Part IB

Numerical Analysis

Paper 1, Section I

5B Numerical Analysis

Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\boldsymbol{y} \in \mathbb{R}^m$ where $m \geqslant n$, consider the problem of finding $\boldsymbol{c}^* \in \mathbb{R}^n$ that minimises $\|A\boldsymbol{c} - \boldsymbol{y}\|_2$ for $\boldsymbol{c} \in \mathbb{R}^n$, where $\|\cdot\|_2$ is the standard Euclidean norm.

(a) Prove that $\boldsymbol{c}^*$ is a solution to the above minimisation problem if and only if $A^T A\boldsymbol{c}^* = A^T\boldsymbol{y}$.

(b) Show that if A is of full rank, then $\boldsymbol{c}^*$ is unique.

Paper 4, Section I

6B Numerical Analysis

Consider the inner product

$$\langle g, h\rangle = \int_a^b g(x)h(x)w(x)\,dx \qquad (*)$$

on $C[a,b]$, where $w(x) > 0$ for $x \in (a,b)$. Define $\|g\|^2 = \langle g, g\rangle$. Let $Q_0, Q_1, Q_2, \ldots$ be orthogonal polynomials with respect to the inner product $(*)$, and let $f \in C[a,b]$.

(a) Prove that the polynomial $p_n^* \in \mathcal{P}_n$ that minimises the squared distance $\|f - p\|^2$ among all $p \in \mathcal{P}_n$ is given by

$$p_n^*(x) = \sum_{k=0}^{n} \frac{\langle f, Q_k\rangle}{\langle Q_k, Q_k\rangle} Q_k(x).$$

(b) Hence, show that

$$\|f\|^2 = \|f - p_n^*\|^2 + \|p_n^*\|^2.$$

Paper 1, Section II

17B Numerical Analysis

Consider the ODE

$$y' = f(y), \quad y(0) = y_0 > 0, \tag{$*$}$$

where $f(y) = -\operatorname{sign}(y)$, $y(t) \in \mathbb{R}$ and $t \in [0, T]$, with $T > y_0$. The sign function is defined as

$$\operatorname{sign}(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y = 0 \\ -1 & \text{for } y < 0. \end{cases}$$

(a) Does the function f satisfy a Lipschitz condition for $y \in \mathbb{R}$? Justify your answer.

(b) Show that there is a unique continuous function $y : [0, T] \to \mathbb{R}$ that is differentiable for all $t \in [0, T]$ except for some $\tilde{t} \in (0, T]$ and satisfies the ODE $(*)$ for all $t \in [0, T] \setminus \tilde{t}$.

(c) The Euler method for $(*)$ produces a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size. Is

$$|y_n - y(nh)| \leqslant \mathcal{O}(h), \qquad \text{for } 0 \leqslant n \leqslant N,$$

where $y(t)$ is the solution described in part (b)? Justify your answer.

Paper 2, Section II

17B Numerical Analysis

Consider an ODE of the form

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}, \tag{$*$}$$

where $y(t)$ exists and is unique for $t \in [0, T]$ and $T > 0$.

(a) For a numerical method approximating the solution of $(*)$, define the linear stability domain. What does it mean for such a numerical method to be A-stable?

(b) Let $a \in \mathbb{R}$ and consider the Runge–Kutta method—producing a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size—defined by

$$\begin{aligned} k_1 &= f\left(y_n + \frac{1}{4}hk_1 + \left(\frac{1}{4} - a\right)hk_2\right), \\ k_2 &= f\left(y_n + \left(\frac{1}{4} + a\right)hk_1 + \frac{1}{4}hk_2\right), \\ y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2), \quad n = 0, 1, \ldots, N-1. \end{aligned}$$

Determine the values of the parameter $a \in \mathbb{R}$ for which the Runge–Kutta method is A-stable.

Paper 3, Section II

17B Numerical Analysis

Consider $C[a,b]$ equipped with the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\mathrm{d}x$, where $w(x) > 0$ for $x \in (a,b)$. Let $\mathcal{P}_n$ denote the set of polynomials of degree less than or equal to n. For $f \in C[a,b]$ consider the quadrature formulas

$$I(f) = \int_a^b f(x)w(x)\mathrm{d}x \approx \sum_{i=0}^{n} a_i^{(n)} f(x_i^{(n)}) = I_n(f), \quad n = 0,1,2,\ldots \tag{$*$}$$

with weights $a_i^{(n)} \in \mathbb{R}$ and nodes $x_i^{(n)} \in [a,b]$, which are exact on all polynomials $q \in \mathcal{P}_n$.

(a) Prove that the quadrature formula $(*)$ is exact for all $q \in \mathcal{P}_{n+1+k}$ if and only if the polynomial $Q_{n+1}(x) = \prod_{i=0}^{n}(x - x_i^{(n)})$ is orthogonal (with respect to $\langle\cdot,\cdot\rangle$) to all polynomials of degree k.

(b) Prove that no quadrature formula $(*)$ could be exact on polynomials of degree $2n+2$.

(c) Prove that if $(*)$ is exact on $\mathcal{P}_{2n}$, then $a_i^{(n)} > 0$.

(d) Show that if $a_i^{(n)} > 0$ for all i and n, then

$$I_n(f) \to I(f), \qquad n \to \infty.$$

[*Hint: Use the Weierstrass theorem: for any* $\epsilon > 0$ *there exists* $n \in \mathbb{N}$ *and a polynomial* $p_n \in \mathcal{P}_n$ *such that* $|f(x) - p_n(x)| < \epsilon$, *for* $x \in [a,b]$.]

Paper 1, Section I
5C Numerical Analysis

Use the Gram–Schmidt algorithm to compute a reduced QR factorization of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & -4 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix},$$

i.e. find a matrix $Q \in \mathbb{R}^{4\times 3}$ with orthonormal columns and an upper triangular matrix $R \in \mathbb{R}^{3\times 3}$ such that $A = QR$.

Paper 4, Section I
6C Numerical Analysis

(a) Suppose that $w(x) > 0$ for all $x \in [a, b]$. The weights $b_1, \ldots, b_n$ and nodes $c_1, \ldots, c_n$ are chosen so that the Gaussian quadrature formula for a function $f \in C[a, b]$

$$\int_a^b w(x) f(x) dx \approx \sum_{k=1}^{n} b_k f(c_k)$$

is exact for every polynomial of degree $2n - 1$. Show that the b_i, $i = 1, \ldots, n$ are all positive.

(b) Evaluate the coefficients b_k and c_k of the Gaussian quadrature of the integral

$$\int_{-1}^{1} x^2 f(x) dx,$$

which uses two evaluations of the function $f(x)$ and is exact for all f that are polynomials of degree 3.

Paper 1, Section II

17C Numerical Analysis

For a function $f \in C^3[-1,1]$ consider the following approximation of $f''(0)$:

$$f''(0) \approx \eta(f) = a_{-1} f(-1) + a_0 f(0) + a_1 f(1)\,,$$

with the error

$$e(f) = f''(0) - \eta(f).$$

We want to find the smallest constant c such that

$$|e(f)| \leqslant c \max_{x \in [-1,1]} \left| f'''(x) \right|. \qquad (\star)$$

(a) State the necessary conditions on the approximation scheme η for the inequality $(\star)$ to be valid with some $c < \infty$. Hence, determine the coefficients a_{-1}, a_0, a_1.

(b) State the Peano kernel theorem and use it to find the smallest constant c in the inequality $(\star)$.

(c) Explain briefly why this constant is sharp.

Paper 2, Section II

17C Numerical Analysis

A scalar, autonomous, ordinary differential equation $y' = f(y)$ is solved using the Runge–Kutta method

$$\begin{aligned} k_1 &= f(y_n)\,, \\ k_2 &= f(y_n + (1-a)hk_1 + ahk_2)\,, \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2)\,, \end{aligned}$$

where h is a step size and a is a real parameter.

(a) Determine the order of the method and its dependence on a.

(b) Find the range of values of a for which the method is A-stable.

Paper 3, Section II
17C Numerical Analysis

(a) The equation $y' = f(t, y)$ is solved using the following multistep method with s steps,

$$\sum_{k=0}^{s} \rho_k y_{n+k} = h \sum_{k=0}^{s} \sigma_k f(t_{n+k}, y_{n+k}),$$

where h is the step size and ρ_k, σ_k are specified constants with $\rho_s = 1$. Prove that this method is of order p if and only if

$$\sum_{k=0}^{s} \rho_k P(t_{n+k}) = h \sum_{k=0}^{s} \sigma_k P'(t_{n+k}),$$

for all polynomials P of degree p.

(b) State the Dahlquist equivalence theorem regarding the convergence of a multistep method. Consider a multistep method

$$y_{n+3} + (2a-3)(y_{n+2} - y_{n+1}) - y_n = ha(f_{n+2} + f_{n+1}),$$

where $a \neq 0$ is a real parameter. Determine the values of a for which this method is convergent, and find its order.

Paper 1, Section I

5B Numerical Analysis

Prove, from first principles, that there is an algorithm that can determine whether any real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite or not, with the computational cost (number of arithmetic operations) bounded by $\mathcal{O}(n^3)$.

[*Hint: Consider the LDL decomposition.*]

Paper 4, Section I

6B Numerical Analysis

(a) Given the data $f(0) = 0$, $f(1) = 4$, $f(2) = 2$, $f(3) = 8$, find the interpolating cubic polynomial $p_3 \in \mathbb{P}_3[x]$ in the Newton form.

