Part IB

Special Relativity

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Paper 1, Section I

4C Special Relativity

Write down the components of the position four-vector x_{μ} . Hence find the components of the four-momentum $p_{\mu} = MU_{\mu}$ of a particle of mass M, where $U_{\mu} = dx_{\mu}/d\tau$, with τ being the proper time.

The particle, viewed in a frame in which it is initially at rest, disintegrates leaving a particle of mass m moving with constant velocity together with other remnants which have a total three-momentum \mathbf{p} and energy E. Show that

$$m = \sqrt{\left(M - \frac{E}{c^2}\right)^2 - \frac{|\mathbf{p}|^2}{c^2}}$$

Paper 2, Section I

7C Special Relativity

Show that the two-dimensional Lorentz transformation relating (ct', x') in frame S' to (ct, x) in frame S, where S' moves relative to S with speed v, can be written in the form

$$x' = x \cosh \phi - ct \sinh \phi$$
$$ct' = -x \sinh \phi + ct \cosh \phi,$$

where the hyperbolic angle ϕ associated with the transformation is given by $\tanh \phi = v/c$. Deduce that

$$x' + ct' = e^{-\phi}(x + ct)$$
$$x' - ct' = e^{\phi}(x - ct).$$

Hence show that if the frame S'' moves with speed v' relative to S' and $\tanh \phi' = v'/c$, then the hyperbolic angle associated with the Lorentz transformation connecting S'' and S is given by

$$\phi'' = \phi' + \phi.$$

Hence find an expression for the speed of S'' as seen from S.

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Paper 4, Section II

17C Special Relativity

A star moves with speed v in the x-direction in a reference frame S. When viewed in its rest frame S' it emits a photon of frequency ν' which propagates along a line making an angle θ' with the x'-axis. Write down the components of the four-momentum of the photon in S'. As seen in S, the photon moves along a line that makes an angle θ with the x-axis and has frequency ν . Using a Lorentz transformation, write down the relationship between the components of the four-momentum of the photon in S' to those in S and show that

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + v\cos\theta'/c}.$$

As viewed in S', the star emits two photons with frequency ν' in opposite directions with $\theta' = \pi/2$ and $\theta' = -\pi/2$, respectively. Show that an observer in S records them as having a combined momentum p directed along the x-axis, where

$$p = \frac{Ev}{c^2\sqrt{1 - v^2/c^2}}$$

and where E is the combined energy of the photons as seen in S'. How is this momentum loss from the star consistent with its maintaining a constant speed as viewed in S?

1/I/4C Special Relativity

In an inertial frame S a photon of energy E is observed to travel at an angle θ relative to the x-axis. The inertial frame S' moves relative to S at velocity v in the x-direction and the x'-axis of S' is taken parallel to the x-axis of S. Observed in S', the photon has energy E' and travels at an angle θ' relative to the x'-axis. Show that

$$E' = \frac{E(1 - \beta \cos \theta)}{\sqrt{1 - \beta^2}}, \qquad \cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta},$$

where $\beta = v/c$.

2/I/7C Special Relativity

A photon of energy E collides with a particle of rest mass m, which is at rest. The final state consists of a photon and a particle of rest mass M, M > m. Show that the minimum value of E for which it is possible for this reaction to take place is

$$E_{\min} = \frac{M^2 - m^2}{2m} c^2 .$$

4/II/17C Special Relativity

Write down the formulae for the one-dimensional Lorentz transformation $(x,t) \rightarrow (x',t')$ for frames moving with relative velocity v along the x-direction. Derive the relativistic formula for the addition of velocities v and u.

A train, of proper length L, travels past a station at velocity v > 0. The origin of the inertial frame S, with coordinates (x, t), in which the train is stationary, is located at the mid-point of the train. The origin of the inertial frame S', with coordinates (x', t'), in which the station is stationary, is located at the mid-point of the platform. Coordinates are chosen such that when the origins coincide then t = t' = 0.

Observers A and B, stationary in S, are located, respectively, at the front and rear of the train. Observer C, stationary in S', is located at the origin of S'. At t' = 0, C sends two signals, which both travel at speed u, where $v < u \leq c$, one directed towards A and the other towards B, who receive the signals at respective times t_A and t_B . C observes these events to occur, respectively, at times t'_A and t'_B . At t' = 0, C also observes that the two ends of the platform coincide with the positions of A and B.

