Part II

Algebraic Topology

Paper 1, Section II
21G Algebraic Topology

State the universal property which characterizes an amalgamated free product of groups. State the Seifert-van Kampen theorem.

Suppose that $\{U_1, U_2\}$ is an open cover of a topological space X, that $U_1 \cap U_2$ is path connected and that $x_0 \in U_1 \cap U_2$. If $i_k : U_k \to X$ is the inclusion, prove that $\pi_1(X, x_0)$ is generated by $i_{1*}(\pi_1(U_1, x_0))$ and $i_{2*}(\pi_1(U_2, x_0))$. [You may use the Lebesgue covering lemma if you state it clearly.]

Consider the Mobius band $M = I^2/\sim$, where $(0, x) \sim (1, 1-x)$. Identify its boundary $\partial M = (I \times \{0, 1\})/\sim$ with S^1. Note that if $f : \partial M \to X$, the space obtained by *attaching a Mobius band to X using f* is $X \cup_f M = (X \amalg M)/\sim$, where now $\sim$ is the smallest equivalence relation containing $x \sim f(x)$ for all $x \in \partial M$. Now let Y be the space obtained by attaching two Mobius bands to $T^2 = S^1 \times S^1$ using the maps $f_1, f_2 : S^1 \to T^2$ given by $f_1(z) = (z, z)$ and $f_2(z) = (z^2, z^3)$. Give a two-generator one-relator presentation of $\pi_1(Y, y_0)$ for some $y_0 \in Y$. Show that this group is non-abelian.

Paper 2, Section II
21G Algebraic Topology

Let $p : \widehat{X} \to X$ be a covering map, and suppose that X and $\widehat{X}$ are path connected and locally path connected topological spaces. If $x_0, x_1 \in X$, show that $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality. [You may use any theorems from the course, as long as you state them clearly.]

Define what it means for p to be a *normal covering map.* State an appropriate lifting theorem and use it to prove that if $p : \widehat{X} \to X$ is a *universal* covering map, then it is normal.

Let Σ_g be a surface of genus g and suppose that $p : \widehat{\Sigma}_g \to \Sigma_g$ is a connected covering map of degree $n \in \mathbb{N}$. For which values of g and n must p be normal? Justify your answer. For those values of g and n for which p need not be normal, give an explicit example of a non-normal covering map p.

Paper 3, Section II
20G Algebraic Topology

Consider the set $X \subset S^3$ given by $X = \{(x_1, x_2, x_3, x_4) \in S^3 \;:\; |x_4| \leqslant \frac{1}{2}\}$ and its boundary $\partial X = \{(x_1, x_2, x_3, x_4) \in S^3 \;:\; |x_4| = \frac{1}{2}\}$. Define Y and ∂Y to be the image of X and ∂X in $\mathbb{RP}^3 = S^3/\sim$, where $x \sim -x$. Show that Y is homotopy equivalent to $\mathbb{RP}^2$. Compute $H_*(\mathbb{RP}^3)$. [You may assume $\mathbb{RP}^3$ admits a triangulation containing Y and ∂Y as subcomplexes, and may use $H_*(\mathbb{RP}^2)$ if you state it precisely.]

Let $f : \partial Y \to \partial Y$ be the identity map, and define Z to be the space obtained by identifying two copies of Y along their boundary: $Z = Y \cup_f Y$. Compute $H_*(Z)$ and $\pi_1(Z, z_0)$, where $z_0 \in Z$. The universal covering space of Z is homeomorphic to a familiar space. What is it?

Paper 4, Section II

21G Algebraic Topology

Suppose that (C, d) and (C', d') are chain complexes, and that $f, g : C \to C'$ are chain maps. Show that f induces a map $f_* : H_*(C) \to H_*(C')$. Define what it means for f and g to be *chain homotopic*. Show that if f and g are chain homotopic, they induce the same map on homology.

Define a chain complex $(M(f), d_f)$ as follows: $M(f)_i = C_{i-1} \oplus C'_i$ and the map $(d_f)_i : M(f)_i \to M(f)_{i-1}$ is given by the matrix

$$\begin{pmatrix} d_{i-1} & 0 \\ (-1)^i f_{i-1} & d'_i \end{pmatrix}.$$

Verify that $(M(f), d_f)$ is a chain complex. Show that there is a long exact sequence

$$\ldots \to H_i(C) \xrightarrow{(-1)^{i+1} f_*} H_i(C') \to H_i(M(f)) \to H_{i-1}(C) \xrightarrow{(-1)^i f_*} H_{i-1}(C') \to \ldots$$

If f is chain homotopic to g, show that $(M(f), d_f)$ and $(M(g), d_g)$ are isomorphic as chain complexes.

Paper 1, Section II
21I Algebraic Topology

Suppose $f, g : C_* \to C'_*$ are chain maps. Define what it means for f and g to be *chain homotopic*. Show that if f and g are chain homotopic then $f_* = g_*$.

Let $C_* = \widetilde{C}_*(\Delta^n)$ be the reduced chain complex of the n-dimensional simplex. Show that id_{C_*} is chain homotopic to 0_{C_*}. Hence compute $H_*(\Delta^n)$.

Now let $K = \Delta^6_2$ be the 2-skeleton of Δ^6. Compute $H_*(K)$. Let $f : K \to K$ be the simplicial map given by $f(e_i) = e_{\sigma(i)}$, where σ is the permutation given in cycle notation by $(0123)(456)$. Compute the trace of the linear map $f_* : H_2(K; \mathbb{Q}) \to H_2(K; \mathbb{Q})$.

Paper 2, Section II
21I Algebraic Topology

State the *snake lemma* and derive the exactness of the Mayer–Vietoris sequence from it.

Suppose that K is a simplicial complex of dimension $n \geqslant 1$, that every $(n-1)$-simplex of K is a face of precisely two n-simplices, and that if σ and σ' are n-simplices of K then there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ of n-simplices in K such that for all i, σ_i and σ_{i+1} have an $(n-1)$-simplex in common. Show that $H_n(K)$ is either trivial or isomorphic to $\mathbb{Z}$.

Now suppose that K is as above and that $H_n(K) \cong \mathbb{Z}$ is generated by $x \in H_n(K)$. If K is the union of subcomplexes L_1 and L_2 such that $L_1 \cap L_2$ has dimension less than n, describe ∂x, where ∂ is the boundary map in the Mayer–Vietoris sequence associated to the decomposition $K = L_1 \cup L_2$. Justify your answer. When is $\partial x \neq 0$?

Finally, suppose that K, L_1 and L_2 are as in the previous paragraph, that K is homeomorphic to S^3, that L_1 is homeomorphic to $S^1 \times D^2$, and that the image of $L_1 \cap L_2$ under this homeomorphism is $S^1 \times S^1 \subset S^1 \times D^2$. Compute $H_*(L_2)$.