(b) We add to the data one more value, $f(-2) = 10$. Find the interpolating quartic polynomial $p_4 \in \mathbb{P}_4[x]$ for the extended data in the Newton form.

Paper 1, Section II

17B Numerical Analysis

For the ordinary differential equation

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \quad \boldsymbol{y}(0) = \tilde{\boldsymbol{y}}_0, \quad t \geqslant 0, \tag{$*$}$$

where $\boldsymbol{y}(t) \in \mathbb{R}^N$ and the function $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is analytic, consider an explicit one-step method described as the mapping

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{\varphi}(t_n, \boldsymbol{y}_n, h). \tag{$\dagger$}$$

Here $\boldsymbol{\varphi} : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$, $n = 0, 1, \ldots$ and $t_n = nh$ with time step $h > 0$, producing numerical approximations $\boldsymbol{y}_n$ to the exact solution $\boldsymbol{y}(t_n)$ of equation $(*)$, with $\boldsymbol{y}_0$ being the initial value of the numerical solution.

(i) Define *the local error* of a one-step method.

(ii) Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ and suppose that

$$\|\boldsymbol{\varphi}(t, \boldsymbol{u}, h) - \boldsymbol{\varphi}(t, \boldsymbol{v}, h)\| \leqslant L\|\boldsymbol{u} - \boldsymbol{v}\|,$$

for all $h > 0$, $t \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, where L is some positive constant. Let $t^* > 0$ be given and $\boldsymbol{e}_0 = \boldsymbol{y}_0 - \boldsymbol{y}(0)$ denote the initial error (potentially non-zero). Show that if the local error of the one-step method $(\dagger)$ is $\mathcal{O}(h^{p+1})$, then

$$\max_{n=0,\ldots,\lfloor t^*/h \rfloor} \|\boldsymbol{y}_n - \boldsymbol{y}(nh)\| \leqslant e^{t^* L}\|\boldsymbol{e}_0\| + \mathcal{O}(h^p), \quad h \to 0. \tag{$\dagger\dagger$}$$

(iii) Let $N = 1$ and consider equation $(*)$ where f is time-independent satisfying $|f(u) - f(v)| \leqslant K|u - v|$ for all $u, v \in \mathbb{R}$, where K is a positive constant. Consider the one-step method given by

$$y_{n+1} = y_n + \frac{1}{4}h(k_1 + 3k_2), \qquad k_1 = f(y_n), \quad k_2 = f(y_n + \frac{2}{3}hk_1).$$

Use part (ii) to show that for this method we have that equation $(\dagger\dagger)$ holds (with a potentially different constant L) for $p = 2$.

Paper 2, Section II
17B Numerical Analysis

(a) Define *Householder reflections* and show that a real Householder reflection is symmetric and orthogonal. Moreover, show that if $H, A \in \mathbb{R}^{n\times n}$, where H is a Householder reflection and A is a full matrix, then the computational cost (number of arithmetic operations) of computing HAH^{-1} can be $\mathcal{O}(n^2)$ operations, as opposed to $\mathcal{O}(n^3)$ for standard matrix products.

(b) Show that for any $A \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$QAQ^T = T = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,n} \\ 0 & t_{3,2} & t_{3,3} & \cdots & t_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{n,n-1} & t_{n,n} \end{bmatrix}.$$

In particular, T has zero entries below the first subdiagonal. Show that one can compute such a T and Q (they may not be unique) using $\mathcal{O}(n^3)$ arithmetic operations.

[*Hint: Multiply A from the left and right with Householder reflections.*]

Paper 3, Section II

17B Numerical Analysis

The functions $p_0, p_1, p_2, \ldots$ are generated by the formula

$$p_n(x) = (-1)^n x^{-1/2} e^x \frac{d^n}{dx^n}\left(x^{n+1/2}e^{-x}\right), \qquad 0 \leqslant x < \infty\,.$$

(a) Show that $p_n(x)$ is a monic polynomial of degree n. Write down the explicit forms of $p_0(x)$, $p_1(x)$, $p_2(x)$.

(b) Demonstrate the orthogonality of these polynomials with respect to the scalar product

$$\langle f, g\rangle = \int_0^\infty x^{1/2}e^{-x} f(x)g(x)\,dx\,,$$

i.e. that $\langle p_n, p_m\rangle = 0$ for $m \neq n$, and show that

$$\langle p_n, p_n\rangle = n!\,\Gamma\left(n+\frac{3}{2}\right)\,,$$

where $\Gamma(y) = \int_0^\infty x^{y-1}e^{-x}\,dx$.

(c) Assuming that a three-term recurrence relation in the form

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad n = 1, 2, \ldots\,,$$

holds, find the explicit expressions for α_n and β_n as functions of n.

[*Hint: you may use the fact that* $\Gamma(y+1) = y\Gamma(y)$.]

Paper 1, Section I

5C Numerical Analysis

(a) Find an LU factorisation of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 2 & 12 \\ 0 & 5 & 7 & 32 \\ 3 & -1 & -1 & -10 \end{bmatrix},$$

where the diagonal elements of L are $L_{11} = L_{44} = 1$, $L_{22} = L_{33} = 2$.

(b) Use this factorisation to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} -3 \\ -12 \\ -30 \\ 13 \end{bmatrix}.$$

Paper 1, Section II

18C Numerical Analysis

(a) Given a set of $n+1$ distinct real points $x_0, x_1, \ldots, x_n$ and real numbers $f_0, f_1, \ldots, f_n$, show that the interpolating polynomial $p_n \in \mathbb{P}_n[x]$, $p_n(x_i) = f_i$, can be written in the form

$$p_n(x) = \sum_{k=0}^{n} a_k \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}\,, \qquad x \in \mathbb{R}\,,$$

where the coefficients a_k are to be determined.

(b) Consider the approximation of the integral of a function $f \in C[a,b]$ by a finite sum,

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{s-1} w_k f(c_k)\,, \tag{1}$$

where the weights $w_0, \ldots, w_{s-1}$ and nodes $c_0, \ldots, c_{s-1} \in [a,b]$ are independent of f. Derive the expressions for the weights w_k that make the approximation (1) exact for f being any polynomial of degree $s-1$, i.e. $f \in \mathbb{P}_{s-1}[x]$.

Show that by choosing $c_0, \ldots, c_{s-1}$ to be zeros of the polynomial $q_s(x)$ of degree s, one of a sequence of orthogonal polynomials defined with respect to the scalar product

$$\langle u, v\rangle = \int_a^b u(x)v(x)dx\,, \tag{2}$$

the approximation (1) becomes exact for $f \in \mathbb{P}_{2s-1}[x]$ (i.e. for all polynomials of degree $2s-1$).

(c) On the interval $[a,b] = [-1,1]$ the scalar product (2) generates orthogonal polynomials given by

$$q_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n\,, \quad n = 0, 1, 2, \ldots\,.$$

Find the values of the nodes c_k for which the approximation (1) is exact for all polynomials of degree 7 (i.e. $f \in \mathbb{P}_7[x]$) but no higher.

Paper 2, Section II

17C Numerical Analysis

Consider a multistep method for numerical solution of the differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$:

$$\mathbf{y}_{n+2} - \mathbf{y}_{n+1} = h\left[\,(1+\alpha)\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}) + \beta\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - (\alpha+\beta)\mathbf{f}(t_n, \mathbf{y}_n)\,\right], \qquad (*)$$

where $n = 0, 1, \ldots$, and α and β are constants.

(a) Define the *order* of a method for numerically solving an ODE.

(b) Show that in general an explicit method of the form $(*)$ has order 1. Determine the values of α and β for which this multistep method is of order 3.

(c) Show that the multistep method $(*)$ is convergent.

Paper 1, Section I

6C Numerical Analysis

Let $[a, b]$ be the smallest interval that contains the $n+1$ distinct real numbers $x_0, x_1, \ldots, x_n$, and let f be a continuous function on that interval.

Define the *divided difference* $f[x_0, x_1, \ldots, x_m]$ of degree $m \leqslant n$.

Prove that the polynomial of degree n that interpolates the function f at the points $x_0, x_1, \ldots, x_n$ is equal to the Newton polynomial

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)\,.$$

Prove the recursive formula

$$f[x_0, x_1, \ldots, x_m] = \frac{f[x_1, x_2, \ldots, x_m] - f[x_0, x_1, \ldots, x_{m-1}]}{x_m - x_0}$$

for $1 \leqslant m \leqslant n$.

Paper 4, Section I

8C Numerical Analysis

Calculate the LU factorization of the matrix

$$A = \begin{pmatrix} 3 & 2 & -3 & -3 \\ 6 & 3 & -7 & -8 \\ 3 & 1 & -6 & -4 \\ -6 & -3 & 9 & 6 \end{pmatrix}.$$

Use this to evaluate $\det(A)$ and to solve the equation

$$A\mathbf{x} = \mathbf{b}$$

with

$$\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -3 \end{pmatrix}.$$

Paper 1, Section II

18C Numerical Analysis

(a) An s-step method for solving the ordinary differential equation

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

is given by

$$\sum_{l=0}^{s} \rho_l \, \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \, \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \qquad n = 0, 1, \ldots,$$

where ρ_l and σ_l $(l = 0, 1, \ldots, s)$ are constant coefficients, with $\rho_s = 1$, and h is the time-step. Prove that the method is of order $p \geqslant 1$ if and only if

$$\rho(e^z) - z\sigma(e^z) = O(z^{p+1})$$

as $z \to 0$, where

$$\rho(w) = \sum_{l=0}^{s} \rho_l w^l, \qquad \sigma(w) = \sum_{l=0}^{s} \sigma_l w^l.$$

(b) Show that the Adams–Moulton method

$$\mathbf{y}_{n+2} = \mathbf{y}_{n+1} + \frac{h}{12}\Big(5\,\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}) + 8\,\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \mathbf{f}(t_n, \mathbf{y}_n)\Big)$$

is of third order and convergent.

