Part II

Asymptotic Methods

Paper 2, Section II

32E Asymptotic Methods

(a) Let $\phi_n(x) > 0$, for $n = 0, 1, 2, \ldots$, be a sequence of real functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < a\}$ which is an asymptotic sequence as $x \to x_0$.

(i) Let $\psi_0(x) = \phi_0(x)$ and

$$\psi_n(x) = \frac{\phi_{n-1}(x)\phi_n(x)}{\phi_{n-1}(x) + \phi_n(x)}, \qquad n = 1, 2, 3, \ldots$$

Show that $(\psi_n(x))_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.

Is it true that $\phi_n(x) \sim \psi_n(x)$ as $x \to x_0$ for every $n = 0, 1, 2, \ldots$? You should either give a proof or a counterexample.

(ii) Let $\chi_0(x) = \phi_0(x)$ and

$$\chi_n(x) = \sqrt{\phi_{n-1}(x)\phi_n(x)}, \qquad n = 1, 2, 3, \ldots$$

Show that $(\chi_n(x))_{n=0}^{\infty}$ is an asymptotic sequence as $x \to x_0$.

Is it true that $\phi_n(x) \sim \chi_n(x)$ as $x \to x_0$ for every $n = 0, 1, 2, \ldots$? You should either give a proof or a counterexample.

(b) Let $(\phi_n(x))_{n=0}^{\infty}$ and $(\psi_n(x))_{n=0}^{\infty}$ be two sequences of real functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < a\}$ which are asymptotic sequences as $x \to x_0$. Suppose that

$$\phi_n(x) \sim \psi_n(x) \quad \text{as} \quad x \to x_0,$$

for $n = 0, 1, 2, \ldots$, and that for some sequence of real numbers $(a_n)_{n=0}^{\infty}$ we have

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{as} \quad x \to x_0.$$

Does there necessarily exist a sequence of real numbers $(b_n)_{n=0}^{\infty}$ such that

$$f(x) \sim \sum_{n=0}^{\infty} b_n \psi_n(x) \quad \text{as} \quad x \to x_0\,?$$

You should either give a proof or a counterexample.

Paper 3, Section II

30E Asymptotic Methods

A stationary Schrödinger equation in one dimension has the form

$$\varepsilon^2 \frac{d^2\psi}{dx^2} = -(E - V(x))\psi\,, \quad \text{for} \quad x \in \mathbb{R}\,, \tag{$*$}$$

where $\varepsilon > 0$ is assumed to be very small and the potential $V(x)$ is given by

$$V(x) = \begin{cases} \frac{1}{4}|x| & \text{for} \quad |x| \leqslant 4 \\ \sqrt{|x|} - 1 & \text{for} \quad |x| \geqslant 4 \end{cases}.$$

The connection formula for the approximate energies E of bound states ψ in $(*)$ is

$$\frac{1}{\varepsilon}\int_a^b (E - V(x))^{1/2}\, dx = (n + \frac{1}{2})\pi\,. \tag{$**$}$$

(a) State the appropriate values of a, b and n.

(b) For $E \geqslant 0$ define

$$f(E) = \int_a^b (E - V(x))^{1/2}\, dx,$$

with a, b as in (a). Find and sketch f, and deduce that for each n and ε, $(**)$ has a unique solution $E = E_n$.

(c) Show that for n fixed and ε sufficiently small, E_n can be determined explicitly and give an expression for it.

(d) Show that as $n \to \infty$ with ε fixed, E_n satisfies

$$E_n \sim c n^{\alpha}\,,$$

and determine the values of c and α.

Paper 4, Section II

31E Asymptotic Methods

Justifying your steps carefully, use the method of steepest descent to find the first term in the asymptotic approximation of the function:

$$I(x) = \int_C \frac{1}{z^2 + 16} e^{x\cosh z}\, dz\,, \quad \text{as} \quad x \to \infty\,,$$

where $x \in \mathbb{R}$ and the integral is over the contour

$$C = \{z \in \mathbb{C} :\ z = p + iq\,, q = 2\arctan p\,,\ p \in \mathbb{R}\},$$

taken in the direction of increasing p.

Paper 2, Section II
32E Asymptotic Methods

(a) Let $n = 1, 2, \ldots$. Which of the following sequences are asymptotic and why?

(i) $\phi_n(x) = \ln(\cos(x^n))$ as $x \to 0$.

(ii) $\psi_n(x) = n^{1/x}$ as $x \to \infty$.

(iii) $\chi_n(x) = \sin(x^n)$ as $x \to \infty$.

(b) Let $\phi_n(x)$ and $\psi_n(x)$, for $n = 0, 1, 2, \ldots$, be two sequences of real positive functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < 1\}$ which are asymptotic sequences as $x \to x_0$.

For $n = 0, 1, 2, \ldots$, show that the sequence

$$\chi_n(x) = \sum_{k=0}^{n} \phi_k(x)\psi_{n-k}(x),$$

is an asymptotic sequence as $x \to x_0$.

Paper 3, Section II
30E Asymptotic Methods

(a) Derive the leading order term of the asymptotic expansion, as $x \to \infty$, for the integral

$$I(x) = \int_0^2 \ln t \; e^{x(t^3 - 2t^2 + t)}\, dt.$$

Justify your steps.

(b) The derivative of the Gamma function has the following integral representation

$$\Gamma'(z) = \int_0^\infty \frac{\ln t}{t} e^{z \ln t - t}\, dt \qquad \text{for} \quad \operatorname{Re} z > 0.$$

In what follows we assume $z \in \mathbb{R}$ and $z > 0$.

(i) Justify briefly why the integral converges. Explain why Laplace's method cannot be used directly to find the leading order behaviour of $\Gamma'(z)$ as $z \to \infty$.

(ii) Now perform the change of variables $t = zs$, then apply Laplace's method to show that

$$\Gamma'(z) \sim \sqrt{\frac{a}{z}} e^{z \ln z - z} \ln z \qquad \text{as} \quad z \to \infty,$$

for a real number a, which you should determine.

Paper 4, Section II

31E Asymptotic Methods

Consider the differential equation

$$x^2 y'' + xy' - \frac{1}{x^2} y = 0\,. \tag{$*$}$$

(i) What type of regular or singular point does equation $(*)$ have at $x = 0$?

(ii) For $x > 0$, find a transformation that maps equation $(*)$ to an equation of the form

$$u'' + q(x)u = 0 \tag{$\dagger$}$$

and compute $q(x)$.

(iii) Determine the leading asymptotic behaviour of the solution u of equation $(\dagger)$, as $x \to 0^+$, using the Liouville-Green method and justifying your assumptions at each stage.

(iv) Conclude from the above an asymptotic expansion of two linearly independent solutions of equation $(*)$, as $x \to 0^+$.

Paper 2, Section II

32A Asymptotic Methods

(a) Let $x(t)$ and $\phi_n(t)$, for $n = 0, 1, 2, \ldots$, be real-valued functions on $\mathbb{R}$.

(i) Define what it means for the sequence $\left\{\phi_n(t)\right\}_{n=0}^{\infty}$ to be an *asymptotic sequence* as $t \to \infty$.

