Part II

Analysis of Functions

Paper 1, Section II

23F Analysis of Functions

(a) State and prove the Sobolev trace theorem that maps $H^s(\mathbb{R}^n)$ into a suitable Sobolev space over $\mathbb{R}^{n-1}$.

(b) Show that there is no bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^{n-1})$ satisfying $Tu = u|_{\mathbb{R}^{n-1}\times\{0\}}$ for all $u \in C(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

(c) For $u \in C_c^\infty(\mathbb{R}^2)$, prove that

$$\int_{\mathbb{R}^2} |u|^4 \, dx \leqslant C \int_{\mathbb{R}^2} |u|^2 \, dx \int_{\mathbb{R}^2} |\nabla u|^2 \, dx,$$

for a constant C independent of u. [*Hint: First show that, for all* $(x, y) \in \mathbb{R}^2$, $|u(x,y)|^2 \leqslant 2 \int_{\mathbb{R}} |u(x,t)||\nabla u(x,t)| \, dt$.]

Paper 2, Section II

23F Analysis of Functions

(a) Let $U \subset \mathbb{R}^n$ be open with finite Lebesgue measure. Let $p \in (1, \infty)$ and let $\Lambda \in L^p(U)'$ be positive. Prove there is $\omega \in L^q(U)$ where $1/p + 1/q = 1$ such that

$$\Lambda(f) = \int_U f\omega \, dx \qquad \text{for all } f \in L^p(U).$$

[You may use without proof that $\|g\|_{L^q(U)} = \sup\{\int_U |fg| \, dx : \|f\|_{L^p(U)} \leqslant 1\}$.]

(b) (i) Define the Fourier transform of $f \in L^1(\mathbb{R}^n)$.

(ii) Let $p, q \in (1, \infty)$, and assume $\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,q} \|f\|_{L^p(\mathbb{R}^n)}$ for all $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Show that q is uniquely determined by p.

(iii) Compute the Fourier transform of $f(x) = |x|^{-1}$ on $\mathbb{R}^3$ up to a multiplicative constant which you do not need to determine. [You may use the Fourier transform of a Gaussian without proof.]

Paper 3, Section II

22F Analysis of Functions

(a) Let $U \subset \mathbb{R}^n$ be bounded and open, and let $m^2 > 0$. Given $f \in L^2(U)$, define what it means for u to be a *weak solution* to

$$\begin{aligned} -\Delta u + m^2 u &= f & &\text{in } U \\ u &= 0 & &\text{on } \partial U. \end{aligned}$$

Show that for any $f \in L^2(U)$ there is a unique weak solution u and let $Tf = u$. Show that $T : L^2(U) \to L^2(U)$ defines a compact operator. [You may use any theorems from the course if you state them carefully.]

(b) Let $U \subset \mathbb{R}^n$ be bounded and open, and let $(u_k) \subset L^2(U)$ be a sequence such that $u_k \rightharpoonup u$ weakly in $L^2(U)$. Assume that $\sup_k \int_{\{|p| \geqslant t\}} (|\hat{u}_k(p)|^2 + |\hat{u}(p)|^2)\, dp \to 0$ as $t \to \infty$. Show that then $u_k \to u$ in $L^2(U)$.

(c) Given $f \in H^r(\mathbb{R}^n)$, assume that $u \in L^2(\mathbb{R}^n)$ satisfies

$$\Delta^{2022} u + u = f \qquad \text{on } \mathbb{R}^n$$

in distributional sense. For which n is u a function that solves the equation in the classical sense? [You may cite any theorems from the course.]

Paper 4, Section II

23F Analysis of Functions

(a) Prove that the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is not compact.

(b) Construct a bounded linear functional on $L^\infty(\mathbb{R}^n)$ that cannot be expressed as $f \in L^\infty(\mathbb{R}^n) \mapsto \int f(x)g(x)\, dx$ for any $g \in L^1(\mathbb{R}^n)$. [You may use theorems from the course if you state them carefully.]

(c) Prove that $H^n(\mathbb{R}^n)$ embeds continuously into $C^{0,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$.

(d) Let θ be the Heaviside function, defined by $\theta(x) = 1_{x \geqslant 0}$, $x \in \mathbb{R}$. Find the Hardy–Littlewood maximal function $M\theta$.

Paper 1, Section II
23G Analysis of Functions

In this question, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and λ is Lebesgue measure on $\mathbb{R}^n$.

