Part IB

Quantum Mechanics

Paper 3, Section I

6D Quantum Mechanics

Consider the one-dimensional, time-independent Schrödinger equation:

$$\frac{d^2\chi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - U(x)]\chi(x) = 0, \qquad x \in \mathbb{R}.$$

(a) Explain the meaning of the functions $\chi(x), U(x)$ and parameters $E, m, \hbar$.

(b) Solutions of this equation describing bound states correspond to $\chi(x) \to 0$ for $x \to \pm\infty$. Are there bound states for a potential that asymptotes to a constant U_0 (that is $U(x) \to U_0$ as $x \to \pm\infty$) for the cases $E > U_0 > 0$ and $0 < E < U_0$?

(c) Show, by contradiction or otherwise, that the energy spectrum of bound states is non-degenerate.

Paper 4, Section I

4D Quantum Mechanics

(a) Prove Ehrenfest's theorem in one-dimensional quantum mechanics:

$$\frac{d}{dt}\langle\hat{O}\rangle_\psi = \frac{i}{\hbar}\langle[\hat{H}, \hat{O}]\rangle_\psi + \langle\frac{\partial\hat{O}}{\partial t}\rangle_\psi,$$

where $\hat{O}$ is a Hermitian operator, $\hat{H}$ is the Hamiltonian and

$$\langle\hat{O}\rangle_\psi = \int \psi^*(x,t)\hat{O}\psi(x,t)dx$$

is the expectation value of the operator $\hat{O}$ in a state determined by the wave function $\psi(x,t)$.

(b) Using Ehrenfest's theorem prove that

$$m\frac{d}{dt}\langle\hat{x}\rangle_\psi = \langle\hat{p}\rangle_\psi, \qquad \frac{d}{dt}\langle\hat{p}\rangle_\psi = -\langle\frac{dU}{dx}\rangle_\psi, \qquad \frac{d}{dt}\langle\hat{H}\rangle_\psi = 0,$$

where $U(x)$ is the scalar potential. Compare with similar expressions in classical mechanics.

Paper 1, Section II

14D Quantum Mechanics

Consider a physical observable O represented by a Hermitian operator $\hat{O}$ acting on a Hilbert space $\mathcal{H}$. We define the uncertainty $\Delta_\psi O$ in a measurement of O on a state ψ as $(\Delta_\psi O)^2 = \langle \hat{O}^2 \rangle_\psi - \langle \hat{O} \rangle_\psi^2$ with the expectation value defined as $\langle \hat{O} \rangle_\psi = (\psi, \hat{O}\psi)$.

(a) Using the Schwartz inequality $|(\phi, \psi)|^2 \leqslant (\phi, \phi)(\psi, \psi)$ for two states ϕ, ψ, prove the generalised uncertainty relation for the observables A, B:

$$(\Delta_\psi A)(\Delta_\psi B) \geqslant \frac{1}{2} \left| \Big(\psi, [\hat{A}, \hat{B}]\psi \Big) \right|, \qquad (\dagger)$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of $\hat{A}$ and $\hat{B}$.

(b) Given the two Hermitian operators $\hat{X}$ and $\hat{Y}$ and a real parameter λ, we define

$$f(\lambda) = \Big\langle \big(\hat{X} - i\lambda \hat{Y} \big) \big(\hat{X} + i\lambda \hat{Y} \big) \Big\rangle_\psi .$$

Minimising $f(\lambda)$ and using the fact that $f(\lambda) \geqslant 0$, provide an alternative derivation of the uncertainty relation ($\dagger$).

(c) For the position and momentum operators, $\hat{x}$ and $\hat{p} = -i\hbar \frac{\partial}{\partial x}$, respectively, find their commutator $[\hat{x}, \hat{p}]$ and derive the Heisenberg uncertainty relation $\Delta_\psi x \Delta_\psi p \geqslant \frac{1}{2}\hbar$.

(d) Show that a Gaussian wave function $\psi(x) = Ce^{-\alpha x^2}$ solves the one-dimensional Schrödinger's equation for a quadratic potential $U(x) = kx^2$ with $k > 0$. Determine the value of the constants α, C and the energy E in terms of k and the particle's mass m. Show that this wave function saturates the Heisenberg uncertainty relation ($\Delta_\psi x \Delta_\psi p = \frac{1}{2}\hbar$). Furthermore, show that in order to saturate this Heisenberg relation, the wave function has to be Gaussian. [*Hint: You may use* $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ *and* $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \sqrt{\frac{\pi}{4a^3}}$.]

Paper 2, Section II

15D Quantum Mechanics

(a) Consider the Schrödinger equation for the wave function $\psi(\mathbf{r},t)$ corresponding to a particle subject to a real potential energy $U(\mathbf{r},t)$. Defining the probability density $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and probability current density

$$\mathbf{J}(\mathbf{r},t) = -\frac{i\hbar}{2m}\left[\psi^*\nabla\psi - (\nabla\psi)^*\,\psi\right],$$

derive and interpret the continuity equation $\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$.

(b) Consider the one-dimensional Schrödinger equation with a step potential

$$U(x) = \begin{cases} 0 & x < -a \\ U_0 & -a < x < a \\ 0 & x > a, \end{cases}$$

where $a > 0, U_0 > 0$.

(i) Using matching conditions at $x = \pm a$, find the transmitted wave function $\psi(x,t)$ and probability density $\rho(x,t)$ in the region $x > a$, for an incident wave corresponding to a particle of mass m and energy $E = U_0/2$ moving towards the potential barrier from $x < -a$. Express the results in terms of the quantity $k = \sqrt{2mE}/\hbar$.

(ii) Compute the ratio between the transmitted and the incident current densities and interpret the result in terms of the continuity equation.

Paper 4, Section II

15D Quantum Mechanics

(a) Using the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$ with $i, j = 1, 2, 3$, show that the angular momentum operators $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ satisfy the commutation relations:

$$[\hat{L}_i, \hat{L}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{L}_k, \qquad [\hat{L}_i, \hat{x}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{x}_k, \qquad [\hat{L}_i, \hat{p}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{p}_k.$$

Using these relations show that $[\hat{L}^2, \hat{L}_i] = 0$ where $\hat{L}^2 = \hat{L}_i\hat{L}_i$. Show further that for a spherically symmetric system $[\hat{L}^2, \hat{H}] = 0$, where the Hamiltonian $\hat{H}$ takes the form $\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{r})$. Can the operators $\hat{H}, \hat{L}^2, \hat{L}_3$ be simultaneously diagonalised? Justify your answer.

(b) Consider the Schrödinger equation for the Hydrogen atom in which the potential energy is $U(r) = -\frac{q^2}{r}$. Concentrating on the wave function with zero eigenvalues for both $\hat{L}_3$ and $\hat{L}^2$, the equation for the radial component of the wave function, $R(r)$, reduces to:

$$R'' + \frac{2}{r}R' + \left(\frac{\beta}{r} - \gamma^2\right) R = 0,$$

where $\beta = \frac{2mq^2}{\hbar^2}$ and $\gamma^2 = -\frac{2mE}{\hbar^2}$, with E denoting the energy.

(i) Considering the $r \to \infty$ limit, explain why $R \sim e^{-\gamma r}$.

(ii) Consider then the series solution

$$R(r) = f(r)e^{-\gamma r}, \qquad f(r) = \sum_n a_n r^n.$$

Derive the recurrence relation

$$a_n = \frac{2\gamma n - \beta}{n(n+1)} a_{n-1},$$

then argue why the energy is quantised and determine the ground state energy.

(iii) Using the ground state wave function $R(r) = Ce^{-\gamma r}$, determine the normalisation factor C and estimate the expectation value of the radius $\langle r \rangle_R$. Compare with the Bohr radius.

Paper 3, Section I
6B Quantum Mechanics

(a) A beam of identical, free particles, each of mass m, moves in one dimension. There is no potential. Show that the wavefunction $\chi(x) = Ae^{ikx}$ is an energy eigenstate for any constants A and k.

What is the energy E and the momentum p in terms of k? What can you say about the sign of E?

(b) Write down expressions for the probability density ρ and the probability current J in terms of the wavefunction $\psi(x,t)$. Use the current conservation equation, i.e.

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

to show that, for a stationary state of fixed energy E, the probability current J is independent of x.

(c) A beam of particles in a stationary state is incident from $x \to -\infty$ upon a potential $U(x)$ with $U(x) \to 0$ as $x \to \pm\infty$. Given the asymptotic behaviour of the form

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}\,, & x \to -\infty\,, \\ \\ Te^{ikx}\,, & x \to \infty\,, \end{cases}$$

show that $|R|^2 + |T|^2 = 1$. Interpret this result.

Paper 4, Section I
4B Quantum Mechanics

The radial wavefunction $g(r)$ for the hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}g(r)\right) - \frac{e^2}{4\pi\epsilon_0 r}g(r) + \frac{l(l+1)\hbar^2}{2mr^2}g(r) = Eg(r)\,. \qquad (\dagger)$$

(a) Explain the origin of each of the terms in ($\dagger$). What are the allowed values of l?

(b) For a given l, the lowest energy bound state solution of ($\dagger$) takes the form $r^a e^{-br}$. Find a, b, and the corresponding value of E, in terms of l.

(c) A hydrogen atom makes a transition between two such states, corresponding to $l+1$ and l. What is the frequency of the photon emitted?

Paper 1, Section II
14B Quantum Mechanics

(a) Write down the time-dependent Schrödinger equation for a harmonic oscillator of mass m, frequency ω and coordinate x.

(b) Show that a wavefunction of the form

$$\psi(x,t) = N(t)\,\exp\Big(-F(t)x^2 + G(t)x\Big)\,,$$

where F, G and N are complex functions of time, is a solution to the Schrödinger equation, provided that F, G, N satisfy certain conditions which you should establish.

(c) Verify that

$$F(t) = A\tanh(a + i\omega t), \quad G(t) = \sqrt{\frac{m\omega}{\hbar}}\,\mathrm{sech}(a + i\omega t)\,,$$

where a is a real positive constant, satisfy the conditions you established in part (b). Hence determine the constant A. [You do not need to find the time-dependent normalization function $N(t)$.]

(d) By completing the square, or otherwise, show that $|\psi(x,t)|^2$ is peaked around a certain position $x = h(t)$ and express $h(t)$ in terms of F and G.

(e) Find $h(t)$ as a function of time and describe its behaviour.

(f) Sketch $|\psi(x,t)|^2$ for a fixed value of t. What is the value of $\langle \hat{x} \rangle_\psi$?

[*You may find the following identities useful:*

$$\cosh(\alpha + i\beta) = \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta\,,$$

$$\sinh(\alpha + i\beta) = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta\,.]$$

Paper 2, Section II
15B Quantum Mechanics

A particle of mass m is confined to the region $0 \leqslant x \leqslant a$ by a potential that is zero inside the region and infinite outside.

(a) Find the energy eigenvalues E_n and the corresponding normalised energy eigenstates $\chi_n(x)$.

(b) At time $t = 0$ the wavefunction $\psi(x,t)$ of the particle is given by

$$\psi(x,0) = f(x)\,,$$

where $f(x)$ is not an energy eigenstate and satisfies the boundary conditions $f(0) = f(a) = 0$.

(i) Express $\psi(x,t)$ in terms of $\chi_n(x)$ and E_n.

(ii) Show that $T = 2ma^2/\pi\hbar$ is the earliest time at which $\psi(a-x,T)$ and $\psi(x,0)$ correspond to physically equivalent states. Thus, determine $\psi(x,2T)$.

Show that if $\psi(x,0) = 0$ for $a/2 \leqslant x \leqslant a$, then the probability of finding the particle in $0 \leqslant x \leqslant a/2$ at $t = T$ is zero.

(iii) For

$$f(x) = \begin{cases} \dfrac{2}{\sqrt{a}} \sin \dfrac{2\pi x}{a}\,, & 0 \leqslant x \leqslant \dfrac{a}{2}\,, \\[2ex] 0\,, & \dfrac{a}{2} \leqslant x \leqslant a\,, \end{cases}$$

find the probability that a measurement of the energy of the particle at time $t = 0$ will yield a value $2\pi^2\hbar^2/ma^2$.

What is the probability if, instead, the same measurement is carried out at time $t = 2T$? What is the probability at $t = T$?

Suppose that the result of the measurement of the energy was indeed $2\pi^2\hbar^2/ma^2$. What is the probability that a subsequent measurement of energy will yield the same result?

Paper 4, Section II
15B Quantum Mechanics

(a) Write down the time-dependent Schrödinger equation for the wavefunction $\psi(x,t)$ of a particle with Hamiltonian $\hat{H}$.

Suppose that A is an observable associated with the operator $\hat{A}$. Show that

$$i\hbar \frac{d\langle \hat{A}\rangle_\psi}{dt} = \langle [\hat{A},\hat{H}]\rangle_\psi + i\hbar \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle_\psi .$$

(b) Consider a particle of mass m subject to a constant gravitational field with potential energy $U(x) = mgx$.

[For the rest of the question you should assume that $\psi(x,t)$ is normalized.]

(i) Find the differential equation satisfied by the function $\Phi(x,t)$ defined by

$$\psi(x,t) = \Phi(x,t) \exp\left[-\frac{im}{\hbar} gt \left(x + \frac{1}{6} gt^2\right)\right] .$$

(ii) Show that $\Theta(X,T) = \Phi(x,t)$, with $X = x + \frac{1}{2}gt^2$ and $T = t$, satisfies the free-particle Schrödinger equation

$$i\hbar \frac{\partial \Theta}{\partial T} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Theta}{\partial X^2} .$$

Hence, show that

$$\frac{d\langle \hat{X}\rangle_\Theta}{dT} = \frac{1}{m} \langle \hat{P}\rangle_\Theta , \qquad \frac{d\langle \hat{P}\rangle_\Theta}{dT} = 0 ,$$

where $\hat{P} = -i\hbar \dfrac{\partial}{\partial X}$.