Paper 3, Section II
20I Algebraic Topology

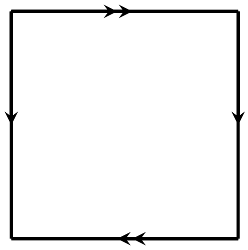
Suppose $f : S^{n-1} \to X$ is a continuous map. Show that f extends to a continuous map $F : D^n \to X$ if and only if f is homotopic to a constant map.

Let X be a path-connected and locally path-connected topological space. Define what it means for a space $\widetilde{X}$ to be a *universal covering space* of X. State a suitable lifting property and use it to prove that any two universal covering spaces of X are homeomorphic.

Now suppose that $\widetilde{X}$ is a universal covering space of X, and that $\widetilde{X}$ is contractible. Let K be a path-connected simplicial complex with 1-skeleton K_1, and let $i : K_1 \to K$ be the inclusion. Given a continuous map $f : |K_1| \to X$, prove that f extends to a continuous map $F : |K| \to X$ if and only if there is a homomorphism $\Phi : \pi_1(|K|, v) \to \pi_1(X, f(v))$ with $f_* = \Phi \circ i_*$, where v is any vertex of K. [*Hint: Induct on the number of simplices in* $K \setminus K_1$.]

Paper 4, Section II
21I Algebraic Topology

Let K be the Klein bottle obtained by identifying the sides of the unit square as shown in the figure, and let $k_0 \in K$ be the image of the corners of the square.



Show that K is the union of two Möbius bands with their boundaries identified. Deduce that $\pi_1(K, k_0)$ has a presentation

$$\pi_1(K, k_0) = \langle a, b \,|\, a^2 b^{-2} \rangle.$$

Show that there is a degree two covering map $p : (T^2, x_0) \to (K, k_0)$. Describe generators α, β for $\pi_1(T^2, x_0)$ and express $p_*(\alpha)$ and $p_*(\beta)$ in terms of a and b.

Let $Y = T^2 \times [0,1)/\sim$, where $\sim$ is the smallest equivalence relation with $(x, 0) \sim (x', 0)$ whenever $p(x) = p(x')$. What is $\pi_1(Y, y_0)$, where y_0 is the image of $(x_0, 0)$ in Y?

Suppose X is a path-connected Hausdorff space, that $U \subset X$ is an open subset, and that U is homeomorphic to Y. Can X be simply connected? Justify your answer.

Paper 1, Section II
21F Algebraic Topology

(a) What does it mean for two spaces X and Y to be *homotopy equivalent*?

(b) What does it mean for a subspace $Y \subseteq X$ to be a *retract* of a space X? What does it mean for a space X to be *contractible*? Show that a retract of a contractible space is contractible.

(c) Let X be a space and $A \subseteq X$ a subspace. We say the pair (X, A) has the *homotopy extension property* if, for any pair of maps $f : X \times \{0\} \to Y$ and $H' : A \times I \to Y$ with

$$f|_{A\times\{0\}} = H'|_{A\times\{0\}},$$

there exists a map $H : X \times I \to Y$ with

$$H|_{X\times\{0\}} = f, \qquad H|_{A\times I} = H'.$$

Now suppose that $A \subseteq X$ is contractible. Denote by X/A the quotient of X by the equivalence relation $x \sim x'$ if and only if $x = x'$ or $x, x' \in A$. Show that, if (X, A) satisfies the homotopy extension property, then X and X/A are homotopy equivalent.

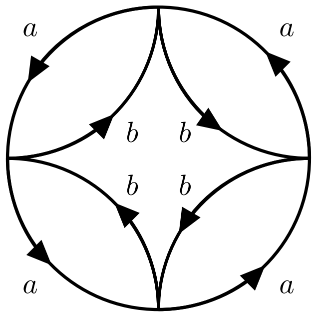
Paper 2, Section II

21F Algebraic Topology

(a) State a suitable version of the Seifert–van Kampen theorem and use it to calculate the fundamental groups of the torus $T^2 := S^1 \times S^1$ and of the real projective plane $\mathbb{RP}^2$.

(b) Show that there are no covering maps $T^2 \to \mathbb{RP}^2$ or $\mathbb{RP}^2 \to T^2$.

(c) Consider the following covering space of $S^1 \vee S^1$:



Here the line segments labelled a and b are mapped to the two different copies of S^1 contained in $S^1 \vee S^1$, with orientations as indicated.

Using the Galois correspondence with basepoints, identify a subgroup of

$$\pi_1(S^1 \vee S^1, x_0) = F_2$$

(where x_0 is the wedge point) that corresponds to this covering space.

Paper 3, Section II

20F Algebraic Topology

Let X be a space. We define the *cone* of X to be

$$CX := (X \times I)/\sim$$

where $(x_1, t_1) \sim (x_2, t_2)$ if and only if either $t_1 = t_2 = 1$ or $(x_1, t_1) = (x_2, t_2)$.

(a) Show that if X is triangulable, so is CX. Calculate $H_i(CX)$. [You may use any results proved in the course.]

(b) Let K be a simplicial complex and $L \subseteq K$ a subcomplex. Let $X = |K|$, $A = |L|$, and let X' be the space obtained by identifying $|L| \subseteq |K|$ with $|L| \times \{0\} \subseteq C|L|$. Show that there is a long exact sequence

$$\cdots \to H_{i+1}(X') \to H_i(A) \to H_i(X) \to H_i(X') \to H_{i-1}(A) \to \cdots$$

$$\cdots \to H_1(X') \to H_0(A) \to \mathbb{Z} \oplus H_0(X) \to H_0(X') \to 0.$$

(c) In part (b), suppose that $X = S^1 \times S^1$ and $A = S^1 \times \{x\} \subseteq X$ for some $x \in S^1$. Calculate $H_i(X')$ for all i.

Paper 4, Section II

21F Algebraic Topology

(a) Define the *Euler characteristic* of a triangulable space X.

(b) Let Σ_g be an orientable surface of genus g. A map $\pi : \Sigma_g \to S^2$ is a *double-branched cover* if there is a set $Q = \{p_1, \ldots, p_n\} \subseteq S^2$ of branch points, such that the restriction $\pi : \Sigma_g \setminus \pi^{-1}(Q) \to S^2 \setminus Q$ is a covering map of degree 2, but for each $p \in Q$, $\pi^{-1}(p)$ consists of one point. By carefully choosing a triangulation of S^2, use the Euler characteristic to find a formula relating g and n.

Paper 1, Section II

21F Algebraic Topology

Let $p : \mathbb{R}^2 \to S^1 \times S^1 =: X$ be the map given by

$$p(r_1, r_2) = \left(e^{2\pi i r_1}, e^{2\pi i r_2}\right),$$

where S^1 is identified with the unit circle in $\mathbb{C}$. [You may take as given that p is a covering map.]