(a) Draw two space-time diagrams, one for S and the other for S', showing the trajectories of the observers and the events that take place.

(b) What is the length of the platform in terms of L? Briefly illustrate your answer by reference to the space-time diagrams.

(c) Calculate the time differences $t_B - t_A$ and $t'_B - t'_A$.

(d) Setting u = c, use this example to discuss briefly the fact that two events observed to be simultaneous in one frame need not be observed to be simultaneous in another.

1/I/4B Special Relativity

Write down the position four-vector. Suppose this represents the position of a particle with rest mass M and velocity \mathbf{v} . Show that the four momentum of the particle is

$$p_a = (M\gamma c, M\gamma \mathbf{v}),$$

where $\gamma = (1 - |\mathbf{v}|^2 / c^2)^{-1/2}$.

For a particle of zero rest mass show that

$$p_a = \left(|\mathbf{p}|, \; \mathbf{p}\right),$$

where \mathbf{p} is the three momentum.

2/I/7B Special Relativity

A particle in inertial frame S has coordinates (t, x), whilst the coordinates are (t', x') in frame S', which moves with relative velocity v in the x direction. What is the relationship between the coordinates of S and S'?

Frame S'', with coordinates (t'', x''), moves with velocity u with respect to S' and velocity V with respect to S. Derive the relativistic formula for V in terms of u and v. Show how the Newtonian limit is recovered.

4/II/17B Special Relativity

(a) A moving π^0 particle of rest-mass m_{π} decays into two photons of zero rest-mass,

$$\pi^0 \to \gamma + \gamma$$
.

Show that

$$\sin\frac{\theta}{2} = \frac{m_\pi c^2}{2\sqrt{E_1 E_2}} \,,$$

where θ is the angle between the three-momenta of the two photons and E_1, E_2 are their energies.

(b) The π^- particle of rest-mass m_{π} decays into an electron of rest-mass m_e and a neutrino of zero rest mass,

$$\pi^- \to e^- + \nu$$
.

Show that v, the speed of the electron in the rest frame of the π^- , is

$$v = c \left[\frac{1 - (m_e/m_\pi)^2}{1 + (m_e/m_\pi)^2} \right] \,.$$

1/I/4B Special Relativity

A ball of clay of mass m travels at speed v in the laboratory frame towards an identical ball at rest. After colliding head-on, the balls stick together, moving in the same direction as the first ball was moving before the collision. Calculate the mass m' and speed v' of the combined lump, justifying your answers carefully.

2/I/7B Special Relativity

 A_1 moves at speed v_1 in the x-direction with respect to A_0 . A_2 moves at speed v_2 in the x-direction with respect to A_1 . By applying a Lorentz transformation between the rest frames of A_0 , A_1 , and A_2 , calculate the speed at which A_0 observes A_2 to travel.

 A_3 moves at speed v_3 in the x-direction with respect to A_2 . Calculate the speed at which A_0 observes A_3 to travel.

4/II/17B Special Relativity

A javelin of length 4 metres is thrown at a speed of $\frac{12}{13}c$ horizontally and lengthwise through a barn of length 3 metres, which is open at both ends. (Here c denotes the speed of light.)

- (a) What is the length of the javelin in the rest frame of the barn?
- (b) What is the length of the barn in the rest frame of the javelin?

(c) Define the rest frame coordinates of the barn and of the javelin such that the point where the trailing end of the javelin enters the barn is the origin in both frames. Draw a space-time diagram in the rest frame coordinates (ct, x) of the barn, showing the world lines of both ends of the javelin and of the front and back of the barn. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the barn.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the barn, and (B) the leading end of the javelin exiting the back of the barn. Give the corresponding (ct, x) and (ct', x') coordinates for (B).

Are the events (A) and (B) space-like, null, or time-like separated?

As the javelin is longer than the barn in one frame and shorter than the barn in another, it might be argued that the javelin is contained entirely within the barn for a period according to an observer in one frame, but not according to an observer in another. Explain how this apparent inconsistency is resolved.

1/I/4G Special Relativity

The four-velocity U_{μ} of a particle of rest mass m is defined by $U_{\mu} = dx_{\mu}/d\tau$, where τ is the proper time (the time as measured in the particle's rest frame). Derive the expression for each of the four components of U_{μ} in terms of the components of the three-velocity and the speed of light, c.

Show that $U \cdot U = c^2$ for an appropriately defined scalar product.

