the action variable $I = \frac{1}{2\pi} \oint p_i dq_i$.]

Paper 2, Section II
15B Classical Dynamics

(i) The action for a system with a generalized coordinate q is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \,.$$

(a) State the Principle of Least Action and derive the Euler–Lagrange equation.

(b) Consider an arbitrary function $f(q,t)$. Show that $L' = L + df/dt$ leads to the same equation of motion.

(ii) A wire frame ABC in a shape of an equilateral triangle with side a rotates in a horizontal plane with constant angular frequency ω about a vertical axis through A. A bead of mass m is threaded on BC and moves without friction. The bead is connected to B and C by two identical light springs of force constant k and equilibrium length $a/2$.

(a) Introducing the displacement η of the particle from the mid point of BC, determine the Lagrangian $L(\eta, \dot{\eta})$.

(b) Derive the equation of motion. Identify the integral of the motion.

(c) Describe the motion of the bead. Find the condition for there to be a stable equilibrium and find the frequency of small oscillations about it when it exists.

Paper 4, Section I

9A Classical Dynamics

Consider a one-dimensional dynamical system with generalized coordinate and momentum (q, p).

(a) Define the Poisson bracket $\{f, g\}$ of two functions $f(q, p, t)$ and $g(q, p, t)$.

(b) Find the Poisson brackets $\{q, q\}$, $\{p, p\}$ and $\{q, p\}$.

(c) Assuming Hamilton's equations of motion prove that

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,.$$

(d) State the condition for a transformation $(q, p) \to (Q, P)$ to be canonical in terms of the Poisson brackets found in (b). Use this to determine whether or not the following transformations are canonical:

(i) $Q = \sin q,\ P = \frac{p-a}{\cos q}$,

(ii) $Q = \cos q,\ P = \frac{p-a}{\sin q}$,

where a is constant.

Paper 3, Section I

9A Classical Dynamics

The motion of a particle of charge q and mass m in an electromagnetic field with scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ is characterized by the Lagrangian

$$L = \frac{m\dot{\mathbf{r}}^2}{2} - q(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}).$$

(a) Show that the Euler–Lagrange equation is invariant under the gauge transformation

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\Lambda,$$

for an arbitrary function $\Lambda(\mathbf{r}, t)$.

(b) Derive the equations of motion in terms of the electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$.

[Recall that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$.]

Paper 2, Section I
9A Classical Dynamics

(a) The action for a system with a generalized coordinate q is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \, .$$

State the Principle of Least Action and state the Euler–Lagrange equation.

(b) Consider a light rigid circular wire of radius a and centre O. The wire lies in a vertical plane, which rotates about the vertical axis through O. At time t the plane containing the wire makes an angle $\phi(t)$ with a fixed vertical plane. A bead of mass m is threaded onto the wire. The bead slides without friction along the wire, and its location is denoted by A. The angle between the line OA and the downward vertical is $\theta(t)$.

Show that the Lagrangian of this system is

$$\frac{ma^2}{2}\dot{\theta}^2 + \frac{ma^2}{2}\dot{\phi}^2 \sin^2\theta + mga\cos\theta \, .$$

Calculate two independent constants of the motion, and explain their physical significance.

Paper 1, Section I
9A Classical Dynamics

Consider a heavy symmetric top of mass M, pinned at point P, which is a distance l from the centre of mass.

(a) Working in the body frame $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ (where $\mathbf{e_3}$ is the symmetry axis of the top) define the *Euler angles* (ψ, θ, ϕ) and show that the components of the angular velocity can be expressed in terms of the Euler angles as

$$\boldsymbol{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi,\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi,\ \dot{\psi} + \dot{\phi}\cos\theta) \, .$$

(b) Write down the Lagrangian of the top in terms of the Euler angles and the principal moments of inertia I_1, I_3.

(c) Find the three constants of motion.

Paper 4, Section II

15A Classical Dynamics

A homogenous thin rod of mass M and length l is constrained to rotate in a horizontal plane about its centre O. A bead of mass m is set to slide along the rod without friction. The bead is attracted to O by a force resulting in a potential $kx^2/2$, where x is the distance from O.

(a) Identify suitable generalized coordinates and write down the Lagrangian of the system.

(b) Identify all conserved quantities.

(c) Derive the equations of motion and show that one of them can be written as

$$m\ddot{x} = -\frac{\partial V_{\text{eff}}(x)}{\partial x}\,,$$

where the form of the effective potential $V_{\text{eff}}(x)$ should be found explicitly.

(d) Sketch the effective potential. Find and characterize all points of equilibrium.

(e) Find the frequencies of small oscillations around the stable equilibria.

Paper 2, Section II

15A Classical Dynamics

Consider a rigid body with principal moments of inertia I_1, I_2, I_3.

(a) Derive Euler's equations of torque-free motion

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,, \end{aligned}$$

with components of the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ given in the body frame.

(b) Show that rotation about the second principal axis is unstable if $(I_2 - I_3)(I_1 - I_2) > 0$.

(c) The principal moments of inertia of a uniform cylinder of radius R, height h and mass M about its centre of mass are

$$I_1 = I_2 = \frac{MR^2}{4} + \frac{Mh^2}{12} \quad ; \quad I_3 = \frac{MR^2}{2}\,.$$

The cylinder has two identical cylindrical holes of radius r drilled along its length. The axes of symmetry of the holes are at a distance a from the axis of symmetry of the cylinder such that $r < R/2$ and $r < a < R - r$. All three axes lie in a single plane.

Compute the principal moments of inertia of the body.

Paper 1, Section I
9C Classical Dynamics

(i) A particle of mass m and charge q, at position $\mathbf{x}$, moves in an electromagnetic field with scalar potential $\phi(\mathbf{x},t)$ and vector potential $\mathbf{A}(\mathbf{x},t)$. Verify that the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q(\phi - \dot{\mathbf{x}}\cdot\mathbf{A})$$

gives the correct equations of motion.