(iii) Express $\langle \hat{X}\rangle_\Theta$ in terms of $\langle \hat{x}\rangle_\psi$. Deduce that

$$\langle \hat{x}\rangle_\psi = a + vt - \tfrac{1}{2} gt^2 ,$$

for some constants a and v. Briefly comment on the physical significance of this result.

Paper 3, Section I

6C Quantum Mechanics

The electron in a hydrogen-like atom moves in a spherically symmetric potential $V(r) = -K/r$ where K is a positive constant and r is the radial coordinate of spherical polar coordinates. The two lowest energy spherically symmetric normalised states of the electron are given by

$$\chi_1(r) = \frac{1}{\sqrt{\pi}\,a^{3/2}}\,e^{-r/a} \quad \text{and} \quad \chi_2(r) = \frac{1}{4\sqrt{2\pi}\,a^{3/2}}\left(2 - \frac{r}{a}\right)\,e^{-r/2a}$$

where $a = \hbar^2/mK$ and m is the mass of the electron. For any spherically symmetric function $f(r)$, the Laplacian is given by $\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$.

(i) Suppose that the electron is in the state $\chi(r) = \frac{1}{2}\chi_1(r) + \frac{\sqrt{3}}{2}\chi_2(r)$ and its energy is measured. Find the expectation value of the result.

(ii) Suppose now that the electron is in state $\chi(r)$ (as above) at time $t = 0$. Let $R(t)$ be the expectation value of a measurement of the electron's radial position r at time t. Show that the value of $R(t)$ oscillates sinusoidally about a constant level and determine the frequency of the oscillation.

Paper 4, Section I

4C Quantum Mechanics

Let $\Psi(x,t)$ be the wavefunction for a particle of mass m moving in one dimension in a potential $U(x)$. Show that, with suitable boundary conditions as $x \to \pm\infty$,

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx \;=\; 0\,.$$

Why is this important for the interpretation of quantum mechanics?

Verify the result above by first calculating $|\Psi(x,t)|^2$ for the free particle solution

$$\Psi(x,t) \;=\; C f(t)^{1/2} \exp\Big(-\frac{1}{2} f(t)\,x^2\Big) \quad \text{with} \quad f(t) = \Big(\alpha + \frac{i\hbar}{m}t\Big)^{-1},$$

where C and $\alpha > 0$ are real constants, and then considering the resulting integral.

Paper 1, Section II

14C Quantum Mechanics

Consider a quantum mechanical particle of mass m in a one-dimensional stepped potential well $U(x)$ given by:

$$U(x) = \begin{cases} \infty & \text{for } x < 0 \text{ and } x > a \\ 0 & \text{for } 0 \leqslant x \leqslant a/2 \\ U_0 & \text{for } a/2 < x \leqslant a \end{cases}$$

where $a > 0$ and $U_0 \geqslant 0$ are constants.

(i) Show that all energy levels E of the particle are non-negative. Show that any level E with $0 < E < U_0$ satisfies

$$\frac{1}{k}\tan\frac{ka}{2} = -\frac{1}{l}\tanh\frac{la}{2}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} > 0 \quad \text{and} \quad l = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} > 0.$$

(ii) Suppose that initially $U_0 = 0$ and the particle is in the ground state of the potential well. U_0 is then changed to a value $U_0 > 0$ (while the particle's wavefunction stays the same) and the energy of the particle is measured. For $0 < E < U_0$, give an expression in terms of E for prob(E), the probability that the energy measurement will find the particle having energy E. The expression may be left in terms of integrals that you need not evaluate.

Paper 2, Section II

15C Quantum Mechanics

(a) Write down the expressions for the probability density ρ and associated current density j of a quantum particle in one dimension with wavefunction $\psi(x,t)$. Show that if ψ is a stationary state then the function j is constant.

For the non-normalisable free particle wavefunction $\psi(x,t) = Ae^{ikx-iEt/\hbar}$ (where E and k are real constants and A is a complex constant) compute the functions ρ and j, and briefly give a physical interpretation of the functions ψ, ρ and j in this case.

(b) A quantum particle of mass m and energy $E > 0$ moving in one dimension is incident from the left in the potential $V(x)$ given by

$$V(x) = \begin{cases} -V_0 & 0 \leqslant x \leqslant a \\ 0 & x < 0 \text{ or } x > a \end{cases}$$

where a and V_0 are positive constants. Write down the form of the wavefunction in the regions $x < 0$, $0 \leqslant x \leqslant a$ and $x > a$.

Suppose now that $V_0 = 3E$. Show that the probability T of transmission of the particle into the region $x > a$ is given by

$$T = \frac{16}{16 + 9\sin^2\left(\frac{a\sqrt{8mE}}{\hbar}\right)}.$$

Paper 4, Section II

15C Quantum Mechanics

(a) Consider the angular momentum operators $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ and $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ where

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \quad \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \text{ and } \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z.$$

Use the standard commutation relations for these operators to show that

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y \qquad \text{obeys} \qquad [\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm \quad \text{and} \quad [\hat{\mathbf{L}}^2, \hat{L}_\pm] = 0\,.$$

Deduce that if φ is a joint eigenstate of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ with angular momentum quantum numbers m and ℓ respectively, then $\hat{L}_\pm\varphi$ are also joint eigenstates, provided they are non-zero, with quantum numbers $m \pm 1$ and ℓ.

(b) A harmonic oscillator of mass M in three dimensions has Hamiltonian

$$\hat{H} \;=\; \frac{1}{2M}(\,\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\,) \;+\; \frac{1}{2}\,M\omega^2(\,\hat{x}^2 + \hat{y}^2 + \hat{z}^2\,)\,.$$

Find eigenstates of $\hat{H}$ in terms of eigenstates ψ_n for an oscillator in one dimension with $n = 0, 1, 2, \ldots$ and eigenvalues $\hbar\omega(n + \frac{1}{2})$; hence determine the eigenvalues E of $\hat{H}$.

Verify that the ground state for $\hat{H}$ is a joint eigenstate of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ with $\ell = m = 0$. At the first excited energy level, find an eigenstate of $\hat{L}_z$ with $m = 0$ and construct from this two eigenstates of $\hat{L}_z$ with $m = \pm 1$.

Why should you expect to find joint eigenstates of $\hat{L}_z$, $\hat{\mathbf{L}}^2$ and $\hat{H}$?

[*The first two eigenstates for an oscillator in one dimension are* $\psi_0(x) = C_0 \exp(-M\omega x^2/2\hbar)$ *and* $\psi_1(x) = C_1 x \exp(-M\omega x^2/2\hbar)$, *where* C_0 *and* C_1 *are normalisation constants.*]

Paper 1, Section I

4A Quantum Mechanics

Define what it means for an operator Q to be *hermitian* and briefly explain the significance of this definition in quantum mechanics.

Define the *uncertainty* $(\Delta Q)_\psi$ of Q in a state ψ. If P is also a hermitian operator, show by considering the state $(Q + i\lambda P)\psi$, where λ is a real number, that

$$\langle Q^2 \rangle_\psi \, \langle P^2 \rangle_\psi \;\geqslant\; \frac{1}{4}\, |\, \langle i[Q,P] \rangle_\psi \,|^2 \, .$$

Hence deduce that

$$(\Delta Q)_\psi \, (\Delta P)_\psi \;\geqslant\; \frac{1}{2} |\, \langle i[Q,P] \rangle_\psi \,| \, .$$

Give a physical interpretation of this result.

Paper 1, Section II

15A Quantum Mechanics

Consider a quantum system with Hamiltonian H and wavefunction Ψ obeying the time-dependent Schrödinger equation. Show that if Ψ is a *stationary state* then $\langle Q \rangle_\Psi$ is independent of time, if the observable Q is independent of time.

A particle of mass m is confined to the interval $0 \leqslant x \leqslant a$ by infinite potential barriers, but moves freely otherwise. Let $\Psi(x,t)$ be the normalised wavefunction for the particle at time t, with

$$\Psi(x,0) = c_1 \psi_1(x) + c_2 \psi_2(x)$$

where

$$\psi_1(x) \;=\; \left(\frac{2}{a}\right)^{1/2} \sin\frac{\pi x}{a} \, , \qquad \psi_2(x) \;=\; \left(\frac{2}{a}\right)^{1/2} \sin\frac{2\pi x}{a}$$

and c_1, c_2 are complex constants. If the energy of the particle is measured at time t, what are the possible results, and what is the probability for each result to be obtained? Give brief justifications of your answers.

Calculate $\langle \hat{x} \rangle_\Psi$ at time t and show that the result oscillates with a frequency ω, to be determined. Show in addition that

$$\left| \langle \hat{x} \rangle_\Psi - \frac{a}{2} \right| \;\leqslant\; \frac{16a}{9\pi^2} \, .$$

Paper 2, Section II

14A Quantum Mechanics

(a) The potential $V(x)$ for a particle of mass m in one dimension is such that $V \to 0$ rapidly as $x \to \pm\infty$. Let $\psi(x)$ be a wavefunction for the particle satisfying the time-independent Schrödinger equation with energy E.

Suppose ψ has the asymptotic behaviour

$$\psi(x) \sim Ae^{ikx} + Be^{-ikx} \quad (x \to -\infty)\,, \qquad \psi(x) \sim Ce^{ikx} \quad (x \to +\infty)\,,$$

where A, B, C are complex coefficients. Explain, in outline, how the probability current $j(x)$ is used in the interpretation of such a solution as a scattering process and how the transmission and reflection probabilities P_{tr} and P_{ref} are found.

Now suppose instead that $\psi(x)$ is a bound state solution. Write down the asymptotic behaviour in this case, relating an appropriate parameter to the energy E.

(b) Consider the potential

$$V(x) \;=\; -\,\frac{\hbar^2}{m}\,\frac{a^2}{\cosh^2 ax}$$

where a is a real, positive constant. Show that

$$\psi(x) \;=\; Ne^{ikx}(a\tanh ax \;-\; ik)\,,$$

where N is a complex coefficient, is a solution of the time-independent Schrödinger equation for any real k and find the energy E. Show that ψ represents a scattering process for which $P_{\mathrm{ref}} = 0$, and find P_{tr} explicitly.

Now let $k = i\lambda$ in the formula for ψ above. Show that this defines a bound state if a certain real positive value of λ is chosen and find the energy of this solution.

Paper 4, Section I

6B Quantum Mechanics

(a) Define the probability density ρ and probability current j for the wavefunction $\Psi(x,t)$ of a particle of mass m. Show that

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,,$$

and deduce that $j = 0$ for a normalizable, stationary state wavefunction. Give an example of a non-normalizable, stationary state wavefunction for which j is non-zero, and calculate the value of j.

(b) A particle has the instantaneous, normalized wavefunction

$$\Psi(x,0) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2 + ikx}\,,$$

where α is positive and k is real. Calculate the expectation value of the momentum for this wavefunction.

Paper 3, Section I

8B Quantum Mechanics

Consider a quantum mechanical particle moving in two dimensions with Cartesian coordinates x, y. Show that, for wavefunctions with suitable decay as $x^2 + y^2 \to \infty$, the operators

$$x \quad \text{and} \quad -i\hbar\frac{\partial}{\partial x}$$

are Hermitian, and similarly

$$y \quad \text{and} \quad -i\hbar\frac{\partial}{\partial y}$$

are Hermitian.

Show that if F and G are Hermitian operators, then

$$\frac{1}{2}(FG + GF)$$

is Hermitian. Deduce that

$$L = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad \text{and} \quad D = -i\hbar\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1\right)$$

are Hermitian. Show that

$$[L, D] = 0.$$

Paper 1, Section II
15B Quantum Mechanics

Starting from the time-dependent Schrödinger equation, show that a stationary state $\psi(x)$ of a particle of mass m in a harmonic oscillator potential in one dimension with frequency ω satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi\,.$$

Find a rescaling of variables that leads to the simplified equation

$$-\frac{d^2\psi}{dy^2}+y^2\psi=\varepsilon\psi\,.$$

Setting $\psi=f(y)e^{-\frac{1}{2}y^2}$, find the equation satisfied by $f(y)$.

Assume now that f is a polynomial

$$f(y)=y^N+a_{N-1}\,y^{N-1}+a_{N-2}\,y^{N-2}+\ldots+a_0\,.$$

Determine the value of ε and deduce the corresponding energy level E of the harmonic oscillator. Show that if N is even then the stationary state $\psi(x)$ has even parity.

Paper 3, Section II
16B Quantum Mechanics

Consider a particle of unit mass in a one-dimensional square well potential

$$V(x)=0\quad\text{for}\quad 0\leqslant x\leqslant\pi\,,$$

with V infinite outside. Find all the stationary states $\psi_n(x)$ and their energies E_n, and write down the general normalized solution of the time-dependent Schrödinger equation in terms of these.

The particle is initially constrained by a barrier to be in the ground state in the narrower square well potential

$$V(x)=0\quad\text{for}\quad 0\leqslant x\leqslant\frac{\pi}{2}\,,$$

with V infinite outside. The barrier is removed at time $t=0$, and the wavefunction is instantaneously unchanged. Show that the particle is now in a superposition of stationary states of the original potential well, and calculate the probability that an energy measurement will yield the result E_n.

Paper 2, Section II

17B Quantum Mechanics

Let x, y, z be Cartesian coordinates in $\mathbb{R}^3$. The angular momentum operators satisfy the commutation relation

$$[L_x, L_y] = i\hbar L_z$$

and its cyclic permutations. Define the *total angular momentum operator* $\mathbf{L}^2$ and show that $[L_z, \mathbf{L}^2] = 0$. Write down the explicit form of L_z.