[*You may assume the Dahlquist equivalence theorem if you state it clearly.*]

Paper 3, Section II

19C Numerical Analysis

(a) Let $w(x)$ be a positive weight function on the interval $[a,b]$. Show that

$$\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx$$

defines an inner product on $C[a,b]$.

(b) Consider the sequence of polynomials $p_n(x)$ defined by the three-term recurrence relation

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x)\,, \qquad n = 1,2,\ldots, \qquad (*)$$

where

$$p_0(x) = 1\,, \qquad p_1(x) = x - \alpha_0\,,$$

and the coefficients α_n (for $n \geqslant 0$) and β_n (for $n \geqslant 1$) are given by

$$\alpha_n = \frac{\langle p_n, xp_n\rangle}{\langle p_n, p_n\rangle}\,, \qquad \beta_n = \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle}\,.$$

Prove that this defines a sequence of monic orthogonal polynomials on $[a,b]$.

(c) The Hermite polynomials $He_n(x)$ are orthogonal on the interval $(-\infty,\infty)$ with weight function $e^{-x^2/2}$. Given that

$$He_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)\,,$$

deduce that the Hermite polynomials satisfy a relation of the form $(*)$ with $\alpha_n = 0$ and $\beta_n = n$. Show that $\langle He_n, He_n\rangle = n!\sqrt{2\pi}$.

(d) State, without proof, how the properties of the Hermite polynomial $He_N(x)$, for some positive integer N, can be used to estimate the integral

$$\int_{-\infty}^{\infty} f(x)\,e^{-x^2/2}\,dx\,,$$

where $f(x)$ is a given function, by the method of Gaussian quadrature. For which polynomials is the quadrature formula exact?

Paper 2, Section II

19C Numerical Analysis

Define the *linear least squares problem* for the equation

$$A\mathbf{x} = \mathbf{b}\,,$$

where A is a given $m \times n$ matrix with $m > n$, $\mathbf{b} \in \mathbb{R}^m$ is a given vector and $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector.

Explain how the linear least squares problem can be solved by obtaining a QR factorization of the matrix A, where Q is an orthogonal $m \times m$ matrix and R is an upper-triangular $m \times n$ matrix in standard form.

Use the Gram–Schmidt method to obtain a QR factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and use it to solve the linear least squares problem in the case

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}\ .$$

Paper 1, Section I

6D Numerical Analysis

The Trapezoidal Rule for solving the differential equation

$$y'(t) = f(t,y), \qquad t \in [0,T], \qquad y(0) = y_0$$

is defined by

$$y_{n+1} = y_n + \tfrac{1}{2}h\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right],$$

where $h = t_{n+1} - t_n$.

Determine the minimum order of convergence k of this rule for general functions f that are sufficiently differentiable. Show with an explicit example that there is a function f for which the local truncation error is Ah^{k+1} for some constant A.

Paper 4, Section I

8D Numerical Analysis

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 5 & 6 \\ 1 & 5 & 13 & 14 \\ 2 & 6 & 14 & \lambda \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 7 \\ \mu \end{bmatrix},$$

where λ and μ are real parameters. Find the LU factorisation of the matrix A. For what values of λ does the equation $Ax = b$ have a unique solution for x?

For $\lambda = 20$, use the LU decomposition with forward and backward substitution to determine a value for μ for which a solution to $Ax = b$ exists. Find the most general solution to the equation in this case.

Paper 1, Section II

18D Numerical Analysis

Show that if $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ then the $m \times m$ matrix transformation

$$H_{\mathbf{u}} = I - 2\frac{\mathbf{u}\mathbf{u}^\top}{\|\mathbf{u}\|^2}$$

is orthogonal. Show further that, for any two vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^m$ of equal length,

$$H_{\mathbf{a}-\mathbf{b}}\mathbf{a} = \mathbf{b}.$$

Explain how to use such transformations to convert an $m \times n$ matrix A with $m \geqslant n$ into the form $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix, and illustrate the method using the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Paper 3, Section II

19D Numerical Analysis

Taylor's theorem for functions $f \in C^{k+1}[a,b]$ is given in the form

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^k}{k!} f^{(k)}(a) + R(x).$$

Use integration by parts to show that

$$R(x) = \frac{1}{k!} \int_a^x (x-\theta)^k f^{(k+1)}(\theta)\, d\theta.$$

Let λ_k be a linear functional on $C^{k+1}[a,b]$ such that $\lambda_k[p] = 0$ for $p \in \mathbb{P}_k$. Show that

$$\lambda_k[f] = \frac{1}{k!} \int_a^b K(\theta) f^{(k+1)}(\theta)\, d\theta, \qquad (\dagger)$$

where the Peano kernel function $K(\theta) = \lambda_k\left[(x-\theta)_+^k\right]$. [*You may assume that the functional commutes with integration over a fixed interval.*]

The error in the mid-point rule for numerical quadrature on $[0,1]$ is given by

$$e[f] = \int_0^1 f(x)dx - f(\tfrac{1}{2}).$$

Show that $e[p] = 0$ if p is a linear polynomial. Find the Peano kernel function corresponding to e explicitly and verify the formula $(\dagger)$ in the case $f(x) = x^2$.

Paper 2, Section II

19D Numerical Analysis

Show that the recurrence relation

$$\begin{aligned} p_0(x) &= 1, \\ p_{n+1}(x) &= q_{n+1}(x) - \sum_{k=0}^{n} \frac{\langle q_{n+1}, p_k \rangle}{\langle p_k, p_k \rangle} p_k(x), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is an inner product on real polynomials, produces a sequence of orthogonal, monic, real polynomials $p_n(x)$ of degree exactly n of the real variable x, provided that q_n is a monic, real polynomial of degree exactly n.

Show that the choice $q_{n+1}(x) = x p_n(x)$ leads to a three-term recurrence relation of the form

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x - \alpha_0, \\ p_{n+1}(x) &= (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \end{aligned}$$

where α_n and β_n are constants that should be determined in terms of the inner products $\langle p_n, p_n \rangle$, $\langle p_{n-1}, p_{n-1} \rangle$ and $\langle p_n, x p_n \rangle$.

Use this recurrence relation to find the first four monic Legendre polynomials, which correspond to the inner product defined by

$$\langle p, q \rangle \equiv \int_{-1}^{1} p(x) q(x) dx.$$

Paper 1, Section I

6C Numerical Analysis

Given $n+1$ real points $x_0 < x_1 < \cdots < x_n$, define the *Lagrange cardinal polynomials* $\ell_i(x)$, $i = 0, 1, \ldots, n$. Let $p(x)$ be the polynomial of degree n that interpolates the function $f \in C^n[x_0, x_n]$ at these points. Express $p(x)$ in terms of the values $f_i = f(x_i)$, $i = 0, 1, \ldots, n$ and the Lagrange cardinal polynomials.

Define the *divided difference* $f[x_0, x_1, \ldots, x_n]$ and give an expression for it in terms of $f_0, f_1, \ldots, f_n$ and $x_0, x_1, \ldots, x_n$. Prove that

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

for some number $\xi \in [x_0, x_n]$.

Paper 4, Section I

8C Numerical Analysis

For the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 5 & 5 \\ 1 & 5 & 14 & 14 \\ 1 & 5 & 14 & \lambda \end{bmatrix}$$

find a factorization of the form

$$A = LDL^\top ,$$

where D is diagonal and L is lower triangular with ones on its diagonal.

For what values of λ is A positive definite?

In the case $\lambda = 30$ find the Cholesky factorization of A.

Paper 1, Section II

18C Numerical Analysis

A three-stage explicit Runge–Kutta method for solving the autonomous ordinary differential equation

$$\frac{dy}{dt} = f(y)$$

is given by

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3)\,,$$

where

$$\begin{aligned} k_1 &= f(y_n)\,, \\ k_2 &= f(y_n + ha_1k_1)\,, \\ k_3 &= f(y_n + h(a_2k_1 + a_3k_2)) \end{aligned}$$

and $h > 0$ is the time-step. Derive sufficient conditions on the coefficients b_1, b_2, b_3, a_1, a_2 and a_3 for the method to be of third order.

Assuming that these conditions hold, verify that $-\frac{5}{2}$ belongs to the linear stability domain of the method.

Paper 2, Section II

19C Numerical Analysis

Define the *linear least-squares problem* for the equation $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix with $m > n$, $\mathbf{b} \in \mathbb{R}^m$ is a given vector and $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector.

If $A = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix in standard form, explain why the least-squares problem is solved by minimizing the Euclidean norm $\|R\mathbf{x} - Q^\top\mathbf{b}\|$.

Using the method of Householder reflections, find a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Hence find the solution of the least-squares problem in the case

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Paper 3, Section II

19C Numerical Analysis

Let $p_n \in \mathbb{P}_n$ be the nth monic orthogonal polynomial with respect to the inner product

$$\langle f, g\rangle = \int_a^b w(x)f(x)g(x)\,dx$$

on $C[a,b]$, where w is a positive weight function.