(a) Using the covering map p, show that $\pi_1(X, x_0)$ is isomorphic to $\mathbb{Z}^2$ as a group, where $x_0 = (1, 1) \in X$.

(b) Let $\mathrm{GL}_2(\mathbb{Z})$ denote the group of 2×2 matrices A with integer entries such that $\det A = \pm 1$. If $A \in \mathrm{GL}_2(\mathbb{Z})$, we obtain a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$. Show that this linear transformation induces a homeomorphism $f_A : X \to X$ with $f_A(x_0) = x_0$ and such that $f_{A*} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ agrees with A as a map $\mathbb{Z}^2 \to \mathbb{Z}^2$.

(c) Let $p_i : \widehat{X}_i \to X$ for $i = 1, 2$ be connected covering maps of degree 2. Show that there exist homeomorphisms $\phi : \widehat{X}_1 \to \widehat{X}_2$ and $\psi : X \to X$ so that the diagram

$$\begin{tikzcd} \widehat{X}_1 \arrow[r, "\phi"] \arrow[d, "p_1"'] & \widehat{X}_2 \arrow[d, "p_2"] \\ X \arrow[r, "\psi"'] & X \end{tikzcd}$$

is commutative.

Paper 2, Section II

21F Algebraic Topology

(a) Let $f : X \to Y$ be a map of spaces. We define the *mapping cylinder* M_f of f to be the space

$$(([0,1] \times X) \sqcup Y)/\sim$$

with $(0, x) \sim f(x)$. Show carefully that the canonical inclusion $Y \hookrightarrow M_f$ is a homotopy equivalence.

(b) Using the Seifert–van Kampen theorem, show that if X is path-connected and $\alpha : S^1 \to X$ is a map, and $x_0 = \alpha(\theta_0)$ for some point $\theta_0 \in S^1$, then

$$\pi_1(X \cup_\alpha D^2, x_0) \cong \pi_1(X, x_0)/\langle\langle[\alpha]\rangle\rangle.$$

Use this fact to construct a connected space X with

$$\pi_1(X) \cong \langle a, b \,|\, a^3 = b^7\rangle.$$

(c) Using a covering space of $S^1 \vee S^1$, give explicit generators of a subgroup of F_2 isomorphic to F_3. Here F_n denotes the free group on n generators.

Paper 3, Section II

20F Algebraic Topology

Let K be a simplicial complex with four vertices $v_1, \ldots, v_4$ with simplices $\langle v_1, v_2, v_3 \rangle$, $\langle v_1, v_4 \rangle$ and $\langle v_2, v_4 \rangle$ and their faces.

(a) Draw a picture of $|K|$, labelling the vertices.

(b) Using the definition of homology, calculate $H_n(K)$ for all n.

(c) Let L be the subcomplex of K consisting of the vertices v_1, v_2, v_4 and the 1-simplices $\langle v_1, v_2 \rangle$, $\langle v_1, v_4 \rangle$, $\langle v_2, v_4 \rangle$. Let $i : L \to K$ be the inclusion. Construct a simplicial map $j : K \to L$ such that the topological realisation $|j|$ of j is a homotopy inverse to $|i|$. Construct an explicit chain homotopy $h : C_\bullet(K) \to C_\bullet(K)$ between $i_\bullet \circ j_\bullet$ and $\mathrm{id}_{C_\bullet(K)}$, and verify that h is a chain homotopy.

Paper 4, Section II

21F Algebraic Topology

In this question, you may assume all spaces involved are triangulable.

(a) (i) State and prove the Mayer–Vietoris theorem. [You may assume the theorem that states that a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.]

(ii) Use Mayer–Vietoris to calculate the homology groups of an oriented surface of genus g.

(b) Let S be an oriented surface of genus g, and let $D_1, \ldots, D_n$ be a collection of mutually disjoint closed subsets of S with each D_i homeomorphic to a two-dimensional disk. Let D_i° denote the interior of D_i, homeomorphic to an open two-dimensional disk, and let

$$T := S \setminus (D_1^\circ \cup \cdots \cup D_n^\circ).$$

Show that

$$H_i(T) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}^{2g+n-1} & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let T be the surface given in (b) when $S = S^2$ and $n = 3$. Let $f : T \to S^1 \times S^1$ be a map. Does there exist a map $g : S^1 \times S^1 \to T$ such that $f \circ g$ is homotopic to the identity map? Justify your answer.

Paper 3, Section II

20F Algebraic Topology

Let K be a simplicial complex, and L a subcomplex. As usual, $C_k(K)$ denotes the group of k-chains of K, and $C_k(L)$ denotes the group of k-chains of L.

(a) Let

$$C_k(K, L) = C_k(K)/C_k(L)$$

for each integer k. Prove that the boundary map of K descends to give $C_\bullet(K, L)$ the structure of a chain complex.

(b) The *homology groups of K relative to L*, denoted by $H_k(K, L)$, are defined to be the homology groups of the chain complex $C_\bullet(K, L)$. Prove that there is a long exact sequence that relates the homology groups of K relative to L to the homology groups of K and the homology groups of L.

(c) Let D_n be the closed n-dimensional disc, and S^{n-1} be the $(n-1)$-dimensional sphere. Exhibit simplicial complexes K_n and subcomplexes L_{n-1} such that $D_n \cong |K_n|$ in such a way that $|L_{n-1}|$ is identified with S^{n-1}.

(d) Compute the relative homology groups $H_k(K_n, L_{n-1})$, for all integers $k \geqslant 0$ and $n \geqslant 2$ where K_n and L_{n-1} are as in (c).

Paper 4, Section II

21F Algebraic Topology

State the *Lefschetz fixed point theorem.*

Let $n \geqslant 2$ be an integer, and $x_0 \in S^2$ a choice of base point. Define a space

$$X := (S^2 \times \mathbb{Z}/n\mathbb{Z})/\sim$$

where $\mathbb{Z}/n\mathbb{Z}$ is discrete and $\sim$ is the smallest equivalence relation such that $(x_0, i) \sim (-x_0, i+1)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Let $\phi : X \to X$ be a homeomorphism without fixed points. Use the Lefschetz fixed point theorem to prove the following facts.

(i) If $\phi^3 = \mathrm{Id}_X$ then n is divisible by 3.

(ii) If $\phi^2 = \mathrm{Id}_X$ then n is even.