(ii) Define what it means for $x(t)$ to have the *asymptotic expansion*

$$x(t) \sim \sum_{n=0}^{\infty} a_n \phi_n(t) \quad \text{as} \quad t \to \infty \,.$$

(b) Use the method of stationary phase to calculate the leading-order asymptotic approximation as $x \to \infty$ of

$$I(x) = \int_0^1 \sin\left(x(2t^4 - t^2)\right) dt \,.$$

[*You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi}\, e^{i\pi/4}$.]

(c) Use Laplace's method to calculate the leading-order asymptotic approximation as $x \to \infty$ of

$$J(x) = \int_0^1 \sinh\left(x(2t^4 - t^2)\right) dt \,.$$

[*In parts (b) and (c) you should include brief qualitative reasons for the origin of the leading-order contributions, but you do not need to give a formal justification.*]

Paper 3, Section II

30A Asymptotic Methods

(a) Carefully state Watson's lemma.

(b) Use the method of steepest descent and Watson's lemma to obtain an infinite asymptotic expansion of the function

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{-x(z^2 - 2iz)}}{1 - iz}\, dz \quad \text{as} \quad x \to \infty \,.$$

Paper 4, Section II

31A Asymptotic Methods

(a) Classify the nature of the point at ∞ for the ordinary differential equation

$$y'' + \frac{2}{x}y' + \left(\frac{1}{x} - \frac{1}{x^2}\right) y = 0\,. \qquad (*)$$

(b) Find a transformation from $(*)$ to an equation of the form

$$u'' + q(x)u = 0\,, \qquad (\dagger)$$

and determine $q(x)$.

(c) Given $u(x)$ satisfies $(\dagger)$, use the Liouville–Green method to find the first three terms in an asymptotic approximation as $x \to \infty$ for $u(x)$, verifying the consistency of any approximations made.

(d) Hence obtain corresponding asymptotic approximations as $x \to \infty$ of two linearly independent solutions $y(x)$ of $(*)$.

Paper 2, Section II

31D Asymptotic Methods

(a) Let $\delta > 0$ and $x_0 \in \mathbb{R}$. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a sequence of (real) functions that are nonzero for all x with $0 < |x - x_0| < \delta$, and let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonzero real numbers. For every $N = 0, 1, 2, \ldots$, the function $f(x)$ satisfies

$$f(x) - \sum_{n=0}^{N} a_n \phi_n(x) = o(\phi_N(x)) \,, \quad \text{as} \quad x \to x_0.$$

(i) Show that $\phi_{n+1}(x) = o(\phi_n(x))$, for all $n = 0, 1, 2, \ldots$; i.e., $\{\phi_n(x)\}_{n=0}^{\infty}$ is an asymptotic sequence.

(ii) Show that for any $N = 0, 1, 2, \ldots$, the functions $\phi_0(x), \phi_1(x), \ldots, \phi_N(x)$ are linearly independent on their domain of definition.

(b) Let

$$I(\varepsilon) = \int_0^{\infty} (1 + \varepsilon t)^{-2} e^{-(1+\varepsilon)t} \, dt \,, \quad \text{for } \varepsilon > 0 \,.$$

(i) Find an asymptotic expansion (not necessarily a power series) of $I(\varepsilon)$, as $\varepsilon \to 0^+$.

(ii) Find the first four terms of the expansion of $I(\varepsilon)$ into an asymptotic power series of ε, that is, with error $o(\varepsilon^3)$ as $\varepsilon \to 0^+$.

Paper 3, Section II

30D Asymptotic Methods

(a) Find the leading order term of the asymptotic expansion, as $x \to \infty$, of the integral

$$I(x) = \int_0^{3\pi} e^{(t + x \cos t)} \, dt \,.$$

(b) Find the first two leading nonzero terms of the asymptotic expansion, as $x \to \infty$, of the integral

$$J(x) = \int_0^{\pi} (1 - \cos t) e^{-x \ln(1+t)} \, dt \,.$$

Paper 4, Section II

31A Asymptotic Methods

Consider the differential equation

$$y'' - y' - \frac{2(x+1)}{x^2}\, y \;=\; 0\,. \tag{†}$$

(i) Classify what type of regularity/singularity equation (†) has at $x = \infty$.

(ii) Find a transformation that maps equation (†) to an equation of the form

$$u'' + q(x)u = 0\,. \tag{$*$}$$

(iii) Find the leading-order term of the asymptotic expansions of the solutions of equation ($*$), as $x \to \infty$, using the Liouville–Green method.

(iv) Derive the leading-order term of the asymptotic expansion of the solutions y of (†). Check that one of them is an exact solution for (†).

Paper 4, Section II

30A Asymptotic Methods

Consider, for small ϵ, the equation

$$\epsilon^2 \frac{d^2\psi}{dx^2} - q(x)\psi = 0. \qquad (*)$$

Assume that $(*)$ has bounded solutions with two turning points a, b where $b > a$, $q'(b) > 0$ and $q'(a) < 0$.

(a) Use the WKB approximation to derive the relationship

$$\frac{1}{\epsilon}\int_a^b |q(\xi)|^{1/2} d\xi = \left(n + \frac{1}{2}\right)\pi \quad \text{with} \quad n = 0, 1, 2, \cdots . \qquad (**)$$

[You may quote without proof any standard results or formulae from WKB theory.]

(b) In suitable units, the radial Schrödinger equation for a spherically symmetric potential given by $V(r) = -V_0/r$, for constant V_0, can be recast in the standard form $(*)$ as:

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + e^{2x}\left[\lambda - V(e^x) - \frac{\hbar^2}{2m}\left(l + \frac{1}{2}\right)^2 e^{-2x}\right]\psi = 0,$$

where $r = e^x$ and $\epsilon = \hbar/\sqrt{2m}$ is a small parameter.

Use result $(**)$ to show that the energies of the bound states (i.e $\lambda = -|\lambda| < 0$) are approximated by the expression:

$$E = -|\lambda| = -\frac{m}{2\hbar^2}\frac{V_0^2}{(n + l + 1)^2}.$$

[*You may use the result*

$$\int_a^b \frac{1}{r}\sqrt{(r-a)(b-r)}\, dr = (\pi/2)\left[\sqrt{b} - \sqrt{a}\right]^2 .]$$

Paper 3, Section II
30A Asymptotic Methods

(a) State *Watson's lemma* for the case when all the functions and variables involved are real, and use it to calculate the asymptotic approximation as $x \to \infty$ for the integral I, where

$$I = \int_0^\infty e^{-xt} \sin(t^2)\, dt.$$

(b) The Bessel function $J_\nu(z)$ of the first kind of order ν has integral representation

$$J_\nu(z) = \frac{1}{\Gamma(\nu + \frac{1}{2})\sqrt{\pi}} \left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{izt}(1-t^2)^{\nu - 1/2}\, dt\ ,$$

where Γ is the Gamma function, $\mathrm{Re}(\nu) > 1/2$ and z is in general a complex variable. The complex version of Watson's lemma is obtained by replacing x with the complex variable z, and is valid for $|z| \to \infty$ and $|\arg(z)| \leqslant \pi/2 - \delta < \pi/2$, for some δ such that $0 < \delta < \pi/2$. Use this version to derive an asymptotic expansion for $J_\nu(z)$ as $|z| \to \infty$. For what values of $\arg(z)$ is this approximation valid?