The four-momentum, $p_{\mu} = m U_{\mu}$, of a particle of rest mass m defines a relativistic generalisation of energy and momentum. Show that the standard non-relativistic expressions for the momentum and kinetic energy of a particle with mass m travelling with velocity v are obtained in the limit $v/c \ll 1$. Show also how the concept of a rest energy equal to mc^2 emerges.

2/I/7G Special Relativity

Bob and Alice are twins. Bob accelerates rapidly away from Earth in a rocket that travels in a straight line until it reaches a velocity v relative to the Earth. It then travels with constant v for a long time before reversing its engines and decelerating rapidly until it is travelling at a velocity -v relative to the Earth. After a further long period of time the rocket returns to Earth, decelerating rapidly until it is at rest. Alice remains on Earth throughout. Sketch the space-time diagram that describes Bob's world-line in Alice's rest frame, assuming that the periods of acceleration and deceleration are negligibly small compared to the total time, explain carefully why Bob ages less than Alice between his departure and his return and show that

$$\Delta t_B = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t_A$$

where Δt_B is the time by which Bob has aged and Δt_A is the time by which Alice has aged.

Indicate on your diagram how Bob sees Alice aging during his voyage.



4/II/17G Special Relativity

Obtain the Lorentz transformations that relate the coordinates of an event measured in one inertial frame (t, x, y, z) to those in another inertial frame moving with velocity v along the x axis. Take care to state the assumptions that lead to your result.

A star is at rest in a three-dimensional coordinate frame S' that is moving at constant velocity v along the x axis of a second coordinate frame S. The star emits light of frequency ν' , which may considered to be a beam of photons. A light ray from the star to the origin in S' is a straight line that makes an angle θ' with the x' axis. Write down the components of the four-momentum of a photon in this light ray.

The star is seen by an observer at rest at the origin of S at time t = t' = 0, when the origins of the coordinate frames S and S' coincide. The light that reaches the observer moves along a line through the origin that makes an angle θ to the x axis and has frequency ν . Make use of the Lorentz transformations between the four-momenta of a photon in these two frames to determine the relation

$$\lambda = \lambda' \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \left(1 + \frac{v}{c} \cos \theta \right) \,.$$

where λ is the observed wavelength of the photon and λ' is the wavelength in the star's rest frame.

Comment on the form of this result in the special cases with $\cos \theta = 1$, $\cos \theta = -1$ and $\cos \theta = 0$.

[You may assume that the energy of a photon of frequency ν is $h\nu$ and its threemomentum is a three-vector of magnitude $h\nu/c$.]

3/I/8B Special Relativity

Write down the Lorentz transformation with one space dimension between two inertial frames S and S' moving relatively to one another at speed V.

A particle moves at velocity u in frame S. Find its velocity u' in frame S' and show that u' is always less than c.

4/I/7D Special Relativity

For a particle with energy E and momentum $(p \cos \theta, p \sin \theta, 0)$, explain why an observer moving in the *x*-direction with velocity v would find

$$E' = \gamma (E - p \cos \theta v), \qquad p' \cos \theta' = \gamma \left(p \cos \theta - E \frac{v}{c^2} \right), \qquad p' \sin \theta' = p \sin \theta,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. What is the relation between *E* and *p* for a photon? Show that the same relation holds for *E'* and *p'* and that

$$\cos \theta' = \frac{\cos \theta - \frac{v}{c}}{1 - \frac{v}{c} \cos \theta} \,.$$

What happens for $v \to c$?

4/II/17D Special Relativity

State how the 4-momentum p_{μ} of a particle is related to its energy and 3-momentum. How is p_{μ} related to the particle mass? For two particles with 4-momenta $p_{1\mu}$ and $p_{2\mu}$ find a Lorentz-invariant expression that gives the total energy in their centre of mass frame.

A photon strikes an electron at rest. What is the minimum energy it must have in order for it to create an electron and positron, of the same mass m_e as the electron, in addition to the original electron? Express the result in units of $m_e c^2$.

[It may be helpful to consider the minimum necessary energy in the centre of mass frame.]



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3/I/10A Special Relativity

What are the momentum and energy of a photon of wavelength λ ?

A photon of wavelength λ is incident on an electron. After the collision, the photon has wavelength λ' . Show that

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta),$$

where θ is the scattering angle and *m* is the electron rest mass.

