Part IB

Metric and Topological Spaces

Paper 3, Section I

3G Metric and Topological Spaces

Let X be a metric space.

(a) What does it mean for X to be *compact*? What does it mean for X to be *sequentially compact*?

(b) Prove that if X is compact then X is sequentially compact.

Paper 2, Section I

4G Metric and Topological Spaces

(a) Let $f : X \to Y$ be a continuous surjection of topological spaces. Prove that, if X is connected, then Y is also connected.

(b) Let $g : [0,1] \to [0,1]$ be a continuous map. Deduce from part (a) that, for every y between $g(0)$ and $g(1)$, there is $x \in [0,1]$ such that $g(x) = y$. [You may not assume the Intermediate Value Theorem, but you may use the fact that suprema exist in $\mathbb{R}$.]

Paper 1, Section II

12G Metric and Topological Spaces

Consider the set of sequences of integers

$$X = \{(x_1, x_2, \ldots) \mid x_n \in \mathbb{Z} \text{ for all } n\}\,.$$

Define

$$n_{\min}((x_n),(y_n)) = \begin{cases} \infty & x_n = y_n \text{ for all } n \\ \min\{n \mid x_n \neq y_n\} & \text{otherwise} \end{cases}$$

for two sequences $(x_n), (y_n) \in X$. Let

$$d((x_n),(y_n)) = 2^{-n_{\min}((x_n),(y_n))}$$

where, as usual, we adopt the convention that $2^{-\infty} = 0$.

(a) Prove that d defines a metric on X.

(b) What does it mean for a metric space to be *complete*? Prove that (X,d) is complete.

(c) Is (X,d) path connected? Justify your answer.

Paper 4, Section II

13G Metric and Topological Spaces

(a) Define the *subspace*, *quotient* and *product topologies*.

(b) Let X be a compact topological space and Y a Hausdorff topological space. Prove that a continuous bijection $f : X \to Y$ is a homeomorphism.

(c) Let $S = [0,1] \times [0,1]$, equipped with the product topology. Let $\sim$ be the smallest equivalence relation on S such that $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$, for all $s,t \in [0,1]$. Let

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2} - 2)^2 + z^2 = 1\}$$

equipped with the subspace topology from $\mathbb{R}^3$. Prove that $S/{\sim}$ and T are homeomorphic.

[*You may assume without proof that S is compact.*]

Paper 3, Section I

3E Metric and Topological Spaces

What does it mean to say that a topological space is *connected*? If X is a topological space and $x \in X$, show that there is a connected subspace K_x of X so that if S is any other connected subspace containing x then $S \subseteq K_x$.

Show that the sets K_x partition X.

Paper 2, Section I

4E Metric and Topological Spaces

What does it mean to say that d is a *metric* on a set X? What does it mean to say that a subset of X is *open* with respect to the metric d? Show that the collection of subsets of X that are open with respect to d satisfies the axioms of a topology.

For $X = C[0,1]$, the set of continuous functions $f : [0,1] \to \mathbb{R}$, show that the metrics

$$d_1(f,g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x$$

$$d_2(f,g) = \left[\int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x \right]^{1/2}$$

give different topologies.

Paper 1, Section II

12E Metric and Topological Spaces

What does it mean to say that a topological space is *compact*? Prove directly from the definition that $[0,1]$ is compact. Hence show that the unit circle $S^1 \subset \mathbb{R}^2$ is compact, proving any results that you use. *[You may use without proof the continuity of standard functions.]*

The set $\mathbb{R}^2$ has a topology $\mathcal{T}$ for which the closed sets are the empty set and the finite unions of vector subspaces. Let X denote the set $\mathbb{R}^2 \backslash \{0\}$ with the subspace topology induced by $\mathcal{T}$. By considering the subspace topology on $S^1 \subset \mathbb{R}^2$, or otherwise, show that X is compact.