Show that a function of the form $(x+iy)^m z^n f(r)$, where $r^2 = x^2 + y^2 + z^2$, is an eigenfunction of L_z and find the eigenvalue. State the analogous result for $(x-iy)^m z^n f(r)$.

There is an energy level for a particle in a certain spherically symmetric potential well that is 6-fold degenerate. A basis for the (unnormalized) energy eigenstates is of the form

$$(x^2-1)f(r),\ (y^2-1)f(r),\ (z^2-1)f(r),\ xyf(r),\ xzf(r),\ yzf(r)\,.$$

Find a new basis that consists of simultaneous eigenstates of L_z and $\mathbf{L}^2$ and identify their eigenvalues.

[*You may quote the range of L_z eigenvalues associated with a particular eigenvalue of* $\mathbf{L}^2$.]

Paper 4, Section I

6B Quantum Mechanics

A particle moving in one space dimension with wavefunction $\Psi(x,t)$ obeys the time-dependent Schrödinger equation. Write down the probability density ρ and current density j in terms of the wavefunction and show that they obey the equation

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0\,.$$

Evaluate $j(x,t)$ in the case that

$$\Psi(x,t) = \left(Ae^{ikx} + Be^{-ikx}\right) e^{-iEt/\hbar}\,,$$

where $E = \hbar^2 k^2/2m$, and A and B are constants, which may be complex.

Paper 3, Section I

8B Quantum Mechanics

What is meant by the statement that an operator is *Hermitian*?

Consider a particle of mass m in a real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is Hermitian.

Starting from the time-dependent Schrödinger equation, show that

$$\frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{m}\langle \hat{p} \rangle, \qquad \frac{d}{dt}\langle \hat{p} \rangle = -\langle V'(\hat{x}) \rangle,$$

where $\hat{p}$ is the momentum operator and $\langle \hat{A} \rangle$ denotes the expectation value of the operator $\hat{A}$.

Paper 1, Section II

15B Quantum Mechanics

The relative motion of a neutron and proton is described by the Schrödinger equation for a single particle of mass m under the influence of the central potential

$$V(r) = \begin{cases} -U & r < a \\ 0 & r > a \end{cases}$$

where U and a are positive constants. Solve this equation for a spherically symmetric state of the deuteron, which is a bound state of a proton and a neutron, giving the condition on U for this state to exist.

[*If ψ is spherically symmetric then $\nabla^2\psi = \frac{1}{r}\,\frac{d^2}{dr^2}\,(r\psi)$.*]

Paper 3, Section II

16B Quantum Mechanics

What is the physical significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x) Q \psi(x) dx$$

of an observable Q in the normalised state $\psi(x)$? Let P and Q be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \frac{1}{4} |\langle [Q, P] \rangle|^2 .$$

Deduce the generalised uncertainty relation

$$\Delta Q \; \Delta P \geqslant \frac{1}{2} |\langle [Q, P] \rangle| ,$$

where the uncertainty ΔQ in the state $\psi(x)$ is defined by

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle .$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m \omega^2 x^2$. By considering the commutator $[x, p]$, show that every energy eigenvalue E satisfies

$$E \geqslant \tfrac{1}{2} \hbar \omega .$$

Paper 2, Section II

17B Quantum Mechanics

For an electron in a hydrogen atom, the stationary-state wavefunctions are of the form $\psi(r, \theta, \phi) = R(r) Y_{lm}(\theta, \phi)$, where in suitable units R obeys the radial equation

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} - \frac{l(l+1)}{r^2} R + 2 \left(E + \frac{1}{r} \right) R = 0 \, .$$

Explain briefly how the terms in this equation arise.

This radial equation has bound-state solutions of energy $E = E_n$, where $E_n = -\frac{1}{2n^2}$ $(n = 1, 2, 3, \dots)$. Show that when $l = n - 1$, there is a solution of the form $R(r) = r^{\alpha} e^{-r/n}$, and determine α. Find the expectation value $\langle r \rangle$ in this state.

Determine the total degeneracy of the energy level with energy E_n.

Paper 4, Section I

6B Quantum Mechanics

(a) Give a physical interpretation of the wavefunction $\phi(x,t) = Ae^{ikx}e^{-iEt/\hbar}$ (where A, k and E are real constants).

(b) A particle of mass m and energy $E > 0$ is incident from the left on the potential step

$$V(x) = \begin{cases} 0 & \text{for } -\infty < x < a \\ V_0 & \text{for } a < x < \infty. \end{cases}$$

with $V_0 > 0$.

State the conditions satisfied by a stationary state at the point $x = a$.

Compute the probability that the particle is reflected as a function of E, and compare your result with the classical case.

Paper 3, Section I

8B Quantum Mechanics

A particle of mass m is confined to a one-dimensional box $0 \leqslant x \leqslant a$. The potential $V(x)$ is zero inside the box and infinite outside.

(a) Find the allowed energies of the particle and the normalised energy eigenstates.

(b) At time $t = 0$ the particle has wavefunction ψ_0 that is uniform in the left half of the box i.e. $\psi_0(x) = \sqrt{\frac{2}{a}}$ for $0 < x < a/2$ and $\psi_0(x) = 0$ for $a/2 < x < a$. Find the probability that a measurement of energy at time $t = 0$ will yield a value less than $5\hbar^2\pi^2/(2ma^2)$.

Paper 1, Section II

15B Quantum Mechanics

Consider the time-independent Schrödinger equation in one dimension for a particle of mass m with potential $V(x)$.

(a) Show that if the potential is an even function then any non-degenerate stationary state has definite parity.

(b) A particle of mass m is subject to the potential $V(x)$ given by

$$V(x) = -\lambda\Big(\delta(x-a) + \delta(x+a)\Big)$$

where λ and a are real positive constants and $\delta(x)$ is the Dirac delta function.

Derive the conditions satisfied by the wavefunction $\psi(x)$ around the points $x = \pm a$.

Show (using a graphical method or otherwise) that there is a bound state of even parity for any $\lambda > 0$, and that there is an odd parity bound state only if $\lambda > \hbar^2/(2ma)$. [*Hint: You may assume without proof that the functions* $x \tanh x$ *and* $x \coth x$ *are monotonically increasing for* $x > 0$.]

Paper 3, Section II

16B Quantum Mechanics

(a) Given the position and momentum operators $\hat{x}_i = x_i$ and $\hat{p}_i = -i\hbar\,\partial/\partial x_i$ (for $i = 1, 2, 3$) in three dimensions, define the angular momentum operators $\hat{L}_i$ and the total angular momentum $\hat{L}^2$.
Show that $\hat{L}_3$ is Hermitian.

(b) Derive the generalised uncertainty relation for the observables $\hat{L}_3$ and $\hat{x}_1$ in the form

$$\Delta_\psi \hat{L}_3 \; \Delta_\psi \hat{x}_1 \geqslant M$$

for any state ψ and a suitable expression M that you should determine. [*Hint: It may be useful to consider the operator* $\hat{L}_3 + i\lambda\hat{x}_1$.]

(c) Consider a particle with wavefunction

$$\psi = K(x_1 + x_2 + 2x_3)e^{-\alpha r}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and K and α are real positive constants.
Show that ψ is an eigenstate of total angular momentum $\hat{L}^2$ and find the corresponding angular momentum quantum number l. Find also the expectation value of a measurement of $\hat{L}_3$ on the state ψ.

Paper 2, Section II

17B Quantum Mechanics

(a) The potential for the one-dimensional harmonic oscillator is $V(x) = \frac{1}{2}m\omega^2 x^2$. By considering the associated time-independent Schrödinger equation for the wavefunction $\psi(x)$ with substitutions

$$\xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} x \quad \text{and} \quad \psi(x) = f(\xi)e^{-\xi^2/2},$$

show that the allowed energy levels are given by $E_n = (n + \frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \ldots$. [You may assume without proof that f must be a polynomial for ψ to be normalisable.]

(b) Consider a particle with charge q and mass $m = 1$ subject to the one-dimensional harmonic oscillator potential $U_0(x) = x^2/2$. You may assume that the normalised ground state of this potential is

$$\psi_0(x) = \left(\frac{1}{\pi\hbar}\right)^{1/4} e^{-x^2/(2\hbar)}.$$

The particle is in the stationary state corresponding to $\psi_0(x)$ when at time $t = t_0$, an electric field of constant strength E is turned on, adding an extra term $U_1(x) = -qEx$ to the harmonic potential.

(i) Using the result of part (a) or otherwise, find the energy levels of the new potential.

(ii) Show that the probability of finding the particle in the ground state immediately after t_0 is given by $e^{-q^2E^2/(2\hbar)}$. [You may assume that $\int_{-\infty}^{\infty} e^{-x^2+2Ax}\,dx = \sqrt{\pi}e^{A^2}$.]

Paper 4, Section I

6B Quantum Mechanics

(a) Define the quantum orbital angular momentum operator $\hat{\boldsymbol{L}} = (\hat{L}_1, \hat{L}_2, \hat{L}_3)$ in three dimensions, in terms of the position and momentum operators.

(b) Show that $[\hat{L}_1, \hat{L}_2] = i\hbar\hat{L}_3$. [You may assume that the position and momentum operators satisfy the canonical commutation relations.]

(c) Let $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$. Show that $\hat{L}_1$ commutes with $\hat{L}^2$.
[In this part of the question you may additionally assume without proof the permuted relations $[\hat{L}_2, \hat{L}_3] = i\hbar\hat{L}_1$ and $[\hat{L}_3, \hat{L}_1] = i\hbar\hat{L}_2$.]
[*Hint: It may be useful to consider the expression* $[\hat{A}, \hat{B}]\,\hat{B} + \hat{B}\,[\hat{A}, \hat{B}]$ *for suitable operators* $\hat{A}$ *and* $\hat{B}$.]

(d) Suppose that $\psi_1(x, y, z)$ and $\psi_2(x, y, z)$ are normalised eigenstates of $\hat{L}_1$ with eigenvalues $\hbar$ and $-\hbar$ respectively. Consider the wavefunction

$$\psi = \frac{1}{2}\,\psi_1\,\cos\omega t + \frac{\sqrt{3}}{2}\,\psi_2\,\sin\omega t\,,$$

with ω being a positive constant. Find the earliest time $t_0 > 0$ such that the expectation value of $\hat{L}_1$ in ψ is zero.

Paper 3, Section I

8B Quantum Mechanics

(a) Consider a quantum particle moving in one space dimension, in a time-independent real potential $V(x)$. For a wavefunction $\psi(x, t)$, define the *probability density* $\rho(x, t)$ and *probability current* $j(x, t)$ and show that

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0.$$

(b) Suppose now that $V(x) = 0$ and $\psi(x, t) = (e^{ikx} + Re^{-ikx})e^{-iEt/\hbar}$, where $E = \hbar^2 k^2/(2m)$, k and m are real positive constants, and R is a complex constant. Compute the probability current for this wavefunction. Interpret the terms in ψ and comment on how this relates to the computed expression for the probability current.

Paper 1, Section II

15B Quantum Mechanics

(a) A particle of mass m in one space dimension is confined to move in a potential $V(x)$ given by

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{for } x < 0 \text{ or } x > a. \end{cases}$$

The normalised initial wavefunction of the particle at time $t = 0$ is

$$\psi_0(x) = \frac{4}{\sqrt{5a}} \sin^3\left(\frac{\pi x}{a}\right).$$

(i) Find the expectation value of the energy at time $t = 0$.

(ii) Find the wavefunction of the particle at time $t = 1$.

[*Hint: It may be useful to recall the identity* $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.]

(b) The right hand wall of the potential is lowered to a finite constant value $U_0 > 0$ giving the new potential:

$$U(x) = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{for } x < 0, \\ U_0 & \text{for } x > a. \end{cases}$$

This potential is set up in the laboratory but the value of U_0 is unknown. The stationary states of the potential are investigated and it is found that there exists exactly one bound state. Show that the value of U_0 must satisfy

$$\frac{\pi^2\hbar^2}{8ma^2} < U_0 < \frac{9\pi^2\hbar^2}{8ma^2}.$$

Paper 3, Section II

16B Quantum Mechanics

The spherically symmetric bound state wavefunctions $\psi(r)$ for the Coulomb potential $V = -e^2/(4\pi\epsilon_0 r)$ are normalisable solutions of the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + \frac{2\lambda}{r}\psi = -\frac{2mE}{\hbar^2}\psi.$$

Here $\lambda = (me^2)/(4\pi\epsilon_0\hbar^2)$ and $E < 0$ is the energy of the state.

(a) By writing the wavefunction as $\psi(r) = f(r)\exp(-Kr)$, for a suitable constant K that you should determine, show that there are normalisable wavefunctions $\psi(r)$ only for energies of the form

$$E = \frac{-me^4}{32\pi^2\epsilon_0^2\hbar^2 N^2},$$

with N being a positive integer.

(b) The energies in (a) reproduce the predictions of the Bohr model of the hydrogen atom. How do the wavefunctions above compare to the assumptions in the Bohr model?

Paper 2, Section II

17B Quantum Mechanics

The one dimensional quantum harmonic oscillator has Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2,$$

where m and ω are real positive constants and $\hat{x}$ and $\hat{p}$ are the standard position and momentum operators satisfying the commutation relation $[\hat{x}, \hat{p}] = i\hbar$. Consider the operators

$$\hat{A} = \hat{p} - im\omega\hat{x} \quad \text{and} \quad \hat{B} = \hat{p} + im\omega\hat{x}.$$

(a) Show that

$$\hat{B}\hat{A} = 2m\left(\hat{H} - \frac{1}{2}\hbar\omega\right) \quad \text{and} \quad \hat{A}\hat{B} = 2m\left(\hat{H} + \frac{1}{2}\hbar\omega\right).$$

(b) Suppose that ϕ is an eigenfunction of $\hat{H}$ with eigenvalue E. Show that $\hat{A}\phi$ is then also an eigenfunction of $\hat{H}$ and that its corresponding eigenvalue is $E - \hbar\omega$.