4/I/9A Special Relativity

Prove that the two-dimensional Lorentz transformation can be written in the form

$$\begin{aligned} x' &= x \cosh \phi - ct \sinh \phi \\ ct' &= -x \sinh \phi + ct \cosh \phi, \end{aligned}$$

where $\tanh \phi = v/c$. Hence, show that

$$x' + ct' = e^{-\phi}(x + ct)$$
$$x' - ct' = e^{\phi}(x - ct).$$

Given that frame S' has speed v with respect to S and S'' has speed v' with respect to S', use this formalism to find the speed v'' of S'' with respect to S.

[*Hint:* rotation through a hyperbolic angle ϕ , followed by rotation through ϕ' , is equivalent to rotation through $\phi + \phi'$.]

4/II/18A Special Relativity

A pion of rest mass M decays at rest into a muon of rest mass m < M and a neutrino of zero rest mass. What is the speed u of the muon?

In the pion rest frame S, the muon moves in the y-direction. A moving observer, in a frame S' with axes parallel to those in the pion rest frame, wishes to take measurements of the decay along the x-axis, and notes that the pion has speed v with respect to the x-axis. Write down the four-dimensional Lorentz transformation relating S' to S and determine the momentum of the muon in S'. Hence show that in S' the direction of motion of the muon makes an angle θ with respect to the y-axis, where

$$\tan \theta = \frac{M^2 + m^2}{M^2 - m^2} \frac{v}{(c^2 - v^2)^{1/2}} \,.$$

3/I/10D Special Relativity

Write down the formulae for a Lorentz transformation with velocity v taking one set of co-ordinates (t, x) to another (t', x').

Imagine you observe a train travelling past Cambridge station at a relativistic speed u. Someone standing still on the train throws a ball in the direction the train is moving, with speed v. How fast do you observe the ball to be moving? Justify your answer.

4/I/9D Special Relativity

A particle with mass M is observed to be at rest. It decays into a particle of mass m < M, and a massless particle. Calculate the energies and momenta of both final particles.

4/II/18D Special Relativity

A javelin of length 2m is thrown horizontally and lengthwise into a shed of length 1.5m at a speed of 0.8c, where c is the speed of light.

- (a) What is the length of the javelin in the rest frame of the shed?
- (b) What is the length of the shed in the rest frame of the javelin?

(c) Draw a space-time diagram in the rest frame coordinates (ct, x) of the shed, showing the world lines of both ends of the javelin, and of the front and back of the shed. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the shed.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the shed, and (B) the leading end of the javelin hitting the back of the shed. Give the corresponding (ct, x) and (ct', x') coordinates for both (A) and (B). Are these two events space-like, null or time-like separated? How does the javelin fit inside the shed, even though it is initially longer than the shed in its own rest frame?

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3/I/10F Special Relativity

A particle of rest mass m and four-momentum $P = (E/c, \mathbf{p})$ is detected by an observer with four-velocity U, satisfying $U \cdot U = c^2$, where the product of two four-vectors $P = (p^0, \mathbf{p})$ and $Q = (q^0, \mathbf{q})$ is $P \cdot Q = p^0 q^0 - \mathbf{p} \cdot \mathbf{q}$.

Show that the speed of the detected particle in the observer's rest frame is

$$v=c~\sqrt{1-\frac{P\cdot Pc^2}{(P\cdot U)^2}}\cdot$$

4/I/9F Special Relativity

What is Einstein's principle of relativity?

Show that a spherical shell expanding at the speed of light, $\mathbf{x}^2 = c^2 t^2$, in one coordinate system (t, \mathbf{x}) , is still spherical in a second coordinate system (t', \mathbf{x}') defined by

$$ct' = \gamma \left(ct - \frac{u}{c}x\right),$$

$$x' = \gamma(x - ut),$$

$$y' = y,$$

$$z' = z,$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$. Why do these equations define a Lorentz transformation?

4/II/18F Special Relativity

A particle of mass M is at rest at x = 0, in coordinates (t, x). At time t = 0 it decays into two particles A and B of equal mass m < M/2. Assume that particle A moves in the *negative* x direction.

(a) Using relativistic energy and momentum conservation compute the energy, momentum and velocity of both particles A and B.

(b) After a proper time τ , measured in its own rest frame, particle A decays. Show that the spacetime coordinates of this event are

$$\begin{split} t = & \frac{M}{2m} \tau, \\ x = & -\frac{MV}{2m} \tau, \end{split}$$

where $V = c\sqrt{1 - 4(m/M)^2}$.