[Note that $\mathbf{E} = -\boldsymbol{\nabla}\phi - \dot{\mathbf{A}}$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$.]

(ii) Consider the case of a constant uniform magnetic field, with $\mathbf{E} = \mathbf{0}$, given by $\phi = 0$, $\mathbf{A} = (0, xB, 0)$, where (x, y, z) are Cartesian coordinates and B is a constant. Find the motion of the particle, and describe it carefully.

Paper 2, Section I
9C Classical Dynamics

Three particles, each of mass m, move along a straight line. Their positions on the line containing the origin, O, are x_1, x_2 and x_3. They are subject to forces derived from the potential energy function

$$V = \frac{1}{2}m\Omega^2\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + x_1^2 + x_2^2 + x_3^2\right].$$

Obtain Lagrange's equations for the system, and show that the frequency, ω, of a normal mode satisfies

$$f^3 - 9f^2 + 24f - 16 = 0\,,$$

where $f = (\omega^2/\Omega^2)$. Find a complete set of normal modes for the system, and draw a diagram indicating the nature of the corresponding motions.

Paper 3, Section I

9C Classical Dynamics

The Lagrangian for a heavy symmetric top is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta \right)^2 - Mgl\cos\theta \, .$$

State Noether's Theorem. Hence, or otherwise, find two conserved quantities linear in momenta, and a third conserved quantity quadratic in momenta.

Writing $\mu = \cos\theta$, deduce that μ obeys an equation of the form

$$\dot{\mu}^2 = F(\mu) \, ,$$

where $F(\mu)$ is cubic in μ. [You need not determine the explicit form of $F(\mu)$.]

Paper 4, Section I

9C Classical Dynamics

(i) A dynamical system is described by the Hamiltonian $H(q_i, p_i)$. Define the Poisson bracket $\{f, g\}$ of two functions $f(q_i, p_i, t)$, $g(q_i, p_i, t)$. Assuming the Hamiltonian equations of motion, find an expression for df/dt in terms of the Poisson bracket.

(ii) A one-dimensional system has the Hamiltonian

$$H = p^2 + \frac{1}{q^2} \, .$$

Show that $u = pq - 2Ht$ is a constant of the motion. Deduce the form of $(q(t), p(t))$ along a classical path, in terms of the constants u and H.

Paper 2, Section II

15C Classical Dynamics

Derive Euler's equations governing the torque-free and force-free motion of a rigid body with principal moments of inertia I_1, I_2 and I_3, where $I_1 < I_2 < I_3$. Identify two constants of the motion. Hence, or otherwise, find the equilibrium configurations such that the angular-momentum vector, as measured with respect to axes fixed in the body, remains constant. Discuss the stability of these configurations.

A spacecraft may be regarded as moving in a torque-free and force-free environment. Nevertheless, flexing of various parts of the frame can cause significant dissipation of energy. How does the angular-momentum vector ultimately align itself within the body?

Paper 4, Section II

15C Classical Dynamics

Given a Hamiltonian system with variables (q_i, p_i), $i = 1, \ldots, n$, state the definition of a canonical transformation

$$(q_i, p_i) \to (Q_i, P_i)\,,$$

where $\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{p}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p}, t)$. Write down a matrix equation that is equivalent to the condition that the transformation is canonical.

Consider a harmonic oscillator of unit mass, with Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)\,.$$

Write down the Hamilton–Jacobi equation for Hamilton's principal function $S(q, E, t)$, and deduce the Hamilton–Jacobi equation

$$\frac{1}{2}\left[\left(\frac{\partial W}{\partial q}\right)^2 + \omega^2 q^2\right] = E \tag{1}$$

for Hamilton's characteristic function $W(q, E)$.

Solve (1) to obtain an integral expression for W, and deduce that, at energy E,

$$S = \sqrt{2E}\int dq \sqrt{\left(1 - \frac{\omega^2 q^2}{2E}\right)} - Et\,. \tag{2}$$

Let $\alpha = E$, and define the angular coordinate

$$\beta = \left(\frac{\partial S}{\partial E}\right)_{q,t}\,.$$

You may assume that (2) implies

$$t + \beta = \left(\frac{1}{\omega}\right)\arcsin\left(\frac{\omega q}{\sqrt{2E}}\right)\,.$$

Deduce that

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{(2E - \omega^2 q^2)}\,,$$

from which

$$p = \sqrt{2E}\cos[\omega(t + \beta)]\,.$$

Hence, or otherwise, show that the transformation from variables (q, p) to (α, β) is canonical.

Paper 1, Section I

9D Classical Dynamics

A system with coordinates q_i, $i = 1, \ldots, n$, has the Lagrangian $L(q_i, \dot{q}_i)$. Define the energy E.

Consider a charged particle, of mass m and charge e, moving with velocity $\mathbf{v}$ in the presence of a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. The usual vector equation of motion can be derived from the Lagrangian

$$L = \frac{1}{2} m \, \mathbf{v}^2 + e \, \mathbf{v} \cdot \mathbf{A} \,,$$

where $\mathbf{A}$ is the vector potential.

The particle moves in the presence of a field such that

$$\mathbf{A} = (0, \, r \, g(z), \, 0) \,, \qquad g(z) > 0 \,,$$

referred to cylindrical polar coordinates (r, ϕ, z). Obtain two constants of the motion, and write down the Lagrangian equations of motion obtained by variation of r, ϕ and z.

Show that, if the particle is projected from the point (r_0, ϕ_0, z_0) with velocity $(0, -2\,(e/m)\, r_0 \, g(z_0), 0)$, it will describe a circular orbit provided that $g'(z_0) = 0$.