State *Lebesgue's differentiation theorem* and the *Radon–Nikodym theorem.* For a set $A \in \mathcal{M}$, and a measure μ defined on $\mathcal{M}$, let the *μ-density* of A at $x \in \mathbb{R}^n$ be

$$\rho_{\mu,A}(x) = \lim_{r \searrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))},$$

whenever the limit exists, where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is the open ball of radius r centred at x.

For each $t \in [0,1]$, give an example of a set $B \subset \mathbb{R}^2$ and point $z \in \mathbb{R}^2$ for which $\rho_{\lambda,B}(z)$ exists and is equal to t.

Show that for λ-almost every $x \in \mathbb{R}^n$, $\rho_{\lambda,A}(x)$ exists and takes the value 0 or 1. Show that $\rho_{\lambda,A}$ vanishes λ-almost everywhere if and only if A has Lebesgue measure zero.

Let ν be a measure on $\mathcal{M}$ such that $\nu \ll \lambda$ and $\lambda \ll \nu$. Show that $\rho_{\nu,A}(x)$ exists and takes the value 0 or 1 at λ-almost every $x \in \mathbb{R}^n$.

Paper 2, Section II
23G Analysis of Functions

Let X be a real vector space. State what it means for a functional $p : X \to \mathbb{R}$ to be *sublinear.*

Let $M \subsetneq X$ be a proper subspace. Suppose that $p : X \to \mathbb{R}$ is sublinear and the linear map $\ell : M \to \mathbb{R}$ satisfies $\ell(y) \leqslant p(y)$ for all $y \in M$. Fix $x \in X \setminus M$ and let $\widetilde{M} = \operatorname{span}\{M, x\}$. Show that there exists a linear map $\tilde{\ell} : \widetilde{M} \to \mathbb{R}$ such that $\tilde{\ell}(z) \leqslant p(z)$ for all $z \in \widetilde{M}$ and $\tilde{\ell}(y) = \ell(y)$ for all $y \in M$.

State the *Hahn–Banach theorem.*

Let $\{z_1, \ldots, z_n\}$ be a set of linearly independent elements of a real Banach space Z. Show that for each $j = 1, \ldots, n$ there exists $\ell_j \in Z'$ with $\ell_j(z_k) = \delta_{jk}$ for all $k = 1, \ldots, n$. Suppose $M \subset Z$ is a finite dimensional subspace. Show that there exists a closed subspace N such that $Z = M \oplus N$.

Paper 3, Section II
22G Analysis of Functions

State and prove the *Riemann–Lebesgue lemma*. State *Parseval's identity*, including any assumptions you make on the functions involved.

Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is given by

$$f(x) = \frac{|x|^a}{(1+|x|^2)^{\frac{b+a}{2}}} \, .$$

Show that if $2a > -n$ and $b > n$ then $\hat{f} \in L^p(\mathbb{R}^n)$ for all $2 \leqslant p \leqslant \infty$, where $\hat{f}$ is the Fourier transform of f.

Paper 4, Section II
23G Analysis of Functions

For $s \in \mathbb{R}$, define the *Sobolev space* $H^s(\mathbb{R}^n)$. Show that for any multi-index α, the map $u \mapsto D^\alpha u$ is a bounded linear map from $H^s(\mathbb{R}^n)$ to $H^{s-|\alpha|}(\mathbb{R}^n)$.

Given $f \in H^s(\mathbb{R}^n)$, show that the PDE

$$-\Delta u + u = f$$

admits a unique solution with $u \in H^{s+2}(\mathbb{R}^n)$. Show that the map taking f to u is a linear isomorphism of $H^s(\mathbb{R}^n)$ onto $H^{s+2}(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider a sequence of functions $(u_j)_{j=1}^\infty$ with $u_j \in C^\infty(\mathbb{R}^n)$, supported in Ω, such that

$$\|\Delta u_j\|_{L^2(\Omega)} + \|u_j\|_{L^2(\Omega)} \leqslant K \, ,$$

for some constant K independent of j. Show that there exists a subsequence $(u_{j_k})_{k=1}^\infty$ which converges strongly in $H^1(\mathbb{R}^n)$.

Paper 1, Section II
23H Analysis of Functions

Below, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and λ is Lebesgue measure.

(a) State the Lebesgue differentiation theorem for an integrable function $f : \mathbb{R}^n \to \mathbb{C}$. Let $g : \mathbb{R} \to \mathbb{C}$ be integrable and define $G : \mathbb{R} \to \mathbb{C}$ by $G(x) := \int_{[a,x]} g \, d\lambda$ for some $a \in \mathbb{R}$. Show that G is differentiable λ-almost everywhere.