Paper 4, Section II

13E Metric and Topological Spaces

Let $X = \{2, 3, 4, 5, 6, 7, 8, \ldots\}$ and for each $n \in X$ let

$$U_n = \{d \in X \mid d \text{ divides } n\}.$$

Prove that the set of unions of the sets U_n forms a topology on X.

Prove or disprove each of the following:

(i) X is Hausdorff;

(ii) X is compact.

If Y and Z are topological spaces, Y is the union of closed subspaces A and B, and $f : Y \to Z$ is a function such that both $f|_A : A \to Z$ and $f|_B : B \to Z$ are continuous, show that f is continuous. Hence show that X is path-connected.

Paper 3, Section I

3E Metric and Topological Spaces

Let X and Y be topological spaces.

(a) Define what is meant by the *product topology* on $X \times Y$. Define the *projection maps* $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ and show they are continuous.

(b) Consider $\Delta = \{(x, x) \mid x \in X\}$ in $X \times X$. Show that X is Hausdorff if and only if Δ is a closed subset of $X \times X$ in the product topology.

Paper 2, Section I

4E Metric and Topological Spaces

Let $f \colon (X, d) \to (Y, e)$ be a function between metric spaces.

(a) Give the ϵ-δ definition for f to be *continuous*. Show that f is continuous if and only if $f^{-1}(U)$ is an open subset of X for each open subset U of Y.

(b) Give an example of f such that f is not continuous but $f(V)$ is an open subset of Y for every open subset V of X.

Paper 1, Section II

12E Metric and Topological Spaces

Consider $\mathbb{R}$ and $\mathbb{R}^2$ with their usual Euclidean topologies.

(a) Show that a non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval. Find a compact subset $K \subset \mathbb{R}$ for which $\mathbb{R} \setminus K$ has infinitely many connected components.

(b) Let T be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus T$ is path-connected. Deduce that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

Paper 4, Section II

13E Metric and Topological Spaces

Let $f\colon X \to Y$ be a continuous map between topological spaces.

(a) Assume X is compact and that $Z \subseteq X$ is a closed subset. Prove that Z and $f(Z)$ are both compact.

(b) Suppose that

(i) $f^{-1}(\{y\})$ is compact for each $y \in Y$, and

(ii) if A is any closed subset of X, then $f(A)$ is a closed subset of Y.

Show that if $K \subseteq Y$ is compact, then $f^{-1}(K)$ is compact.

[*Hint: Given an open cover* $f^{-1}(K) \subseteq \bigcup_{i \in I} U_i$, *find a finite subcover, say* $f^{-1}(\{y\}) \subseteq \bigcup_{i \in I_y} U_i$, *for each* $y \in K$*; use closedness of* $X \setminus \bigcup_{i \in I_y} U_i$ *and property* (ii) *to produce an open cover of* K.]

Paper 3, Section I

3E Metric and Topological Spaces

Let X be a topological space and $A \subseteq X$ be a subset. A *limit point* of A is a point $x \in X$ such that any open neighbourhood U of x intersects A. Show that A is closed if and only if it contains all its limit points. Explain what is meant by the *interior* $\mathrm{Int}(A)$ and the *closure* $\overline{A}$ of A. Show that if A is connected, then $\overline{A}$ is connected.

Paper 2, Section I

4E Metric and Topological Spaces

Consider $\mathbb{R}$ and $\mathbb{Q}$ with their usual topologies.

(a) Show that compact subsets of a Hausdorff topological space are closed. Show that compact subsets of $\mathbb{R}$ are closed and bounded.

(b) Show that there exists a complete metric space (X, d) admitting a surjective continuous map $f\colon X \to \mathbb{Q}$.

Paper 1, Section II

12E Metric and Topological Spaces

Let p be a prime number. Define what is meant by the *p-adic metric* d_p on $\mathbb{Q}$. Show that for $a, b, c \in \mathbb{Q}$ we have

$$d_p(a, b) \leqslant \max\{d_p(a, c), d_p(c, b)\}.$$

Show that the sequence (a_n), where $a_n = 1 + p + \cdots + p^{n-1}$, converges to some element in $\mathbb{Q}$.