Paper 2, Section I

9D Classical Dynamics

Given the form

$$T = \frac{1}{2} \, T_{ij} \, \dot{q}_i \, \dot{q}_j \,, \qquad V = \frac{1}{2} \, V_{ij} \, q_i \, q_j \,,$$

for the kinetic energy T and potential energy V of a mechanical system, deduce Lagrange's equations of motion.

A light elastic string of length $4b$, fixed at both ends, has three particles, each of mass m, attached at distances $b, 2b, 3b$ from one end. Gravity can be neglected. The particles vibrate with small oscillations transversely to the string, the tension S in the string providing the restoring force. Take the displacements of the particles, q_i, $i = 1, 2, 3$, to be the generalized coordinates. Take units such that $m = 1$, $S/b = 1$ and show that

$$V = \frac{1}{2} \left[{q_1}^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + {q_3}^2 \right] .$$

Find the normal–mode frequencies for this system.

Paper 3, Section I

9D Classical Dynamics

Euler's equations for the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ of a rigid body, viewed in the body frame, are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\, \omega_2\, \omega_3$$

and cyclic permutations, where the principal moments of inertia are assumed to obey $I_1 < I_2 < I_3$.

Write down two quadratic first integrals of the motion.

There is a family of solutions $\boldsymbol{\omega}(t)$, unique up to time–translations $t \to (t - t_0)$, which obey the boundary conditions $\boldsymbol{\omega} \to (0, \Omega, 0)$ as $t \to -\infty$ and $\boldsymbol{\omega} \to (0, -\Omega, 0)$ as $t \to \infty$, for a given positive constant Ω. Show that, for such a solution, one has

$$\mathbf{L}^2 = 2EI_2\,,$$

where $\mathbf{L}$ is the angular momentum and E is the kinetic energy.

By eliminating ω_1 and ω_3 in favour of ω_2, or otherwise, show that, in this case, the second Euler equation reduces to

$$\frac{ds}{d\tau} = 1 - s^2\,,$$

where $s = \omega_2/\Omega$ and $\tau = \Omega t\, \big[(I_1 - I_2)(I_2 - I_3)/I_1 I_3\big]^{1/2}$. Find the general solution $s(\tau)$.

[You are not expected to calculate $\omega_1(t)$ or $\omega_3(t)$.]

Paper 4, Section I

9D Classical Dynamics

A system with one degree of freedom has Lagrangian $L(q, \dot{q})$. Define the canonical momentum p and the energy E. Show that E is constant along any classical path.

Consider a classical path $q_c(t)$ with the boundary–value data

$$q_c(0) = q_I \,, \qquad q_c(T) = q_F \,, \quad T > 0 \,.$$

Define the action $S_c(q_I, q_F, T)$ of the path. Show that the total derivative dS_c/dT along the classical path obeys

$$\frac{dS_c}{dT} = L \,.$$

Using Lagrange's equations, or otherwise, deduce that

$$\frac{\partial S_c}{\partial q_F} = p_F \,, \qquad \frac{\partial S_c}{\partial T} = -E \,,$$

where p_F is the final momentum.

Paper 2, Section II

15D Classical Dynamics

An axially–symmetric top of mass m is free to rotate about a fixed point O on its axis. The principal moments of inertia about O are A, A, C, and the centre of gravity G is at a distance ℓ from O. Define Euler angles θ, ϕ and ψ which specify the orientation of the top, where θ is the inclination of OG to the upward vertical. Show that there are three conserved quantities for the motion, and give their physical meaning.

Initially, the top is spinning with angular velocity n about OG, with G vertically above O, before being disturbed slightly. Show that, in the subsequent motion, θ will remain close to zero provided $C^2n^2 > 4mg\ell A$, but that if $C^2n^2 < 4mg\ell A$, then θ will attain a maximum value given by

$$\cos\theta \simeq (C^2n^2/\, 2mg\ell A) - 1 \,.$$

Paper 4, Section II

15D Classical Dynamics

A system is described by the Hamiltonian $H(q,p)$. Define the *Poisson bracket* $\{f,g\}$ of two functions $f(q,p,t)$, $g(q,p,t)$, and show from Hamilton's equations that

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}\,.$$

Consider the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right),$$

and define

$$a = (p - i\omega q)/(2\omega)^{1/2}\,, \qquad a^* = (p + i\omega q)/(2\omega)^{1/2}\,,$$

where $i = \sqrt{-1}$. Evaluate $\{a,a\}$ and $\{a,a^*\}$, and show that $\{a,H\} = -i\omega a$ and $\{a^*,H\} = i\omega a^*$. Show further that, when $f(q,p,t)$ is regarded as a function of the independent complex variables a, a^* and of t, one has

$$\frac{df}{dt} = i\omega\left(a^*\frac{\partial f}{\partial a^*} - a\,\frac{\partial f}{\partial a}\right) + \frac{\partial f}{\partial t}\,.$$

Deduce that both $\log a^* - i\omega t$ and $\log a + i\omega t$ are constant during the motion.

Paper 1, Section I

9E Classical Dynamics

Lagrange's equations for a system with generalized coordinates $q_i(t)$ are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

where L is the Lagrangian. The Hamiltonian is given by

$$H = \sum_j p_j \dot{q}_j - L,$$

where the momentum conjugate to q_j is

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Derive Hamilton's equations in the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Explain what is meant by the statement that q_k is an ignorable coordinate and give an associated constant of the motion in this case.

The Hamiltonian for a particle of mass m moving on the surface of a sphere of radius a under a potential $V(\theta)$ is given by

$$H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + V(\theta),$$

where the generalized coordinates are the spherical polar angles (θ, ϕ). Write down two constants of the motion and show that it is possible for the particle to move with constant θ provided that

$${p_\phi}^2 = \left(\frac{ma^2 \sin^3\theta}{\cos\theta}\right)\frac{dV}{d\theta}.$$

Paper 2, Section I

9E Classical Dynamics

A system of three particles of equal mass m moves along the x axis with x_i denoting the x coordinate of particle i. There is an equilibrium configuration for which $x_1 = 0$, $x_2 = a$ and $x_3 = 2a$.