[*Hint: You may find the substitution* $t = 2\tau - 1$ *useful.*]

Paper 2, Section II
30A Asymptotic Methods

(a) Define formally what it means for a real valued function $f(x)$ to have an *asymptotic expansion* about x_0, given by

$$f(x) \sim \sum_{n=0}^{\infty} f_n (x - x_0)^n \quad \text{as} \quad x \to x_0\ .$$

Use this definition to prove the following properties.

(i) If both $f(x)$ and $g(x)$ have asymptotic expansions about x_0, then $h(x) = f(x) + g(x)$ also has an asymptotic expansion about x_0.

(ii) If $f(x)$ has an asymptotic expansion about x_0 and is integrable, then

$$\int_{x_0}^{x} f(\xi)\, d\xi \sim \sum_{n=0}^{\infty} \frac{f_n}{n+1}(x - x_0)^{n+1} \quad \text{as} \quad x \to x_0\ .$$

(b) Obtain, with justification, the first three terms in the asymptotic expansion as $x \to \infty$ of the complementary error function, $\mathrm{erfc}(x)$, defined as

$$\mathrm{erfc}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2}\, dt.$$

Paper 2, Section II

31B Asymptotic Methods

Given that $\int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$ obtain the value of $\lim_{R\to+\infty} \int_{-R}^{+R} e^{-itu^2} du$ for real positive t. Also obtain the value of $\lim_{R\to+\infty} \int_0^R e^{-itu^3} du$, for real positive t, in terms of $\Gamma(\frac{4}{3}) = \int_0^{+\infty} e^{-u^3} du$.

For $\alpha > 0$, $x > 0$, let

$$Q_\alpha(x) = \frac{1}{\pi} \int_0^\pi \cos\big(x \sin\theta - \alpha\theta\big)\, d\theta\,.$$

Find the leading terms in the asymptotic expansions as $x \to +\infty$ of (i) $Q_\alpha(x)$ with α fixed, and (ii) of $Q_x(x)$.

Paper 3, Section II

31B Asymptotic Methods

(a) Find the curves of steepest descent emanating from $t = 0$ for the integral

$$J_x(x) \;=\; \frac{1}{2\pi i} \int_C e^{x(\sinh t - t)}\, dt\,,$$

for $x > 0$ and determine the angles at which they meet at $t = 0$, and their asymptotes at infinity.

(b) An integral representation for the Bessel function $K_\nu(x)$ for real $x > 0$ is

$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{\nu h(t)}\, dt \quad , \qquad h(t) = t \;-\; \left(\frac{x}{\nu}\right) \cosh t \;\;.$$

Show that, as $\nu \to +\infty$, with x fixed,

$$K_\nu(x) \sim \left(\frac{\pi}{2\nu}\right)^{\frac{1}{2}} \left(\frac{2\nu}{ex}\right)^{\nu}\,.$$

Paper 4, Section II

31B Asymptotic Methods

Show that

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\,d\theta$$

is a solution to the equation

$$xy'' + y' - xy = 0\,,$$

and obtain the first two terms in the asymptotic expansion of $I_0(x)$ as $x \to +\infty$.

For $x > 0$, define a new dependent variable $w(x) = x^{\frac{1}{2}}y(x)$, and show that if y solves the preceding equation then

$$w'' + \left(\frac{1}{4x^2} - 1\right) w = 0\,.$$

Obtain the Liouville–Green approximate solutions to this equation for large positive x, and compare with your asymptotic expansion for $I_0(x)$ at the leading order.

Paper 2, Section II

29E Asymptotic Methods

Consider the function

$$f_\nu(x) \equiv \frac{1}{2\pi} \int_C \exp\big[-ix\sin z + i\nu z\big]\, dz,$$

where the contour C is the boundary of the half-strip $\{z : -\pi < \mathrm{Re}\, z < \pi \text{ and } \mathrm{Im}\, z > 0\}$, taken anti-clockwise.

Use integration by parts and the method of stationary phase to:

(i) Obtain the leading term for $f_\nu(x)$ coming from the vertical lines $z = \pm\pi + iy$ $(0 < y < +\infty)$ for large $x > 0$.

(ii) Show that the leading term in the asymptotic expansion of the function $f_\nu(x)$ for large positive x is

$$\sqrt{\frac{2}{\pi x}} \cos\Big(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\Big),$$

and obtain an estimate for the remainder as $O(x^{-a})$ for some a to be determined.

Paper 3, Section II

29E Asymptotic Methods

Consider the integral representation for the modified Bessel function

$$I_0(x) = \frac{1}{2\pi i} \oint_C t^{-1} \exp\left[\frac{ix}{2}\left(t - \frac{1}{t}\right)\right] dt,$$

where C is a simple closed contour containing the origin, taken anti-clockwise.

Use the method of steepest descent to determine the full asymptotic expansion of $I_0(x)$ for large real positive x.

Paper 4, Section II

30E Asymptotic Methods

Consider solutions to the equation

$$\frac{d^2y}{dx^2} = \left(\frac{1}{4} + \frac{\mu^2 - \frac{1}{4}}{x^2}\right) y \qquad (\star)$$

of the form

$$y(x) \;=\; \exp\Big[S_0(x) + S_1(x) + S_2(x) + \ldots\Big]\,,$$

with the assumption that, for large positive x, the function $S_j(x)$ is small compared to $S_{j-1}(x)$ for all $j = 1, 2 \ldots$

Obtain equations for the $S_j(x)$, $j = 0, 1, 2 \ldots$, which are formally equivalent to $(\star)$. Solve explicitly for S_0 and S_1. Show that it is consistent to assume that $S_j(x) = c_j x^{-(j-1)}$ for some constants c_j. Give a recursion relation for the c_j.

Deduce that there exist two linearly independent solutions to $(\star)$ with asymptotic expansions as $x \to +\infty$ of the form

$$y_{\pm}(x) \sim e^{\pm x/2}\Big(1 + \sum_{j=1}^{\infty} A_j^{\pm} x^{-j}\Big)\,.$$

Determine a recursion relation for the $A_j^{\pm}$. Compute $A_1^{\pm}$ and $A_2^{\pm}$.

Paper 3, Section II

28C Asymptotic Methods

Consider the integral

$$I(x) = \int_0^1 \frac{1}{\sqrt{t(1-t)}} \exp[ixf(t)]\,dt$$

for real $x > 0$, where $f(t) = t^2 + t$. Find and sketch, in the complex t-plane, the paths of steepest descent through the endpoints $t = 0$ and $t = 1$ and through any saddle point(s). Obtain the leading order term in the asymptotic expansion of $I(x)$ for large positive x. What is the order of the next term in the expansion? Justify your answer.

Paper 2, Section II

29C Asymptotic Methods

What is meant by the asymptotic relation

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0\,,\ \mathrm{Arg}\,(z - z_0) \in (\theta_0, \theta_1)\,?$$

Show that

$$\sinh(z^{-1}) \sim \frac{1}{2}\exp(z^{-1}) \quad \text{as} \quad z \to 0\,,\ \mathrm{Arg}\,z \in (-\pi/2, \pi/2)\,,$$

and find the corresponding result in the sector $\mathrm{Arg}\,z \in (\pi/2, 3\pi/2)$.