For $a \in \mathbb{Q}$ define $|a|_p = d_p(a, 0)$. Show that if $a, b \in \mathbb{Q}$ and if $|a|_p \neq |b|_p$, then

$$|a + b|_p = \max\{|a|_p, |b|_p\}.$$

Let $a \in \mathbb{Q}$ and let $B(a, \delta)$ be the open ball with centre a and radius $\delta > 0$, with respect to the metric d_p. Show that $B(a, \delta)$ is a closed subset of $\mathbb{Q}$ with respect to the topology induced by d_p.

Paper 4, Section II

13E Metric and Topological Spaces

(a) Let X be a topological space. Define what is meant by a *quotient* of X and describe the *quotient topology* on the quotient space. Give an example in which X is Hausdorff but the quotient space is not Hausdorff.

(b) Let T^2 be the 2-dimensional torus considered as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, and let $\pi : \mathbb{R}^2 \to T^2$ be the quotient map.

(i) Let $B(u,r)$ be the open ball in $\mathbb{R}^2$ with centre u and radius $r < 1/2$. Show that $U = \pi(B(u,r))$ is an open subset of T^2 and show that $\pi^{-1}(U)$ has infinitely many connected components. Show each connected component is homeomorphic to $B(u,r)$.

(ii) Let $\alpha \in \mathbb{R}$ be an irrational number and let $L \subset \mathbb{R}^2$ be the line given by the equation $y = \alpha x$. Show that $\pi(L)$ is dense in T^2 but $\pi(L) \neq T^2$.

Paper 3, Section I
3E Metric and Topological Spaces

Define what it means for a topological space X to be (i) *connected* (ii) *path-connected*.

Prove that any path-connected space X is connected. [You may assume the interval $[0,1]$ is connected.]

Give a counterexample (without justification) to the converse statement.

Paper 2, Section I
4E Metric and Topological Spaces

Let X and Y be topological spaces and $f : X \to Y$ a continuous map. Suppose H is a subset of X such that $f(\overline{H})$ is closed (where $\overline{H}$ denotes the closure of H). Prove that $f(\overline{H}) = \overline{f(H)}$.

Give an example where f, X, Y and H are as above but $f(\overline{H})$ is not closed.

Paper 1, Section II
12E Metric and Topological Spaces

Give the definition of a *metric* on a set X and explain how this defines a topology on X.

Suppose (X,d) is a metric space and U is an open set in X. Let $x, y \in X$ and $\epsilon > 0$ such that the open ball $B_\epsilon(y) \subseteq U$ and $x \in B_{\epsilon/2}(y)$. Prove that $y \in B_{\epsilon/2}(x) \subseteq U$.

Explain what it means (i) for a set S to be *dense* in X, (ii) to say $\mathcal{B}$ is a *base* for a topology $\mathcal{T}$.

Prove that any metric space which contains a countable dense set has a countable basis.

Paper 4, Section II

13E Metric and Topological Spaces

Explain what it means for a metric space (M,d) to be (i) *compact*, (ii) *sequentially compact*. Prove that a compact metric space is sequentially compact, stating clearly any results that you use.

Let (M,d) be a compact metric space and suppose $f\colon M \to M$ satisfies $d(f(x),f(y)) = d(x,y)$ for all $x,y \in M$. Prove that f is surjective, stating clearly any results that you use. [*Hint: Consider the sequence* $(f^n(x))$ *for* $x \in M$*.*]

Give an example to show that the result does not hold if M is not compact.

Paper 3, Section I

3E Metric and Topological Spaces

Suppose (X, d) is a metric space. Do the following necessarily define a metric on X? Give proofs or counterexamples.

(i) $d_1(x, y) = kd(x, y)$ for some constant $k > 0$, for all $x, y \in X$.

(ii) $d_2(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$.