Particles 1 and 2, and particles 2 and 3, are connected by springs with spring constant μ that provide restoring forces when the respective particle separations deviate from their equilibrium values. In addition, particle 1 is connected to the origin by a spring with spring constant $16\mu/3$. The Lagrangian for the system is

$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{\eta}_1^2 + \dot{\eta}_2^2\right) - \frac{\mu}{2}\left(\frac{16}{3}x_1^2 + (\eta_1 - x_1)^2 + (\eta_2 - \eta_1)^2\right),$$

where the generalized coordinates are x_1, $\eta_1 = x_2 - a$ and $\eta_2 = x_3 - 2a$.

Write down the equations of motion. Show that the generalized coordinates can oscillate with a period $P = 2\pi/\omega$, where

$$\omega^2 = \frac{\mu}{3m},$$

and find the form of the corresponding normal mode in this case.

Paper 3, Section I

9E Classical Dynamics

(a) Show that the principal moments of inertia of a uniform circular cylinder of radius a, length h and mass M about its centre of mass are $I_1 = I_2 = M(a^2/4 + h^2/12)$ and $I_3 = Ma^2/2$, with the x_3 axis being directed along the length of the cylinder.

(b) Euler's equations governing the angular velocity $(\omega_1, \omega_2, \omega_3)$ of an arbitrary rigid body as viewed in the body frame are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3,$$

$$I_2 \frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1$$

and

$$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2.$$

Show that, for the cylinder of part (a), ω_3 is constant. Show further that, when $\omega_3 \neq 0$, the angular momentum vector precesses about the x_3 axis with angular velocity Ω given by

$$\Omega = \left(\frac{3a^2 - h^2}{3a^2 + h^2}\right)\omega_3.$$

Paper 4, Section I

9E Classical Dynamics

(a) A Hamiltonian system with n degrees of freedom has the Hamiltonian $H(\mathbf{p}, \mathbf{q})$, where $\mathbf{q} = (q_1, q_2, q_3, \ldots, q_n)$ are the coordinates and $\mathbf{p} = (p_1, p_2, p_3, \ldots, p_n)$ are the momenta.

A second Hamiltonian system has the Hamiltonian $G = G(\mathbf{p}, \mathbf{q})$. Neither H nor G contains the time explicitly. Show that the condition for $H(\mathbf{p}, \mathbf{q})$ to be invariant under the evolution of the coordinates and momenta generated by the Hamiltonian $G(\mathbf{p}, \mathbf{q})$ is that the Poisson bracket $[H, G]$ vanishes. Deduce that G is a constant of the motion for evolution under H.

Show that, when $G = \alpha \sum_{k=1}^{n} p_k$, where α is constant, the motion it generates is a translation of each q_k by an amount αt, while the corresponding p_k remains fixed. What do you infer is conserved when H is invariant under this transformation?

(b) When $n = 3$ and H is a function of $p_1^2 + p_2^2 + p_3^2$ and $q_1^2 + q_2^2 + q_3^2$ only, find $[H, L_i]$ when

$$L_i = \epsilon_{ijk} q_j p_k.$$

Paper 2, Section II

15E Classical Dynamics

A symmetric top of unit mass moves under the action of gravity. The Lagrangian is given by

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - gl\,\cos\theta\,,$$

where the generalized coordinates are the Euler angles (θ, ϕ, ψ), the principal moments of inertia are I_1 and I_3 and the distance from the centre of gravity of the top to the origin is l.

Show that $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ and $p_\phi = I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta$ are constants of the motion. Show further that, when $p_\phi = I_3\omega_3$, with $\omega_3 > 0$, the equation of motion for θ is

$$\frac{d^2\theta}{dt^2} = \frac{gl\sin\theta}{I_1}\left(1 - \frac{I_3^2\omega_3^2}{4I_1 gl\cos^4(\theta/2)}\right)\,.$$

Find the possible equilibrium values of θ in the two cases:

(i) $I_3^2\,\omega_3^2 \;>\; 4I_1\,gl$,

(ii) $I_3^2\,\omega_3^2 \;<\; 4I_1\,gl$.

By considering linear perturbations in the neighbourhoods of the equilibria in each case, find which are unstable and give expressions for the periods of small oscillations about the stable equilibria.

Paper 4, Section II

15E Classical Dynamics

The Hamiltonian for a particle of mass m, charge e and position vector $\mathbf{q} = (x, y, z)$, moving in an electromagnetic field, is given by

$$H(\mathbf{p}, \mathbf{q}, t) = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2,$$

where $\mathbf{A}(\mathbf{q}, t)$ is the vector potential. Write down Hamilton's equations and use them to derive the equations of motion for the charged particle.

Show that, when $\mathbf{A} = (-yB_0(z,t), 0, 0)$, there are solutions for which $p_x = 0$ and for which the particle motion is such that

$$\frac{d^2y}{dt^2} = -\Omega^2 y,$$

where $\Omega = eB_0/(mc)$. Show in addition that the Hamiltonian may be written as

$$H = \frac{m}{2}\left(\frac{dz}{dt}\right)^2 + E',$$

where

$$E' = \frac{m}{2}\left(\left(\frac{dy}{dt}\right)^2 + \Omega^2 y^2\right).$$

Assuming that B_0 is constant, find the action

$$I(E', B_0) = \frac{1}{2\pi}\oint m\left(\frac{dy}{dt}\right) dy$$

associated with the y motion.

It is now supposed that B_0 varies on a time-scale much longer than Ω^{-1} and thus is slowly varying. Show by applying the theory of adiabatic invariance that the motion in the z direction takes place under an effective potential and give an expression for it.