(b) Suppose $h : \mathbb{R} \to \mathbb{R}$ is strictly increasing, continuous, and maps sets of λ-measure zero to sets of λ-measure zero. Show that we can define a measure ν on $\mathcal{M}$ by setting $\nu(A) := \lambda\big(h(A)\big)$ for $A \in \mathcal{M}$, and establish that $\nu \ll \lambda$. Deduce that h is differentiable λ-almost everywhere. Does the result continue to hold if h is assumed to be non-decreasing rather than strictly increasing?

[*You may assume without proof that a strictly increasing, continuous, function* $w : \mathbb{R} \to \mathbb{R}$ *is injective, and* $w^{-1} : w(\mathbb{R}) \to \mathbb{R}$ *is continuous.*]

Paper 2, Section II
23H Analysis of Functions

Define the *Schwartz space*, $\mathscr{S}(\mathbb{R}^n)$, and the *space of tempered distributions*, $\mathscr{S}'(\mathbb{R}^n)$, stating what it means for a sequence to converge in each space.

For a C^k function $f : \mathbb{R}^n \to \mathbb{C}$, and non-negative integers N, k, we say $f \in X_{N,k}$ if

$$\|f\|_{N,k} := \sup_{x \in \mathbb{R}^n; |\alpha| \leqslant k} \left| \big(1 + |x|^2\big)^{\frac{N}{2}} D^\alpha f(x) \right| < \infty.$$

You may assume that $X_{N,k}$ equipped with $\| \cdot \|_{N,k}$ is a Banach space in which $\mathscr{S}(\mathbb{R}^n)$ is dense.

(a) Show that if $u \in \mathscr{S}'(\mathbb{R}^n)$ there exist $N, k \in \mathbb{Z}_{\geqslant 0}$ and $C > 0$ such that

$$|u[\phi]| \leqslant C\|\phi\|_{N,k} \text{ for all } \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Deduce that there exists a unique $\tilde{u} \in X'_{N,k}$ such that $\tilde{u}[\phi] = u[\phi]$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(b) Recall that $v \in \mathscr{S}'(\mathbb{R}^n)$ is *positive* if $v[\phi] \geqslant 0$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ satisfying $\phi \geqslant 0$. Show that if $v \in \mathscr{S}'(\mathbb{R}^n)$ is positive, then there exist $M \in \mathbb{Z}_{\geqslant 0}$ and $K > 0$ such that

$$|v[\phi]| \leqslant K\|\phi\|_{M,0}, \quad \text{for all } \phi \in \mathscr{S}(\mathbb{R}^n).$$

$\Big[$*Hint: Note that* $|\phi(x)| \leqslant \|\phi\|_{M,0}\big(1 + |x|^2\big)^{-\frac{M}{2}}$. $\Big]$

Paper 3, Section II
22H Analysis of Functions

(a) State the Riemann–Lebesgue lemma. Show that the Fourier transform maps $\mathscr{S}(\mathbb{R}^n)$ to itself continuously.

(b) For some $s \geqslant 0$, let $f \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$. Consider the following system of equations for $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = f, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{0}.$$

Show that there exists a unique $\mathbf{B} = (B_1, B_2, B_3)$ solving the equations with $B_j \in H^{s+1}(\mathbb{R}^3)$ for $j = 1, 2, 3$. You need not find $\mathbf{B}$ explicitly, but should give an expression for the Fourier transform of B_j. Show that there exists a constant $C > 0$ such that

$$\|B_j\|_{H^{s+1}} \leqslant C\big(\|f\|_{L^1} + \|f\|_{H^s}\big), \qquad j = 1, 2, 3.$$

For what values of s can we conclude that $B_j \in C^1(\mathbb{R}^n)$?

Paper 4, Section II
23H Analysis of Functions

Fix $1 < p < \infty$ and let q satisfy $p^{-1} + q^{-1} = 1$.