(iii) $d_3(x, y) = (d(x, y))^2$ for all $x, y \in X$.

Paper 2, Section I

4E Metric and Topological Spaces

Let X and Y be topological spaces. What does it mean to say that a function $f : X \to Y$ is *continuous*?

Are the following statements true or false? Give proofs or counterexamples.

(i) Every continuous function $f : X \to Y$ is an open map, i.e. if U is open in X then $f(U)$ is open in Y.

(ii) If $f : X \to Y$ is continuous and bijective then f is a homeomorphism.

(iii) If $f : X \to Y$ is continuous, open and bijective then f is a homeomorphism.

Paper 1, Section II

12E Metric and Topological Spaces

Define what it means for a topological space to be *compact*. Define what it means for a topological space to be *Hausdorff*.

Prove that a compact subspace of a Hausdorff space is closed. Hence prove that if C_1 and C_2 are compact subspaces of a Hausdorff space X then $C_1 \cap C_2$ is compact.

A subset U of $\mathbb{R}$ is open in the *cocountable topology* if U is empty or its complement in $\mathbb{R}$ is countable. Is $\mathbb{R}$ Hausdorff in the cocountable topology? Which subsets of $\mathbb{R}$ are compact in the cocountable topology?

Paper 4, Section II

13E Metric and Topological Spaces

Explain what it means for a metric space to be *complete.*

Let X be a metric space. We say the subsets A_i of X, with $i \in \mathbb{N}$, form a *descending sequence* in X if $A_1 \supset A_2 \supset A_3 \supset \cdots$.

Prove that the metric space X is complete if and only if any descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of X, such that the diameters of the subsets A_i converge to zero, has an intersection $\bigcap_{i=1}^{\infty} A_i$ that is non-empty.

[Recall that the diameter $\operatorname{diam}(S)$ of a set S is the supremum of the set $\{d(x,y) : x, y \in S\}$.]

Give examples of

(i) a metric space X, and a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of X, with $\operatorname{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(ii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty sets in $\mathbb{R}$ with $\operatorname{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(iii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed sets in $\mathbb{R}$ with $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

Paper 3, Section I

3G Metric and Topological Spaces

Let X be a metric space with the metric $d : X \times X \to \mathbb{R}$.

(i) Show that if X is compact as a topological space, then X is complete.

(ii) Show that the completeness of X is not a topological property, i.e. give an example of two metrics d, d' on a set X, such that the associated topologies are the same, but (X, d) is complete and (X, d') is not.

Paper 2, Section I

4G Metric and Topological Spaces

Let X be a topological space. Prove or disprove the following statements.

(i) If X is discrete, then X is compact if and only if it is a finite set.

(ii) If Y is a subspace of X and X, Y are both compact, then Y is closed in X.

Paper 1, Section II

12G Metric and Topological Spaces

Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, a subset of $\mathbb{R}^3$, as a subspace of $\mathbb{R}^3$ with the Euclidean metric.

(i) Show that S^2 is compact and Hausdorff as a topological space.

(ii) Let $X = S^2/\sim$ be the quotient set with respect to the equivalence relation identifying the antipodes, i.e.

$$(x, y, z) \sim (x', y', z') \iff (x', y', z') = (x, y, z) \text{ or } (-x, -y, -z).$$

Show that X is compact and Hausdorff with respect to the quotient topology.

Paper 4, Section II

13G Metric and Topological Spaces

Let X be a topological space. A *connected component* of X means an equivalence class with respect to the equivalence relation on X defined as:

$$x \sim y \iff x, y \text{ belong to some connected subspace of } X.$$

(i) Show that every connected component is a connected and closed subset of X.

(ii) If X, Y are topological spaces and $X \times Y$ is the product space, show that every connected component of $X \times Y$ is a direct product of connected components of X and Y.

Paper 3, Section I

3F Metric and Topological Spaces

Define the notion of a *connected component* of a space X.