1/I/9A **Classical Dynamics**

The action for a system with generalized coordinates $q_i(t)$ for a time interval $[t_1, t_2]$ is given by

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt,$$

where L is the Lagrangian. The end point values $q_i(t_1)$ and $q_i(t_2)$ are fixed.

Derive Lagrange's equations from the principle of least action by considering the variation of S for all possible paths.

Define the momentum p_i conjugate to q_i. Derive a condition for p_i to be a constant of the motion.

A symmetric top moves under the action of a potential $V(\theta)$. The Lagrangian is given by

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - V,$$

where the generalized coordinates are the Euler angles (θ, ϕ, ψ) and the principal moments of inertia are I_1 and I_3.

Show that $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ is a constant of the motion and give expressions for two others. Show further that it is possible for the top to move with both θ and $\dot{\phi}$ constant provided these satisfy the condition

$$I_1 \dot{\phi}^2 \sin\theta\cos\theta - I_3 \omega_3 \dot{\phi} \sin\theta = \frac{dV}{d\theta}.$$

2/II/15B Classical Dynamics

A particle of mass m, charge e and position vector $\mathbf{r} = (x_1, x_2, x_3) \equiv \mathbf{q}$ moves in a magnetic field whose vector potential is $\mathbf{A}$. Its Hamiltonian is given by

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2m}\left(\mathbf{p} - e\frac{\mathbf{A}}{c}\right)^2 .$$

Write down Hamilton's equations and use them to derive the equations of motion for the charged particle.

Define the Poisson bracket $[F, G]$ for general $F(\mathbf{p}, \mathbf{q})$ and $G(\mathbf{p}, \mathbf{q})$. Show that for motion governed by the above Hamiltonian

$$[m\dot{x}_i, x_j] = -\delta_{ij}, \qquad \text{and} \qquad [m\dot{x}_i, m\dot{x}_j] = \frac{e}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right).$$

Consider the vector potential to be given by $\mathbf{A} = (0, 0, F(r))$, where $r = \sqrt{x_1^2 + x_2^2}$. Use Hamilton's equations to show that p_3 is constant and that circular motion at radius r with angular frequency Ω is possible provided that

$$\Omega^2 = -\left(p_3 - \frac{eF}{c}\right)\frac{e}{m^2 cr}\frac{dF}{dr}.$$

2/I/9A **Classical Dynamics**

A system of N particles $i = 1, 2, 3, \ldots, N$, with mass m_i, moves around a circle of radius a. The angle between the radius to particle i and a fixed reference radius is θ_i. The interaction potential for the system is

$$V = \frac{1}{2} k \sum_{j=1}^{N} (\theta_{j+1} - \theta_j)^2,$$

where k is a constant and $\theta_{N+1} = \theta_1 + 2\pi$.

The Lagrangian for the system is

$$L = \frac{1}{2} a^2 \sum_{j=1}^{N} m_j \dot{\theta}_j^2 - V.$$

Write down the equation of motion for particle i and show that the system is in equilibrium when the particles are equally spaced around the circle.

Show further that the system always has a normal mode of oscillation with zero frequency. What is the form of the motion associated with this?

Find all the frequencies and modes of oscillation when $N = 2$, $m_1 = km/a^2$ and $m_2 = 2km/a^2$, where m is a constant.

3/I/9E **Classical Dynamics**

Writing $\mathbf{x} = (p_1, p_2, p_3, \ldots, p_n, q_1, q_2, q_3, \ldots, q_n)$, Hamilton's equations may be written in the form

$$\dot{\mathbf{x}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{x}},$$

where the $2n \times 2n$ matrix

$$\mathbf{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and I and 0 denote the $n \times n$ unit and zero matrices respectively.

Explain what is meant by the statement that the transformation $\mathbf{x} \to \mathbf{y}$,

$$(p_1, p_2, p_3, \ldots, p_n, q_1, q_2, q_3, \ldots, q_n) \to (P_1, P_2, P_3, \ldots, P_n, Q_1, Q_2, Q_3, \ldots, Q_n),$$

is canonical, and show that the condition for this is that

$$\mathbf{J} = \mathcal{J}\mathbf{J}\mathcal{J}^T,$$

where $\mathcal{J}$ is the Jacobian matrix with elements

$$\mathcal{J}_{ij} = \frac{\partial y_i}{\partial x_j}.$$

Use this condition to show that for a system with $n = 1$ the transformation given by

$$P = p + 2q, \quad Q = \frac{1}{2}q - \frac{1}{4}p$$

is canonical.

4/II/15B **Classical Dynamics**

(a) A Hamiltonian system with n degrees of freedom has Hamiltonian $H = H(\mathbf{p}, \mathbf{q})$, where the coordinates $\mathbf{q} = (q_1, q_2, q_3, \ldots, q_n)$ and the momenta $\mathbf{p} = (p_1, p_2, p_3, \ldots, p_n)$ respectively.

Show from Hamilton's equations that when H does not depend on time explicitly, for any function $F = F(\mathbf{p}, \mathbf{q})$,

$$\frac{dF}{dt} = [F, H]\,,$$

where $[F, H]$ denotes the Poisson bracket.

For a system of N interacting vortices

$$H(\mathbf{p}, \mathbf{q}) = -\frac{\kappa}{4} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \ln\left[(p_i - p_j)^2 + (q_i - q_j)^2\right],$$

where κ is a constant. Show that the quantity defined by

$$F = \sum_{j=1}^{N} (q_j^2 + p_j^2)$$

is a constant of the motion.