Paper 2, Section II

21F Algebraic Topology

Let $T = S^1 \times S^1$, $U = S^1 \times D^2$ and $V = D^2 \times S^1$. Let $i : T \to U$, $j : T \to V$ be the natural inclusion maps. Consider the space $S := U \cup_T V$; that is,

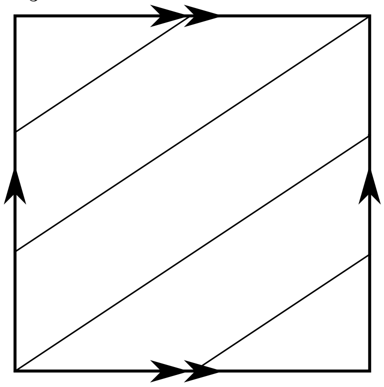
$$S := (U \sqcup V)/\sim$$

where $\sim$ is the smallest equivalence relation such that $i(x) \sim j(x)$ for all $x \in T$.

(a) Prove that S is homeomorphic to the 3-sphere S^3.

[*Hint: It may help to think of S^3 as contained in $\mathbb{C}^2$.*]

(b) Identify T as a quotient of the square $I \times I$ in the usual way. Let K be the circle in T given by the equation $y = \frac{2}{3}x \mod 1$. K is illustrated in the figure below.



Compute a presentation for $\pi_1(S - K)$, where $S - K$ is the complement of K in S, and deduce that $\pi_1(S - K)$ is non-abelian.

Paper 1, Section II

21F Algebraic Topology

In this question, X and Y are path-connected, locally simply connected spaces.

(a) Let $f : Y \to X$ be a continuous map, and $\widehat{X}$ a path-connected covering space of X. State and prove a uniqueness statement for lifts of f to $\widehat{X}$.

(b) Let $p : \widehat{X} \to X$ be a covering map. A covering transformation of p is a homeomorphism $\phi : \widehat{X} \to \widehat{X}$ such that $p \circ \phi = p$. For each integer $n \geqslant 3$, give an example of a space X and an n-sheeted covering map $p_n : \widehat{X}_n \to X$ such that the only covering transformation of p_n is the identity map. Justify your answer. *[Hint: Take X to be a wedge of two circles.]*

(c) Is there a space X and a 2-sheeted covering map $p_2 : \widehat{X}_2 \to X$ for which the only covering transformation of p_2 is the identity? Justify your answer briefly.

Paper 3, Section II

20H Algebraic Topology

(a) State a version of the Seifert–van Kampen theorem for a cell complex X written as the union of two subcomplexes Y, Z.

(b) Let

$$X_n = \underbrace{S^1 \vee \ldots \vee S^1}_{n} \vee \mathbb{R}P^2$$

for $n \geqslant 1$, and take any $x_0 \in X_n$. Write down a presentation for $\pi_1(X_n, x_0)$.

(c) By computing a homology group of a suitable four-sheeted covering space of X_n, prove that X_n is not homotopy equivalent to a compact, connected surface whenever $n \geqslant 1$.

Paper 2, Section II

21H Algebraic Topology

(a) Define the *first barycentric subdivision* K' of a simplicial complex K. Hence define the r^{th} *barycentric subdivision* $K^{(r)}$. [You do not need to prove that K' is a simplicial complex.]

(b) Define the *mesh* $\mu(K)$ of a simplicial complex K. State a result that describes the behaviour of $\mu(K^{(r)})$ as $r \to \infty$.

(c) Define a *simplicial approximation* to a continuous map of polyhedra

$$f : |K| \to |L|.$$

Prove that, if g is a simplicial approximation to f, then the realisation $|g| : |K| \to |L|$ is homotopic to f.

(d) State and prove the simplicial approximation theorem. [You may use the Lebesgue number lemma without proof, as long as you state it clearly.]

(e) Prove that every continuous map of spheres $S^n \to S^m$ is homotopic to a constant map when $n < m$.

Paper 1, Section II

21H Algebraic Topology

(a) Let V be the vector space of 3-dimensional upper-triangular matrices with real entries:

$$V = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

Let Γ be the set of elements of V for which x, y, z are integers. Notice that Γ is a subgroup of $GL_3(\mathbb{R})$; let Γ act on V by left-multiplication and let $N = \Gamma \backslash V$. Show that the quotient map $V \to N$ is a covering map.

(b) Consider the unit circle $S^1 \subseteq \mathbb{C}$, and let $T = S^1 \times S^1$. Show that the map $f : T \to T$ defined by

$$f(z, w) = (zw, w)$$

is a homeomorphism.

(c) Let $M = [0, 1] \times T / \sim$, where $\sim$ is the smallest equivalence relation satisfying

$$(1, x) \sim (0, f(x))$$

for all $x \in T$. Prove that N and M are homeomorphic by exhibiting a homeomorphism $M \to N$. [You may assume without proof that N is Hausdorff.]

(d) Prove that $\pi_1(M) \cong \Gamma$.

Paper 4, Section II

21H Algebraic Topology

(a) State the Mayer–Vietoris theorem for a union of simplicial complexes

$$K = M \cup N$$

with $L = M \cap N$.

(b) Construct the map $\partial_* : H_k(K) \to H_{k-1}(L)$ that appears in the statement of the theorem. [You do not need to prove that the map is well defined, or a homomorphism.]

(c) Let K be a simplicial complex with $|K|$ homeomorphic to the n-dimensional sphere S^n, for $n \geqslant 2$. Let $M \subseteq K$ be a subcomplex with $|M|$ homeomorphic to $S^{n-1} \times [-1, 1]$. Suppose that $K = M \cup N$, such that $L = M \cap N$ has polyhedron $|L|$ identified with $S^{n-1} \times \{-1, 1\} \subseteq S^{n-1} \times [-1, 1]$. Prove that $|N|$ has two path components.

Paper 3, Section II

18I Algebraic Topology

The n-torus is the product of n circles:

$$T^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} .$$

For all $n \geqslant 1$ and $0 \leqslant k \leqslant n$, compute $H_k(T^n)$.

[You may assume that relevant spaces are triangulable, but you should state carefully any version of any theorem that you use.]

Paper 2, Section II

19I Algebraic Topology

(a) (i) Define the *push-out* of the following diagram of groups.

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"] & G_1 \\ G_2 & \end{tikzcd}$$

When is a push-out a *free product with amalgamation*?

(ii) State the Seifert–van Kampen theorem.

(b) Let $X = \mathbb{R}P^2 \vee S^1$ (recalling that $\mathbb{R}P^2$ is the real projective plane), and let $x \in X$.

(i) Compute the fundamental group $\pi_1(X, x)$ of the space X.

(ii) Show that there is a surjective homomorphism $\phi : \pi_1(X, x) \to S_3$, where S_3 is the symmetric group on three elements.

(c) Let $\widehat{X} \to X$ be the covering space corresponding to the kernel of ϕ.

(i) Draw $\widehat{X}$ and justify your answer carefully.

(ii) Does $\widehat{X}$ retract to a graph? Justify your answer briefly.

(iii) Does $\widehat{X}$ deformation retract to a graph? Justify your answer briefly.