What is meant by the asymptotic expansion

$$f(z) \sim \sum_{j=0}^{\infty} c_j (z - z_0)^j \quad \text{as} \quad z \to z_0\,,\ \mathrm{Arg}\,(z - z_0) \in (\theta_0, \theta_1)\,?$$

Show that the coefficients $\{c_j\}_{j=0}^{\infty}$ are determined uniquely by f. Show that if f is analytic at z_0, then its Taylor series is an asymptotic expansion for f as $z \to z_0$ (for any $\mathrm{Arg}\,(z - z_0)$).

Show that

$$u(x,t) = \int_{-\infty}^{\infty} \exp(-ik^2 t + ikx)\, f(k)\, dk$$

defines a solution of the equation $i\,\partial_t u + \partial_x^2 u = 0$ for any smooth and rapidly decreasing function f. Use the method of stationary phase to calculate the leading-order behaviour of $u(\lambda t, t)$ as $t \to +\infty$, for fixed λ.

Paper 4, Section II

29C Asymptotic Methods

Consider the equation

$$\epsilon^2 \frac{d^2y}{dx^2} = Q(x)y\,, \tag{1}$$

where $\epsilon > 0$ is a small parameter and $Q(x)$ is smooth. Search for solutions of the form

$$y(x) = \exp\left[\frac{1}{\epsilon}\Big(S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots\Big)\right],$$

and, by equating powers of ϵ, obtain a collection of equations for the $\{S_j(x)\}_{j=0}^{\infty}$ which is formally equivalent to (1). By solving explicitly for S_0 and S_1 derive the Liouville–Green approximate solutions $y^{LG}(x)$ to (1).

For the case $Q(x) = -V(x)$, where $V(x) \geqslant V_0$ and V_0 is a positive constant, consider the eigenvalue problem

$$\frac{d^2y}{dx^2} + E\,V(x)y = 0\,, \qquad y(0) = y(\pi) = 0\,. \tag{2}$$

Show that any eigenvalue E is necessarily positive. Solve the eigenvalue problem exactly when $V(x) = V_0$.

Obtain Liouville–Green approximate eigenfunctions $y_n^{LG}(x)$ for (2) with $E \gg 1$, and give the corresponding Liouville–Green approximation to the eigenvalues E_n^{LG}. Compare your results to the exact eigenvalues and eigenfunctions in the case $V(x) = V_0$, and comment on this.

Paper 4, Section II

27C Asymptotic Methods

Consider the ordinary differential equation

$$\frac{d^2u}{dz^2} + f(z)\frac{du}{dz} + g(z)u = 0\,,$$

where

$$f(z) \sim \sum_{m=0}^{\infty} \frac{f_m}{z^m}\,, \qquad g(z) \sim \sum_{m=0}^{\infty} \frac{g_m}{z^m}\,, \qquad z \to \infty\,,$$

and f_m, g_m are constants. Look for solutions in the asymptotic form

$$u(z) = e^{\lambda z} z^{\mu} \left[1 + \frac{a}{z} + \frac{b}{z^2} + O\left(\frac{1}{z^3}\right) \right]\,, \qquad z \to \infty\,,$$

and determine λ in terms of (f_0, g_0), as well as μ in terms of (λ, f_0, f_1, g_1).

Deduce that the Bessel equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) u = 0\,,$$

where ν is a complex constant, has two solutions of the form

$$u^{(1)}(z) = \frac{e^{iz}}{z^{1/2}} \left[1 + \frac{a^{(1)}}{z} + O\left(\frac{1}{z^2}\right) \right]\,, \quad z \to \infty\,,$$

$$u^{(2)}(z) = \frac{e^{-iz}}{z^{1/2}} \left[1 + \frac{a^{(2)}}{z} + O\left(\frac{1}{z^2}\right) \right]\,, \quad z \to \infty\,,$$

and determine $a^{(1)}$ and $a^{(2)}$ in terms of ν.

Can the above asymptotic expansions be valid for all $\arg(z)$, or are they valid only in certain domains of the complex z-plane? Justify your answer briefly.

Paper 3, Section II
27C Asymptotic Methods

Show that

$$\int_0^1 e^{ikt^3} dt = I_1 - I_2\,, \qquad k > 0\,,$$

where I_1 is an integral from 0 to ∞ along the line $\arg(z) = \frac{\pi}{6}$ and I_2 is an integral from 1 to ∞ along a steepest-descent contour C which you should determine.

By employing in the integrals I_1 and I_2 the changes of variables $u = -iz^3$ and $u = -i(z^3 - 1)$, respectively, compute the first two terms of the large k asymptotic expansion of the integral above.

Paper 1, Section II
27C Asymptotic Methods

(a) State the integral expression for the gamma function $\Gamma(z)$, for $\mathrm{Re}(z) > 0$, and express the integral

$$\int_0^\infty t^{\gamma-1} e^{it} dt\,, \qquad 0 < \gamma < 1\,,$$

in terms of $\Gamma(\gamma)$. Explain why the constraints on γ are necessary.

(b) Show that

$$\int_0^\infty \frac{e^{-kt^2}}{(t^2+t)^{\frac{1}{4}}} dt \sim \sum_{m=0}^\infty \frac{a_m}{k^{\alpha+\beta m}}\,, \qquad k \to \infty\,,$$

for some constants a_m, α and β. Determine the constants α and β, and express a_m in terms of the gamma function.

State without proof the basic result needed for the rigorous justification of the above asymptotic formula.

[You may use the identity:

$$(1+z)^\alpha = \sum_{m=0}^\infty c_m z^m, \qquad c_m = \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha+1-m)}\,, \quad |z| < 1\,.]$$

Paper 4, Section II

31C Asymptotic Methods

Derive the leading-order Liouville–Green (or WKBJ) solution for $\epsilon \ll 1$ to the ordinary differential equation

$$\epsilon^2 \frac{d^2 f}{dy^2} + \Phi(y) f = 0\,,$$

where $\Phi(y) > 0$.

The function $f(y;\epsilon)$ satisfies the ordinary differential equation

$$\epsilon^2 \frac{d^2 f}{dy^2} + \left(1 + \frac{1}{y} - \frac{2\epsilon^2}{y^2}\right) f = 0\,, \tag{1}$$

subject to the boundary condition $f''(0) = 2$. Show that the Liouville–Green solution of (1) for $\epsilon \ll 1$ takes the asymptotic forms

$$f \sim \alpha_1 y^{\frac{1}{4}} \exp(2i\sqrt{y}/\epsilon) + \alpha_2 y^{\frac{1}{4}} \exp(-2i\sqrt{y}/\epsilon) \qquad \text{for} \quad \epsilon^2 \ll y \ll 1$$

and

$$f \sim B \cos\left[\theta_2 + (y + \log\sqrt{y})/\epsilon\right] \qquad \text{for} \quad y \gg 1\,,$$

where α_1, α_2, B and θ_2 are constants.