If $A_\alpha \subset X$ are connected subsets of X such that $\bigcap_\alpha A_\alpha \neq \emptyset$, show that $\bigcup_\alpha A_\alpha$ is connected.

Prove that any point $x \in X$ is contained in a unique connected component.

Let $X \subset \mathbb{R}$ consist of the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$. What are the connected components of X?

Paper 2, Section I

4F Metric and Topological Spaces

For each case below, determine whether the given metrics d_1 and d_2 induce the same topology on X. Justify your answers.

(i) $X = \mathbb{R}^2$, $d_1((x_1, y_1), (x_2, y_2)) = \sup\{|x_1 - x_2|, |y_1 - y_2|\}$

$$d_2((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

(ii) $X = C[0,1]$, $d_1(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$

$$d_2(f, g) = \int_0^1 |f(t) - g(t)|\, dt.$$

Paper 1, Section II

12F Metric and Topological Spaces

A topological space X is said to be *normal* if each point of X is a closed subset of X and for each pair of closed sets $C_1, C_2 \subset X$ with $C_1 \cap C_2 = \emptyset$ there are open sets $U_1, U_2 \subset X$ so that $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. In this case we say that the U_i *separate* the C_i.

Show that a compact Hausdorff space is normal. [*Hint: first consider the case where C_2 is a point.*]

For $C \subset X$ we define an equivalence relation $\sim_C$ on X by $x \sim_C y$ for all $x, y \in C$, $x \sim_C x$ for $x \notin C$. If C, C_1 and C_2 are pairwise disjoint closed subsets of a normal space X, show that C_1 and C_2 may be separated by open subsets U_1 and U_2 such that $U_i \cap C = \emptyset$. Deduce that the quotient space $X/\sim_C$ is also normal.

Paper 4, Section II

13F Metric and Topological Spaces

Suppose A_1 and A_2 are topological spaces. Define the product topology on $A_1 \times A_2$. Let $\pi_i : A_1 \times A_2 \to A_i$ be the projection. Show that a map $F : X \to A_1 \times A_2$ is continuous if and only if $\pi_1 \circ F$ and $\pi_2 \circ F$ are continuous.

Prove that if A_1 and A_2 are connected, then $A_1 \times A_2$ is connected.

Let X be the topological space whose underlying set is $\mathbb{R}$, and whose open sets are of the form (a, ∞) for $a \in \mathbb{R}$, along with the empty set and the whole space. Describe the open sets in $X \times X$. Are the maps $f, g : X \times X \to X$ defined by $f(x, y) = x + y$ and $g(x, y) = xy$ continuous? Justify your answers.

Paper 2, Section I

4G Metric and Topological Spaces

(i) Let $t > 0$. For $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y') \in \mathbb{R}^2$, let

$$d(\mathbf{x}, \mathbf{x}') = |x' - x| + t|y' - y|,$$

$$\delta(\mathbf{x}, \mathbf{x}') = \sqrt{(x' - x)^2 + (y' - y)^2}.$$

(δ is the usual Euclidean metric on $\mathbb{R}^2$.) Show that d is a metric on $\mathbb{R}^2$ and that the two metrics d, δ give rise to the same topology on $\mathbb{R}^2$.

(ii) Give an example of a topology on $\mathbb{R}^2$, different from the one in (i), whose induced topology (subspace topology) on the x-axis is the usual topology (the one defined by the metric $d(x, x') = |x' - x|$). Justify your answer.

Paper 3, Section I

3G Metric and Topological Spaces

Let X, Y be topological spaces, and suppose Y is Hausdorff.

(i) Let $f, g : X \to Y$ be two continuous maps. Show that the set

$$E(f, g) := \{x \in X \mid f(x) = g(x)\} \subset X$$

is a closed subset of X.

(ii) Let W be a dense subset of X. Show that a continuous map $f : X \to Y$ is determined by its restriction $f|_W$ to W.