Paper 1, Section II

20I Algebraic Topology

Let X be a topological space and let x_0 and x_1 be points of X.

(a) Explain how a path $u : [0,1] \to X$ from x_0 to x_1 defines a map $u_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

(b) Prove that $u_\#$ is an isomorphism of groups.

(c) Let $\alpha, \beta : (S^1, 1) \to (X, x_0)$ be based loops in X. Suppose that α, β are homotopic as *unbased* maps, i.e. the homotopy is not assumed to respect basepoints. Show that the corresponding elements of $\pi_1(X, x_0)$ are conjugate.

(d) Take X to be the 2-torus $S^1 \times S^1$. If α, β are homotopic as unbased loops as in part (c), then *exhibit* a based homotopy between them. Interpret this fact algebraically.

(e) Exhibit a pair of elements in the fundamental group of $S^1 \vee S^1$ which are homotopic as unbased loops but not as based loops. Justify your answer.

Paper 4, Section II

20I Algebraic Topology

Recall that $\mathbb{R}P^n$ is real projective n-space, the quotient of S^n obtained by identifying antipodal points. Consider the standard embedding of S^n as the unit sphere in $\mathbb{R}^{n+1}$.

(a) For n odd, show that there exists a continuous map $f : S^n \to S^n$ such that $f(x)$ is orthogonal to x, for all $x \in S^n$.

(b) Exhibit a triangulation of $\mathbb{R}P^n$.

(c) Describe the map $H_n(S^n) \to H_n(S^n)$ induced by the antipodal map, justifying your answer.

(d) Show that, for n even, there is no continuous map $f : S^n \to S^n$ such that $f(x)$ is orthogonal to x for all $x \in S^n$.

Paper 3, Section II

18G Algebraic Topology

Construct a space X as follows. Let Z_1, Z_2, Z_3 each be homeomorphic to the standard 2-sphere $S^2 \subseteq \mathbb{R}^3$. For each i, let $x_i \in Z_i$ be the North pole $(1,0,0)$ and let $y_i \in Z_i$ be the South pole $(-1,0,0)$. Then

$$X = (Z_1 \sqcup Z_2 \sqcup Z_3)/\sim$$

where $x_{i+1} \sim y_i$ for each i (and indices are taken modulo 3).

(a) Describe the universal cover of X.

(b) Compute the fundamental group of X (giving your answer as a well-known group).

(c) Show that X is not homotopy equivalent to the circle S^1.

Paper 2, Section II

19G Algebraic Topology

(a) Let K, L be simplicial complexes, and $f : |K| \to |L|$ a continuous map. What does it mean to say that $g : K \to L$ is a *simplicial approximation* to f?

(b) Define the *barycentric subdivision* of a simplicial complex K, and state the Simplicial Approximation Theorem.

(c) Show that if g is a simplicial approximation to f then $f \simeq |g|$.

(d) Show that the natural inclusion $|K^{(1)}| \to |K|$ induces a surjective map on fundamental groups.

Paper 1, Section II

20G Algebraic Topology

Let $T = S^1 \times S^1$ be the 2-dimensional torus. Let $\alpha : S^1 \to T$ be the inclusion of the coordinate circle $S^1 \times \{1\}$, and let X be the result of attaching a 2-cell along α.

(a) Write down a presentation for the fundamental group of X (with respect to some basepoint), and identify it with a well-known group.

(b) Compute the simplicial homology of any triangulation of X.

(c) Show that X is not homotopy equivalent to any compact surface.

Paper 4, Section II

20G Algebraic Topology

Let $T = S^1 \times S^1$ be the 2-dimensional torus, and let X be constructed from T by removing a small open disc.

(a) Show that X is homotopy equivalent to $S^1 \vee S^1$.

(b) Show that the universal cover of X is homotopy equivalent to a tree.

(c) Exhibit (finite) cell complexes X, Y, such that X and Y are not homotopy equivalent but their universal covers $\widetilde{X}, \widetilde{Y}$ are.

[*State carefully any results from the course that you use.*]

Paper 3, Section II
17H Algebraic Topology

Let K and L be simplicial complexes. Explain what is meant by a *simplicial approximation* to a continuous map $f : |K| \to |L|$. State the simplicial approximation theorem, and define the homomorphism induced on homology by a continuous map between triangulable spaces. [You do not need to show that the homomorphism is well-defined.]

Let $h : S^1 \to S^1$ be given by $z \mapsto z^n$ for a positive integer n, where S^1 is considered as the unit complex numbers. Compute the map induced by h on homology.

Paper 4, Section II
18H Algebraic Topology

State the Mayer–Vietoris theorem for a simplicial complex K which is the union of two subcomplexes M and N. Explain briefly how the connecting homomorphism $\partial_n : H_n(K) \to H_{n-1}(M \cap N)$ is defined.

If K is the union of subcomplexes $M_1, M_2, \ldots, M_n$, with $n \geqslant 2$, such that each intersection

$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_k}, \qquad 1 \leqslant k \leqslant n,$$

is either empty or has the homology of a point, then show that

$$H_i(K) = 0 \quad \text{for} \quad i \geqslant n-1.$$

Construct examples for each $n \geqslant 2$ showing that this is sharp.

Paper 2, Section II
18H Algebraic Topology

Define what it means for $p : \widetilde{X} \to X$ to be a covering map, and what it means to say that p is a universal cover.

Let $p : \widetilde{X} \to X$ be a universal cover, $A \subset X$ be a locally path connected subspace, and $\widetilde{A} \subset p^{-1}(A)$ be a path component containing a point $\tilde{a}_0$ with $p(\tilde{a}_0) = a_0$. Show that the restriction $p|_{\widetilde{A}} : \widetilde{A} \to A$ is a covering map, and that under the Galois correspondence it corresponds to the subgroup

$$\mathrm{Ker}\big(\, \pi_1(A, a_0) \to \pi_1(X, a_0) \,\big)$$

of $\pi_1(A, a_0)$.

Paper 1, Section II

18H Algebraic Topology

State carefully a version of the Seifert–van Kampen theorem for a cover of a space by two closed sets.

Let X be the space obtained by gluing together a Möbius band M and a torus $T = S^1 \times S^1$ along a homeomorphism of the boundary of M with $S^1 \times \{1\} \subset T$. Find a presentation for the fundamental group of X, and hence show that it is infinite and non-abelian.

Paper 3, Section II
20F Algebraic Topology

Let K be a simplicial complex in $\mathbb{R}^N$, which we may also consider as lying in $\mathbb{R}^{N+1}$ using the first N coordinates. Write $c = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{N+1}$. Show that if $\langle v_0, v_1, \ldots, v_n \rangle$ is a simplex of K then $\langle v_0, v_1, \ldots, v_n, c \rangle$ is a simplex in $\mathbb{R}^{N+1}$.