(c) Show that for any normalisable wavefunctions χ and ψ,

$$\int_{-\infty}^{\infty} \chi^* \,(\hat{A}\psi)\, dx = \int_{-\infty}^{\infty} (\hat{B}\chi)^* \,\psi\, dx.$$

[You may assume that the operators $\hat{x}$ and $\hat{p}$ are Hermitian.]

(d) With ϕ as in (b), obtain an expression for the norm of $\hat{A}\phi$ in terms of E and the norm of ϕ. [The squared norm of any wavefunction ψ is $\int_{-\infty}^{\infty} |\psi|^2\, dx$.]

(e) Show that all eigenvalues of $\hat{H}$ are non-negative.

(f) Using the above results, deduce that each eigenvalue E of $\hat{H}$ must be of the form $E = (n + \frac{1}{2})\hbar\omega$ for some non-negative integer n.

Paper 4, Section I

6D Quantum Mechanics

The radial wavefunction $R(r)$ for an electron in a hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right)+\frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r)-\frac{e^2}{4\pi\epsilon_0 r}R(r)=E\,R(r) \qquad (*)$$

Briefly explain the origin of each term in this equation.

The wavefunctions for the ground state and the first radially excited state, both with $\ell=0$, can be written as

$$\begin{aligned} R_1(r) &= N_1 e^{-\alpha r} \\ R_2(r) &= N_2\left(1-\tfrac{1}{2}r\alpha\right)e^{-\frac{1}{2}\alpha r} \end{aligned}$$

where N_1 and N_2 are normalisation constants. Verify that $R_1(r)$ is a solution of $(*)$, determining α and finding the corresponding energy eigenvalue E_1. Assuming that $R_2(r)$ is a solution of $(*)$, compare coefficients of the dominant terms when r is large to determine the corresponding energy eigenvalue E_2. [You do *not* need to find N_1 or N_2, nor show that R_2 is a solution of $(*)$.]

A hydrogen atom makes a transition from the first radially excited state to the ground state, emitting a photon. What is the angular frequency of the emitted photon?

Paper 3, Section I

8D Quantum Mechanics

A quantum-mechanical system has normalised energy eigenstates χ_1 and χ_2 with non-degenerate energies E_1 and E_2 respectively. The observable A has normalised eigenstates,

$$\begin{aligned} \phi_1 &= C(\chi_1+2\chi_2)\,, &\quad \text{eigenvalue} = a_1\,, \\ \phi_2 &= C(2\chi_1-\chi_2)\,, &\quad \text{eigenvalue} = a_2\,, \end{aligned}$$

where C is a positive real constant. Determine C.

Initially, at time $t=0$, the state of the system is ϕ_1. Write down an expression for $\psi(t)$, the state of the system with $t \geqslant 0$. What is the probability that a measurement of energy at time t will yield E_2?

For the same initial state, determine the probability that a measurement of A at time $t>0$ will yield a_1 and the probability that it will yield a_2.

Paper 1, Section II

15D Quantum Mechanics

Write down expressions for the probability density $\rho(x,t)$ and the probability current $j(x,t)$ for a particle in one dimension with wavefunction $\Psi(x,t)$. If $\Psi(x,t)$ obeys the time-dependent Schrödinger equation with a real potential, show that

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0\,.$$

Consider a stationary state, $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$, with

$$\psi(x) \sim \begin{cases} e^{ik_1 x} + Re^{-ik_1 x} & x \to -\infty \\ Te^{ik_2 x} & x \to +\infty \end{cases}\,,$$

where E, k_1, k_2 are real. Evaluate $j(x,t)$ for this state in the regimes $x \to +\infty$ and $x \to -\infty$.

Consider a real potential,

$$V(x) = -\alpha\delta(x) + U(x)\,, \qquad U(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}\,,$$

where $\delta(x)$ is the Dirac delta function, $V_0 > 0$ and $\alpha > 0$. Assuming that $\psi(x)$ is continuous at $x = 0$, derive an expression for

$$\lim_{\epsilon \to 0}\left[\psi'(\epsilon) - \psi'(-\epsilon)\right]\,.$$

Hence calculate the reflection and transmission probabilities for a particle incident from $x = -\infty$ with energy $E > V_0$.

Paper 3, Section II
16D Quantum Mechanics

Define the angular momentum operators $\hat{L}_i$ for a particle in three dimensions in terms of the position and momentum operators $\hat{x}_i$ and $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$. Write down an expression for $[\hat{L}_i, \hat{L}_j]$ and use this to show that $[\hat{L}^2, \hat{L}_i] = 0$ where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. What is the significance of these two commutation relations?

Let $\psi(x, y, z)$ be both an eigenstate of $\hat{L}_z$ with eigenvalue zero and an eigenstate of $\hat{L}^2$ with eigenvalue $\hbar^2 l(l+1)$. Show that $(\hat{L}_x + i\hat{L}_y)\psi$ is also an eigenstate of both $\hat{L}_z$ and $\hat{L}^2$ and determine the corresponding eigenvalues.

Find real constants A and B such that

$$\phi(x, y, z) = \left(Az^2 + By^2 - r^2\right) e^{-r}\,, \qquad r^2 = x^2 + y^2 + z^2\,,$$

is an eigenfunction of $\hat{L}_z$ with eigenvalue zero and an eigenfunction of $\hat{L}^2$ with an eigenvalue which you should determine. [*Hint: You might like to show that* $\hat{L}_i\, f(r) = 0$.]

Paper 2, Section II
17D Quantum Mechanics

A quantum-mechanical harmonic oscillator has Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{1}{2}k^2\hat{x}^2\,. \qquad (*)\,,$$

where k is a positive real constant. Show that $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ are Hermitian operators.

The eigenfunctions of $(*)$ can be written as

$$\psi_n(x) = h_n\left(x\sqrt{k/\hbar}\right)\ \exp\left(-\frac{kx^2}{2\hbar}\right)\,,$$

where h_n is a polynomial of degree n with even (odd) parity for even (odd) n and $n = 0, 1, 2, \ldots$. Show that $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$ for all of the states ψ_n.

State the Heisenberg uncertainty principle and verify it for the state ψ_0 by computing (Δx) and (Δp). [*Hint: You should properly normalise the state.*]

The oscillator is in its ground state ψ_0 when the potential is suddenly changed so that $k \to 4k$. If the wavefunction is expanded in terms of the energy eigenfunctions of the new Hamiltonian, ϕ_n, what can be said about the coefficient of ϕ_n for odd n? What is the probability that the particle is in the new ground state just after the change?

[*Hint: You may assume that if* $I_n = \int_{-\infty}^{\infty} e^{-ax^2} x^n\, dx$ *then* $I_0 = \sqrt{\frac{\pi}{a}}$ *and* $I_2 = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$.]

Paper 4, Section I
6A Quantum Mechanics

For some quantum mechanical observable Q, prove that its uncertainty (ΔQ) satisfies

$$(\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2.$$

A quantum mechanical harmonic oscillator has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2},$$

where $m > 0$. Show that (in a stationary state of energy E)

$$E \geqslant \frac{(\Delta p)^2}{2m} + \frac{m\omega^2 (\Delta x)^2}{2}.$$

Write down the Heisenberg uncertainty relation. Then, use it to show that

$$E \geqslant \frac{1}{2}\hbar\omega$$

for our stationary state.

Paper 3, Section I
8A Quantum Mechanics

The wavefunction of a normalised Gaussian wavepacket for a particle of mass m in one dimension with potential $V(x) = 0$ is given by

$$\psi(x,t) = B\sqrt{A(t)}\exp\left(\frac{-x^2 A(t)}{2}\right),$$

where $A(0) = 1$. Given that $\psi(x,t)$ is a solution of the time-dependent Schrödinger equation, find the complex-valued function $A(t)$ and the real constant B.

[You may assume that $\int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx = \sqrt{\pi}/\sqrt{\lambda}$.]

Paper 1, Section II

15A Quantum Mechanics

Consider a particle confined in a one-dimensional infinite potential well: $V(x) = \infty$ for $|x| \geqslant a$ and $V(x) = 0$ for $|x| < a$. The normalised stationary states are

$$\psi_n(x) = \begin{cases} \alpha_n \sin\left(\dfrac{\pi n(x+a)}{2a}\right) & \text{for} \quad |x| < a \\ 0 & \text{for} \quad |x| \geqslant a \end{cases}$$

where $n = 1, 2, \ldots$.

(i) Determine the α_n and the stationary states' energies E_n.

(ii) A state is prepared within this potential well: $\psi(x) \propto x$ for $0 < x < a$, but $\psi(x) = 0$ for $x \leqslant 0$ or $x \geqslant a$. Find an explicit expansion of $\psi(x)$ in terms of $\psi_n(x)$.

(iii) If the energy of the state is then immediately measured, show that the probability that it is *greater* than $\frac{\hbar^2\pi^2}{ma^2}$ is

$$\sum_{n=0}^{4} \frac{b_n}{\pi^n},$$

where the b_n are integers which you should find.

(iv) By considering the normalisation condition for $\psi(x)$ in terms of the expansion in $\psi_n(x)$, show that

$$\frac{\pi^2}{3} = \sum_{p=1}^{\infty} \frac{A}{p^2} + \frac{B}{(2p-1)^2}\left(1 + \frac{C(-1)^p}{(2p-1)\pi}\right)^2,$$

where A, B and C are integers which you should find.

Paper 3, Section II

16A Quantum Mechanics

The Hamiltonian of a two-dimensional isotropic harmonic oscillator is given by

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2)\,,$$

where x and y denote position operators and p_x and p_y the corresponding momentum operators.

State without proof the commutation relations between the operators x, y, p_x, p_y. From these commutation relations, write $[x^2,\ p_x]$ and $[x,\ p_x^2]$ in terms of a single operator. Now consider the observable

$$L = xp_y - yp_x.$$

Ehrenfest's theorem states that, for some observable Q with expectation value $\langle Q\rangle$,

$$\frac{d\langle Q\rangle}{dt} = \frac{1}{i\hbar}\langle[Q,\ H]\rangle + \langle\frac{\partial Q}{\partial t}\rangle.$$

Use it to show that the expectation value of L is constant with time.

Given two states

$$\psi_1 = \alpha x \exp\left(-\beta(x^2+y^2)\right) \text{ and } \psi_2 = \alpha y \exp\left(-\beta(x^2+y^2)\right),$$

where α and β are constants, find a normalised linear combination of ψ_1 and ψ_2 that is an eigenstate of L, and the corresponding L eigenvalue. [You may assume that α correctly normalises both ψ_1 and ψ_2.] If a quantum state is prepared in the linear combination you have found at time $t = 0$, what is the expectation value of L at a later time t?

Paper 2, Section II

17A Quantum Mechanics

For an electron of mass m in a hydrogen atom, the time-independent Schrödinger equation may be written as

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{2mr^2}L^2\psi-\frac{e^2}{4\pi\epsilon_0 r}\psi=E\psi.$$

Consider normalised energy eigenstates of the form

$$\psi_{lm}(r,\ \theta,\ \phi)=R(r)Y_{lm}(\theta,\ \phi)$$

where Y_{lm} are orbital angular momentum eigenstates:

$$L^2Y_{lm}=\hbar^2 l(l+1)Y_{lm},\qquad L_3Y_{lm}=\hbar mY_{lm},$$

where $l=1,\ 2,\ \ldots$ and $m=0,\ \pm1,\ \pm2,\ \ldots\pm l$. The Y_{lm} functions are normalised with $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}|Y_{lm}|^2\ \sin\theta\ d\theta\ d\phi=1$.

(i) Write down the resulting equation satisfied by $R(r)$, for fixed l. Show that it has solutions of the form

$$R(r)=Ar^l\exp\left(-\frac{r}{a(l+1)}\right),$$

where a is a constant which you should determine. Show that

$$E=-\frac{e^2}{D\pi\epsilon_0 a},$$

where D is an integer which you should find (in terms of l). Also, show that

$$|A|^2=\frac{2^{2l+3}}{a^F G!(l+1)^H},$$

where F, G and H are integers that you should find in terms of l.

(ii) Given the radius of the proton $r_p \ll a$, show that the probability of the electron being found within the proton is approximately

$$\frac{2^{2l+3}}{C}\left(\frac{r_p}{a}\right)^{2l+3}\left[1+\mathcal{O}\left(\frac{r_p}{a}\right)\right],$$

finding the integer C in terms of l.

[You may assume that $\int_0^\infty t^l e^{-t}dt=l!$.]

Paper 4, Section I

6B Quantum Mechanics

The components of the three-dimensional angular momentum operator $\hat{\mathbf{L}}$ are defined as follows:

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad \hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \quad \hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Given that the wavefunction

$$\psi = (f(x) + iy)z$$

is an eigenfunction of $\hat{L}_z$, find all possible values of $f(x)$ and the corresponding eigenvalues of ψ. Letting $f(x) = x$, show that ψ is an eigenfunction of $\hat{\mathbf{L}}^2$ and calculate the corresponding eigenvalue.

Paper 3, Section I

8B Quantum Mechanics

If α, β and γ are linear operators, establish the identity

$$[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta.$$

In what follows, the operators x and p are Hermitian and represent position and momentum of a quantum mechanical particle in one-dimension. Show that

$$[x^n, p] = i\hbar n x^{n-1}$$

and

$$[x, p^m] = i\hbar m p^{m-1}$$

where $m, n \in \mathbb{Z}^+$. Assuming $[x^n, p^m] \neq 0$, show that the operators x^n and p^m are Hermitian but their product is not. Determine whether $x^n p^m + p^m x^n$ is Hermitian.