Paper 1, Section II

12G Metric and Topological Spaces

Let X be a metric space with the distance function $d : X \times X \to \mathbb{R}$. For a subset Y of X, its *diameter* is defined as $\delta(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$.

Show that, if X is compact and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of X, then there exists an $\epsilon > 0$ such that every subset $Y \subset X$ with $\delta(Y) < \epsilon$ is contained in some U_λ.

Paper 4, Section II

13G Metric and Topological Spaces

Let X, Y be topological spaces and $X \times Y$ their product set. Let $p_Y : X \times Y \to Y$ be the projection map.

(i) Define the product topology on $X \times Y$. Prove that if a subset $Z \subset X \times Y$ is open then $p_Y(Z)$ is open in Y.

(ii) Give an example of X, Y and a closed set $Z \subset X \times Y$ such that $p_Y(Z)$ is not closed.

(iii) When X is compact, show that if a subset $Z \subset X \times Y$ is closed then $p_Y(Z)$ is closed.

Paper 2, Section I

4H Metric and Topological Spaces

On the set $\mathbb{Q}$ of rational numbers, the 3-*adic metric* d_3 is defined as follows: for $x, y \in \mathbb{Q}$, define $d_3(x, x) = 0$ and $d_3(x, y) = 3^{-n}$, where n is the integer satisfying $x - y = 3^n u$ where u is a rational number whose denominator and numerator are both prime to 3.

(1) Show that this is indeed a metric on $\mathbb{Q}$.

(2) Show that in $(\mathbb{Q}, d_3)$, we have $3^n \to 0$ as $n \to \infty$ while $3^{-n} \not\to 0$ as $n \to \infty$. Let d be the usual metric $d(x, y) = |x - y|$ on $\mathbb{Q}$. Show that neither the identity map $(\mathbb{Q}, d) \to (\mathbb{Q}, d_3)$ nor its inverse is continuous.

Paper 3, Section I

3H Metric and Topological Spaces

Let X be a topological space and Y be a set. Let $p : X \to Y$ be a surjection. The *quotient topology* on Y is defined as follows: a subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in X.

(1) Show that this does indeed define a topology on Y, and show that p is continuous when we endow Y with this topology.

(2) Let Z be another topological space and $f : Y \to Z$ be a map. Show that f is continuous if and only if $f \circ p : X \to Z$ is continuous.

Paper 1, Section II

12H Metric and Topological Spaces

Let $f : X \to Y$ and $g : Y \to X$ be continuous maps of topological spaces with $f \circ g = \mathrm{id}_Y$.

(1) Suppose that (i) Y is path-connected, and (ii) for every $y \in Y$, its inverse image $f^{-1}(y)$ is path-connected. Prove that X is path-connected.

(2) Prove the same statement when "path-connected" is everywhere replaced by "connected".

Paper 4, Section II

13H Metric and Topological Spaces

(1) Prove that if X is a compact topological space, then a closed subset Y of X endowed with the subspace topology is compact.

(2) Consider the following equivalence relation on $\mathbb{R}^2$:

$$(x_1, y_1) \sim (x_2, y_2) \iff (x_1 - x_2,\ y_1 - y_2) \in \mathbb{Z}^2.$$

Let $X = \mathbb{R}^2/\sim$ be the quotient space endowed with the quotient topology, and let $p : \mathbb{R}^2 \to X$ be the canonical surjection mapping each element to its equivalence class. Let $Z = \{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{2}x\}$.

(i) Show that X is compact.

(ii) Assuming that $p(Z)$ is dense in X, show that $p|_Z : Z \to p(Z)$ is bijective but not homeomorphic.

Paper 2, Section I

4F Metric and Topological Spaces

Explain what is meant by a Hausdorff (topological) space, and prove that every compact subset of a Hausdorff space is closed.

Let X be an uncountable set, and consider the topology $\mathcal{T}$ on X defined by

$$U \in \mathcal{T} \Leftrightarrow \text{ either } U = \emptyset \text{ or } X \setminus U \text{ is countable.}$$

Is $(X, \mathcal{T})$ Hausdorff? Is every compact subset of X closed? Justify your answers.