Let $L \leqslant K$ be a subcomplex and let $\overline{K}$ be the collection

$$K \cup \{\langle v_0, v_1, \ldots, v_n, c \rangle \mid \langle v_0, v_1, \ldots, v_n \rangle \in L\} \cup \{\langle c \rangle\}$$

of simplices in $\mathbb{R}^{N+1}$. Show that $\overline{K}$ is a simplicial complex.

If $|K|$ is a Möbius band, and $|L|$ is its boundary, show that

$$H_i(\overline{K}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2 & \text{if } i = 1 \\ 0 & \text{if } i \geqslant 2. \end{cases}$$

Paper 4, Section II
21F Algebraic Topology

State the Lefschetz fixed point theorem.

Let X be an orientable surface of genus g (which you may suppose has a triangulation), and let $f : X \to X$ be a continuous map such that

1. $f^3 = \mathrm{Id}_X$,
2. f has no fixed points.

By considering the eigenvalues of the linear map $f_* : H_1(X; \mathbb{Q}) \to H_1(X; \mathbb{Q})$, and their multiplicities, show that g must be congruent to 1 modulo 3.

Paper 2, Section II
21F Algebraic Topology

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with integer entries. Considering S^1 as the quotient space $\mathbb{R}/\mathbb{Z}$, show that the function

$$\begin{aligned} \varphi_A : S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ ([x],[y]) &\longmapsto ([ax+by],[cx+dy]) \end{aligned}$$

is well-defined and continuous. If in addition $\det(A) = \pm 1$, show that φ_A is a homeomorphism.

State the Seifert–van Kampen theorem. Let X_A be the space obtained by gluing together two copies of $S^1 \times D^2$ along their boundaries using the homeomorphism φ_A. Show that the fundamental group of X_A is cyclic and determine its order.

Paper 1, Section II
21F Algebraic Topology

Define what it means for a map $p : \widetilde{X} \to X$ to be a *covering space*. State the homotopy lifting lemma.

Let $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a based covering space and let $f : (Y, y_0) \to (X, x_0)$ be a based map from a path-connected and locally path-connected space. Show that there is a based lift $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ of f if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.

Paper 3, Section II
20G Algebraic Topology

(i) State, but do not prove, the Mayer–Vietoris theorem for the homology groups of polyhedra.

(ii) Calculate the homology groups of the n-sphere, for every $n \geqslant 0$.

(iii) Suppose that $a \geqslant 1$ and $b \geqslant 0$. Calculate the homology groups of the subspace X of $\mathbb{R}^{a+b}$ defined by $\displaystyle\sum_{i=1}^{a} x_i^2 - \sum_{j=a+1}^{a+b} x_j^2 = 1$.

Paper 4, Section II
21G Algebraic Topology

(i) State, but do not prove, the Lefschetz fixed point theorem.

(ii) Show that if n is even, then for every map $f : S^n \to S^n$ there is a point $x \in S^n$ such that $f(x) = \pm x$. Is this true if n is odd? [Standard results on the homology groups for the n-sphere may be assumed without proof, provided they are stated clearly.]

Paper 2, Section II
21G Algebraic Topology

(i) State the Seifert–van Kampen theorem.

(ii) Assuming any standard results about the fundamental group of a circle that you wish, calculate the fundamental group of the n-sphere, for every $n \geqslant 2$.

(iii) Suppose that $n \geqslant 3$ and that X is a path-connected topological n-manifold. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X - \{P\}, x_0)$ for any $P \in X - \{x_0\}$.

Paper 1, Section II

21G Algebraic Topology

(i) Define the notion of the fundamental group $\pi_1(X, x_0)$ of a path-connected space X with base point x_0.

(ii) Prove that if a group G acts freely and properly discontinuously on a simply connected space Z, then $\pi_1(G\backslash Z, x_0)$ is isomorphic to G. [You may assume the homotopy lifting property, provided that you state it clearly.]

(iii) Suppose that p, q are distinct points on the 2-sphere S^2 and that $X = S^2/(p \sim q)$. Exhibit a simply connected space Z with an action of a group G as in (ii) such that $X = G\backslash Z$, and calculate $\pi_1(X, x_0)$.

Paper 3, Section II

20G Algebraic Topology

State the Mayer–Vietoris Theorem for a simplicial complex K expressed as the union of two subcomplexes L and M. Explain briefly how the connecting homomorphism $\delta_* : H_n(K) \to H_{n-1}(L \cap M)$, which appears in the theorem, is defined. [You should include a proof that δ_* is well-defined, but need not verify that it is a homomorphism.]

Now suppose that $|K| \cong S^3$, that $|L|$ is a solid torus $S^1 \times B^2$, and that $|L \cap M|$ is the boundary torus of $|L|$. Show that $\delta_* : H_3(K) \to H_2(L \cap M)$ is an isomorphism, and hence calculate the homology groups of M. [You may assume that a generator of $H_3(K)$ may be represented by a 3-cycle which is the sum of all the 3-simplices of K, with 'matching' orientations.]

Paper 4, Section II

21G Algebraic Topology

State and prove the Lefschetz fixed-point theorem. Hence show that the n-sphere S^n does not admit a topological group structure for any even $n > 0$. [The existence and basic properties of simplicial homology with rational coefficients may be assumed.]

Paper 2, Section II

21G Algebraic Topology

State the Seifert–Van Kampen Theorem. Deduce that if $f : S^1 \to X$ is a continuous map, where X is path-connected, and $Y = X \cup_f B^2$ is the space obtained by adjoining a disc to X via f, then $\Pi_1(Y)$ is isomorphic to the quotient of $\Pi_1(X)$ by the smallest normal subgroup containing the image of $f_* : \Pi_1(S^1) \to \Pi_1(X)$.

State the classification theorem for connected triangulable 2-manifolds. Use the result of the previous paragraph to obtain a presentation of $\Pi_1(M_g)$, where M_g denotes the compact orientable 2-manifold of genus $g > 0$.

Paper 1, Section II

21G Algebraic Topology

Define the notions of *covering projection* and of *locally path-connected space*. Show that a locally path-connected space is path-connected if it is connected.

Suppose $f\colon Y \to X$ and $g\colon Z \to X$ are continuous maps, the space Y is connected and locally path-connected and that g is a covering projection. Suppose also that we are given base-points x_0, y_0, z_0 satisfying $f(y_0) = x_0 = g(z_0)$. Show that there is a continuous $\tilde{f} : Y \to Z$ satisfying $\tilde{f}(y_0) = z_0$ and $g\tilde{f} = f$ if and only if the image of $f_*\colon \Pi_1(Y, y_0) \to \Pi_1(X, x_0)$ is contained in that of $g_*\colon \Pi_1(Z, z_0) \to \Pi_1(X, x_0)$. [You may assume the path-lifting and homotopy-lifting properties of covering projections.]