(b) The action for a Hamiltonian system with one degree of freedom with $H = H(p, q)$ for which the motion is periodic is

$$I = \frac{1}{2\pi} \oint p(H, q) dq.$$

Show without assuming any specific form for H that the period of the motion T is given by

$$\frac{2\pi}{T} = \frac{dH}{dI}.$$

Suppose now that the system has a parameter that is allowed to vary slowly with time. Explain briefly what is meant by the statement that the action is an adiabatic invariant. Suppose that when this parameter is fixed, $H = 0$ when $I = 0$. Deduce that, if T decreases on an orbit with any I when the parameter is slowly varied, then H increases.

4/I/9B **Classical Dynamics**

(a) Show that the principal moments of inertia for an infinitesimally thin uniform rectangular sheet of mass M with sides of length a and b (with $b < a$) about its centre of mass are $I_1 = Mb^2/12$, $I_2 = Ma^2/12$ and $I_3 = M(a^2 + b^2)/12$.

(b) Euler's equations governing the angular velocity $(\omega_1, \omega_2, \omega_3)$ of the sheet as viewed in the body frame are

$$I_1 \frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3,$$

$$I_2 \frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1,$$

and

$$I_3 \frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2.$$

A possible solution of these equations is such that the sheet rotates with $\omega_1 = \omega_3 = 0$, and $\omega_2 = \Omega =$ constant.

By linearizing, find the equations governing small motions in the neighbourhood of this solution that have $(\omega_1, \omega_3) \neq 0$. Use these to show that there are solutions corresponding to instability such that ω_1 and ω_3 are both proportional to $\exp(\beta\Omega t)$, with $\beta = \sqrt{(a^2 - b^2)/(a^2 + b^2)}$.

1/I/9C **Classical Dynamics**

The action for a system with generalized coordinates, $q_i(t)$, for a time interval $[t_1, t_2]$ is given by

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt \ ,$$

where L is the Lagrangian, and where the end point values $q_i(t_1)$ and $q_i(t_2)$ are fixed at specified values. Derive Lagrange's equations from the principle of least action by considering the variation of S for all possible paths.

What is meant by the statement that a particular coordinate q_j is ignorable? Show that there is an associated constant of the motion, to be specified in terms of L.

A particle of mass m is constrained to move on the surface of a sphere of radius a under a potential, $V(\theta)$, for which the Lagrangian is given by

$$L = \frac{m}{2}\, a^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - V(\theta) \ .$$

Identify an ignorable coordinate and find the associated constant of the motion, expressing it as a function of the generalized coordinates. Evaluate the quantity

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

in terms of the same generalized coordinates, for this case. Is H also a constant of the motion? If so, why?

2/II/15C **Classical Dynamics**

(a) A Hamiltonian system with n degrees of freedom is described by the phase space coordinates $(q_1, q_2, ..., q_n)$ and momenta $(p_1, p_2, ..., p_n)$. Show that the phase-space volume element

$$d\tau \; = \; dq_1 dq_2 .....dq_n dp_1 dp_2 .....dp_n$$

is conserved under time evolution.

(b) The Hamiltonian, H , for the system in part (a) is independent of time. Show that if $F(q_1, ..., q_n, p_1, ..., p_n)$ is a constant of the motion, then the Poisson bracket $[F, H]$ vanishes. Evaluate $[F, H]$ when

$$F \; = \; \sum_{k=1}^{n} p_k$$

and

$$H \; = \; \sum_{k=1}^{n} p_k^2 + V(q_1, q_2, ..., q_n) \; ,$$

where the potential V depends on the q_k $(k = 1, 2, ..., n)$ only through quantities of the form $q_i - q_j$ for $i \neq j$.

(c) For a system with one degree of freedom, state what is meant by the transformation

$$(q, p) \rightarrow \big(Q(q, p),\; P(q, p)\big)$$

being canonical. Show that the transformation is canonical if and only if the Poisson bracket $[Q, P] = 1$.

2/I/9C **Classical Dynamics**

The Lagrangian for a particle of mass m and charge e moving in a magnetic field with position vector $\mathbf{r} = (x, y, z)$ is given by

$$L = \tfrac{1}{2} m \dot{\mathbf{r}}^2 + e \frac{\dot{\mathbf{r}} \cdot \mathbf{A}}{c} ,$$

where the vector potential $\mathbf{A}(\mathbf{r})$, which does not depend on time explicitly, is related to the magnetic field $\mathbf{B}$ through

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

Write down Lagrange's equations and use them to show that the equation of motion of the particle can be written in the form

$$m\ddot{\mathbf{r}} = e \frac{\dot{\mathbf{r}} \times \mathbf{B}}{c} .$$

Deduce that the kinetic energy, T, is constant.

When the magnetic field is of the form $\mathbf{B} = (0, 0, dF/dx)$ for some specified function $F(x)$, show further that

$$\dot{x}^2 = \frac{2T}{m} - \frac{\big(eF(x) + C\big)^2}{m^2 c^2} + D ,$$

where C and D are constants.

3/I/9C **Classical Dynamics**

A particle of mass m_1 is constrained to move in the horizontal (x, y) plane, around a circle of fixed radius r_1 whose centre is at the origin of a Cartesian coordinate system (x, y, z). A second particle of mass m_2 is constrained to move around a circle of fixed radius r_2 that also lies in a horizontal plane, but whose centre is at $(0, 0, a)$. It is given that the Lagrangian L of the system can be written as

$$L = \frac{m_1}{2} r_1^2 \dot{\phi}_1^2 + \frac{m_2}{2} r_2^2 \dot{\phi}_2^2 + \omega^2 r_1 r_2 \cos(\phi_2 - \phi_1) ,$$

using the particles' cylindrical polar angles ϕ_1 and ϕ_2 as generalized coordinates. Deduce the equations of motion and use them to show that $m_1 r_1^2 \dot{\phi}_1 + m_2 r_2^2 \dot{\phi}_2$ is constant, and that $\psi = \phi_2 - \phi_1$ obeys an equation of the form

$$\ddot{\psi} = -k^2 \sin\psi ,$$

where k is a constant to be determined.