$\left[\textit{Hint : You may assume that } \int_0^y \sqrt{1+u^{-1}}\, du = \sqrt{y(1+y)} + \sinh^{-1}\sqrt{y}\,. \right]$

Explain, showing the relevant change of variables, why the leading-order asymptotic behaviour for $0 \leqslant y \ll 1$ can be obtained from the reduced equation

$$\frac{d^2 f}{dx^2} + \left(\frac{1}{x} - \frac{2}{x^2}\right) f = 0\,. \tag{2}$$

The unique solution to (2) with $f''(0) = 2$ is $f = x^{1/2} J_3(2x^{1/2})$, where the Bessel function $J_3(z)$ is known to have the asymptotic form

$$J_3(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{7\pi}{4}\right) \text{ as } z \to \infty\,.$$

Hence find the values of α_1 and α_2.

Paper 3, Section II

31C Asymptotic Methods

(a) Find the Stokes ray for the function $f(z)$ as $z \to 0$ with $0 < \arg z < \pi$, where

$$f(z) = \sinh(z^{-1}).$$

(b) Describe how the leading-order asymptotic behaviour as $x \to \infty$ of

$$I(x) = \int_a^b f(t)e^{ixg(t)}dt$$

may be found by the method of stationary phase, where f and g are real functions and the integral is taken along the real line. You should consider the cases for which:

(i) $g'(t)$ is non-zero in $[a, b)$ and has a simple zero at $t = b$.

(ii) $g'(t)$ is non-zero apart from having one simple zero at $t = t_0$, where $a < t_0 < b$.

(iii) $g'(t)$ has more than one simple zero in (a, b) with $g'(a) \neq 0$ and $g'(b) \neq 0$.

Use the method of stationary phase to find the leading-order asymptotic form as $x \to \infty$ of

$$J(x) = \int_0^1 \cos\left(x(t^4 - t^2)\right) dt.$$

[*You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi}e^{i\pi/4}$.]

Paper 1, Section II

31C Asymptotic Methods

(a) Consider the integral

$$I(k) = \int_0^\infty f(t)e^{-kt}\, dt, \quad k > 0.$$

Suppose that $f(t)$ possesses an asymptotic expansion for $t \to 0^+$ of the form

$$f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}, \qquad \alpha > -1, \;\; \beta > 0,$$

where a_n are constants. Derive an asymptotic expansion for $I(k)$ as $k \to \infty$ in the form

$$I(k) \sim \sum_{n=0}^\infty \frac{A_n}{k^{\gamma+\beta n}},$$

giving expressions for A_n and γ in terms of α, β, n and the gamma function. Hence establish the asymptotic approximation as $k \to \infty$

$$I_1(k) = \int_0^1 e^{kt} t^{-a}(1-t^2)^{-b} dt \sim 2^{-b}\Gamma(1-b)e^k k^{b-1}\left(1 + \frac{(a+b/2)(1-b)}{k}\right),$$

where $a < 1, b < 1$.

(b) Using Laplace's method, or otherwise, find the leading-order asymptotic approximation as $k \to \infty$ for

$$I_2(k) = \int_0^\infty e^{-(2k^2/t + t^2/k)}\, dt\,.$$

[*You may assume that* $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt$ *for* $\mathrm{Re}\, z > 0$,

and that $\int_{-\infty}^\infty e^{-qt^2} dt = \sqrt{\pi/q}$ *for* $q > 0$.]

Paper 4, Section II
31B Asymptotic Methods

Show that the equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left(\frac{1}{x^2} - 1\right) y = 0$$

has an irregular singular point at infinity. Using the Liouville–Green method, show that one solution has the asymptotic expansion

$$y(x) \sim \frac{1}{x} e^x \left(1 + \frac{1}{2x} + \ldots\right)$$

as $x \to \infty$.

Paper 3, Section II
31B Asymptotic Methods

Let

$$I(x) = \int_0^\pi f(t) e^{ix\psi(t)} dt\,,$$

where $f(t)$ and $\psi(t)$ are smooth, and $\psi'(t) \neq 0$ for $t > 0$; also $f(0) \neq 0$, $\psi(0) = a$, $\psi'(0) = \psi''(0) = 0$ and $\psi'''(0) = 6b > 0$. Show that, as $x \to +\infty$,

$$I(x) \sim f(0) e^{i(xa+\pi/6)} \left(\frac{1}{27bx}\right)^{1/3} \Gamma(1/3)\ .$$

Consider the Bessel function

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\, dt\ .$$

Show that, as $n \to +\infty$,

$$J_n(n) \sim \frac{\Gamma(1/3)}{\pi} \frac{1}{(48)^{1/6}} \frac{1}{n^{1/3}}\,.$$

Paper 1, Section II

31B Asymptotic Methods

Suppose $\alpha > 0$. Define what it means to say that

$$F(x) \sim \frac{1}{\alpha x} \sum_{n=0}^{\infty} n! \left(\frac{-1}{\alpha x} \right)^n$$

is an asymptotic expansion of $F(x)$ as $x \to \infty$. Show that $F(x)$ has no other asymptotic expansion in inverse powers of x as $x \to \infty$.

To estimate the value of $F(x)$ for large x, one may use an *optimal truncation* of the asymptotic expansion. Explain what is meant by this, and show that the error is an exponentially small quantity in x.

Derive an integral respresentation for a function $F(x)$ with the above asymptotic expansion.

Paper 4, Section II

31B Asymptotic Methods

The stationary Schrödinger equation in one dimension has the form

$$\epsilon^2 \frac{d^2\psi}{dx^2} = -(E - V(x))\,\psi,$$

where ϵ can be assumed to be small. Using the Liouville–Green method, show that two approximate solutions in a region where $V(x) < E$ are

$$\psi(x) \sim \frac{1}{(E - V(x))^{1/4}} \exp\left\{ \pm \frac{i}{\epsilon} \int_c^x (E - V(x'))^{1/2} dx' \right\},$$

where c is suitably chosen.

Without deriving connection formulae in detail, describe how one obtains the condition

$$\frac{1}{\epsilon} \int_a^b (E - V(x'))^{1/2}\, dx' = \left(n + \frac{1}{2}\right)\pi \qquad (*)$$

for the approximate energies E of bound states in a smooth potential well. State the appropriate values of a, b and n.

Estimate the range of n for which $(*)$ gives a good approximation to the true bound state energies in the cases

(i) $V(x) = |x|$,

(ii) $V(x) = x^2 + \lambda x^6$ with λ small and positive,

(iii) $V(x) = x^2 - \lambda x^6$ with λ small and positive.

Paper 3, Section II

31B Asymptotic Methods

Find the two leading terms in the asymptotic expansion of the Laplace integral

$$I(x) = \int_0^1 f(t) e^{xt^4}\, dt$$

as $x \to \infty$, where $f(t)$ is smooth and positive on $[0, 1]$.

Paper 1, Section II

31B Asymptotic Methods

What precisely is meant by the statement that

$$f(x) \sim \sum_{n=0}^{\infty} d_n\, x^n \qquad (*)$$

as $x \to 0$?