(a) Let (f_j) be a sequence of functions in $L^p(\mathbb{R}^n)$. For $f \in L^p(\mathbb{R}^n)$, what is meant by (i) $f_j \to f$ in $L^p(\mathbb{R}^n)$ and (ii) $f_j \rightharpoonup f$ in $L^p(\mathbb{R}^n)$? Show that if $f_j \rightharpoonup f$, then

$$\|f\|_{L^p} \leqslant \liminf_{j\to\infty} \|f_j\|_{L^p} \ .$$

(b) Suppose that (g_j) is a sequence with $g_j \in L^p(\mathbb{R}^n)$, and that there exists $K > 0$ such that $\|g_j\|_{L^p} \leqslant K$ for all j. Show that there exists $g \in L^p(\mathbb{R}^n)$ and a subsequence $\big(g_{j_k}\big)_{k=1}^{\infty}$, such that for any sequence (h_k) with $h_k \in L^q(\mathbb{R}^n)$ and $h_k \to h \in L^q(\mathbb{R}^n)$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} g_{j_k} h_k \, dx = \int_{\mathbb{R}^n} gh \, dx.$$

Give an example to show that the result need not hold if the condition $h_k \to h$ is replaced by $h_k \rightharpoonup h$ in $L^q(\mathbb{R}^n)$.

Paper 1, Section II

23I Analysis of Functions

Let $\mathbb{R}^n$ be equipped with the σ-algebra of Lebesgue measurable sets, and Lebesgue measure.

(a) Given $f \in L^\infty(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, define the *convolution* $f \star g$, and show that it is a bounded, continuous function. [You may use without proof continuity of translation on $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.]

Suppose $A \subset \mathbb{R}^n$ is a measurable set with $0 < |A| < \infty$ where $|A|$ denotes the Lebesgue measure of A. By considering the convolution of $f(x) = \mathbb{1}_A(x)$ and $g(x) = \mathbb{1}_A(-x)$, or otherwise, show that the set $A - A = \{x - y : x, y \in A\}$ contains an open neighbourhood of 0. Does this still hold if $|A| = \infty$?

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a measurable function satisfying

$$f(x+y) = f(x) + f(y), \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Let $B_r = \{y \in \mathbb{R}^m : |y| < r\}$. Show that for any $\epsilon > 0$:

(i) $f^{-1}(B_\epsilon) - f^{-1}(B_\epsilon) \subset f^{-1}(B_{2\epsilon})$,

(ii) $f^{-1}(B_{k\epsilon}) = kf^{-1}(B_\epsilon)$ for all $k \in \mathbb{N}$, where for $\lambda > 0$ and $A \subset \mathbb{R}^n$, λA denotes the set $\{\lambda x : x \in A\}$.

Show that f is continuous at 0 and hence deduce that f is continuous everywhere.

Paper 3, Section II

22I Analysis of Functions

Let X be a Banach space.

(a) Define the *dual space* X', giving an expression for $\|\Lambda\|_{X'}$ for $\Lambda \in X'$. If $Y = L^p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$, identify Y' giving an expression for a general element of Y'. [You need not prove your assertion.]

(b) For a sequence $(\Lambda_i)_{i=1}^\infty$ with $\Lambda_i \in X'$, what is meant by: (i) $\Lambda_i \to \Lambda$, (ii) $\Lambda_i \rightharpoonup \Lambda$ (iii) $\Lambda_i \overset{*}{\rightharpoonup} \Lambda$? Show that (i) $\implies$ (ii) $\implies$ (iii). Find a sequence $(f_i)_{i=1}^\infty$ with $f_i \in L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))'$ such that, for some $f, g \in L^\infty(\mathbb{R}^n)$:

$$f_i \overset{*}{\rightharpoonup} f, \qquad f_i^2 \overset{*}{\rightharpoonup} g, \qquad g \neq f^2.$$

(c) For $f \in C_c^0(\mathbb{R}^n)$, let $\Lambda : C_c^0(\mathbb{R}^n) \to \mathbb{C}$ be the map $\Lambda f = f(0)$. Show that Λ may be extended to a continuous linear map $\tilde{\Lambda} : L^\infty(\mathbb{R}^n) \to \mathbb{C}$, and deduce that $(L^\infty(\mathbb{R}^n))' \neq L^1(\mathbb{R}^n)$. For which $1 \leqslant p \leqslant \infty$ is $L^p(\mathbb{R}^n)$ reflexive? [You may use without proof the Hahn–Banach theorem].

Paper 4, Section II

23I Analysis of Functions

(a) Define the Sobolev space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$.

(b) Let k be a non-negative integer and let $s > k + \frac{n}{2}$. Show that if $u \in H^s(\mathbb{R}^n)$ then there exists $u^* \in C^k(\mathbb{R}^n)$ with $u = u^*$ almost everywhere.