Paper 1, Section II
15B Quantum Mechanics

A particle with momentum $\hat{p}$ moves in a one-dimensional real potential with Hamiltonian given by

$$\hat{H} = \frac{1}{2m}(\hat{p} + isA)(\hat{p} - isA), \quad -\infty < x < \infty$$

where A is a real function and $s \in \mathbb{R}^+$. Obtain the potential energy of the system. Find $\chi(x)$ such that $(\hat{p} - isA)\chi(x) = 0$. Now, putting $A = x^n$, for $n \in \mathbb{Z}^+$, show that $\chi(x)$ can be normalised only if n is odd. Letting $n = 1$, use the inequality

$$\int_{-\infty}^{\infty} \psi^*(x)\hat{H}\psi(x)dx \geqslant 0$$

to show that

$$\Delta x \Delta p \geqslant \frac{\hbar}{2}$$

assuming that both $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$ vanish.

Paper 3, Section II
16B Quantum Mechanics

Obtain, with the aid of the time-dependent Schrödinger equation, the conservation equation

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = 0$$

where $\rho(\mathbf{x}, t)$ is the probability density and $\mathbf{j}(\mathbf{x}, t)$ is the probability current. What have you assumed about the potential energy of the system?

Show that if the potential $U(\mathbf{x}, t)$ is complex the conservation equation becomes

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = \frac{2}{\hbar}\rho(\mathbf{x}, t)\,\mathrm{Im}U(\mathbf{x}, t).$$

Take the potential to be time-independent. Show, with the aid of the divergence theorem, that

$$\frac{d}{dt}\int_{\mathbb{R}^3} \rho(\mathbf{x}, t)\,dV = \frac{2}{\hbar}\int_{\mathbb{R}^3} \rho(\mathbf{x}, t)\,\mathrm{Im}U(\mathbf{x})\,dV.$$

Assuming the wavefunction $\psi(\mathbf{x}, 0)$ is normalised to unity, show that if $\rho(\mathbf{x}, t)$ is expanded about $t = 0$ so that $\rho(\mathbf{x}, t) = \rho_0(\mathbf{x}) + t\rho_1(\mathbf{x}) + \cdots$, then

$$\int_{\mathbb{R}^3} \rho(\mathbf{x}, t)\,dV = 1 + \frac{2t}{\hbar}\int_{\mathbb{R}^3} \rho_0(\mathbf{x})\,\mathrm{Im}U(\mathbf{x})\,dV + \cdots .$$

As time increases, how does the quantity on the left of this equation behave if $\mathrm{Im}U(\mathbf{x}) < 0$?

Paper 2, Section II
17B Quantum Mechanics

(i) Consider a particle of mass m confined to a one-dimensional potential well of depth $U > 0$ and potential

$$V(x) = \begin{cases} -U, & |x| < l \\ 0, & |x| > l. \end{cases}$$

If the particle has energy E where $-U \leqslant E < 0$, show that for even states

$$\alpha \tan \alpha l = \beta$$

where $\alpha = [\frac{2m}{\hbar^2}(U+E)]^{1/2}$ and $\beta = [-\frac{2m}{\hbar^2}E]^{1/2}$.

(ii) A particle of mass m that is incident from the left scatters off a one-dimensional potential given by

$$V(x) = k\delta(x)$$

where $\delta(x)$ is the Dirac delta. If the particle has energy $E > 0$ and $k > 0$, obtain the reflection and transmission coefficients R and T, respectively. Confirm that $R + T = 1$.

For the case $k < 0$ and $E < 0$ show that the energy of the only even parity bound state of the system is given by

$$E = -\frac{mk^2}{2\hbar^2}.$$

Use part (i) to verify this result by taking the limit $U \to \infty,\ l \to 0$ with Ul fixed.

Paper 4, Section I

6C Quantum Mechanics

In terms of quantum states, what is meant by *energy degeneracy*?

A particle of mass m is confined within the box $0 < x < a$, $0 < y < a$ and $0 < z < c$. The potential vanishes inside the box and is infinite outside. Find the allowed energies by considering a stationary state wavefunction of the form

$$\chi(x, y, z) = X(x)\, Y(y)\, Z(z)\,.$$

Write down the normalised ground state wavefunction. Assuming that $c < a < \sqrt{2}c$, give the energies of the first three excited states.

Paper 3, Section I

8C Quantum Mechanics

A one-dimensional quantum mechanical particle has normalised bound state energy eigenfunctions $\chi_n(x)$ and corresponding non-degenerate energy eigenvalues E_n. At $t = 0$ the normalised wavefunction $\psi(x, t)$ is given by

$$\psi(x, 0) = \sqrt{\frac{5}{6}}\, e^{ik_1}\chi_1(x) + \sqrt{\frac{1}{6}}\, e^{ik_2}\chi_2(x)$$

where k_1 and k_2 are real constants. Write down the expression for $\psi(x, t)$ at a later time t and give the probability that a measurement of the particle's energy will yield a value of E_2.

Show that the expectation value of x at time t is given by

$$\langle x \rangle = \frac{5}{6}\langle x \rangle_{11} + \frac{1}{6}\langle x \rangle_{22} + \frac{\sqrt{5}}{3}\mathrm{Re}\Big[\langle x \rangle_{12}\, e^{i(k_2 - k_1) - i(E_2 - E_1)t/\hbar}\Big]$$

where $\langle x \rangle_{ij} = \int_{-\infty}^{\infty} \chi_i^*(x)\, x\, \chi_j(x)\, dx$.

Paper 1, Section II

15C Quantum Mechanics

Show that if the energy levels are discrete, the general solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x})\psi$$

is a linear superposition of stationary states

$$\psi(\mathbf{x},t) = \sum_{n=1}^{\infty} a_n\chi_n(\mathbf{x})\exp(-iE_nt/\hbar)\,,$$

where $\chi_n(\mathbf{x})$ is a solution of the time-independent Schrödinger equation and a_n are complex coefficients. Can this general solution be considered to be a stationary state? Justify your answer.

A linear operator $\hat{O}$ acts on the orthonormal energy eigenfunctions χ_n as follows:

$$\begin{aligned}\hat{O}\chi_1 &= \chi_1+\chi_2\\ \hat{O}\chi_2 &= \chi_1+\chi_2\\ \hat{O}\chi_n &= 0, \quad n\geqslant 3.\end{aligned}$$

Obtain the eigenvalues of $\hat{O}$. Hence, find the normalised eigenfunctions of $\hat{O}$. In an experiment a measurement is made of $\hat{O}$ at $t=0$ yielding an eigenvalue of 2. What is the probability that a measurement at some later time t will yield an eigenvalue of 2?

Paper 3, Section II

16C Quantum Mechanics

State the condition for a linear operator $\hat{O}$ to be Hermitian.

Given the position and momentum operators $\hat{x}_i$ and $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$, define the angular momentum operators $\hat{L}_i$. Establish the commutation relations

$$[\hat{L}_i,\hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$$

and use these relations to show that $\hat{L}_3$ is Hermitian assuming $\hat{L}_1$ and $\hat{L}_2$ are.

Consider a wavefunction of the form

$$\chi(\mathbf{x}) = x_3(x_1+kx_2)e^{-r}$$

where $r=|\mathbf{x}|$ and k is some constant. Show that $\chi(\mathbf{x})$ is an eigenstate of the total angular momentum operator $\hat{\mathbf{L}}^2$ for all k, and calculate the corresponding eigenvalue. For what values of k is $\chi(\mathbf{x})$ an eigenstate of $\hat{L}_3$? What are the corresponding eigenvalues?

Paper 2, Section II

17C Quantum Mechanics

Consider a quantum mechanical particle in a one-dimensional potential $V(x)$, for which $V(x) = V(-x)$. Prove that when the energy eigenvalue E is non-degenerate, the energy eigenfunction $\chi(x)$ has definite parity.

Now assume the particle is in the double potential well

$$V(x) = \begin{cases} U\,, & 0 \leqslant |x| \leqslant l_1 \\ 0\,, & l_1 < |x| \leqslant l_2 \\ \infty\,, & l_2 < |x|\,, \end{cases}$$

where $0 < l_1 < l_2$ and $0 < E < U$ (U being large and positive). Obtain general expressions for the even parity energy eigenfunctions $\chi^+(x)$ in terms of trigonometric and hyperbolic functions. Show that

$$-\tan[k(l_2 - l_1)] = \frac{k}{\kappa}\coth(\kappa l_1)\,,$$

where $k^2 = \dfrac{2mE}{\hbar^2}$ and $\kappa^2 = \dfrac{2m(U-E)}{\hbar^2}$.

Paper 3, Section I
8C Quantum Mechanics

A particle of mass m and energy E, incident from $x = -\infty$, scatters off a delta function potential at $x = 0$. The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\delta(x)\psi = E\psi$$

where U is a positive constant. Find the reflection and transmission probabilities.

Paper 4, Section I
6C Quantum Mechanics

Consider the 3-dimensional oscillator with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}(x^2 + y^2 + 4z^2)\,.$$

Find the ground state energy and the spacing between energy levels. Find the degeneracies of the lowest three energy levels.

[You may assume that the energy levels of the 1-dimensional harmonic oscillator with Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2$$

are $(n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$.]

Paper 1, Section II

15C Quantum Mechanics

For a quantum mechanical particle moving freely on a circle of length 2π, the wavefunction $\psi(t,x)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

on the interval $0\leqslant x\leqslant 2\pi$, and also the periodicity conditions $\psi(t,2\pi)=\psi(t,0)$, and $\dfrac{\partial\psi}{\partial x}(t,2\pi)=\dfrac{\partial\psi}{\partial x}(t,0)$. Find the allowed energy levels of the particle, and their degeneracies.

The current is defined as

$$j=\frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x}-\psi^*\frac{\partial\psi}{\partial x}\right)$$

where ψ is a normalized state. Write down the general normalized state of the particle when it has energy $2\hbar^2/m$, and show that in any such state the current j is independent of x and t. Find a state with this energy for which the current has its maximum positive value, and find a state with this energy for which the current vanishes.

Paper 2, Section II

17C Quantum Mechanics

The quantum mechanical angular momentum operators are

$$L_i=-i\hbar\,\epsilon_{ijk}\,x_j\,\frac{\partial}{\partial x_k}\qquad(i=1,2,3).$$

Show that each of these is hermitian.

The total angular momentum operator is defined as $\mathbf{L}^2=L_1^2+L_2^2+L_3^2$. Show that $\langle\mathbf{L}^2\rangle\geqslant\langle L_3^2\rangle$ in any state, and show that the only states where $\langle\mathbf{L}^2\rangle=\langle L_3^2\rangle$ are those with no angular dependence. Verify that the eigenvalues of the operators $\mathbf{L}^2$ and L_3^2 (whose values you may quote without proof) are consistent with these results.

Paper 3, Section II

16C Quantum Mechanics

For an electron in a hydrogen atom, the stationary state wavefunctions are of the form $\psi(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi)$, where in suitable units R obeys the radial equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0\,.$$

Explain briefly how the terms in this equation arise.

This radial equation has bound state solutions of energy $E = E_n$, where $E_n = -\frac{1}{2n^2}$ $(n = 1,2,3,\dots)$. Show that when $l = n-1$, there is a solution of the form $R(r) = r^{\alpha}e^{-r/n}$, and determine α. Find the expectation value $\langle r\rangle$ in this state.

What is the total degeneracy of the energy level with energy E_n?

Paper 3, Section I

8D Quantum Mechanics

Write down the commutation relations between the components of position $\mathbf{x}$ and momentum $\mathbf{p}$ for a particle in three dimensions.

A particle of mass m executes simple harmonic motion with Hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + \frac{m\omega^2}{2}\mathbf{x}^2,$$

and the orbital angular momentum operator is defined by

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}.$$

Show that the components of $\mathbf{L}$ are observables commuting with H. Explain briefly why the components of $\mathbf{L}$ are not simultaneous observables. What are the implications for the labelling of states of the three-dimensional harmonic oscillator?

Paper 4, Section I

6D Quantum Mechanics

Determine the possible values of the energy of a particle free to move inside a cube of side a, confined there by a potential which is infinite outside and zero inside.

What is the degeneracy of the lowest-but-one energy level?

Paper 1, Section II
15D Quantum Mechanics

A particle of unit mass moves in one dimension in a potential

$$V = \frac{1}{2}\omega^2 x^2.$$

Show that the stationary solutions can be written in the form

$$\psi_n(x) = f_n(x)\exp(-\alpha x^2).$$

You should give the value of α and derive any restrictions on $f_n(x)$. Hence determine the possible energy eigenvalues E_n.

The particle has a wave function $\psi(x,t)$ which is even in x at $t = 0$. Write down the general form for $\psi(x,0)$, using the fact that $f_n(x)$ is an even function of x only if n is even. Hence write down $\psi(x,t)$ and show that its probability density is periodic in time with period π/ω.

Paper 2, Section II
17D Quantum Mechanics

A particle of mass m moves in a one-dimensional potential defined by

$$V(x) = \begin{cases} \infty & \text{for } x < 0, \\ 0 & \text{for } 0 \leqslant x \leqslant a, \\ V_0 & \text{for } a < x, \end{cases}$$

where a and V_0 are positive constants. Defining $c = [2m(V_0 - E)]^{1/2}/\hbar$ and $k = (2mE)^{1/2}/\hbar$, show that for any allowed positive value E of the energy with $E < V_0$ then

$$c + k\cot ka = 0.$$

Find the minimum value of V_0 for this equation to have a solution.

Find the normalized wave function for the particle. Write down an expression for the expectation value of x in terms of two integrals, which you need not evaluate. Given that

$$\langle x \rangle = \frac{1}{2k}(ka - \tan ka),$$

discuss briefly the possibility of $\langle x \rangle$ being greater than a. [Hint: consider the graph of $-ka\cot ka$ against ka.]