Consider the Stieltjes integral

$$I(x) = \int_1^\infty \frac{\rho(t)}{1+xt} dt\,,$$

where $\rho(t)$ is bounded and decays rapidly as $t \to \infty$, and $x > 0$. Find an asymptotic series for $I(x)$ of the form $(*)$, as $x \to 0$, and prove that it has the asymptotic property.

In the case that $\rho(t) = e^{-t}$, show that the coefficients d_n satisfy the recurrence relation

$$d_n = (-1)^n \frac{1}{e} - n\, d_{n-1} \qquad (n \geqslant 1)$$

and that $d_0 = \dfrac{1}{e}$. Hence find the first three terms in the asymptotic series.

Paper 1, Section II
31A Asymptotic Methods

A function $f(n)$, defined for positive integer n, has an asymptotic expansion for large n of the following form:

$$f(n) \sim \sum_{k=0}^{\infty} a_k \frac{1}{n^{2k}}, \qquad n \to \infty. \qquad (*)$$

What precisely does this mean?

Show that the integral

$$I(n) = \int_0^{2\pi} \frac{\cos nt}{1+t^2}\, dt$$

has an asymptotic expansion of the form $(*)$. [The Riemann–Lebesgue lemma may be used without proof.] Evaluate the coefficients a_0, a_1 and a_2.

Paper 3, Section II
31A Asymptotic Methods

Let

$$I_0 = \int_{C_0} e^{x\phi(z)} dz\,,$$

where $\phi(z)$ is a complex analytic function and C_0 is a steepest descent contour from a simple saddle point of $\phi(z)$ at z_0. Establish the following leading asymptotic approximation, for large real x:

$$I_0 \sim i\sqrt{\frac{\pi}{2\phi''(z_0)x}}\, e^{x\phi(z_0)}\,.$$

Let n be a positive integer, and let

$$I = \int_C e^{-t^2 - 2n \ln t}\, dt\,,$$

where C is a contour in the upper half t-plane connecting $t = -\infty$ to $t = \infty$, and $\ln t$ is real on the positive t-axis with a branch cut along the negative t-axis. Using the method of steepest descent, find the leading asymptotic approximation to I for large n.

Paper 4, Section II

31A Asymptotic Methods

Determine the range of the integer n for which the equation

$$\frac{d^2y}{dz^2} = z^n y$$

has an essential singularity at $z = \infty$.

Use the Liouville–Green method to find the leading asymptotic approximation to two independent solutions of

$$\frac{d^2y}{dz^2} = z^3 y\,,$$

for large $|z|$. Find the Stokes lines for these approximate solutions. For what range of $\arg z$ is the approximate solution which decays exponentially along the positive z-axis an asymptotic approximation to an exact solution with this exponential decay?

Paper 1, Section II

31C Asymptotic Methods

For $\lambda > 0$ let

$$I(\lambda) = \int_0^b f(x)\, \mathrm{e}^{-\lambda x}\, dx\,, \qquad \text{with} \quad 0 < b < \infty\,.$$

Assume that the function $f(x)$ is continuous on $0 < x \leqslant b$, and that

$$f(x) \sim x^{\alpha} \sum_{n=0}^{\infty} a_n\, x^{n\beta},$$

as $x \to 0_+$, where $\alpha > -1$ and $\beta > 0$.

(a) Explain briefly why in this case straightforward partial integrations in general cannot be applied for determining the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$.

(b) Derive with proof an asymptotic expansion for $I(\lambda)$ as $\lambda \to \infty$.

(c) For the function

$$B(s,t) = \int_0^1 u^{s-1}\, (1-u)^{t-1}\, du\,, \qquad s, t > 0\,,$$

obtain, using the substitution $u = e^{-x}$, the first two terms in an asymptotic expansion as $s \to \infty$. What happens as $t \to \infty$?

[*Hint: The following formula may be useful*

$$\Gamma(y) = \int_0^{\infty} x^{y-1}\, \mathrm{e}^{-x}\, dt\,, \qquad \text{for} \quad x > 0\,.$$

]

Paper 3, Section II

31C Asymptotic Methods

Consider the ordinary differential equation

$$y'' = (|x| - E)\, y\,,$$

subject to the boundary conditions $y(\pm\infty) = 0$. Write down the general form of the Liouville-Green solutions for this problem for $E > 0$ and show that asymptotically the eigenvalues E_n, $n \in \mathbb{N}$ and $E_n < E_{n+1}$, behave as $E_n = \mathrm{O}(n^{2/3})$ for large n.

Paper 4, Section II

31C Asymptotic Methods

(a) Consider for $\lambda > 0$ the Laplace type integral

$$I(\lambda) = \int_a^b f(t)\,\mathrm{e}^{-\lambda\phi(t)}\,dt\,,$$

for some finite $a, b \in \mathbb{R}$ and smooth, real-valued functions $f(t), \phi(t)$. Assume that the function $\phi(t)$ has a single minimum at $t = c$ with $a < c < b$. Give an account of Laplace's method for finding the leading order asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$ and briefly discuss the difference if instead $c = a$ or $c = b$, i.e. when the minimum is attained at the boundary.

(b) Determine the leading order asymptotic behaviour of

$$I(\lambda) = \int_{-2}^{1} \cos t\,\mathrm{e}^{-\lambda t^2}\,dt\,, \qquad (*)$$

as $\lambda \to \infty$.

(c) Determine also the leading order asymptotic behaviour when $\cos t$ is replaced by $\sin t$ in $(*)$.

Paper 1, Section II

31A Asymptotic Methods

Consider the integral

$$I(\lambda) = \int_0^A \mathrm{e}^{-\lambda t} f(t)\, dt, \qquad A > 0,$$

in the limit $\lambda \to \infty$, given that $f(t)$ has the asymptotic expansion

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{n\beta}$$

as $t \to 0_+$, where $\beta > 0$. State Watson's lemma.

Now consider the integral

$$J(\lambda) = \int_a^b \mathrm{e}^{\lambda\phi(t)} F(t)\, dt,$$

where $\lambda \gg 1$ and the real function $\phi(t)$ has a unique maximum in the interval $[a, b]$ at c, with $a < c < b$, such that

$$\phi'(c) = 0\,,\ \phi''(c) < 0.$$

By making a monotonic change of variable from t to a suitable variable ζ (Laplace's method), or otherwise, deduce the existence of an asymptotic expansion for $J(\lambda)$ as $\lambda \to \infty$. Derive the leading term

$$J(\lambda) \sim \mathrm{e}^{\lambda\phi(c)} F(c) \left(\frac{2\pi}{\lambda|\phi''(c)|}\right)^{\frac{1}{2}}.$$

The gamma function is defined for $x > 0$ by

$$\Gamma(x+1) = \int_0^\infty \exp\left(x \log t - t\right)\, dt.$$

By means of the substitution $t = xs$, or otherwise, deduce Stirling's formula

$$\Gamma(x+1) \sim x^{(x+\frac{1}{2})}\, \mathrm{e}^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \cdots\right)$$

as $x \to \infty$.

Paper 3, Section II

31A Asymptotic Methods

Consider the contour-integral representation

$$J_0(x) = \mathrm{Re}\, \frac{1}{i\pi} \int_C e^{ix\cosh t}\, dt$$

of the Bessel function J_0 for real x, where C is any contour from $-\infty - \frac{i\pi}{2}$ to $+\infty + \frac{i\pi}{2}$.