Find two values of ψ corresponding to equilibria, and show that one of the two equilibria is stable. Find the period of small oscillations about the stable equilibrium.

4/II/15C **Classical Dynamics**

The Hamiltonian for an oscillating particle with one degree of freedom is

$$H = \frac{p^2}{2m} + V(q,\lambda) .$$

The mass m is a constant, and λ is a function of time t alone. Write down Hamilton's equations and use them to show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} .$$

Now consider a case in which λ is constant and the oscillation is exactly periodic. Denote the constant value of H in that case by E. Consider the quantity $I = (2\pi)^{-1}\oint p\,dq$, where the integral is taken over a single oscillation cycle. For any given function $V(q,\lambda)$ show that I can be expressed as a function of E and λ alone, namely

$$I = I(E,\lambda) = \frac{(2m)^{1/2}}{2\pi}\oint \big(E - V(q,\lambda)\big)^{1/2} dq ,$$

where the sign of the integrand alternates between the two halves of the oscillation cycle. Let τ be the period of oscillation. Show that the function $I(E,\lambda)$ has partial derivatives

$$\frac{\partial I}{\partial E} = \frac{\tau}{2\pi} \qquad \text{and} \qquad \frac{\partial I}{\partial \lambda} = -\frac{1}{2\pi}\oint \frac{\partial V}{\partial \lambda}\,dt .$$

You may assume without proof that $\partial/\partial E$ and $\partial/\partial\lambda$ may be taken inside the integral.

Now let λ change very slowly with time t, by a negligible amount during an oscillation cycle. Assuming that, to sufficient approximation,

$$\frac{d\langle H\rangle}{dt} = \frac{\partial\langle H\rangle}{\partial\lambda}\frac{d\lambda}{dt}$$

where $\langle H\rangle$ is the average value of H over an oscillation cycle, and that

$$\frac{dI}{dt} = \frac{\partial I}{\partial E}\frac{d\langle H\rangle}{dt} + \frac{\partial I}{\partial\lambda}\frac{d\lambda}{dt} ,$$

deduce that $dI/dt = 0$, carefully explaining your reasoning.

When

$$V(q,\lambda) = \lambda q^{2n}$$

with n a positive integer and λ positive, deduce that

$$\langle H\rangle = C\lambda^{1/(n+1)}$$

for slowly-varying λ, where C is a constant.

[*Do not try to solve Hamilton's equations. Rather, consider the form taken by* I*.*]

4/I/9C **Classical Dynamics**

(a) Show that the principal moments of inertia for the oblate spheroid of mass M defined by

$$\frac{(x_1^2+x_2^2)}{a^2}+\frac{x_3^2}{a^2(1-e^2)}\leqslant 1$$

are given by $(I_1, I_2, I_3) = \frac{2}{5}Ma^2\,(1-\frac{1}{2}e^2,\, 1-\frac{1}{2}e^2,\, 1)$. Here a is the semi-major axis and e is the eccentricity.

[*You may assume that a sphere of radius a has principal moments of inertia* $\frac{2}{5}Ma^2$.]

(b) The spheroid in part (a) rotates about an axis that is not a principal axis. Euler's equations governing the angular velocity $(\omega_1, \omega_2, \omega_3)$ as viewed in the body frame are

$$I_1\frac{d\omega_1}{dt} = (I_2 - I_3)\omega_2\omega_3 \ ,$$

$$I_2\frac{d\omega_2}{dt} = (I_3 - I_1)\omega_3\omega_1 \ ,$$

and

$$I_3\frac{d\omega_3}{dt} = (I_1 - I_2)\omega_1\omega_2 \ .$$

Show that ω_3 is constant. Show further that the angular momentum vector precesses around the x_3 axis with period

$$P = \frac{2\pi(2-e^2)}{e^2\omega_3} \ .$$

1/I/9C **Classical Dynamics**

Hamilton's equations for a system with n degrees of freedom can be written in vector form as

$$\dot{\mathbf{x}} = J \frac{\partial H}{\partial \mathbf{x}}$$

where $\mathbf{x} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ is a $2n$-vector and the $2n \times 2n$ matrix J takes the form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

where 1 is the $n \times n$ identity matrix. Derive the condition for a transformation of the form $x_i \to y_i(\mathbf{x})$ to be canonical. For a system with a single degree of freedom, show that the following transformation is canonical for all nonzero values of α:

$$Q = \tan^{-1}\left(\frac{\alpha q}{p}\right) , \qquad P = \tfrac{1}{2}\left(\alpha q^2 + \frac{p^2}{\alpha}\right) .$$

1/II/15C **Classical Dynamics**

(a) In the Hamiltonian framework, the action is defined as

$$S = \int \Big(p_a \dot{q}_a - H(q_a, p_a, t) \Big) \, dt .$$

Derive Hamilton's equations from the principle of least action. Briefly explain how the functional variations in this derivation differ from those in the derivation of Lagrange's equations from the principle of least action. Show that H is a constant of the motion whenever $\partial H/\partial t = 0$.

(b) What is the invariant quantity arising in Liouville's theorem? Does the theorem depend on assuming $\partial H/\partial t = 0$? State and prove Liouville's theorem for a system with a single degree of freedom.

(c) A particle of mass m bounces elastically along a perpendicular between two parallel walls a distance b apart. Sketch the path of a single cycle in phase space, assuming that the velocity changes discontinuously at the wall. Compute the action $I = \oint p \, dq$ as a function of the energy E and the constants m, b. Verify that the period of oscillation T is given by $T = dI/dE$. Suppose now that the distance b changes slowly. What is the relevant adiabatic invariant? How does E change as a function of b?