Paper 3, Section II

16D Quantum Mechanics

A π^- (a particle of the same charge as the electron but 270 times more massive) is bound in the Coulomb potential of a proton. Assuming that the wave function has the form $ce^{-r/a}$, where c and a are constants, determine the normalized wave function of the lowest energy state of the π^-, assuming it to be an S-wave (i.e. the state with $l = 0$). (You should treat the proton as fixed in space.)

Calculate the probability of finding the π^- inside a sphere of radius R in terms of the ratio $\mu = R/a$, and show that this probability is given by $4\mu^3/3 + O(\mu^4)$ if μ is very small. Would the result be larger or smaller if the π^- were in a P-wave ($l = 1$) state? Justify your answer very briefly.

[Hint: in spherical polar coordinates,

$$\nabla^2\psi(\mathbf{r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.\Bigg]$$

Paper 3, Section I

7B Quantum Mechanics

The motion of a particle in one dimension is described by the time-independent hermitian Hamiltonian operator H whose normalized eigenstates $\psi_n(x)$, $n = 0, 1, 2, \ldots$, satisfy the Schrödinger equation

$$H\psi_n = E_n\psi_n \ ,$$

with $E_0 < E_1 < E_2 < \cdots < E_n < \cdots$. Show that

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn} \ .$$

The particle is in a state represented by the wavefunction $\Psi(x,t)$ which, at time $t = 0$, is given by

$$\Psi(x,0) = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{n+1} \psi_n(x) \ .$$

Write down an expression for $\Psi(x,t)$ and show that it is normalized to unity.

Derive an expression for the expectation value of the energy for this state and show that it is independent of time.

Calculate the probability that the particle has energy E_m for a given integer $m \geqslant 0$, and show that this also is time-independent.

Paper 4, Section I

6B Quantum Mechanics

The wavefunction of a Gaussian wavepacket for a particle of mass m moving in one dimension is

$$\psi(x,t) = \frac{1}{\pi^{1/4}} \sqrt{\frac{1}{1 + i\hbar t/m}} \exp\left(-\frac{x^2}{2(1 + i\hbar t/m)}\right) \ .$$

Show that $\psi(x,t)$ satisfies the appropriate time-dependent Schrödinger equation.

Show that $\psi(x,t)$ is normalized to unity and calculate the uncertainty in measurement of the particle position, $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

Is $\psi(x,t)$ a stationary state? Give a reason for your answer.

$\left[\textit{You may assume that} \int_{-\infty}^{\infty} e^{-\lambda x^2} \, dx = \sqrt{\frac{\pi}{\lambda}}. \right]$

Paper 1, Section II

15B Quantum Mechanics

A particle of mass m moves in one dimension in a potential $V(x)$ which satisfies $V(x) = V(-x)$. Show that the eigenstates of the Hamiltonian H can be chosen so that they are also eigenstates of the parity operator P. For eigenstates with odd parity $\psi^{odd}(x)$, show that $\psi^{odd}(0) = 0$.

A potential $V(x)$ is given by

$$V(x) = \begin{cases} \kappa\delta(x) & |x| < a \\ \infty & |x| > a \end{cases} .$$

State the boundary conditions satisfied by $\psi(x)$ at $|x| = a$, and show also that

$$\frac{\hbar^2}{2m} \lim_{\epsilon \to 0} \left[\left. \frac{d\psi}{dx} \right|_{\epsilon} - \left. \frac{d\psi}{dx} \right|_{-\epsilon} \right] = \kappa\psi(0) .$$

Let the energy eigenstates of even parity be given by

$$\psi^{even}(x) = \begin{cases} A\cos\lambda x \; + \; B\sin\lambda x & -a < x < 0 \\ A\cos\lambda x \; - \; B\sin\lambda x & 0 < x < a \\ 0 & \text{otherwise} \end{cases} .$$

Verify that $\psi^{even}(x)$ satisfies

$$P\psi^{even}(x) = \psi^{even}(x) .$$

By demanding that $\psi^{even}(x)$ satisfy the relevant boundary conditions show that

$$\tan\lambda a = -\frac{\hbar^2}{m}\frac{\lambda}{\kappa} .$$

For $\kappa > 0$ show that the energy eigenvalues E_n^{even}, $n = 0, 1, 2, \ldots$, with $E_n^{even} < E_{n+1}^{even}$, satisfy

$$\eta_n = E_n^{even} - \frac{1}{2m}\left[\frac{(2n+1)\hbar\pi}{2a}\right]^2 > 0.$$

Show also that

$$\lim_{n\to\infty} \eta_n = 0,$$

and give a physical explanation of this result.

Show that the energy eigenstates with odd parity and their energy eigenvalues do not depend on κ.

Paper 2, Section II

16B Quantum Mechanics

Write down the expressions for the probability density ρ and the associated current density j for a particle with wavefunction $\psi(x,t)$ moving in one dimension. If $\psi(x,t)$ obeys the time-dependent Schrödinger equation show that ρ and j satisfy

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0\,.$$

Give an interpretation of $\psi(x,t)$ in the case that

$$\psi(x,t) = (e^{ikx} + Re^{-ikx})e^{-iEt/\hbar}\,,$$

and show that $E = \dfrac{\hbar^2 k^2}{2m}$ and $\dfrac{\partial \rho}{\partial t} = 0$.

A particle of mass m and energy $E > 0$ moving in one dimension is incident from the left on a potential $V(x)$ given by

$$V(x) = \begin{cases} -V_0 & 0 < x < a \\ \quad 0 & x < 0,\; x > a \end{cases}\,,$$

where V_0 is a positive constant. What conditions must be imposed on the wavefunction at $x = 0$ and $x = a$? Show that when $3E = V_0$ the probability of transmission is

$$\left[1 + \frac{9}{16}\sin^2\frac{a\sqrt{8mE}}{\hbar}\right]^{-1}\,.$$

For what values of a does this agree with the classical result?

Paper 3, Section II

16B Quantum Mechanics

If A, B, and C are operators establish the identity

$$[AB, C] = A[B, C] + [A, C]B \ .$$

A particle moves in a two-dimensional harmonic oscillator potential with Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(x^2 + y^2) \ .$$

The angular momentum operator is defined by

$$L = xp_y - yp_x \ .$$

Show that L is hermitian and hence that its eigenvalues are real. Establish the commutation relation $[L, H] = 0$. Why does this ensure that eigenstates of H can also be chosen to be eigenstates of L?

Let $\phi_0(x, y) = e^{-(x^2+y^2)/2\hbar}$, and show that ϕ_0, $\phi_x = x\phi_0$ and $\phi_y = y\phi_0$ are all eigenstates of H, and find their respective eigenvalues. Show that

$$L\phi_0 = 0, \quad L\phi_x = i\hbar\phi_y, \quad L\phi_y = -i\hbar\phi_x \ ,$$

and hence, by taking suitable linear combinations of ϕ_x and ϕ_y, find two states, ψ_1 and ψ_2, satisfying

$$L\psi_j = \lambda_j\psi_j, \quad H\psi_j = E_j\psi_j \qquad j = 1, 2 \ .$$

Show that ψ_1 and ψ_2 are orthogonal, and find $\lambda_1, \lambda_2, E_1$ and E_2.

The particle has charge e, and an electric field of strength $\mathcal{E}$ is applied in the x-direction so that the Hamiltonian is now H', where

$$H' = H - e\mathcal{E}x \ .$$

Show that $[L, H'] = -i\hbar e\mathcal{E}y$. Why does this mean that L and H' cannot have simultaneous eigenstates?

By making the change of coordinates $x' = x - e\mathcal{E}$, $y' = y$, show that $\psi_1(x', y')$ and $\psi_2(x', y')$ are eigenstates of H' and write down the corresponding energy eigenvalues.

Find a modified angular momentum operator L' for which $\psi_1(x', y')$ and $\psi_2(x', y')$ are also eigenstates.

1/II/15A **Quantum Mechanics**

The radial wavefunction $g(r)$ for the hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dg(r)}{dr}\right)-\frac{e^2g(r)}{4\pi\epsilon_0 r}+\hbar^2\frac{\ell(\ell+1)}{2mr^2}g(r)=Eg(r)\,. \qquad (*)$$

With reference to the general form for the time-independent Schrödinger equation, explain the origin of each term. What are the allowed values of ℓ?

The lowest-energy bound-state solution of $(*)$, for given ℓ, has the form $r^\alpha e^{-\beta r}$. Find α and β and the corresponding energy E in terms of ℓ.

A hydrogen atom makes a transition between two such states corresponding to $\ell+1$ and ℓ. What is the frequency of the emitted photon?

2/II/16A **Quantum Mechanics**

Give the physical interpretation of the expression

$$\langle A\rangle_\psi=\int\psi(x)^*\hat{A}\psi(x)dx$$

for an observable A, where $\hat{A}$ is a Hermitian operator and ψ is normalised. By considering the norm of the state $(A+i\lambda B)\psi$ for two observables A and B, and real values of λ, show that

$$\langle A^2\rangle_\psi\langle B^2\rangle_\psi\geqslant\frac{1}{4}|\langle[A,B]\rangle_\psi|^2\,.$$

Deduce the uncertainty relation

$$\Delta A\Delta B\geqslant\frac{1}{2}|\langle[A,B]\rangle_\psi|\,,$$

where ΔA is the uncertainty of A.

A particle of mass m moves in one dimension under the influence of potential $\frac{1}{2}m\omega^2x^2$. By considering the commutator $[x,p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H\rangle_\psi\geqslant\frac{1}{2}\hbar\omega\,.$$

3/I/7A Quantum Mechanics

Write down a formula for the orbital angular momentum operator $\hat{\mathbf{L}}$. Show that its components satisfy

$$[L_i, L_j] = i\hbar\, \epsilon_{ijk}\, L_k\,.$$

If $L_3\psi = 0$, show that $(L_1 \pm iL_2)\psi$ are also eigenvectors of L_3, and find their eigenvalues.

3/II/16A Quantum Mechanics

What is the probability current for a particle of mass m, wavefunction ψ, moving in one dimension?

A particle of energy E is incident from $x < 0$ on a barrier given by

$$V(x) = \begin{cases} 0 & x \leqslant 0 \\ V_1 & 0 < x < a \\ V_0 & x \geqslant a \end{cases}$$

where $V_1 > V_0 > 0$. What are the conditions satisfied by ψ at $x = 0$ and $x = a$? Write down the form taken by the wavefunction in the regions $x \leqslant 0$ and $x \geqslant a$ distinguishing between the cases $E > V_0$ and $E < V_0$. For both cases, use your expressions for ψ to calculate the probability currents in these two regions.

Define the reflection and transmission coefficients, R and T. Using current conservation, show that the expressions you have derived satisfy $R + T = 1$. Show that $T = 0$ if $0 < E < V_0$.

4/I/6A Quantum Mechanics

What is meant by a stationary state? What form does the wavefunction take in such a state? A particle has wavefunction $\psi(x,t)$, such that

$$\psi(x,0) = \sqrt{\frac{1}{2}}\,(\chi_1(x) + \chi_2(x))\ ,$$

where χ_1 and χ_2 are normalised eigenstates of the Hamiltonian with energies E_1 and E_2. Write down $\psi(x,t)$ at time t. Show that the expectation value of A at time t is

$$\langle A\rangle_\psi = \frac{1}{2}\int_{-\infty}^{\infty}\left(\chi_1^*\hat{A}\chi_1 + \chi_2^*\hat{A}\chi_2\right)dx + Re\left(e^{i(E_1-E_2)t/\hbar}\int_{-\infty}^{\infty}\chi_1^*\hat{A}\chi_2\,dx\right)\,.$$

1/II/15B **Quantum Mechanics**

The relative motion of a neutron and proton is described by the Schrödinger equation for a single particle of mass m under the influence of the central potential

$$V(r) = \begin{cases} -U & r < a \\ 0 & r > a, \end{cases}$$

where U and a are positive constants. Solve this equation for a spherically symmetric state of the deuteron, which is a bound state of a proton and neutron, giving the condition on U for this state to exist.

[*If* ψ *is spherically symmetric then* $\nabla^2\psi = \frac{1}{r}\,\frac{d^2}{dr^2}\,(r\psi)$.]

2/II/16B **Quantum Mechanics**

Write down the angular momentum operators L_1, L_2, L_3 in terms of the position and momentum operators, $\mathbf{x}$ and $\mathbf{p}$, and the commutation relations satisfied by $\mathbf{x}$ and $\mathbf{p}$.

Verify the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k\,.$$

Further, show that

$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k\,.$$

A wave-function $\Psi_0(r)$ is spherically symmetric. Verify that

$$\mathbf{L}\Psi_0(r) = 0\,.$$

Consider the vector function $\mathbf{\Phi} = \nabla\Psi_0(r)$. Show that Φ_3 and $\Phi_1 \pm i\Phi_2$ are eigenfunctions of L_3 with eigenvalues $0, \pm\hbar$ respectively.

3/I/7B Quantum Mechanics

The quantum mechanical harmonic oscillator has Hamiltonian

$$H = \frac{1}{2m}\, p^2 + \frac{1}{2}\, m\omega^2 x^2 \,,$$

and is in a stationary state of energy $< H >= E$. Show that

$$E \geqslant \frac{1}{2m}\, (\Delta p)^2 + \frac{1}{2}\, m\omega^2 (\Delta x)^2 \,,$$

where $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ and $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$. Use the Heisenberg Uncertainty Principle to show that

$$E \geqslant \frac{1}{2}\hbar\omega \,.$$

3/II/16B Quantum Mechanics

A quantum system has a complete set of orthonormal eigenstates, $\psi_n(x)$, with non-degenerate energy eigenvalues, E_n, where $n = 1, 2, 3 \ldots$. Write down the wave-function, $\Psi(x,t), t \geqslant 0$ in terms of the eigenstates.