Prove that, for $n \geqslant 1$, p_n has n distinct zeros in the interval (a,b).

Let $c_1, c_2, \ldots, c_n \in [a,b]$ be n distinct points. Show that the quadrature formula

$$\int_a^b w(x)f(x)\,dx \approx \sum_{i=1}^{n} b_i f(c_i)$$

is exact for all $f \in \mathbb{P}_{n-1}$ if the weights b_i are chosen to be

$$b_i = \int_a^b w(x) \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - c_j}{c_i - c_j}\,dx\,.$$

Show further that the quadrature formula is exact for all $f \in \mathbb{P}_{2n-1}$ if the nodes c_i are chosen to be the zeros of p_n (Gaussian quadrature). [*Hint: Write f as $qp_n + r$, where $q, r \in \mathbb{P}_{n-1}$.*]

Use the Peano kernel theorem to write an integral expression for the approximation error of Gaussian quadrature for sufficiently differentiable functions. (You should give a formal expression for the Peano kernel but are *not* required to evaluate it.)

Paper 1, Section I

6D Numerical Analysis

(a) What are real *orthogonal polynomials* defined with respect to an inner product $\langle \cdot, \cdot \rangle$? What does it mean for such polynomials to be *monic*?

(b) Real monic orthogonal polynomials, $p_n(x)$, of degree $n = 0, 1, 2, \ldots$, are defined with respect to the inner product,

$$\langle p, q \rangle = \int_{-1}^{1} w(x) p(x) q(x) \, dx,$$

where $w(x)$ is a positive weight function. Show that such polynomials obey the three-term recurrence relation,

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x) ,$$

for appropriate α_n and β_n which should be given in terms of inner products.

Paper 4, Section I

8D Numerical Analysis

(a) Define the *linear stability domain* for a numerical method to solve $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. What is meant by an *A-stable* method?

(b) A two-stage Runge–Kutta scheme is given by

$$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n), \qquad \mathbf{k}_2 = \mathbf{f}(t_n + \tfrac{h}{2}, \mathbf{y}_n + \tfrac{h}{2}\mathbf{k}_1), \qquad \mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{k}_2 ,$$

where h is the step size and $t_n = nh$. Show that the order of this scheme is at least two. For this scheme, find the intersection of the linear stability domain with the real axis. Hence show that this method is *not* A-stable.

Paper 1, Section II

18D Numerical Analysis

(a) Consider a method for numerically solving an ordinary differential equation (ODE) for an initial value problem, $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. What does it mean for a method to *converge* over $t \in [0, T]$ where $T \in \mathbb{R}$? What is the definition of the *order* of a method?

(b) A general multistep method for the numerical solution of an ODE is

$$\sum_{l=0}^{s} \rho_l \, \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \, \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \qquad n = 0, 1, \ldots\,,$$

where s is a fixed positive integer. Show that this method is at least of order $p \geqslant 1$ if and only if

$$\sum_{l=0}^{s} \rho_l = 0 \qquad \text{and} \qquad \sum_{l=0}^{s} l^k \, \rho_l = k \sum_{l=0}^{s} l^{k-1} \, \sigma_l, \qquad k = 1, \ldots, p\,.$$

(c) State the Dahlquist equivalence theorem regarding the convergence of a multistep method.

(d) Consider the multistep method,

$$\mathbf{y}_{n+2} + \theta \, \mathbf{y}_{n+1} + a \, \mathbf{y}_n = h\big[\sigma_0 \mathbf{f}(t_n, \mathbf{y}_n) + \sigma_1 \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \sigma_2 \mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})\big]\,.$$

Determine the values of σ_i and a (in terms of the real parameter θ) such that the method is at least third order. For what values of θ does the method converge?

Paper 3, Section II

19D Numerical Analysis

(a) Determine real quadratic functions $a(x), b(x), c(x)$ such that the interpolation formula,

$$f(x) \approx a(x)f(0) + b(x)f(2) + c(x)f(3)\,,$$

is exact when $f(x)$ is any real polynomial of degree 2.

(b) Use this formula to construct approximations for $f(5)$ and $f'(1)$ which are exact when $f(x)$ is any real polynomial of degree 2. Calculate these approximations for $f(x) = x^3$ and comment on your answers.

(c) State the Peano kernel theorem and define the *Peano kernel* $K(\theta)$. Use this theorem to find the minimum values of the constants α and β such that

$$\left| f(1) - \tfrac{1}{3}\left[f(0) + 3f(2) - f(3)\right] \right| \leqslant \alpha \max_{\xi\in[0,3]} \left| f^{(2)}(\xi) \right|,$$

and

$$\left| f(1) - \tfrac{1}{3}\left[f(0) + 3f(2) - f(3)\right] \right| \leqslant \beta\, \|f^{(2)}\|_1\,,$$

where $f \in C^2[0,3]$. Check that these inequalities hold for $f(x) = x^3$.

Paper 2, Section II

19D Numerical Analysis

(a) Define a *Givens rotation* $\Omega^{[p,q]} \in \mathbb{R}^{m\times m}$ and show that it is an orthogonal matrix.

(b) Define a QR *factorization* of a matrix $\mathsf{A} \in \mathbb{R}^{m\times n}$ with $m \geqslant n$. Explain how Givens rotations can be used to find $\mathsf{Q} \in \mathbb{R}^{m\times m}$ and $\mathsf{R} \in \mathbb{R}^{m\times n}$.

(c) Let

$$\mathsf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3/4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 98/25 \\ 25 \\ 25 \\ 0 \end{bmatrix}.$$

(i) Find a QR factorization of A using Givens rotations.

(ii) Hence find the vector $\mathbf{x}^* \in \mathbb{R}^3$ which minimises $\|\mathsf{A}\mathbf{x} - \mathbf{b}\|$, where $\|\cdot\|$ is the Euclidean norm. What is $\|\mathsf{A}\mathbf{x}^* - \mathbf{b}\|$?

Paper 1, Section I

6D Numerical Analysis

Let

$$\mathsf{A} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 17 & 13 & 11 \\ 3 & 13 & 13 & 12 \\ 2 & 11 & 12 & \lambda \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix},$$

where λ is a real parameter. Find the LU factorization of the matrix A. Give the constraint on λ for A to be positive definite.

For $\lambda = 18$, use this factorization to solve the system $\mathsf{A}x = b$ via forward and backward substitution.

Paper 4, Section I

8D Numerical Analysis

Given $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$, let $p_n \in \mathbb{P}_n$ be the real polynomial of degree n that interpolates a continuous function f at these points. State the *Lagrange interpolation formula*.

Prove that p_n can be written in the *Newton form*

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i),$$

where $f[x_0, \ldots, x_k]$ is the *divided difference*, which you should define. [An explicit expression for the divided difference is *not* required.]

Explain why it can be more efficient to use the Newton form rather than the Lagrange formula.

Paper 1, Section II

18D Numerical Analysis

Determine the real coefficients b_1, b_2, b_3 such that

$$\int_{-2}^{2} f(x)dx = b_1 f(-1) + b_2 f(0) + b_3 f(1)\,,$$

is exact when $f(x)$ is any real polynomial of degree 2. Check explicitly that the quadrature is exact for $f(x) = x^2$ with these coefficients.

State the *Peano kernel theorem* and define the *Peano kernel* $K(\theta)$. Use this theorem to show that if $f \in C^3[-2,2]$, and b_1, b_2, b_3 are chosen as above, then

$$\left|\int_{-2}^{2} f(x)dx - b_1 f(-1) - b_2 f(0) - b_3 f(1)\right| \leqslant \frac{4}{9} \max_{\xi\in[-2,2]} \left|f^{(3)}(\xi)\right|\,.$$

Paper 3, Section II

19D Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A. Explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $||\mathsf{A}x - b||$, where $b \in \mathbb{R}^m$, $m > n$, and $||\cdot||$ is the Euclidean norm.

Explain how to construct Q and R by the Gram-Schmidt procedure. Why is this procedure not useful for numerical factorization of large matrices?

Let

$$\mathsf{A} = \begin{bmatrix} 5 & 6 & -14 \\ 5 & 4 & 4 \\ -5 & 2 & -8 \\ 5 & 12 & -18 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Using the Gram-Schmidt procedure find a QR decomposition of A. Hence solve the least squares problem giving both x^* and $||\mathsf{A}x^* - b||$.

Paper 2, Section II

19D Numerical Analysis

Define the *linear stability domain* for a numerical method to solve $y' = f(t, y)$. What is meant by an *A-stable* method? Briefly explain the relevance of these concepts in the numerical solution of ordinary differential equations.

Consider

$$y_{n+1} = y_n + h\left[\theta f(t_n, y_n) + (1 - \theta) f(t_{n+1}, y_{n+1})\right] ,$$

where $\theta \in [0, 1]$. What is the order of this method?

Find the linear stability domain of this method. For what values of θ is the method A-stable?