Now suppose X is locally path-connected, and both $f\colon Y \to X$ and $g\colon Z \to X$ are covering projections with connected domains. Show that Y and Z are homeomorphic as spaces over X if and only if the images of their fundamental groups under f_* and g_* are conjugate subgroups of $\Pi_1(X, x_0)$.

Paper 1, Section II

21H Algebraic Topology

Are the following statements true or false? Justify your answers.

(i) If x and y lie in the same path-component of X, then $\Pi_1(X,x) \cong \Pi_1(X,y)$.

(ii) If x and y are two points of the Klein bottle K, and u and v are two paths from x to y, then u and v induce the same isomorphism from $\Pi_1(K,x)$ to $\Pi_1(K,y)$.

(iii) $\Pi_1(X \times Y, (x,y))$ is isomorphic to $\Pi_1(X,x) \times \Pi_1(Y,y)$ for any two spaces X and Y.

(iv) If X and Y are connected polyhedra and $H_1(X) \cong H_1(Y)$, then $\Pi_1(X) \cong \Pi_1(Y)$.

Paper 2, Section II

21H Algebraic Topology

Explain what is meant by a covering projection. State and prove the path-lifting property for covering projections, and indicate briefly how it generalizes to a lifting property for homotopies between paths. [You may assume the Lebesgue Covering Theorem.]

Let X be a simply connected space, and let G be a subgroup of the group of all homeomorphisms $X \to X$. Suppose that, for each $x \in X$, there exists an open neighbourhood U of x such that $U \cap g[U] = \emptyset$ for each $g \in G$ other than the identity. Show that the projection $p\colon X \to X/G$ is a covering projection, and deduce that $\Pi_1(X/G) \cong G$.

By regarding S^3 as the set of all quaternions of modulus 1, or otherwise, show that there is a quotient space of S^3 whose fundamental group is a non-abelian group of order 8.

Paper 3, Section II

20H Algebraic Topology

Let K and L be (finite) simplicial complexes. Explain carefully what is meant by a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$. Indicate briefly how the cartesian product $|K| \times |L|$ may be triangulated.

Two simplicial maps $g, h\colon K \to L$ are said to be *contiguous* if, for each simplex σ of K, there exists a simplex $\sigma*$ of L such that both $g(\sigma)$ and $h(\sigma)$ are faces of $\sigma*$. Show that:

(i) any two simplicial approximations to a given map $f\colon |K| \to |L|$ are contiguous;

(ii) if g and h are contiguous, then they induce homotopic maps $|K| \to |L|$;

(iii) if f and g are homotopic maps $|K| \to |L|$, then for some subdivision $K^{(n)}$ of K there exists a sequence $(h_1, h_2, \ldots, h_m)$ of simplicial maps $K^{(n)} \to L$ such that h_1 is a simplicial approximation to f, h_m is a simplicial approximation to g and each pair (h_i, h_{i+1}) is contiguous.

Paper 4, Section II

21H Algebraic Topology

State the Mayer–Vietoris theorem, and use it to calculate, for each integer $q > 1$, the homology group of the space X_q obtained from the unit disc $B^2 \subseteq \mathbb{C}$ by identifying pairs of points (z_1, z_2) on its boundary whenever $z_1^q = z_2^q$. [You should construct an explicit triangulation of X_q.]

Show also how the theorem may be used to calculate the homology groups of the suspension SK of a connected simplicial complex K in terms of the homology groups of K, and of the wedge union $X \vee Y$ of two connected polyhedra. Hence show that, for any finite sequence $(G_1, G_2, \ldots, G_n)$ of finitely-generated abelian groups, there exists a polyhedron X such that $H_0(X) \cong \mathbb{Z}$, $H_i(X) \cong G_i$ for $1 \leqslant i \leqslant n$ and $H_i(X) = 0$ for $i > n$. [You may assume the structure theorem which asserts that any finitely-generated abelian group is isomorphic to a finite direct sum of (finite or infinite) cyclic groups.]

Paper 1, Section II

21H Algebraic Topology

State the path lifting and homotopy lifting lemmas for covering maps. Suppose that X is path connected and locally path connected, that $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are covering maps, and that Y_1 and Y_2 are simply connected. Using the lemmas you have stated, but without assuming the correspondence between covering spaces and subgroups of π_1, prove that Y_1 is homeomorphic to Y_2.

Paper 2, Section II

21H Algebraic Topology

Let G be the finitely presented group $G = \langle a, b \,|\, a^2\, b^3\, a^3\, b^2 = 1 \rangle$. Construct a path connected space X with $\pi_1(X, x) \cong G$. Show that X has a unique connected double cover $\pi : Y \to X$, and give a presentation for $\pi_1(Y, y)$.

Paper 3, Section II

20H Algebraic Topology

Suppose X is a finite simplicial complex and that $H_*(X)$ is a free abelian group for each value of $*$. Using the Mayer-Vietoris sequence or otherwise, compute $H_*(S^1 \times X)$ in terms of $H_*(X)$. Use your result to compute $H_*(T^n)$.

[Note that $T^n = S^1 \times \ldots \times S^1$, where there are n factors in the product.]

Paper 4, Section II

21H Algebraic Topology

State the Snake Lemma. Explain how to define the boundary map which appears in it, and check that it is well-defined. Derive the Mayer-Vietoris sequence from the Snake Lemma.

Given a chain complex C, let $A \subset C$ be the span of all elements in C with grading greater than or equal to n, and let $B \subset C$ be the span of all elements in C with grading less than n. Give a short exact sequence of chain complexes relating A, B, and C. What is the boundary map in the corresponding long exact sequence?

Paper 1, Section II

21G Algebraic Topology

Let X be the space obtained by identifying two copies of the Möbius strip along their boundary. Use the Seifert–Van Kampen theorem to find a presentation of the fundamental group $\pi_1(X)$. Show that $\pi_1(X)$ is an infinite non-abelian group.

Paper 2, Section II

21G Algebraic Topology

Let $p : X \to Y$ be a connected covering map. Define the notion of a *deck transformation* (also known as *covering transformation*) for p. What does it mean for p to be a *regular* (*normal*) covering map?

If $p^{-1}(y)$ contains n points for each $y \in Y$, we say p is n-to-1. Show that p is regular under either of the following hypotheses:

(1) p is 2-to-1,

(2) $\pi_1(Y)$ is abelian.

Give an example of a 3-to-1 cover of $S^1 \vee S^1$ which is regular, and one which is not regular.

Paper 3, Section II

20G Algebraic Topology

(i) Suppose that (C, d) and (C', d') are chain complexes, and $f, g : C \to C'$ are chain maps. Define what it means for f and g to be *chain homotopic*.