(c) Show that if $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$, there exists a unique $u \in H^{s+4}(\mathbb{R}^n)$ which solves:

$$\Delta\Delta u + \Delta u + u = f,$$

in a distributional sense. Prove that there exists a constant $C > 0$, independent of f, such that:

$$\|u\|_{H^{s+4}} \leqslant C\|f\|_{H^s}.$$

For which s will u be a classical solution?

Paper 3, Section II

22H Analysis of Functions

(a) Prove that in a finite-dimensional normed vector space the weak and strong topologies coincide.

(b) Prove that in a normed vector space X, a weakly convergent sequence is bounded. [Any form of the Banach–Steinhaus theorem may be used, as long as you state it clearly.]

(c) Let ℓ^1 be the space of real-valued absolutely summable sequences. Suppose (a^k) is a weakly convergent sequence in ℓ^1 which does not converge strongly. Show there is a constant $\varepsilon > 0$ and a sequence (x^k) in ℓ^1 which satisfies $x^k \rightharpoonup 0$ and $\|x^k\|_{\ell^1} \geqslant \varepsilon$ for all $k \geqslant 1$.

With (x^k) as above, show there is some $y \in \ell^\infty$ and a subsequence (x^{k_n}) of (x^k) with $\langle x^{k_n}, y\rangle \geqslant \varepsilon/3$ for all n. Deduce that every weakly convergent sequence in ℓ^1 is strongly convergent.

[*Hint: Define y so that $y_i = \operatorname{sign} x_i^{k_n}$ for $b_{n-1} < i \leqslant b_n$, where the sequence of integers b_n should be defined inductively along with x^{k_n}.*]

(d) Is the conclusion of part (c) still true if we replace ℓ^1 by $L^1([0, 2\pi])$?

Paper 4, Section II

23H Analysis of Functions

(a) Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear map. If B is continuous prove that there is an $M > 0$ such that $|B(u, v)| \leqslant M\|u\|\|v\|$ for all $u, v \in \mathcal{H}$. [You may use any form of the Banach–Steinhaus theorem as long as you state it clearly.]

(b) Now suppose that B defined as above is bilinear and continuous, and assume also that it is coercive: i.e. there is a $C > 0$ such that $B(u, u) \geqslant C\|u\|^2$ for all $u \in \mathcal{H}$. Prove that for any $f \in \mathcal{H}$, there exists a unique $v_f \in \mathcal{H}$ such that $B(u, v_f) = \langle u, f\rangle$ for all $u \in \mathcal{H}$.

[*Hint: show that there is a bounded invertible linear operator L with bounded inverse so that $B(u, v) = \langle u, Lv\rangle$ for all $u, v \in \mathcal{H}$. You may use any form of the Riesz representation theorem as long as you state it clearly.*]

(c) Define the *Sobolev space* $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open and bounded.

(d) Suppose $f \in L^2(\Omega)$ and $A \in \mathbb{R}^d$ with $|A|_2 < 2$, where $|\cdot|_2$ is the Euclidean norm on $\mathbb{R}^d$. Consider the Dirichlet problem

$$-\Delta v + v + A \cdot \nabla v = f \quad \text{in } \Omega, \qquad v = 0 \quad \text{in } \partial\Omega.$$

Using the result of part (b), prove there is a unique weak solution $v \in H_0^1(\Omega)$.

(e) Now assume that Ω is the open unit disk in $\mathbb{R}^2$ and g is a smooth function on $\mathbb{S}^1$. Sketch how you would solve the following variant:

$$-\Delta v + v + A \cdot \nabla v = 0 \quad \text{in } \Omega, \qquad v = g \quad \text{in } \partial\Omega.$$

[*Hint: Reduce to the result of part (d).*]

Paper 1, Section II

23H Analysis of Functions

(a) Consider the topology $\mathcal{T}$ on the natural numbers $\mathbb{N} \subset \mathbb{R}$ induced by the standard topology on $\mathbb{R}$. Prove it is the discrete topology; i.e. $\mathcal{T} = \mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$.

(b) Describe the corresponding Borel sets on $\mathbb{N}$ and prove that any function $f : \mathbb{N} \to \mathbb{R}$ or $f : \mathbb{N} \to [0, +\infty]$ is measurable.

(c) Using Lebesgue integration theory, define $\sum_{n \geqslant 1} f(n) \in [0, +\infty]$ for a function $f : \mathbb{N} \to [0, +\infty]$ and then $\sum_{n \geqslant 1} f(n) \in \mathbb{C}$ for $f : \mathbb{N} \to \mathbb{C}$. State any condition needed for the sum of the latter series to be defined. What is a simple function in this setting, and which simple functions have finite sum?