Paper 1, Section I

6C Numerical Analysis

(i) A general multistep method for the numerical approximation to the scalar differential equation $y' = f(t, y)$ is given by

$$\sum_{\ell=0}^{s} \rho_\ell \, y_{n+\ell} = h \sum_{\ell=0}^{s} \sigma_\ell f_{n+\ell}, \qquad n = 0, 1, \ldots$$

where $f_{n+\ell} = f(t_{n+\ell}, y_{n+\ell})$. Show that this method is of order $p \geqslant 1$ if and only if

$$\rho(\mathrm{e}^z) - z\sigma(\mathrm{e}^z) = \mathcal{O}(z^{p+1}) \quad \text{as} \quad z \to 0$$

where

$$\rho(w) = \sum_{\ell=0}^{s} \rho_\ell \, w^\ell \quad \text{and} \quad \sigma(w) = \sum_{\ell=0}^{s} \sigma_\ell \, w^\ell \,.$$

(ii) A particular three-step implicit method is given by

$$y_{n+3} + (a-1) y_{n+1} - a y_n = h \left(f_{n+3} + \sum_{\ell=0}^{2} \sigma_\ell f_{n+\ell} \right).$$

where the σ_ℓ are chosen to make the method third order. [The σ_ℓ need not be found.] For what values of a is the method convergent?

Paper 4, Section I

8C Numerical Analysis

Consider the quadrature given by

$$\int_0^\pi w(x) f(x) dx \approx \sum_{k=1}^{\nu} b_k f(c_k)$$

for $\nu \in \mathbb{N}$, disjoint $c_k \in (0, \pi)$ and $w > 0$. Show that it is not possible to make this quadrature exact for all polynomials of order 2ν.

For the case that $\nu = 2$ and $w(x) = \sin x$, by considering orthogonal polynomials find suitable b_k and c_k that make the quadrature exact on cubic polynomials.

[*Hint:* $\int_0^\pi x^2 \sin x \, dx = \pi^2 - 4$ *and* $\int_0^\pi x^3 \sin x \, dx = \pi^3 - 6\pi$.]

Paper 1, Section II

18C Numerical Analysis

Define a Householder transformation H and show that it is an orthogonal matrix. Briefly explain how these transformations can be used for QR factorisation of an $m \times n$ matrix.

Using Householder transformations, find a QR factorisation of

$$\mathsf{A} = \begin{bmatrix} 2 & 5 & 4 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \\ 2 & -1 & 16 \end{bmatrix}.$$

Using this factorisation, find the value of λ for which

$$\mathsf{A}\,x = \begin{bmatrix} 1+\lambda \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

has a unique solution $x \in \mathbb{R}^3$.

Paper 3, Section II

19C Numerical Analysis

A Runge–Kutta scheme is given by

$$k_1 = hf(y_n), \quad k_2 = hf(y_n + [(1-a)k_1 + ak_2]), \quad y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

for the solution of an autonomous differential equation $y' = f(y)$, where a is a real parameter. What is the order of the scheme? Identify all values of a for which the scheme is A-stable. Determine the linear stability domain for this range.

Paper 2, Section II

19C Numerical Analysis

A linear functional acting on $f \in C^{k+1}[a,b]$ is approximated using a linear scheme $L(f)$. The approximation is exact when f is a polynomial of degree k. The error is given by $\lambda(f)$. Starting from the Taylor formula for $f(x)$ with an integral remainder term, show that the error can be written in the form

$$\lambda(f) = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta) d\theta$$

subject to a condition on λ that you should specify. Give an expression for $K(\theta)$.

Find c_0, c_1 and c_3 such that the approximation scheme

$$f''(2) \approx c_0 f(0) + c_1 f(1) + c_3 f(3)$$

is exact for all f that are polynomials of degree 2. Assuming $f \in C^3[0,3]$, apply the Peano kernel theorem to the error. Find and sketch $K(\theta)$ for $k = 2$.

Find the minimum values for the constants r and s for which

$$|\lambda(f)| \leqslant r\|f^{(3)}\|_1 \quad \text{and} \quad |\lambda(f)| \leqslant s\|f^{(3)}\|_\infty$$

and show explicitly that both error bounds hold for $f(x) = x^3$.

Paper 1, Section I

6C Numerical Analysis

Determine the nodes x_1, x_2 of the two-point Gaussian quadrature

$$\int_0^1 f(x)w(x)\,dx \approx a_1 f(x_1) + a_2 f(x_2), \qquad w(x) = x,$$

and express the coefficients a_1, a_2 in terms of x_1, x_2. [You don't need to find numerical values of the coefficients.]

Paper 4, Section I

8C Numerical Analysis

For a continuous function f, and $k+1$ distinct points $\{x_0, x_1, \ldots, x_k\}$, define the divided difference $f[x_0, \ldots, x_k]$ of order k.

Given $n+1$ points $\{x_0, x_1, \ldots, x_n\}$, let $p_n \in \mathbb{P}_n$ be the polynomial of degree n that interpolates f at these points. Prove that p_n can be written in the Newton form

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i)\,.$$

Paper 1, Section II

18C Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $\|Ax - b\|$, where $b \in \mathbb{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Givens rotation $\Omega^{[p,q]}$ and show that it is an orthogonal matrix.

Using a Givens rotation, solve the least squares problem for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

Paper 3, Section II

19C Numerical Analysis

Let

$$f'(0) \approx a_0 f(0) + a_1 f(1) + a_2 f(2) =: \lambda(f)$$

be a formula of numerical differentiation which is exact on polynomials of degree 2, and let

$$e(f) = f'(0) - \lambda(f)$$

be its error.

Find the values of the coefficients a_0, a_1, a_2.

Using the Peano kernel theorem, find the least constant c such that, for all functions $f \in C^3[0,2]$, we have

$$|e(f)| \leqslant c \, \|f'''\|_\infty \, .$$

Paper 2, Section II

19C Numerical Analysis

Explain briefly what is meant by the convergence of a numerical method for solving the ordinary differential equation

$$y'(t) = f(t, y), \qquad t \in [0, T], \qquad y(0) = y_0 \, .$$

Prove from first principles that if the function f is sufficiently smooth and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leqslant L \, |x - y|, \qquad x, y \in \mathbb{R}, \qquad t \in [0, T],$$

for some $L > 0$, then the backward Euler method

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \, ,$$

converges and find the order of convergence.

Find the linear stability domain of the backward Euler method.

Paper 1, Section I

6D Numerical Analysis

Let

$$A=\begin{bmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{bmatrix}, \qquad b=\begin{bmatrix} \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \gamma = 1-a^4 \neq 0\,.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax=b$ via forward and backward substitution. [Other methods of solution are not acceptable.]

Paper 4, Section I

8D Numerical Analysis

State the Dahlquist equivalence theorem regarding convergence of a multistep method.

The multistep method, with a real parameter a,

$$y_{n+3}+(2a-3)(y_{n+2}-y_{n+1})-y_n = ha\,(f_{n+2}-f_{n+1})$$

is of order 2 for any a, and also of order 3 for $a=6$. Determine all values of a for which the method is convergent, and find the order of convergence.

Paper 1, Section II

18D Numerical Analysis

For a numerical method for solving $y'=f(t,y)$, define the linear stability domain, and state when such a method is A-stable.

Determine all values of the real parameter a for which the Runge-Kutta method

$$\begin{aligned} k_1 &= f\Big(t_n+(\tfrac{1}{2}-a)h, y_n+\big(\tfrac{1}{4}hk_1+(\tfrac{1}{4}-a)hk_2\big)\Big), \\ k_2 &= f\Big(t_n+(\tfrac{1}{2}+a)h, y_n+\big((\tfrac{1}{4}+a)hk_1+\tfrac{1}{4}hk_2\big)\Big), \\ y_{n+1} &= y_n+\tfrac{1}{2}h(k_1+k_2) \end{aligned}$$

is A-stable.

Paper 3, Section II
19D Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $\|Ax^* - b\|$, where $b \in \mathbb{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Householder transformation H and show that it is an orthogonal matrix.

Using a Householder transformation, solve the least squares problem for

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

Paper 2, Section II
19D Numerical Analysis

Let $\{P_n\}_{n=0}^{\infty}$ be the sequence of monic polynomials of degree n orthogonal on the interval $[-1, 1]$ with respect to the weight function w.

Prove that each P_n has n distinct zeros in the interval $(-1, 1)$.

Let $P_0(x) = 1$, $P_1(x) = x - a_1$, and let P_n satisfy the following three-term recurrence relation:

$$P_n(x) = (x - a_n)P_{n-1}(x) - b_n^2 P_{n-2}(x), \qquad n \geqslant 2.$$

Set

$$A_n = \begin{bmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{bmatrix}.$$

Prove that $P_n(x) = \det(xI - A_n)$, $n \geqslant 1$, and deduce that all the eigenvalues of A_n are distinct and reside in $(-1, 1)$.

Paper 1, Section I

6B Numerical Analysis

Orthogonal monic polynomials $p_0, p_1, \ldots, p_n, \ldots$ are defined with respect to the inner product $\langle p, q\rangle = \int_{-1}^{1} w(x)p(x)q(x)\,dx$, where p_n is of degree n. Show that such polynomials obey a three-term recurrence relation

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x)$$

for appropriate choices of α_n and β_n.

Now suppose that $w(x)$ is an even function of x. Show that the p_n are even or odd functions of x according to whether n is even or odd.

Paper 4, Section I

8B Numerical Analysis

Consider the multistep method for numerical solution of the differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$:

$$\sum_{l=0}^{s} \rho_l \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \quad n = 0, 1, \ldots.$$

What does it mean to say that the method is of order p, and that the method is convergent?

Show that the method is of order p if

$$\sum_{l=0}^{s} \rho_l = 0, \quad \sum_{l=0}^{s} l^k \rho_l = k \sum_{l=0}^{s} l^{k-1}\sigma_l, \quad k = 1, 2, \ldots, p,$$

and give the conditions on $\rho(w) = \sum_{l=0}^{s} \rho_l w^l$ that ensure convergence.