A linear operator acts on the system such that

$$\begin{aligned} A\psi_1 &= 2\psi_1 - \psi_2 \\ A\psi_2 &= 2\psi_2 - \psi_1 \;. \\ A\psi_n &= 0,\; n \geqslant 3 \end{aligned}$$

Find the eigenvalues of A and obtain a complete set of normalised eigenfunctions, ϕ_n, of A in terms of the ψ_n.

At time $t = 0$ a measurement is made and it is found that the observable corresponding to A has value 3. After time t, A is measured again. What is the probability that the value is found to be 1?

4/I/6B Quantum Mechanics

A particle moving in one space dimension with wave-function $\Psi(x,t)$ obeys the time-dependent Schrödinger equation. Write down the probability density, ρ, and current density, j, in terms of the wave-function and show that they obey the equation

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \,.$$

The wave-function is

$$\Psi(x,t) = \left(e^{ikx} + R\, e^{-ikx}\right) e^{-iEt/\hbar} \,,$$

where $E = \hbar^2 k^2/2m$ and R is a constant, which may be complex. Evaluate j.

1/II/15B **Quantum Mechanics**

Let $V_1(x)$ and $V_2(x)$ be two real potential functions of one space dimension, and let a be a positive constant. Suppose also that $V_1(x) \leqslant V_2(x) \leqslant 0$ for all x and that $V_1(x) = V_2(x) = 0$ for all x such that $|x| \geqslant a$. Consider an incoming beam of particles described by the plane wave $\exp(ikx)$, for some $k > 0$, scattering off one of the potentials $V_1(x)$ or $V_2(x)$. Let p_i be the probability that a particle in the beam is reflected by the potential $V_i(x)$. Is it necessarily the case that $p_1 \leqslant p_2$? Justify your answer carefully, either by giving a rigorous proof or by presenting a counterexample with explicit calculations of p_1 and p_2.

2/II/16B **Quantum Mechanics**

The spherically symmetric bound state wavefunctions $\psi(r)$, where $r = |\mathbf{x}|$, for an electron orbiting in the Coulomb potential $V(r) = -e^2/(4\pi\epsilon_0 r)$ of a hydrogen atom nucleus, can be modelled as solutions to the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + \frac{a}{r}\psi(r) - b^2\psi(r) = 0$$

for $r \geqslant 0$, where $a = e^2 m/(2\pi\epsilon_0\hbar^2)$, $b = \sqrt{-2mE}/\hbar$, and E is the energy of the corresponding state. Show that there are normalisable and continuous wavefunctions $\psi(r)$ satisfying this equation with energies

$$E = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 N^2}$$

for all integers $N \geqslant 1$.

3/I/7B **Quantum Mechanics**

Define the quantum mechanical operators for the angular momentum $\hat{\mathbf{L}}$ and the total angular momentum $\hat{L}^2$ in terms of the operators $\hat{\mathbf{x}}$ and $\boldsymbol{\nabla}$. Calculate the commutators $[\hat{L}_i, \hat{L}_j]$ and $[\hat{L}^2, \hat{L}_i]$.

3/II/16B **Quantum Mechanics**

The expression $\Delta_\psi A$ denotes the uncertainty of a quantum mechanical observable A in a state with normalised wavefunction ψ. Prove that the Heisenberg uncertainty principle

$$(\Delta_\psi x)(\Delta_\psi p) \geqslant \frac{\hbar}{2}$$

holds for all normalised wavefunctions $\psi(x)$ of one spatial dimension.

[*You may quote Schwarz's inequality without proof.*]

A Gaussian wavepacket evolves so that at time t its wavefunction is

$$\psi(x,t) = (2\pi)^{-\frac{1}{4}} \Big(1 + i\hbar t\Big)^{-\frac{1}{2}} \exp\Big(-\frac{x^2}{4(1+i\hbar t)}\Big)\,.$$

Calculate the uncertainties $\Delta_\psi x$ and $\Delta_\psi p$ at each time t, and hence verify explicitly that the uncertainty principle holds at each time t.

$\Big[$*You may quote without proof the results that if* $\mathrm{Re}(a) > 0$ *then*

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^*}\right) x^2 \exp\left(-\frac{x^2}{a}\right) dx = \frac{1}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{|a|^3}{(\mathrm{Re}(a))^{\frac{3}{2}}}$$

and

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx}\exp\left(-\frac{x^2}{a^*}\right)\right)\left(\frac{d}{dx}\exp\left(-\frac{x^2}{a}\right)\right) dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{|a|}{(\mathrm{Re}(a))^{\frac{3}{2}}}\,.\Big]$$

4/I/6B **Quantum Mechanics**

(a) Define the probability density $\rho(\mathbf{x},t)$ and the probability current $\mathbf{J}(\mathbf{x},t)$ for a quantum mechanical wave function $\psi(\mathbf{x},t)$, where the three dimensional vector $\mathbf{x}$ defines spatial coordinates.

Given that the potential $V(\mathbf{x})$ is real, show that

$$\boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0\,.$$

(b) Write down the standard integral expressions for the expectation value $\langle A\rangle_\psi$ and the uncertainty $\Delta_\psi A$ of a quantum mechanical observable A in a state with wavefunction $\psi(\mathbf{x})$. Give an expression for $\Delta_\psi A$ in terms of $\langle A^2\rangle_\psi$ and $\langle A\rangle_\psi$, and justify your answer.

1/II/15G **Quantum Mechanics**

The wave function of a particle of mass m that moves in a one-dimensional potential well satisfies the Schrödinger equation with a potential that is zero in the region $-a \leq x \leq a$ and infinite elsewhere,

$$V(x) = 0 \quad \text{for} \quad |x| \leq a\,, \qquad\qquad V(x) = \infty \quad \text{for} \quad |x| > a\,.$$

Determine the complete set of normalised energy eigenfunctions for the particle and show that the energy eigenvalues are

$$E = \frac{\hbar^2 \pi^2 n^2}{8ma^2}\,,$$

where n is a positive integer.

At time $t = 0$ the wave function is

$$\psi(x) = \frac{1}{\sqrt{5a}} \cos\left(\frac{\pi x}{2a}\right) + \frac{2}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right)\,,$$

in the region $-a \leq x \leq a$, and zero otherwise. Determine the possible results for a measurement of the energy of the system and the relative probabilities of obtaining these energies.

In an experiment the system is measured to be in its lowest possible energy eigenstate. The width of the well is then doubled while the wave function is unaltered. Calculate the probability that a later measurement will find the particle to be in the lowest energy state of the new potential well.

2/II/16G **Quantum Mechanics**

A particle of mass m moving in a one-dimensional harmonic oscillator potential satisfies the Schrödinger equation

$$H\,\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\,\Psi(x,t)\,,$$

where the Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\,\omega^2\,x^2\,.$$

The operators a and $a^\dagger$ are defined by

$$a = \frac{1}{\sqrt{2}}\left(\beta x + \frac{i}{\beta\hbar}p\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\beta x - \frac{i}{\beta\hbar}p\right),$$

where $\beta = \sqrt{m\omega/\hbar}$ and $p = -i\hbar\partial/\partial x$ is the usual momentum operator. Show that $[a, a^\dagger] = 1$.

Express x and p in terms of a and $a^\dagger$ and, hence or otherwise, show that H can be written in the form

$$H = \left(a^\dagger a + \tfrac{1}{2}\right)\,\hbar\omega\,.$$

Show, for an arbitrary wave function Ψ, that $\int dx\,\Psi^*\,H\,\Psi \geq \frac{1}{2}\hbar\omega$ and hence that the energy of any state satisfies the bound

$$E \geq \tfrac{1}{2}\hbar\omega\,.$$

Hence, or otherwise, show that the ground state wave function satisfies $a\Psi_0 = 0$ and that its energy is given by

$$E_0 = \tfrac{1}{2}\hbar\omega\,.$$

By considering H acting on $a^\dagger\,\Psi_0$, $(a^\dagger)^2\,\Psi_0$, and so on, show that states of the form

$$(a^\dagger)^n\,\Psi_0$$

$(n > 0)$ are also eigenstates and that their energies are given by $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

3/I/7G **Quantum Mechanics**

The wave function $\Psi(x,t)$ is a solution of the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$,

$$H\,\Psi(x,t) = i\hbar\,\frac{\partial}{\partial t}\Psi(x,t)\,,$$

where H is the Hamiltonian. Define the expectation value, $\langle\mathcal{O}\rangle$, of any operator $\mathcal{O}$.

At time $t=0$, $\Psi(x,t)$ can be written as a sum of the form

$$\Psi(x,0) = \sum_n a_n\,u_n(x)\,,$$

where u_n is a complete set of normalized eigenfunctions of the Hamiltonian with energy eigenvalues E_n and a_n are complex coefficients that satisfy $\sum_n a_n^* a_n = 1$. Find $\Psi(x,t)$ for $t>0$. What is the probability of finding the system in a state with energy E_p at time t?

Show that the expectation value of the energy is independent of time.

3/II/16G **Quantum Mechanics**

A particle of mass μ moves in two dimensions in an axisymmetric potential. Show that the time-independent Schrödinger equation can be separated in polar coordinates. Show that the angular part of the wave function has the form $e^{im\phi}$, where ϕ is the angular coordinate and m is an integer. Suppose that the potential is zero for $r<a$, where r is the radial coordinate, and infinite otherwise. Show that the radial part of the wave function satisfies

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(1 - \frac{m^2}{\rho^2}\right)R = 0\,,$$

where $\rho = r\left(2\mu E/\hbar^2\right)^{1/2}$. What conditions must R satisfy at $r=0$ and $R=a$?

Show that, when $m=0$, the equation has the solution $R(\rho) = \sum_{k=0}^{\infty} A_k\,\rho^k$, where $A_k = 0$ if k is odd and

$$A_{k+2} = -\frac{A_k}{(k+2)^2}\,,$$

if k is even.

Deduce the coefficients A_2 and A_4 in terms of A_0. By truncating the series expansion at order ρ^4, estimate the smallest value of ρ at which the R is zero. Hence give an estimate of the ground state energy.

[*You may use the fact that the Laplace operator is given in polar coordinates by the expression*

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\,.\quad]$$

4/I/6G **Quantum Mechanics**

Define the commutator $[A\,,B]$ of two operators, A and B. In three dimensions angular momentum is defined by a vector operator $\mathbf{L}$ with components

$$L_x = y\,p_z - z\,p_y \qquad L_y = z\,p_x - x\,p_z \qquad L_z = x\,p_y - y\,p_x\,.$$

Show that $[L_x\,,L_y] = i\,\hbar\,L_z$ and use this, together with permutations, to show that $[\mathbf{L^2}\,,L_w] = 0$, where w denotes any of the directions x, y, z.

At a given time the wave function of a particle is given by

$$\psi = (x+y+z)\,\exp\left(-\sqrt{x^2+y^2+z^2}\right)\,.$$

Show that this is an eigenstate of $\mathbf{L}^2$ with eigenvalue equal to $2\hbar^2$.

1/I/8D **Quantum Mechanics**

From the time-dependent Schrödinger equation for $\psi(x,t)$, derive the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

for $\rho(x,t) = \psi^*(x,t)\psi(x,t)$ and some suitable $j(x,t)$.

Show that $\psi(x,t) = e^{i(kx-\omega t)}$ is a solution of the time-dependent Schrödinger equation with zero potential for suitable $\omega(k)$ and calculate ρ and j. What is the interpretation of this solution?

1/II/19D **Quantum Mechanics**

The angular momentum operators are $\mathbf{L} = (L_1, L_2, L_3)$. Write down their commutation relations and show that $[L_i, \mathbf{L}^2] = 0$. Let

$$L_\pm = L_1 \pm iL_2\,,$$

and show that

$$\mathbf{L}^2 = L_- L_+ + {L_3}^2 + \hbar L_3\,.$$

Verify that $\mathbf{L} f(r) = 0$, where $r^2 = x_i x_i$, for any function f. Show that

$$L_3 (x_1 + ix_2)^n f(r) = n\hbar (x_1 + ix_2)^n f(r)\,, \qquad L_+ (x_1 + ix_2)^n f(r) = 0\,,$$

for any integer n. Show that $(x_1 + ix_2)^n f(r)$ is an eigenfunction of $\mathbf{L}^2$ and determine its eigenvalue. Why must $L_-(x_1 + ix_2)^n f(r)$ be an eigenfunction of $\mathbf{L}^2$? What is its eigenvalue?

2/I/8D **Quantum Mechanics**

A quantum mechanical system is described by vectors $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$. The energy eigenvectors are

$$\psi_0 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\,, \qquad \psi_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\,,$$

with energies E_0, E_1 respectively. The system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at time $t = 0$. What is the probability of finding it in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at a later time t?

2/II/19D **Quantum Mechanics**

Consider a Hamiltonian of the form

$$H = \frac{1}{2m}\big(p + if(x)\big)\big(p - if(x)\big), \qquad -\infty < x < \infty\,,$$

where $f(x)$ is a real function. Show that this can be written in the form $H = p^2/(2m) + V(x)$, for some real $V(x)$ to be determined. Show that there is a wave function $\psi_0(x)$, satisfying a first-order equation, such that $H\psi_0 = 0$. If f is a polynomial of degree n, show that n must be odd in order for ψ_0 to be normalisable. By considering $\int \mathrm{d}x\, \psi^* H \psi$ show that all energy eigenvalues other than that for ψ_0 must be positive.