(d) State and prove the *Beppo Levi theorem* (also known as the monotone convergence theorem).

(e) Consider $f : \mathbb{R} \times \mathbb{N} \to [0, +\infty]$ such that for any $n \in \mathbb{N}$, the function $t \mapsto f(t, n)$ is non-decreasing. Prove that

$$\lim_{t \to \infty} \sum_{n \geqslant 1} f(t, n) = \sum_{n \geqslant 1} \lim_{t \to \infty} f(t, n).$$

Show that this need not be the case if we drop the hypothesis that $t \mapsto f(t, n)$ is non-decreasing, even if all the relevant limits exist.

Paper 3, Section II

22F Analysis of Functions

(a) Let $(X, \mathcal{A}, \mu)$ be a measure space. Define the spaces $L^p(X)$ for $p \in [1, \infty]$. Prove that if $\mu(X) < \infty$ then $L^q(X) \subset L^p(X)$ for all $1 \leqslant p < q \leqslant \infty$.

(b) Now let $X = \mathbb{R}^n$ endowed with Borel sets and Lebesgue measure. Describe the dual spaces of $L^p(X)$ for $p \in [1, \infty)$. Define *reflexivity* and say which $L^p(X)$ are reflexive. Prove that $L^1(X)$ is not the dual space of $L^\infty(X)$.

(c) Now let $X \subset \mathbb{R}^n$ be a Borel subset and consider the measure space $(X, \mathcal{A}, \mu)$ induced from Borel sets and Lebesgue measure on $\mathbb{R}^n$.

(i) Given any $p \in [1, \infty]$, prove that any sequence (f_n) in $L^p(X)$ converging in $L^p(X)$ to some $f \in L^p(X)$ admits a subsequence converging almost everywhere to f.

(ii) Prove that if $L^q(X) \subset L^p(X)$ for $1 \leqslant p < q \leqslant \infty$ then $\mu(X) < \infty$. [*Hint: You might want to prove first that the inclusion is continuous with the help of one of the corollaries of Baire's category theorem.*]

Paper 4, Section II

23F Analysis of Functions

Here and below, $\Phi : \mathbb{R} \to \mathbb{R}$ is smooth such that $\int_{\mathbb{R}} e^{-\Phi(x)}\,\mathrm{d}x = 1$ and

$$\lim_{|x|\to+\infty} \left(\frac{|\Phi'(x)|^2}{4} - \frac{\Phi''(x)}{2} \right) = \ell \in (0,+\infty).$$

$C_c^1(\mathbb{R})$ denotes the set of continuously differentiable complex-valued functions with compact support on $\mathbb{R}$.

(a) Prove that there are constants $R_0 > 0$, $\lambda_1 > 0$ and $K_1 > 0$ so that for any $R \geqslant R_0$ and $h \in C_c^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \left|h'(x)\right|^2 e^{-\Phi(x)}\,dx \geqslant \lambda_1 \int_{\{|x|\geqslant R\}} |h(x)|^2 e^{-\Phi(x)}\,dx - K_1 \int_{\{|x|\leqslant R\}} |h(x)|^2 e^{-\Phi(x)}\,dx.$$

[*Hint: Denote* $g := he^{-\Phi/2}$*, expand the square and integrate by parts.*]

(b) Prove that, given any $R > 0$, there is a $C_R > 0$ so that for any $h \in C^1([-R,R])$ with $\int_{-R}^{+R} h(x)e^{-\Phi(x)}dx = 0$:

$$\max_{x\in[-R,R]} |h(x)| + \sup_{\{x,y\in[-R,R],\ x\neq y\}} \frac{|h(x)-h(y)|}{|x-y|^{1/2}} \leqslant C_R \left(\int_{-R}^{+R} |h'(x)|^2 e^{-\Phi(x)}\,dx \right)^{1/2}.$$

[*Hint: Use the fundamental theorem of calculus to control the second term of the left-hand side, and then compare* h *to its weighted mean to control the first term of the left-hand side.*]