Hence determine for what values of θ and the σ_i the two-step method

$$\mathbf{y}_{n+2} - (1-\theta)\mathbf{y}_{n+1} - \theta\mathbf{y}_n = h[\sigma_0 \mathbf{f}(t_n, \mathbf{y}_n) + \sigma_1 \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \sigma_2 \mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})]$$

is (a) convergent, and (b) of order 3.

Paper 1, Section II

18B Numerical Analysis

Consider a function $f(x)$ defined on the domain $x \in [0,1]$. Find constants α, β, γ so that for any fixed $\xi \in [0,1]$,

$$f''(\xi) = \alpha f(0) + \beta f'(0) + \gamma f(1)$$

is exactly satisfied for polynomials of degree less than or equal to two.

By using the Peano kernel theorem, or otherwise, show that

$$\begin{aligned} f'(\xi) - f'(0) - \xi\left(\alpha f(0) + \beta f'(0) + \gamma f(1)\right) &= \int_0^\xi (\xi - \theta) H_1(\theta) f'''(\theta)\, d\theta \\ &+ \int_0^\xi \theta H_2(\theta) f'''(\theta)\, d\theta + \int_\xi^1 \xi H_2(\theta) f'''(\theta)\, d\theta, \end{aligned}$$

where $H_1(\theta) = 1 - (1-\theta)^2 \geqslant 0,\ H_2(\theta) = -(1-\theta)^2 \leqslant 0$. Thus show that

$$\left| f'(\xi) - f'(0) - \xi(\alpha f(0) + \beta f'(0) + \gamma f(1)) \right| \leqslant \frac{1}{6}(2\xi - 3\xi^2 + 4\xi^3 - \xi^4) \left|\left| f''' \right|\right|_\infty .$$

Paper 2, Section II

19B Numerical Analysis

What is the QR-decomposition of a matrix A? Explain how to construct the matrices Q and R by the Gram-Schmidt procedure, and show how the decomposition can be used to solve the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ when A is a square matrix.

Why is this procedure not useful for numerical decomposition of large matrices? Give a brief description of an alternative procedure using Givens rotations.

Find a QR-decomposition for the matrix

$$\mathsf{A} = \begin{bmatrix} 3 & 4 & 7 & 13 \\ -6 & -8 & -8 & -12 \\ 3 & 4 & 7 & 11 \\ 0 & 2 & 5 & 7 \end{bmatrix}.$$

Is your decomposition unique? Use the decomposition you have found to solve the equation

$$\mathsf{A}\mathbf{x} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 9 \end{bmatrix}.$$

Paper 3, Section II

19B Numerical Analysis

A Gaussian quadrature formula provides an approximation to the integral

$$\int_{-1}^{1} (1-x^2) f(x)\,dx \approx \sum_{k=1}^{\nu} b_k f(c_k)$$

which is exact for all $f(x)$ that are polynomials of degree $\leqslant (2\nu - 1)$.

Write down explicit expressions for the b_k in terms of integrals, and explain why it is necessary that the c_k are the zeroes of a (monic) polynomial p_ν of degree ν that satisfies $\int_{-1}^{1}(1-x^2)p_\nu(x)q(x)\,dx = 0$ for any polynomial $q(x)$ of degree less than ν.

The first such polynomials are $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 1/5$, $p_3 = x^3 - 3x/7$. Show that the Gaussian quadrature formulae for $\nu = 2, 3$ are

$$\begin{aligned} \nu = 2: &\quad \frac{2}{3}\left[f(-\tfrac{1}{\sqrt{5}}) + f(\tfrac{1}{\sqrt{5}})\right], \\ \nu = 3: &\quad \frac{14}{45}\left[f(-\sqrt{\tfrac{3}{7}}) + f(\sqrt{\tfrac{3}{7}})\right] + \frac{32}{45} f(0). \end{aligned}$$

Verify the result for $\nu = 3$ by considering $f(x) = 1, x^2, x^4$.

Paper 1, Section I

6C Numerical Analysis

Obtain the Cholesky decompositions of

$$H_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \lambda \end{pmatrix}.$$

What is the minimum value of λ for H_4 to be positive definite? Verify that if $\lambda = \frac{1}{7}$ then H_4 is positive definite.

Paper 4, Section I

8C Numerical Analysis

Suppose $x_0, x_1, \ldots, x_n \in [a, b] \subset \mathbf{R}$ are pointwise distinct and $f(x)$ is continuous on $[a, b]$. For $k = 1, 2, \ldots, n$ define

$$I_{0,k}(x) = \frac{f(x_0)(x_k - x) - f(x_k)(x_0 - x)}{x_k - x_0},$$

and for $k = 2, 3, \ldots, n$

$$I_{0,1,\ldots,k-2,k-1,k}(x) = \frac{I_{0,1,\ldots,k-2,k-1}(x)(x_k - x) - I_{0,1,\ldots,k-2,k}(x)(x_{k-1} - x)}{x_k - x_{k-1}}.$$

Show that $I_{0,1,\ldots,k-2,k-1,k}(x)$ is a polynomial of order k which interpolates $f(x)$ at $x_0, x_1, \ldots, x_k$.

Given $x_k = \{-1, 0, 2, 5\}$ and $f(x_k) = \{33, 5, 9, 1335\}$, determine the interpolating polynomial.

Paper 1, Section II

18C Numerical Analysis

Let

$$\langle f, g\rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)\, g(x)\, dx\,,$$

be an inner product. The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \ldots$ are polynomials in x of degree n with leading term $2^n x^n$ which are orthogonal with respect to the inner product, with

$$\langle H_m, H_n\rangle = \begin{cases} \gamma_m > 0 & \text{if } m = n\,, \\ 0 & \text{otherwise,} \end{cases}$$

and $H_0(x) = 1$. Find a three-term recurrence relation which is satisfied by $H_n(x)$ and γ_n for $n = 1, 2, 3$. [*You may assume without proof that*

$$\langle 1, 1\rangle = \sqrt{\pi}\,, \quad \langle x, x\rangle = \tfrac{1}{2}\sqrt{\pi}\,, \quad \langle x^2, x^2\rangle = \tfrac{3}{4}\sqrt{\pi}\,, \quad \langle x^3, x^3\rangle = \tfrac{15}{8}\sqrt{\pi}\,.]$$

Next let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of $H_{k+1}(x)$ and for $i, j = 0, 1, \ldots, k$ define the Lagrangian polynomials

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

associated with these points. Prove that $\langle L_i, L_j\rangle = 0$ if $i \neq j$.

Paper 2, Section II

19C Numerical Analysis

Consider the initial value problem for an autonomous differential equation

$$y'(t) = f(y(t)), \qquad y(0) = y_0 \text{ given},$$

and its approximation on a grid of points $t_n = nh$, $n = 0, 1, 2, \ldots$. Writing $y_n = y(t_n)$, it is proposed to use one of two Runge–Kutta schemes defined by

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(y_n)$ and

$$k_2 = \begin{cases} hf(y_n + k_1) & \text{scheme I}\,, \\ hf(y_n + \tfrac{1}{2}(k_1 + k_2)) & \text{scheme II}\,. \end{cases}$$

What is the order of each scheme? Determine the A-stability of each scheme.

Paper 3, Section II

19C Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the $x^* \in \mathbf{R}^n$ which minimises $\|Ax - b\|$ where $b \in \mathbf{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Householder (reflection) transformation H and show that it is an orthogonal matrix.

Using a Householder reflection, solve the least squares problem for

$$A = \begin{pmatrix} 2 & 4 & 7 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 9 \\ -7 \\ 3 \\ 1 \\ -1 \end{pmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

Paper 1, Section I

6C Numerical Analysis

The real non-singular matrix $A \in \mathbb{R}^{m\times m}$ is written in the form $A = A_D + A_U + A_L$, where the matrices $A_D, A_U, A_L \in \mathbb{R}^{m\times m}$ are diagonal and non-singular, strictly upper-triangular and strictly lower-triangular respectively.

Given $b \in \mathbb{R}^m$, the Jacobi iteration for solving $Ax = b$ is

$$A_D x_n = -(A_U + A_L)x_{n-1} + b, \qquad n = 1, 2...$$

where the nth iterate is $x_n \in \mathbb{R}^m$. Show that the iteration converges to the solution x of $Ax = b$, independent of the starting choice x_0, if and only if the spectral radius $\rho(H)$ of the matrix $H = -A_D^{-1}(A_U + A_L)$ is less than 1.

Hence find the range of values of the real number μ for which the iteration will converge when

$$A = \begin{bmatrix} 1 & 0 & -\mu \\ -\mu & 3 & -\mu \\ -4\mu & 0 & 4 \end{bmatrix}.$$

Paper 4, Section I

8C Numerical Analysis

Suppose that $w(x) > 0$ for all $x \in (a, b)$. The weights $b_1, ..., b_n$ and nodes $x_1, ..., x_n$ are chosen so that the Gaussian quadrature formula

$$\int_a^b w(x) f(x) dx \sim \sum_{k=1}^{n} b_k f(x_k)$$

is exact for every polynomial of degree $2n-1$. Show that the $b_i, i = 1, ..., n$ are all positive.

When $w(x) = 1 + x^2$, $a = -1$ and $b = 1$, the first three underlying orthogonal polynomials are $p_0(x) = 1, p_1(x) = x$, and $p_2(x) = x^2 - 2/5$. Find x_1, x_2 and b_1, b_2 when $n = 2$.