For $f(x) = kx$, use these results to find the lowest energy and corresponding wave function for the harmonic oscillator Hamiltonian

$$H_{\text{oscillator}} = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2\,.$$

3/I/9D **Quantum Mechanics**

Write down the expressions for the classical energy and angular momentum for an electron in a hydrogen atom. In the Bohr model the angular momentum L is quantised so that

$$L = n\hbar\,,$$

for integer n. Assuming circular orbits, show that the radius of the n'th orbit is

$$r_n = n^2 a\,,$$

and determine a. Show that the corresponding energy is then

$$E_n = -\frac{e^2}{8\pi\epsilon_0\, r_n}\,.$$

3/II/20D **Quantum Mechanics**

A one-dimensional system has the potential

$$V(x) = \begin{cases} 0 & x < 0\,, \\ \frac{\hbar^2 U}{2m} & 0 < x < L\,, \\ 0 & x > L\,. \end{cases}$$

For energy $E = \hbar^2 \epsilon/(2m)$, $\epsilon < U$, the wave function has the form

$$\psi(x) = \begin{cases} a\, e^{ikx} + c\, e^{-ikx} & x < 0\,, \\ e \cosh Kx + f \sinh Kx & 0 < x < L\,, \\ d\, e^{ik(x-L)} + b\, e^{-ik(x-L)} & x > L\,. \end{cases}$$

By considering the relation between incoming and outgoing waves explain why we should expect

$$|c|^2 + |d|^2 = |a|^2 + |b|^2\,.$$

Find four linear relations between a, b, c, d, e, f. Eliminate d, e, f and show that

$$c = \frac{1}{D}\left[b + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big) \sinh KL\ a \right],$$

where $D = \cosh KL - \frac{1}{2}\big(\lambda + \frac{1}{\lambda}\big) \sinh KL$ and $\lambda = K/(ik)$. By using the result for c, or otherwise, explain why the solution for d is

$$d = \frac{1}{D}\left[a + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big) \sinh KL\ b \right].$$

For $b = 0$ define the transmission coefficient T and show that, for large L,

$$T \approx 16\, \frac{\epsilon(U - \epsilon)}{U^2}\, e^{-2\sqrt{U-\epsilon}\, L}\,.$$

1/I/9A **Quantum Mechanics**

A particle of mass m is confined inside a one-dimensional box of length a. Determine the possible energy eigenvalues.

1/II/18A **Quantum Mechanics**

What is the significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x)\ Q\ \psi(x) dx$$

of an observable Q in the normalized state $\psi(x)$? Let Q and P be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \tfrac{1}{4} |\langle [Q, P] \rangle|^2 .$$

The uncertainty ΔQ of Q in the state $\psi(x)$ is defined as

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle .$$

Deduce the generalized uncertainty relation,

$$\Delta Q \Delta P \geqslant \tfrac{1}{2} |\langle [Q, P] \rangle| .$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m\omega^2 x^2$. By considering the commutator $[x, p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H \rangle \geqslant \tfrac{1}{2} \hbar\omega .$$

2/I/9A **Quantum Mechanics**

What is meant by the statement than an operator is *hermitian*?

A particle of mass m moves in the real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is hermitian.

Show that

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle p \rangle ,$$
$$\frac{d}{dt}\langle p \rangle = \langle -V'(x) \rangle ,$$

where p is the momentum operator and $\langle A \rangle$ denotes the expectation value of the operator A.

2/II/18A Quantum Mechanics

A particle of mass m and energy E moving in one dimension is incident from the left on a potential barrier $V(x)$ given by

$$V(x) = \begin{cases} V_0 & 0 \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

with $V_0 > 0$.

In the limit $V_0 \to \infty, a \to 0$ with $V_0 a = U$ held fixed, show that the transmission probability is

$$T = \left(1 + \frac{mU^2}{2E\hbar^2}\right)^{-1}.$$

3/II/20A Quantum Mechanics

The radial wavefunction for the hydrogen atom satisfies the equation

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r) - \frac{e^2}{4\pi\epsilon_0 r}R(r) = ER(r)\,.$$

Explain the origin of each term in this equation.

The wavefunctions for the ground state and first radially excited state, both with $\ell = 0$, can be written as

$$R_1(r) = N_1 \exp(-\alpha r)$$
$$R_2(r) = N_2(r+b)\exp(-\beta r)$$

respectively, where N_1 and N_2 are normalization constants. Determine α, β, b and the corresponding energy eigenvalues E_1 and E_2.

A hydrogen atom is in the first radially excited state. It makes the transition to the ground state, emitting a photon. What is the frequency of the emitted photon?

1/I/9D **Quantum Mechanics**

Consider a quantum mechanical particle of mass m moving in one dimension, in a potential well

$$V(x) = \begin{cases} \infty, & x < 0, \\ 0, & 0 < x < a, \\ V_0, & x > a. \end{cases}$$

Sketch the ground state energy eigenfunction $\chi(x)$ and show that its energy is $E = \frac{\hbar^2 k^2}{2m}$, where k satisfies

$$\tan ka = -\frac{k}{\sqrt{\frac{2mV_0}{\hbar^2} - k^2}}.$$

[*Hint: You may assume that* $\chi(0) = 0$.]

1/II/18D **Quantum Mechanics**

A quantum mechanical particle of mass M moves in one dimension in the presence of a negative delta function potential

$$V = -\frac{\hbar^2}{2M\Delta}\delta(x),$$

where Δ is a parameter with dimensions of length.

(a) Write down the time-independent Schrödinger equation for energy eigenstates $\chi(x)$, with energy E. By integrating this equation across $x = 0$, show that the gradient of the wavefunction jumps across $x = 0$ according to

$$\lim_{\epsilon \to 0} \left(\frac{d\chi}{dx}(\epsilon) - \frac{d\chi}{dx}(-\epsilon) \right) = -\frac{1}{\Delta}\chi(0).$$

[*You may assume that* χ *is continuous across* $x = 0$.]

(b) Show that there exists a negative energy solution and calculate its energy.

(c) Consider a double delta function potential

$$V(x) = -\frac{\hbar^2}{2M\Delta}[\delta(x+a) + \delta(x-a)].$$

For sufficiently small Δ, this potential yields a negative energy solution of odd parity, i.e. $\chi(-x) = -\chi(x)$. Show that its energy is given by

$$E = -\frac{\hbar^2}{2M}\lambda^2, \quad \text{where} \quad \tanh \lambda a = \frac{\lambda\Delta}{1 - \lambda\Delta}.$$

[*You may again assume* χ *is continuous across* $x = \pm a$.]

2/I/9D **Quantum Mechanics**

From the expressions

$$L_x = yP_z - zP_y, \quad L_y = zP_x - xP_z, \quad L_z = xP_y - yP_x,$$

show that

$$(x + iy)z$$

is an eigenfunction of $\mathbf{L}^2$ and L_z, and compute the corresponding eigenvalues.

2/II/18D **Quantum Mechanics**

Consider a quantum mechanical particle moving in an upside-down harmonic oscillator potential. Its wavefunction $\Psi(x,t)$ evolves according to the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{2}x^2\Psi. \qquad (1)$$

(a) Verify that

$$\Psi(x,t) = A(t)e^{-B(t)x^2} \qquad (2)$$

is a solution of equation (1), provided that

$$\frac{dA}{dt} = -i\hbar\, AB,$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2. \qquad (3)$$

(b) Verify that $B = \frac{1}{2\hbar}\tan(\phi - it)$ provides a solution to (3), where ϕ is an arbitrary real constant.

(c) The expectation value of an operator $\mathcal{O}$ at time t is

$$\langle\mathcal{O}\rangle(t) \equiv \int_{-\infty}^{\infty} dx\Psi^*(x,t)\mathcal{O}\Psi(x,t),$$

where $\Psi(x,t)$ is the normalised wave function. Show that for $\Psi(x,t)$ given by (2),

$$\langle x^2\rangle = \frac{1}{4Re(B)}, \quad \langle p^2\rangle = 4\hbar^2|B|^2\langle x^2\rangle.$$

Hence show that as $t\to\infty$,

$$\langle x^2\rangle \approx \langle p^2\rangle \approx \frac{\hbar}{4\sin 2\phi}e^{2t}.$$

[*Hint: You may use*

$$\frac{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}} = \frac{1}{2C}.]$$

3/II/20D **Quantum Mechanics**

A quantum mechanical system has two states χ_0 and χ_1, which are normalised energy eigenstates of a Hamiltonian H_3, with

$$H_3\chi_0 = -\chi_0, \quad H_3\chi_1 = +\chi_1.$$

A general time-dependent state may be written

$$\Psi(t) = a_0(t)\chi_0 + a_1(t)\chi_1, \tag{1}$$

where $a_0(t)$ and $a_1(t)$ are complex numbers obeying $|a_0(t)|^2 + |a_1(t)|^2 = 1$.

(a) Write down the time-dependent Schrödinger equation for $\Psi(t)$, and show that if the Hamiltonian is H_3, then

$$i\hbar\frac{da_0}{dt} = -a_0, \qquad i\hbar\frac{da_1}{dt} = +a_1.$$

For the general state given in equation (1) above, write down the probability to observe the system, at time t, in a state $\alpha\chi_0 + \beta\chi_1$, properly normalised so that $|\alpha|^2 + |\beta|^2 = 1$.

(b) Now consider starting the system in the state χ_0 at time $t = 0$, and evolving it with a different Hamiltonian H_1, which acts on the states χ_0 and χ_1 as follows:

$$H_1\chi_0 = \chi_1, \qquad H_1\chi_1 = \chi_0.$$

By solving the time-dependent Schrödinger equation for the Hamiltonian H_1, find $a_0(t)$ and $a_1(t)$ in this case. Hence determine the shortest time $T > 0$ such that $\Psi(T)$ is an eigenstate of H_3 with eigenvalue $+1$.

(c) Now consider taking the state $\Psi(T)$ from part (b), and evolving it for further length of time T, with Hamiltonian H_2, which acts on the states χ_0 and χ_1 as follows:

$$H_2\chi_0 = -i\chi_1, \qquad H_2\chi_1 = i\chi_0.$$

What is the final state of the system? Is this state observationally distinguishable from the original state χ_0?

1/I/9F **Quantum Mechanics**

A quantum mechanical particle of mass m and energy E encounters a potential step,

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geqslant 0. \end{cases}$$

Calculate the probability P that the particle is reflected in the case $E > V_0$.

If V_0 is positive, what is the limiting value of P when E tends to V_0? If V_0 is negative, what is the limiting value of P as V_0 tends to $-\infty$ for fixed E?

1/II/18F **Quantum Mechanics**

Consider a quantum-mechanical particle of mass m moving in a potential well,

$$V(x) = \begin{cases} 0, & -a < x < a, \\ \infty, & \text{elsewhere.} \end{cases}$$

(a) Verify that the set of normalised energy eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{1}{a}} \ \sin\left(\frac{n\pi(x+a)}{2a}\right), \qquad n = 1, 2, \ldots,$$

and evaluate the corresponding energy eigenvalues E_n.

(b) At time $t = 0$ the wavefunction for the particle is only nonzero in the positive half of the well,

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \ \sin\left(\frac{\pi x}{a}\right), & 0 < x < a, \\ 0, & \text{elsewhere.} \end{cases}$$

Evaluate the expectation value of the energy, first using

$$\langle E \rangle = \int_{-a}^{a} \psi H \psi dx,$$

and secondly using

$$\langle E \rangle = \sum_n |a_n|^2 E_n,$$

where the a_n are the expansion coefficients in

$$\psi(x) = \sum_n a_n \psi_n(x).$$

Hence, show that

$$1 = \frac{1}{2} + \frac{8}{\pi^2} \sum_{p=0}^{\infty} \frac{(2p+1)^2}{[(2p+1)^2 - 4]^2} \ .$$

2/I/9F Quantum Mechanics

Consider a solution $\psi(x,t)$ of the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$. The expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\ \mathcal{O}\ \rangle = \int dx\ \psi^*(x,t)\ \mathcal{O}\ \psi(x,t).$$

Show that

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m},$$

where

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x},$$

and that

$$\frac{d}{dt}\langle p\rangle = \left\langle -\frac{\partial V}{\partial x}(x)\right\rangle .$$

[*You may assume that $\psi(x,t)$ vanishes as $x \to \pm\infty$.*]

2/II/18F Quantum Mechanics

(a) Write down the angular momentum operators L_1, L_2, L_3 in terms of x_i and

$$p_i = -i\hbar\frac{\partial}{\partial x_i},\ i = 1,2,3.$$

Verify the commutation relation

$$[L_1, L_2] = i\hbar L_3.$$

Show that this result and its cyclic permutations imply

$$[L_3, L_1 \pm iL_2] = \pm\hbar\ (L_1 \pm iL_2),$$
$$[\mathbf{L}^2, L_1 \pm iL_2] = 0.$$

(b) Consider a wavefunction of the form $\psi = (x_3^2 + ar^2)f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that for a particular value of a, ψ is an eigenfunction of both $\mathbf{L}^2$ and L_3. What are the corresponding eigenvalues?

3/II/20F **Quantum Mechanics**

A quantum system has a complete set of orthonormalised energy eigenfunctions $\psi_n(x)$ with corresponding energy eigenvalues $E_n,\ n = 1, 2, 3, \ldots$.

(a) If the time-dependent wavefunction $\psi(x,t)$ is, at $t = 0$,

$$\psi(x,0) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

determine $\psi(x,t)$ for all $t > 0$.

(b) A linear operator $\mathcal{S}$ acts on the energy eigenfunctions as follows:

$$\begin{aligned} \mathcal{S}\psi_1 &= 7\psi_1 + 24\psi_2, \\ \mathcal{S}\psi_2 &= 24\psi_1 - 7\psi_2, \\ \mathcal{S}\psi_n &= 0, \quad n \geqslant 3. \end{aligned}$$

Find the eigenvalues of $\mathcal{S}$. At time $t = 0,\ \mathcal{S}$ is measured and its lowest eigenvalue is found. At time $t > 0,\ \mathcal{S}$ is measured again. Show that the probability for obtaining the lowest eigenvalue again is

$$\frac{1}{625}\left(337 + 288\cos(\omega t)\right),$$

where $\omega = (E_1 - E_2)/\hbar$.