Show that if f and g are chain homotopic, and $f_*, g_* : H_*(C) \to H_*(C')$ are the induced maps, then $f_* = g_*$.

(ii) Define the *Euler characteristic* of a finite chain complex.

Given that one of the sequences below is exact and the others are not, which is the exact one?

$$0 \to \mathbb{Z}^{11} \to \mathbb{Z}^{24} \to \mathbb{Z}^{20} \to \mathbb{Z}^{13} \to \mathbb{Z}^{20} \to \mathbb{Z}^{25} \to \mathbb{Z}^{11} \to 0,$$

$$0 \to \mathbb{Z}^{11} \to \mathbb{Z}^{24} \to \mathbb{Z}^{20} \to \mathbb{Z}^{13} \to \mathbb{Z}^{20} \to \mathbb{Z}^{24} \to \mathbb{Z}^{11} \to 0,$$

$$0 \to \mathbb{Z}^{11} \to \mathbb{Z}^{24} \to \mathbb{Z}^{19} \to \mathbb{Z}^{13} \to \mathbb{Z}^{20} \to \mathbb{Z}^{23} \to \mathbb{Z}^{11} \to 0.$$

Justify your choice.

Paper 4, Section II

21G Algebraic Topology

Let X be the subset of $\mathbb{R}^4$ given by $X = A \cup B \cup C \subset \mathbb{R}^4$, where A, B and C are defined as follows:

$$
\begin{aligned}
A &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\},\\
B &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_1 = x_2 = 0, x_3^2 + x_4^2 \leqslant 1\},\\
C &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ x_3 = x_4 = 0, x_1^2 + x_2^2 \leqslant 1\}.
\end{aligned}
$$

Compute $H_*(X)$.

1/II/21F **Algebraic Topology**

(i) State the van Kampen theorem.

(ii) Calculate the fundamental group of the wedge $S^2 \vee S^1$.

(iii) Let $X = \mathbb{R}^3 \setminus A$ where A is a circle. Calculate the fundamental group of X.

2/II/21F **Algebraic Topology**

Prove the Borsuk–Ulam theorem in dimension 2: there is no map $f : S^2 \to S^1$ such that $f(-x) = -f(x)$ for every $x \in S^2$. Deduce that S^2 is not homeomorphic to any subset of $\mathbb{R}^2$.

3/II/20F **Algebraic Topology**

Let X be the quotient space obtained by identifying one pair of antipodal points on S^2. Using the Mayer–Vietoris exact sequence, calculate the homology groups and the Betti numbers of X.

4/II/21F **Algebraic Topology**

Let X and Y be topological spaces.

(i) Show that a map $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq 1_Y$ and $hf \simeq 1_X$. More generally, show that a map $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that fg and hf are homotopy equivalences.

(ii) Suppose that $\tilde{X}$ and $\tilde{Y}$ are universal covering spaces of the path-connected, locally path-connected spaces X and Y. Using path-lifting properties, show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$.

1/II/21H Algebraic Topology

(i) Compute the fundamental group of the Klein bottle. Show that this group is not abelian, for example by defining a suitable homomorphism to the symmetric group S_3.

(ii) Let X be the closed orientable surface of genus 2. How many (connected) double coverings does X have? Show that the fundamental group of X admits a homomorphism onto the free group on 2 generators.

2/II/21H Algebraic Topology

State the Mayer–Vietoris sequence for a simplicial complex X which is a union of two subcomplexes A and B. Define the homomorphisms in the sequence (but do *not* check that they are well-defined). Prove exactness of the sequence at the term $H_i(A \cap B)$.

3/II/20H Algebraic Topology

Define what it means for a group G to act on a topological space X. Prove that, if G acts freely, in a sense that you should specify, then the quotient map $X \to X/G$ is a covering map and there is a surjective group homomorphism from the fundamental group of X/G to G.

4/II/21H Algebraic Topology

Compute the homology of the space obtained from the torus $S^1 \times S^1$ by identifying $S^1 \times \{p\}$ to a point and $S^1 \times \{q\}$ to a point, for two distinct points p and q in S^1.

1/II/21H Algebraic Topology

Compute the homology groups of the "pinched torus" obtained by identifying a meridian circle $S^1 \times \{p\}$ on the torus $S^1 \times S^1$ to a point, for some point $p \in S^1$.

2/II/21H Algebraic Topology

State the simplicial approximation theorem. Compute the number of 0-simplices (vertices) in the barycentric subdivision of an n-simplex and also compute the number of n-simplices. Finally, show that there are at most countably many homotopy classes of continuous maps from the 2-sphere to itself.

3/II/20H Algebraic Topology

Let X be the union of two circles identified at a point: the "figure eight". Classify all the connected double covering spaces of X. If we view these double coverings just as topological spaces, determine which of them are homeomorphic to each other and which are not.

4/II/21H Algebraic Topology

Fix a point p in the torus $S^1 \times S^1$. Let G be the group of homeomorphisms f from the torus $S^1 \times S^1$ to itself such that $f(p) = p$. Determine a non-trivial homomorphism ϕ from G to the group $\mathrm{GL}(2, \mathbb{Z})$.
[*The group* $\mathrm{GL}(2,\mathbb{Z})$ *consists of* 2×2 *matrices with integer coefficients that have an inverse which also has integer coefficients.*]

Establish whether each f in the kernel of ϕ is homotopic to the identity map.

1/II/21H **Algebraic Topology**

(i) Show that if $E \to T$ is a covering map for the torus $T = S^1 \times S^1$, then E is homeomorphic to one of the following: the plane $\mathbb{R}^2$, the cylinder $\mathbb{R} \times S^1$, or the torus T.

(ii) Show that any continuous map from a sphere S^n ($n \geqslant 2$) to the torus T is homotopic to a constant map.

[*General theorems from the course may be used without proof, provided that they are clearly stated.*]

2/II/21H **Algebraic Topology**

State the Van Kampen Theorem. Use this theorem and the fact that $\pi_1 S^1 = \mathbb{Z}$ to compute the fundamental groups of the torus $T = S^1 \times S^1$, the punctured torus $T \setminus \{p\}$, for some point $p \in T$, and the connected sum $T \# T$ of two copies of T.

3/II/20H **Algebraic Topology**

Let X be a space that is triangulable as a simplicial complex with no n-simplices. Show that any continuous map from X to S^n is homotopic to a constant map.

[*General theorems from the course may be used without proof, provided they are clearly stated.*]

4/II/21H **Algebraic Topology**

Let X be a simplicial complex. Suppose $X = B \cup C$ for subcomplexes B and C, and let $A = B \cap C$. Show that the inclusion of A in B induces an isomorphism $H_* A \to H_* B$ if and only if the inclusion of C in X induces an isomorphism $H_* C \to H_* X$.