Paper 2, Section II

18C Numerical Analysis

The real orthogonal matrix $\Omega^{[p,q]} \in \mathbb{R}^{m\times m}$ with $1 \leqslant p < q \leqslant m$ is a Givens rotation with rotation angle θ. Write down the form of $\Omega^{[p,q]}$.

Show that for any matrix $A \in \mathbb{R}^{m\times m}$ it is possible to choose θ such that the matrix $\Omega^{[p,q]}A$ satisfies $(\Omega^{[p,q]}A)_{q,j} = 0$ for any j, where $1 \leqslant j \leqslant m$.

Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \\ \sqrt{2} & 7/\sqrt{2} & 4\sqrt{2} \end{bmatrix}.$$

By applying a sequence of Givens rotations of the form $\Omega^{[1,3]}\Omega^{[1,2]}$, chosen to reduce the elements in the first column below the main diagonal to zero, find a factorisation of the matrix $A \in \mathbb{R}^{3\times 3}$ of the form $A = QR$, where $Q \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix and $R \in \mathbb{R}^{3\times 3}$ is an upper-triangular matrix for which the leading non-zero element in each row is positive.

Paper 3, Section II

19C Numerical Analysis

Starting from Taylor's theorem with integral form of the remainder, prove the Peano kernel theorem: the error of an approximant $L(f)$ applied to $f(x) \in C^{k+1}[a,b]$ can be written in the form

$$L(f) = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta) d\theta.$$

You should specify the form of $K(\theta)$. Here it is assumed that $L(f)$ is identically zero when $f(x)$ is a polynomial of degree k. State any other necessary conditions.

Setting $a = 0$ and $b = 2$, find $K(\theta)$ and show that it is negative for $0 < \theta < 2$ when

$$L(f) = \int_0^2 f(x)dx - \frac{1}{3}\left(f(0) + 4f(1) + f(2)\right) \quad \text{for} \quad f(x) \in C^4[0,2].$$

Hence determine the minimum value of ρ for which

$$|L(f)| \leqslant \rho \|f^{(4)}\|_\infty,$$

holds for all $f(x) \in C^4[0,2]$.

1/I/6D Numerical Analysis

Show that if $A = LDL^T$, where $L \in \mathbb{R}^{m\times m}$ is a lower triangular matrix with all elements on the main diagonal being unity and $D \in \mathbb{R}^{m\times m}$ is a diagonal matrix with positive elements, then A is positive definite. Find L and the corresponding D when

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

2/II/18D Numerical Analysis

(a) A Householder transformation (reflection) is given by

$$H = I - \frac{2uu^T}{\|u\|^2},$$

where $H \in \mathbb{R}^{m\times m}$, $u \in \mathbb{R}^m$, and I is the $m \times m$ unit matrix and u is a non-zero vector which has norm $\|u\| = (\sum_{i=1}^m u_i^2)^{1/2}$. Show that H is orthogonal.

(b) Suppose that $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ with $n < m$. Show that if x minimises $\|Ax - b\|^2$ then it also minimises $\|QAx - Qb\|^2$, where Q is an arbitrary $m \times m$ orthogonal matrix.

(c) Using Householder reflection, find the x that minimises $\|Ax - b\|^2$ when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

3/II/19D Numerical Analysis

Starting from the Taylor formula for $f(x) \in C^{k+1}[a,b]$ with an integral remainder term, show that the error of an approximant $L(f)$ can be written in the form (Peano kernel theorem)

$$L(f) = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta) d\theta,$$

when $L(f)$, which is identically zero if $f(x)$ is a polynomial of degree k, satisfies conditions that you should specify. Give an expression for $K(\theta)$.

Hence determine the minimum value of c in the inequality

$$|L(f)| \le c\|f'''\|_\infty\,,$$

when

$$L(f) = f'(1) - \frac{1}{2}\left(f(2) - f(0)\right) \quad \text{for} \quad f(x) \in C^3[0,2].$$

4/I/8D Numerical Analysis

Show that the Chebyshev polynomials, $T_n(x) = \cos(n\cos^{-1}x)\,, n = 0,\,1,\,2,\,\ldots$ obey the orthogonality relation

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\delta_{n,m}(1+\delta_{n,0}).$$

State briefly how an optimal choice of the parameters a_k, x_k, $k = 1, 2 \ldots n$ is made in the Gaussian quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \sim \sum_{k=1}^{n} a_k f(x_k).$$

Find these parameters for the case $n = 3$.

1/I/6F Numerical Analysis

Solve the least squares problem

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

using QR method with Householder transformation. (A solution using normal equations is *not* acceptable.)

2/II/18F Numerical Analysis

For a symmetric, positive definite matrix A with the spectral radius $\rho(A)$, the linear system $Ax = b$ is solved by the iterative procedure

$$x^{(k+1)} = x^{(k)} - \tau(Ax^{(k)} - b), \qquad k \geq 0\,,$$

where τ is a real parameter. Find the range of τ that guarantees convergence of $x^{(k)}$ to the exact solution for any choice of $x^{(0)}$.

3/II/19F Numerical Analysis

Prove that the monic polynomials Q_n, $n \geq 0$, orthogonal with respect to a given weight function $w(x) > 0$ on $[a, b]$, satisfy the three-term recurrence relation

$$Q_{n+1}(x) = (x - a_n)Q_n(x) - b_n Q_{n-1}(x), \quad n \geq 0\,.$$

where $Q_{-1}(x) \equiv 0$, $Q_0(x) \equiv 1$. Express the values a_n and b_n in terms of Q_n and Q_{n-1} and show that $b_n > 0$.

4/I/8F Numerical Analysis

Given $f \in C^3[0, 2]$, we approximate $f'(0)$ by the linear combination

$$\mu(f) = -\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2)\,.$$

Using the Peano kernel theorem, determine the least constant c in the inequality

$$|f'(0) - \mu(f)| \leq c\,\|f'''\|_\infty\,,$$

and give an example of f for which the inequality turns into equality.

1/I/6D **Numerical Analysis**

(a) Perform the LU-factorization with column pivoting of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

(b) Explain briefly why every nonsingular matrix A admits an LU-factorization with column pivoting.

2/II/18D **Numerical Analysis**

(a) For a positive weight function w, let

$$\int_{-1}^{1} f(x)w(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

be the corresponding Gaussian quadrature with $n+1$ nodes. Prove that all the coefficients a_i are positive.

(b) The integral

$$I(f) = \int_{-1}^{1} f(x)w(x)\,dx$$

is approximated by a quadrature

$$I_n(f) = \sum_{i=0}^{n} a_i^{(n)} f(x_i^{(n)})$$

which is exact on polynomials of degree $\leqslant n$ and has positive coefficients $a_i^{(n)}$. Prove that, for any f continuous on $[-1,1]$, the quadrature converges to the integral, i.e.,

$$|I(f) - I_n(f)| \to 0 \quad \text{as} \quad n \to \infty.$$

[*You may use the Weierstrass theorem: for any f continuous on* $[-1,1]$, *and for any* $\epsilon > 0$, *there exists a polynomial Q of degree $n = n(\epsilon, f)$ such that* $\max_{x\in[-1,1]} |f(x) - Q(x)| < \epsilon$.]

3/II/19D Numerical Analysis

(a) Define the QR factorization of a rectangular matrix and explain how it can be used to solve the least squares problem of finding an $x^* \in \mathbb{R}^n$ such that

$$\|Ax^* - b\| = \min_{x \in \mathbb{R}^n} \|Ax - b\|, \quad \text{where} \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad m \geqslant n,$$

and the norm is the Euclidean distance $\|y\| = \sqrt{\sum_{i=1}^m |y_i|^2}$.

(b) Define a Householder transformation (reflection) H and prove that H is an orthogonal matrix.

(c) Using Householder reflection, solve the least squares problem for the case

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix},$$

and find the value of $\|Ax^* - b\| = \min_{x \in \mathbb{R}^2} \|Ax - b\|$.

4/I/8D Numerical Analysis

(a) Given the data

x_i	-1	0	1	3
$f(x_i)$	-7	-3	-3	9

find the interpolating cubic polynomial $p \in \mathcal{P}_3$ in the Newton form, and transform it to the power form.

(b) We add to the data one more value $f(x_i)$ at $x_i = 2$. Find the power form of the interpolating quartic polynomial $q \in \mathcal{P}_4$ to the extended data

x_i	-1	0	1	2	3
$f(x_i)$	-7	-3	-3	-7	9

1/I/6F Numerical Analysis

Determine the Cholesky factorization (without pivoting) of the matrix

$$A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 10+\lambda & 2+3\lambda \\ 2 & 2+3\lambda & 23+9\lambda \end{bmatrix}$$

where λ is a real parameter. Hence, find the range of values of λ for which the matrix A is positive definite.

2/II/18F Numerical Analysis

(a) Let $\{Q_n\}_{n\geqslant 0}$ be a set of polynomials orthogonal with respect to some inner product $(\,\cdot\,,\,\cdot\,)$ in the interval $[a,b]$. Write explicitly the least-squares approximation to $f \in C[a,b]$ by an nth-degree polynomial in terms of the polynomials $\{Q_n\}_{n\geqslant 0}$.

(b) Let an inner product be defined by the formula

$$(g,h) = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} g(x)h(x)dx.$$

Determine the nth degree polynomial approximation of $f(x) = (1-x^2)^{\frac{1}{2}}$ with respect to this inner product as a linear combination of the underlying orthogonal polynomials.