Writing $t = u + iv$, give in terms of the real quantities u, v the equation of the steepest-descent contour from $-\infty - \frac{i\pi}{2}$ to $+\infty + \frac{i\pi}{2}$ which passes through $t = 0$.

Deduce the leading term in the asymptotic expansion of $J_0(x)$, valid as $x \to \infty$

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\, \cos\left(x - \frac{\pi}{4}\right).$$

Paper 4, Section II

31A Asymptotic Methods

The differential equation

$$f'' = Q(x) f \qquad (*)$$

has a singular point at $x = \infty$. Assuming that $Q(x) > 0$, write down the Liouville–Green lowest approximations $f_\pm(x)$ for $x \to \infty$, with $f_-(x) \to 0$.

The Airy function $\mathrm{Ai}(x)$ satisfies $(*)$ with

$$Q(x) = x,$$

and $\mathrm{Ai}(x) \to 0$ as $x \to \infty$. Writing

$$\mathrm{Ai}(x) = w(x) f_-(x),$$

show that $w(x)$ obeys

$$x^2 w'' - \left(2x^{5/2} + \frac{1}{2}x\right) w' + \frac{5}{16} w = 0\,.$$

Derive the expansion

$$w \sim c\left(1 - \frac{5}{48} x^{-3/2}\right) \quad \text{as} \quad x \to \infty,$$

where c is a constant.

1/II/30A **Asymptotic Methods**

Obtain an expression for the nth term of an asymptotic expansion, valid as $\lambda \to \infty$, for the integral

$$I(\lambda) = \int_0^1 t^{2\alpha} e^{-\lambda(t^2+t^3)}\, dt \qquad (\alpha > -1/2).$$

Estimate the value of n for the term of least magnitude.

Obtain the first two terms of an asymptotic expansion, valid as $\lambda \to \infty$, for the integral

$$J(\lambda) = \int_0^1 t^{2\alpha} e^{-\lambda(t^2-t^3)}\, dt \qquad (-1/2 < \alpha < 0)\,.$$

[*Hint:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt\,.]$$

[Stirling's formula may be quoted.]

3/II/30A **Asymptotic Methods**

Describe how the leading-order approximation may be found by the method of stationary phase of

$$I(\lambda) = \int_a^b f(t)\, \exp\big(i\lambda\, g(t)\big)\, dt,$$

for $\lambda \gg 1$, where λ, f and g are real. You should consider the cases for which:

(a) $g'(t)$ has one simple zero at $t = t_0$, where $a < t_0 < b$;

(b) $g'(t)$ has more than one simple zero in the region $a < t < b$; and

(c) $g'(t)$ has only a simple zero at $t = b$.

What is the order of magnitude of $I(\lambda)$ if $g'(t)$ is non zero for $a \leqslant t \leqslant b$?

Use the method of stationary phase to find the leading-order approximation for $\lambda \gg 1$ to

$$J(\lambda) = \int_0^1 \sin\left(\lambda\left(t^3 - t\right)\right) dt.$$

[*Hint:*

$$\int_{-\infty}^{\infty} \exp\left(iu^2\right) du = \sqrt{\pi} e^{i\pi/4}\,.]$$

4/II/31A **Asymptotic Methods**

The Bessel equation of order n is

$$z^2 y'' + z y' + \left(z^2 - n^2\right) y = 0 . \qquad (1)$$

Here, n is taken to be an integer, with $n \geqslant 0$. The transformation $w(z) = z^{\frac{1}{2}} y(z)$ converts (1) to the form

$$w'' + q(z)\, w = 0 , \qquad (2)$$

where

$$q(z) = 1 - \frac{\left(n^2 - \frac{1}{4}\right)}{z^2} .$$

Find two linearly independent solutions of the form

$$w = e^{sz} \sum_{k=0}^{\infty} c_k z^{\rho - k} , \qquad (3)$$

where c_k are constants, with $c_0 \neq 0$, and s and ρ are to be determined. Find recurrence relationships for the c_k.

Find the first two terms of two linearly independent Liouville–Green solutions of (2) for $w(z)$ valid in a neighbourhood of $z = \infty$. Relate these solutions to those of the form (3).

1/II/30B **Asymptotic Methods**

State Watson's lemma, describing the asymptotic behaviour of the integral

$$I(\lambda) = \int_0^A e^{-\lambda t} f(t) dt, \quad A > 0,$$

as $\lambda \to \infty$, given that $f(t)$ has the asymptotic expansion

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n \, t^{n\beta}$$

as $t \to 0_+$, where $\beta > 0$ and $\alpha > -1$.

Give an account of Laplace's method for finding asymptotic expansions of integrals of the form

$$J(z) = \int_{-\infty}^{\infty} e^{-zp(t)} \, q(t) \, dt$$

for large real z, where $p(t)$ is real for real t.

Deduce the following asymptotic expansion of the contour integral

$$\int_{-\infty - i\pi}^{\infty + i\pi} \exp\left(z \cosh t\right) dt \;=\; 2^{1/2} i e^{z} \, \Gamma\left(\tfrac{1}{2}\right) \left[z^{-1/2} + \tfrac{1}{8} \, z^{-3/2} + O\left(z^{-5/2}\right) \right]$$

as $z \to \infty$.

3/II/30B **Asymptotic Methods**

Explain the method of stationary phase for determining the behaviour of the integral

$$I(x) = \int_a^b du \; e^{ixf(u)}$$

for large x. Here, the function $f(u)$ is real and differentiable, and a, b and x are all real.

Apply this method to show that the first term in the asymptotic behaviour of the function

$$\Gamma(m+1) = \int_0^{\infty} du \; u^m \, e^{-u} \; ,$$

where $m = i\, n$ with $n > 0$ and real, is

$$\Gamma(i\,n + 1) \quad \sim \quad \sqrt{2\pi} \; e^{-i\,n} \exp\left[\left(i\,n + \tfrac{1}{2}\right) \left(\frac{i\pi}{2} + \log n \right) \right]$$

as $n \to \infty$.

4/II/31B **Asymptotic Methods**

Consider the time-independent Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \lambda^2 q(x)\psi(x) = 0,$$

where $\lambda \gg 1$ denotes $\hbar^{-1}$ and $q(x)$ denotes $2m[E - V(x)]$. Suppose that

$$\begin{aligned} & q(x) > 0 \quad \text{for} \quad a < x < b, \\ \text{and} \quad & q(x) < 0 \quad \text{for} \quad -\infty < x < a \;\text{ and }\; b < x < \infty \end{aligned}$$

and consider a bound state $\psi(x)$. Write down the possible Liouville–Green approximate solutions for $\psi(x)$ in each region, given that $\psi \to 0$ as $|x| \to \infty$.