2/I/9C **Classical Dynamics**

Two point masses, each of mass m, are constrained to lie on a straight line and are connected to each other by a spring of force constant k. The left-hand mass is also connected to a wall on the left by a spring of force constant j. The right-hand mass is similarly connected to a wall on the right, by a spring of force constant ℓ, so that the potential energy is

$$V \;=\; \tfrac{1}{2}k(\eta_1 - \eta_2)^2 \;+\; \tfrac{1}{2}j\eta_1^2 \;+\; \tfrac{1}{2}\ell\eta_2^2 \;,$$

where η_i is the distance from equilibrium of the i^{th} mass. Derive the equations of motion. Find the frequencies of the normal modes.

3/I/9C **Classical Dynamics**

A pendulum of length ℓ oscillates in the xy plane, making an angle $\theta(t)$ with the vertical y axis. The pivot is attached to a moving lift that descends with constant acceleration a, so that the position of the bob is

$$x = \ell \sin\theta \;, \qquad y = \tfrac{1}{2}at^2 + \ell\cos\theta \;.$$

Given that the Lagrangian for an unconstrained particle is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy \;,$$

determine the Lagrangian for the pendulum in terms of the generalized coordinate θ. Derive the equation of motion in terms of θ. What is the motion when $a = g$?

Find the equilibrium configurations for arbitrary a. Determine which configuration is stable when

(i) $a < g$

and when

(ii) $a > g$.

3/II/15C **Classical Dynamics**

A particle of mass m is constrained to move on the surface of a sphere of radius ℓ. The Lagrangian is given in spherical polar coordinates by

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mg\ell\cos\theta \ ,$$

where gravity g is constant. Find the two constants of the motion.

The particle is projected horizontally with velocity v from a point whose depth below the centre is $\ell\cos\theta = D$. Find v such that the particle trajectory

(i) just grazes the horizontal equatorial plane $\theta = \pi/2$;

(ii) remains at depth D for all time t.

4/I/9C **Classical Dynamics**

Calculate the principal moments of inertia for a uniform cylinder, of mass M, radius R and height $2h$, about its centre of mass. For what height-to-radius ratio does the cylinder spin like a sphere?

1/I/9C Classical Dynamics

A particle of mass m_1 is constrained to move on a circle of radius r_1, centre $x = y = 0$ in a horizontal plane $z = 0$. A second particle of mass m_2 moves on a circle of radius r_2, centre $x = y = 0$ in a horizontal plane $z = c$. The two particles are connected by a spring whose potential energy is

$$V = \tfrac{1}{2}\omega^2 d^2,$$

where d is the distance between the particles. How many degrees of freedom are there? Identify suitable generalized coordinates and write down the Lagrangian of the system in terms of them.

1/II/15C Classical Dynamics

(i) The action for a system with generalized coordinates (q_a) is given by

$$S = \int_{t_1}^{t_2} L(q_a, \dot{q}_b)\, dt.$$

Derive Lagrange's equations from the principle of least action by considering all paths with fixed endpoints, $\delta q_a(t_1) = \delta q_a(t_2) = 0$.

(ii) A pendulum consists of a point mass m at the end of a light rod of length l. The pivot of the pendulum is attached to a mass M which is free to slide without friction along a horizontal rail. Choose as generalized coordinates the position x of the pivot and the angle θ that the pendulum makes with the vertical.

Write down the Lagrangian and derive the equations of motion.

Find the frequency of small oscillations around the stable equilibrium.

Now suppose that a force acts on the pivot causing it to travel with constant acceleration in the x-direction. Find the equilibrium angle θ of the pendulum.

2/I/9C **Classical Dynamics**

A rigid body has principal moments of inertia I_1, I_2 and I_3 and is moving under the action of no forces with angular velocity components $(\omega_1, \omega_2, \omega_3)$. Its motion is described by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0\,. \end{aligned}$$

Are the components of the angular momentum to be evaluated in the body frame or the space frame?

Now suppose that an asymmetric body is moving with constant angular velocity $(\Omega, 0, 0)$. Show that this motion is stable if and only if I_1 is the largest or smallest principal moment.

3/I/9C **Classical Dynamics**

Define the Poisson bracket $\{f, g\}$ between two functions $f(q_a, p_a)$ and $g(q_a, p_a)$ on phase space. If $f(q_a, p_a)$ has no explicit time dependence, and there is a Hamiltonian H, show that Hamilton's equations imply

$$\frac{df}{dt} = \{f, H\}\,.$$

A particle with position vector $\mathbf{x}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Compute $\{p_a, L_b\}$ and $\{L_a, L_b\}$.

3/II/15C **Classical Dynamics**

(i) A point mass m with position q and momentum p undergoes one-dimensional periodic motion. Define the action variable I in terms of q and p. Prove that an orbit of energy E has period

$$T = 2\pi \frac{dI}{dE}\,.$$

(ii) Such a system has Hamiltonian

$$H(q,p) = \frac{p^2+q^2}{\mu^2-q^2}\,,$$

where μ is a positive constant and $|q| < \mu$ during the motion. Sketch the orbits in phase space both for energies $E \gg 1$ and $E \ll 1$. Show that the action variable I is given in terms of the energy E by

$$I = \frac{\mu^2}{2}\,\frac{E}{\sqrt{E+1}}\,.$$

Hence show that for $E \gg 1$ the period of the orbit is $T \approx \frac{1}{2}\pi\mu^3/p_0$, where p_0 is the greatest value of the momentum during the orbit.

4/I/9C **Classical Dynamics**

Define a canonical transformation for a one-dimensional system with coordinates $(q,p) \to (Q,P)$. Show that if the transformation is canonical then $\{Q,P\} = 1$.

Find the values of constants α and β such that the following transformations are canonical:

(i) $Q = pq^\beta,\ P = \alpha q^{-1}\,.$

(ii) $Q = q^\alpha \cos(\beta p),\ P = q^\alpha \sin(\beta p).$