3/II/19F Numerical Analysis

Given real $\mu \neq 0$, we consider the matrix

$$A = \begin{bmatrix} \frac{1}{\mu} & 1 & 0 & 0 \\ -1 & \frac{1}{\mu} & 1 & 0 \\ 0 & -1 & \frac{1}{\mu} & 1 \\ 0 & 0 & -1 & \frac{1}{\mu} \end{bmatrix}.$$

Construct the Jacobi and Gauss–Seidel iteration matrices originating in the solution of the linear system $Ax = b$.

Determine the range of real $\mu \neq 0$ for which each iterative procedure converges.

4/I/8F Numerical Analysis

Define Gaussian quadrature.

Evaluate the coefficients of the Gaussian quadrature of the integral

$$\int_{-1}^{1} (1-x^2)f(x)dx$$

which uses two function evaluations.

2/I/9A **Numerical Analysis**

Determine the coefficients of Gaussian quadrature for the evaluation of the integral

$$\int_0^1 f(x)x\,dx$$

that uses two function evaluations.

2/II/20A **Numerical Analysis**

Given an $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$, prove that the vector $\mathbf{x} \in \mathbb{R}^n$ is the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$ if and only if $A^T(A\mathbf{x} - \mathbf{b}) = \mathbf{0}$. Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$

Determine the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$.

3/I/11A **Numerical Analysis**

The linear system

$$\begin{bmatrix} \alpha & 2 & 1 \\ 1 & \alpha & 2 \\ 2 & 1 & \alpha \end{bmatrix} \mathbf{x} = \mathbf{b},$$

where real $\alpha \neq 0$ and $\mathbf{b} \in \mathbb{R}^3$ are given, is solved by the iterative procedure

$$\mathbf{x}^{(k+1)} = -\frac{1}{\alpha} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \frac{1}{\alpha}\mathbf{b}, \qquad k \geqslant 0.$$

Determine the conditions on α that guarantee convergence.

3/II/22A **Numerical Analysis**

Given $f \in C^3[0,1]$, we approximate $f'(\frac{1}{3})$ by the linear combination

$$\mathcal{T}[f] = -\frac{5}{3}f(0) + \frac{4}{3}f(\tfrac{1}{2}) + \frac{1}{3}f(1)\,.$$

By finding the Peano kernel, determine the least constant c such that

$$\left|\mathcal{T}[f] - f'(\tfrac{1}{3})\right| \leq c\left\|f'''\right\|_\infty.$$

2/I/5B **Numerical Analysis**

Let

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \gamma \\ 0 \\ 0 \\ \gamma a \end{pmatrix}, \quad \gamma = 1 - a^4 \neq 0.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax = b$.

2/II/14B **Numerical Analysis**

Let

$$f''(0) \approx a_0 f(-1) + a_1 f(0) + a_2 f(1) = \mu(f)$$

be an approximation of the second derivative which is exact for $f \in \mathcal{P}_2$, the set of polynomials of degree ≤ 2, and let

$$e(f) = f''(0) - \mu(f)$$

be its error.

(a) Determine the coefficients a_0, a_1, a_2.

(b) Using the Peano kernel theorem prove that, for $f \in C^3[-1,1]$, the set of three-times continuously differentiable functions, the error satisfies the inequality

$$|e(f)| \leq \frac{1}{3} \max_{x \in [-1,1]} |f'''(x)| \, .$$

3/I/6B **Numerical Analysis**

Given $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$, let

$$\ell_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

be the fundamental Lagrange polynomials of degree n, let

$$\omega(x) = \prod_{i=0}^{n} (x - x_i),$$

and let p be any polynomial of degree $\le n$.

(a) Prove that $\sum_{i=0}^{n} p(x_i)\ell_i(x) \equiv p(x)$.

(b) Hence or otherwise derive the formula

$$\frac{p(x)}{\omega(x)} = \sum_{i=0}^{n} \frac{A_i}{x - x_i}, \quad A_i = \frac{p(x_i)}{\omega'(x_i)},$$

which is the decomposition of $p(x)/\omega(x)$ into partial fractions.

3/II/16B **Numerical Analysis**

The functions $H_0, H_1, \ldots$ are generated by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

(a) Show that H_n is a polynomial of degree n, and that the H_n are orthogonal with respect to the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2}\, dx.$$

(b) By induction or otherwise, prove that the H_n satisfy the three-term recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

[*Hint: you may need to prove the equality* $H_n'(x) = 2nH_{n-1}(x)$ *as well.*]

2/I/5B **Numerical Analysis**

Applying the Gram–Schmidt orthogonalization, compute a “skinny” QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 6 \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix},$$

i.e. find a 4×3 matrix Q with orthonormal columns and an upper triangular 3×3 matrix R such that $A = QR$.

2/II/14B **Numerical Analysis**

Let $f \in C[a,b]$ and let $n+1$ distinct points $x_0,\ldots,x_n \in [a,b]$ be given.

(a) Define the divided difference $f[x_0,\ldots,x_n]$ of order n in terms of interpolating polynomials. Prove that it is a symmetric function of the variables x_i, $i=0,\ldots,n$.

(b) Prove the recurrence relation

$$f[x_0,\ldots,x_n] = \frac{f[x_1,\ldots,x_n] - f[x_0,\ldots,x_{n-1}]}{x_n - x_0}.$$

(c) Hence or otherwise deduce that, for any $i \neq j$, we have

$$f[x_0,\ldots,x_n] = \frac{f[x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n] - f[x_0,\ldots,x_{j-1},x_{j+1},\ldots,x_n]}{x_j - x_i}.$$

(d) From the formulas above, show that, for any $i = 1,\ldots,n-1$,

$$f[x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n] = \gamma f[x_0,\ldots,x_{n-1}] + (1-\gamma) f[x_1,\ldots,x_n],$$

where $\gamma = \dfrac{x_i - x_0}{x_n - x_0}$.

3/I/6B **Numerical Analysis**

For numerical integration, a quadrature formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^n a_i f(x_i)$$

is applied which is exact on $\mathcal{P}_n$, i.e., for all polynomials of degree n.

Prove that such a formula is exact for all $f \in \mathcal{P}_{2n+1}$ if and only if x_i, $i=0,\ldots,n$, are the zeros of an orthogonal polynomial $p_{n+1} \in \mathcal{P}_{n+1}$ which satisfies $\int_a^b p_{n+1}(x) r(x)\,dx = 0$ for all $r \in \mathcal{P}_n$. [*You may assume that p_{n+1} has $(n+1)$ distinct zeros.*]

3/II/16B Numerical Analysis

(a) Consider a system of linear equations $Ax = b$ with a non-singular square $n \times n$ matrix A. To determine its solution $x = x^*$ we apply the iterative method

$$x^{k+1} = Hx^k + v.$$

Here $v \in \mathbb{R}^n$, while the matrix $H \in \mathbb{R}^{n \times n}$ is such that $x^* = Hx^* + v$ implies $Ax^* = b$. The initial vector $x^0 \in \mathbb{R}^n$ is arbitrary. Prove that, if the matrix H possesses n linearly independent eigenvectors $w_1, \ldots, w_n$ whose corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy $\max_i |\lambda_i| < 1$, then the method converges for any choice of x^0, i.e. $x^k \to x^*$ as $k \to \infty$.

(b) Describe the Jacobi iteration method for solving $Ax = b$. Show directly from the definition of the method that, if the matrix A is strictly diagonally dominant by rows, i.e.

$$|a_{ii}|^{-1} \sum_{j=1, j \neq i}^{n} |a_{ij}| \leq \gamma < 1, \quad i = 1, \ldots, n,$$

then the method converges.

2/I/5E Numerical Analysis

Find an LU factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 \\ -4 & 3 & -4 & -2 \\ 4 & -2 & 3 & 6 \\ -6 & 5 & -8 & 1 \end{pmatrix},$$

and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 4 \\ 11 \end{pmatrix}.$$

2/II/14E Numerical Analysis

(a) Let B be an $n \times n$ positive-definite, symmetric matrix. Define the Cholesky factorization of B and prove that it is unique.

(b) Let A be an $m \times n$ matrix, $m \geqslant n$, such that $\operatorname{rank} A = n$. Prove the uniqueness of the "skinny QR factorization"

$$A = QR,$$

where the matrix Q is $m \times n$ with orthonormal columns, while R is an $n \times n$ upper-triangular matrix with positive diagonal elements.

[*Hint: Show that you may choose R as a matrix that features in the Cholesky factorization of $B = A^T A$.*]

3/I/6E Numerical Analysis

Given $f \in C^{n+1}[a,b]$, let the nth-degree polynomial p interpolate the values $f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct. Given $x \in [a,b]$, find the error $f(x) - p(x)$ in terms of a derivative of f.

3/II/16E **Numerical Analysis**

Let the monic polynomials p_n, $n \geqslant 0$, be orthogonal with respect to the weight function $w(x) > 0$, $a < x < b$, where the degree of each p_n is exactly n.

(a) Prove that each p_n, $n \geqslant 1$, has n distinct zeros in the interval (a, b).

(b) Suppose that the p_n satisfy the three-term recurrence relation

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n^2 p_{n-2}(x), \quad n \geqslant 2,$$

where $p_0(x) \equiv 1$, $p_1(x) = x - a_1$. Set

$$A_n = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}, \quad n \geqslant 2.$$

Prove that $p_n(x) = \det(xI - A_n)$, $n \geqslant 2$, and deduce that all the eigenvalues of A_n reside in (a, b).