Assume that $q(x)$ may be approximated by $q'(a)(x-a)$ near $x = a$, where $q'(a) > 0$, and by $q'(b)(x-b)$ near $x = b$, where $q'(b) < 0$. The Airy function $\mathrm{Ai}(z)$ satisfies

$$\frac{d^2(\mathrm{Ai})}{dz^2} - z(\mathrm{Ai}) = 0$$

and has the asymptotic expansions

$$\mathrm{Ai}(z) \quad \sim \quad \tfrac{1}{2}\pi^{-1/2} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \qquad \text{as} \qquad z \to +\infty\,,$$

and

$$\mathrm{Ai}(z) \quad \sim \quad \pi^{-1/2}|z|^{-1/4} \cos\left[\left(\frac{2}{3}|z|^{3/2}\right) - \frac{\pi}{4}\right] \qquad \text{as} \qquad z \to -\infty\,.$$

Deduce that the energies E of bound states are given approximately by the WKB condition:

$$\lambda \int_a^b q^{1/2}(x)\, dx = \left(n + \tfrac{1}{2}\right)\pi \qquad (n = 0, 1, 2, \ldots).$$

1/II/30B **Asymptotic Methods**

Two real functions $p(t), q(t)$ of a real variable t are given on an interval $[0, b]$, where $b > 0$. Suppose that $q(t)$ attains its minimum precisely at $t = 0$, with $q'(0) = 0$, and that $q''(0) > 0$. For a real argument x, define

$$I(x) = \int_0^b p(t) e^{-xq(t)}\, dt.$$

Explain how to obtain the leading asymptotic behaviour of $I(x)$ as $x \to +\infty$ (Laplace's method).

The modified Bessel function $I_\nu(x)$ is defined for $x > 0$ by:

$$I_\nu(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos\theta} \cos(\nu\theta)\, d\theta - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-x(\cosh t) - \nu t}\, dt.$$

Show that

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

as $x \to \infty$ with ν fixed.

3/II/30B **Asymptotic Methods**

The Airy function $\mathrm{Ai}(z)$ is defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_C \exp\left(-\frac{1}{3}t^3 + zt\right)\, dt\ ,$$

where the contour C begins at infinity along the ray $\arg(t) = 4\pi/3$ and ends at infinity along the ray $\arg(t) = 2\pi/3$. Restricting attention to the case where z is real and positive, use the method of steepest descent to obtain the leading term in the asymptotic expansion for $\mathrm{Ai}(z)$ as $z \to \infty$:

$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}z^{3/2}\right)}{2\pi^{1/2} z^{1/4}}\ .$$

[*Hint: put* $t = z^{1/2}\tau$.]

4/II/31B **Asymptotic Methods**

(a) Outline the Liouville–Green approximation to solutions $w(z)$ of the ordinary differential equation

$$\frac{d^2w}{dz^2} = f(z)w$$

in a neighbourhood of infinity, in the case that, near infinity, $f(z)$ has the convergent series expansion

$$f(z) = \sum_{s=0}^{\infty} \frac{f_s}{z^s} \ ,$$

with $f_0 \neq 0$.

In the case

$$f(z) = 1 + \frac{1}{z} + \frac{2}{z^2} \ ,$$

explain why you expect a basis of two asymptotic solutions $w_1(z)$, $w_2(z)$, with

$$\begin{aligned} w_1(z) &\sim z^{\frac{1}{2}} e^z \left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right), \\ w_2(z) &\sim z^{-\frac{1}{2}} e^{-z} \left(1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots\right), \end{aligned}$$

as $z \to +\infty$, and show that $a_1 = -\frac{9}{8}$.

(b) Determine, at leading order in the large positive real parameter λ, an approximation to the solution $u(x)$ of the eigenvalue problem:

$$u''(x) + \lambda^2 g(x)u(x) = 0; \quad u(0) = u(1) = 0;$$

where $g(x)$ is greater than a positive constant for $x \in [0,1]$.

1/II/30A Asymptotic Methods

Explain what is meant by an asymptotic power series about $x = a$ for a real function $f(x)$ of a real variable. Show that a convergent power series is also asymptotic.

Show further that an asymptotic power series is unique (assuming that it exists).

Let the function $f(t)$ be defined for $t \geqslant 0$ by

$$f(t) = \frac{1}{\pi^{1/2}} \int_0^\infty \frac{e^{-x}}{x^{1/2}(1+2xt)} dx .$$

By suitably expanding the denominator of the integrand, or otherwise, show that, as $t \to 0_+$,

$$f(t) \sim \sum_{k=0}^{\infty} (-1)^k 1.3 \ldots (2k-1) t^k$$

and that the error, when the series is stopped after n terms, does not exceed the absolute value of the $(n+1)$th term of the series.

3/II/30A Asymptotic Methods

Explain, without proof, how to obtain an asymptotic expansion, as $x \to \infty$, of

$$I(x) = \int_0^\infty e^{-xt} f(t) dt ,$$

if it is known that $f(t)$ possesses an asymptotic power series as $t \to 0$.

Indicate the modification required to obtain an asymptotic expansion, under suitable conditions, of

$$\int_{-\infty}^{\infty} e^{-xt^2} f(t)\ dt .$$

Find an asymptotic expansion as $z \to \infty$ of the function defined by

$$I(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(z-t)} dt \qquad (\mathrm{Im}(z) < 0)$$

and its analytic continuation to $\mathrm{Im}(z) \geqslant 0$. Where are the Stokes lines, that is, the critical lines separating the Stokes regions?

4/II/31A **Asymptotic Methods**

Consider the differential equation

$$\frac{d^2w}{dx^2} = q(x)w\,,$$

where $q(x) \geqslant 0$ in an interval (a, ∞). Given a solution $w(x)$ and a further smooth function $\xi(x)$, define

$$W(x) = [\xi'(x)]^{1/2} w(x)\,.$$

Show that, when ξ is regarded as the independent variable, the function $W(\xi)$ obeys the differential equation

$$\frac{d^2W}{d\xi^2} = \left\{ \dot{x}^2 q(x) + \dot{x}^{1/2} \frac{d^2}{d\xi^2}[\dot{x}^{-1/2}] \right\} W, \qquad (*)$$

where $\dot{x}$ denotes $dx/d\xi$.

Taking the choice

$$\xi(x) = \int q^{1/2}(x)dx\,,$$

show that equation $(*)$ becomes

$$\frac{d^2W}{d\xi^2} = (1+\phi)W\,,$$

where

$$\phi = -\frac{1}{q^{3/4}} \frac{d^2}{dx^2}\left(\frac{1}{q^{1/4}}\right).$$

In the case that ϕ is negligible, deduce the Liouville–Green approximate solutions

$$w_{\pm} = q^{-1/4} \exp\left(\pm \int q^{1/2} dx\right).$$

Consider the Whittaker equation

$$\frac{d^2w}{dx^2} = \left[\frac{1}{4} + \frac{s(s-1)}{x^2}\right] w\,,$$

where s is a real constant. Show that the Liouville–Green approximation suggests the existence of solutions $w_{A,B}(x)$ with asymptotic behaviour of the form

$$w_A \sim \exp(x/2)\left(1 + \sum_{n=1}^{\infty} a_n x^{-n}\right), \qquad w_B \sim \exp(-x/2)\left(1 + \sum_{n=1}^{\infty} b_n x^{-n}\right)$$

as $x \to \infty$.

Given that these asymptotic series may be differentiated term-by-term, show that

$$a_n = \frac{(-1)^n}{n!}(s-n)(s-n+1)\dots(s+n-1)\,.$$