(c) Prove that, given any $R > 0$, there is a $\lambda_R > 0$ so that for any $h \in C^1([-R,R])$:

$$\int_{-R}^{+R} \left|h'(x)\right|^2 e^{-\Phi(x)}\,dx \geqslant \lambda_R \int_{-R}^{+R} \left| h(x) - \frac{\int_{-R}^{+R} h(y)e^{-\Phi(y)}\,dy}{\int_{-R}^{+R} e^{-\Phi(y)}\,dy} \right|^2 e^{-\Phi(x)}\,dx.$$

[*Hint: Show first that one can reduce to the case* $\int_{-R}^{+R} he^{-\Phi} = 0$*. Then argue by contradiction with the help of the Arzelà–Ascoli theorem and part (b).*]

(d) Deduce that there is a $\lambda_0 > 0$ so that for any $h \in C_c^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \left|h'(x)\right|^2 e^{-\Phi(x)}\,dx \geqslant \lambda_0 \int_{\mathbb{R}} \left| h(x) - \left(\int_{\mathbb{R}} h(y)e^{-\Phi(y)}\,dy \right) \right|^2 e^{-\Phi(x)}\,dx.$$

[*Hint: Show first that one can reduce to the case* $\int_{\mathbb{R}} he^{-\Phi} = 0$*. Then combine the inequality (a), multiplied by a constant of the form* $\epsilon = \epsilon_0\lambda_R$ *(where* $\epsilon_0 > 0$ *is chosen so that* ϵ *be sufficiently small), and the inequality (c).*]

Paper 1, Section II

23F Analysis of Functions

(a) Consider a measure space $(X, \mathcal{A}, \mu)$ and a complex-valued measurable function F on X. Prove that for any $\varphi : [0, +\infty) \to [0, +\infty)$ differentiable and increasing such that $\varphi(0) = 0$, then

$$\int_X \varphi(|F(x)|) \, \mathrm{d}\mu(x) = \int_0^{+\infty} \varphi'(s) \mu(\{|F| > s\}) \, \mathrm{d}\lambda(s)$$

where λ is the Lebesgue measure.

(b) Consider a complex-valued measurable function $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and its maximal function $Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f| \, \mathrm{d}\lambda$. Prove that for $p \in (1, +\infty)$ there is a constant $c_p > 0$ such that $\|Mf\|_{L^p(\mathbb{R}^n)} \leqslant c_p \|f\|_{L^p(\mathbb{R}^n)}$.

[*Hint: Split* $f = f_0 + f_1$ *with* $f_0 = f\chi_{\{|f|>s/2\}}$ *and* $f_1 = f\chi_{\{|f|\leqslant s/2\}}$ *and prove that* $\lambda(\{Mf > s\}) \leqslant \lambda(\{Mf_0 > s/2\})$. *Then use the maximal inequality* $\lambda(\{Mf > s\}) \leqslant \frac{C_1}{s} \|f\|_{L^1(\mathbb{R}^n)}$ *for some constant* $C_1 > 0$.]

(c) Consider $p, q \in (1, +\infty)$ with $p < q$ and $\alpha \in (0, n)$ such that $1/q = 1/p - \alpha/n$. Define $I_\alpha|f|(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, \mathrm{d}\lambda(y)$ and prove $I_\alpha|f|(x) \leqslant \|f\|_{L^p(\mathbb{R}^n)}^{\alpha p/n} Mf(x)^{1-\alpha p/n}$.

[*Hint: Split the integral into* $|x - y| \geqslant r$ *and* $|x - y| \in [2^{-k-1}r, 2^{-k}r)$ *for all* $k \geqslant 0$, *given some suitable* $r > 0$.]

Paper 3, Section II

20F Analysis of Functions

Denote by $C_0(\mathbb{R}^n)$ the space of continuous complex-valued functions on $\mathbb{R}^n$ converging to zero at infinity. Denote by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\,dx$ the Fourier transform of $f \in L^1(\mathbb{R}^n)$.

(i) Prove that the image of $L^1(\mathbb{R}^n)$ under $\mathcal{F}$ is included and dense in $C_0(\mathbb{R}^n)$, and that $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ is injective. [Fourier inversion can be used without proof when properly stated.]

(ii) Calculate the Fourier transform of $\chi_{[a,b]}$, the characteristic function of $[a,b] \subset \mathbb{R}$.

(iii) Prove that $g_n := \chi_{[-n,n]} * \chi_{[-1,1]}$ belongs to $C_0(\mathbb{R})$ and is the Fourier transform of a function $h_n \in L^1(\mathbb{R})$, which you should determine.

(iv) Using the functions h_n, g_n and the open mapping theorem, deduce that the Fourier transform is not surjective from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$.

Paper 4, Section II

22F Analysis of Functions

Consider $\mathbb{R}^n$ with the Lebesgue measure. Denote by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\,dx$ the Fourier transform of $f \in L^1(\mathbb{R}^n)$ and by $\hat{f}$ the Fourier–Plancherel transform of $f \in L^2(\mathbb{R}^n)$. Let $\chi_R(\xi) := \left(1 - \dfrac{|\xi|}{R}\right)\chi_{|\xi|\leqslant R}$ for $R > 0$ and define for $s \in \mathbb{R}_+$

$$H^s(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) \;\middle|\; (1+|\cdot|^2)^{s/2}\hat{f}(\cdot) \in L^2(\mathbb{R}^n) \right\}.$$

(i) Prove that $H^s(\mathbb{R}^n)$ is a vector subspace of $L^2(\mathbb{R}^n)$, and is a Hilbert space for the inner product $\langle f, g\rangle := \displaystyle\int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

(ii) Construct a function $f \in H^s(\mathbb{R})$, $s \in (0, 1/2)$, that is not almost everywhere equal to a continuous function.

(iii) For $f \in L^1(\mathbb{R}^n)$, prove that $F_R : x \mapsto \displaystyle\int_{\mathbb{R}^n} \mathcal{F}f(\xi)\chi_R(\xi)e^{2i\pi x\cdot\xi}\,d\xi$ is a well-defined function and that $F_R \in L^1(\mathbb{R}^n)$ converges to f in $L^1(\mathbb{R}^n)$ as $R \to +\infty$.

[*Hint: Prove that* $F_R = K_R * f$ *where* K_R *is an approximation of the unit as* $R \to +\infty$.]

(iv) Deduce that if $f \in L^1(\mathbb{R}^n)$ and $(1+|\cdot|^2)^{s/2}\mathcal{F}f(\cdot) \in L^2(\mathbb{R}^n)$ then $f \in H^s(\mathbb{R}^n)$.

[*Hint: Prove that: (1) there is a sequence* $R_k \to +\infty$ *such that* $K_{R_k} * f$ *converges to* f *almost everywhere; (2)* $K_R * f$ *is uniformly bounded in* $L^2(\mathbb{R}^n)$ *as* $R \to +\infty$.]

Paper 1, Section II

22F Analysis of Functions

Consider a sequence $f_n : \mathbb{R} \to \mathbb{R}$ of measurable functions converging pointwise to a function $f : \mathbb{R} \to \mathbb{R}$. The Lebesgue measure is denoted by λ.

(a) Consider a Borel set $A \subset \mathbb{R}$ with finite Lebesgue measure $\lambda(A) < +\infty$. Define for $k, n \geqslant 1$ the sets

$$E_n^{(k)} := \bigcap_{m \geqslant n} \left\{ x \in A \;\middle|\; |f_m(x) - f(x)| \leqslant \frac{1}{k} \right\}.$$

Prove that for any $k, n \geqslant 1$, one has $E_n^{(k)} \subset E_{n+1}^{(k)}$ and $E_n^{(k+1)} \subset E_n^{(k)}$. Prove that for any $k \geqslant 1$, $A = \cup_{n \geqslant 1} E_n^{(k)}$.

(b) Consider a Borel set $A \subset \mathbb{R}$ with finite Lebesgue measure $\lambda(A) < +\infty$. Prove that for any $\varepsilon > 0$, there is a Borel set $A_\varepsilon \subset A$ for which $\lambda(A \setminus A_\varepsilon) \leqslant \varepsilon$ and such that f_n converges to f uniformly on A_ε as $n \to +\infty$. Is the latter still true when $\lambda(A) = +\infty$?

(c) Assume additionally that $f_n \in L^p(\mathbb{R})$ for some $p \in (1, +\infty]$, and there exists an $M \geqslant 0$ for which $\|f_n\|_{L^p(\mathbb{R})} \leqslant M$ for all $n \geqslant 1$. Prove that $f \in L^p(\mathbb{R})$.

(d) Let f_n and f be as in part (c). Consider a Borel set $A \subset \mathbb{R}$ with finite Lebesgue measure $\lambda(A) < +\infty$. Prove that f_n, f are integrable on A and $\int_A f_n \, d\lambda \to \int_A f \, d\lambda$ as $n \to \infty$. Deduce that f_n converges weakly to f in $L^p(\mathbb{R})$ when $p < +\infty$. Does the convergence have to be strong?