Paper 3, Section I

4F Metric and Topological Spaces

Are the following statements true or false? Give brief justifications for your answers.

(i) If X is a connected open subset of $\mathbb{R}^n$ for some n, then X is path-connected.

(ii) A cartesian product of two connected spaces is connected.

(iii) If X is a Hausdorff space and the only connected subsets of X are singletons $\{x\}$, then X is discrete.

Paper 1, Section II

12F Metric and Topological Spaces

Given a function $f : X \to Y$ between metric spaces, we write Γ_f for the subset $\{(x, f(x)) \mid x \in X\}$ of $X \times Y$.

(i) If f is continuous, show that Γ_f is closed in $X \times Y$.

(ii) If Y is compact and Γ_f is closed in $X \times Y$, show that f is continuous.

(iii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that Γ_f is closed but f is not continuous.

Paper 4, Section II

14F Metric and Topological Spaces

A nonempty subset A of a topological space X is called *irreducible* if, whenever we have open sets U and V such that $U \cap A$ and $V \cap A$ are nonempty, then we also have $U \cap V \cap A \neq \emptyset$. Show that the closure of an irreducible set is irreducible, and deduce that the closure of any singleton set $\{x\}$ is irreducible.

X is said to be a *sober* topological space if, for any irreducible closed $A \subseteq X$, there is a unique $x \in X$ such that $A = \overline{\{x\}}$. Show that any Hausdorff space is sober, but that an infinite set with the cofinite topology is not sober.

Given an arbitrary topological space $(X, \mathcal{T})$, let $\widehat{X}$ denote the set of all irreducible closed subsets of X, and for each $U \in \mathcal{T}$ let

$$\widehat{U} = \{A \in \widehat{X} \mid U \cap A \neq \emptyset\} \; .$$

Show that the sets $\{\widehat{U} \mid U \in \mathcal{T}\}$ form a topology $\widehat{\mathcal{T}}$ on $\widehat{X}$, and that the mapping $U \mapsto \widehat{U}$ is a bijection from $\mathcal{T}$ to $\widehat{\mathcal{T}}$. Deduce that $(\widehat{X}, \widehat{\mathcal{T}})$ is sober. [*Hint: consider the complement of an irreducible closed subset of* $\widehat{X}$.]

1/II/12F Metric and Topological Spaces

Write down the definition of a topology on a set X.

For each of the following families $\mathcal{T}$ of subsets of $\mathbb{Z}$, determine whether $\mathcal{T}$ is a topology on $\mathbb{Z}$. In the cases where the answer is 'yes', determine also whether $(\mathbb{Z}, \mathcal{T})$ is a Hausdorff space and whether it is compact.

(*a*) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{ either } U \text{ is finite or } 0 \in U\}$.

(*b*) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{ either } \mathbb{Z} \setminus U \text{ is finite or } 0 \notin U\}$.

(*c*) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{ there exists } k > 0 \text{ such that, for all } n,\ n \in U \Leftrightarrow n + k \in U\}$.

(*d*) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{ for all } n \in U, \text{ there exists } k > 0 \text{ such that } \{n+km : m \in \mathbb{Z}\} \subseteq U\}$.

2/I/4F Metric and Topological Spaces

Stating carefully any results on compactness which you use, show that if X is a compact space, Y is a Hausdorff space and $f \colon X \to Y$ is bijective and continuous, then f is a homeomorphism.

Hence or otherwise show that the unit circle $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the quotient space $[0, 1]/\sim$, where $\sim$ is the equivalence relation defined by

$$x \sim y \Leftrightarrow \text{ either } x = y \text{ or } \{x, y\} = \{0, 1\} .$$

3/I/4F Metric and Topological Spaces

Explain what it means for a topological space to be connected.

Are the following subspaces of the unit square $[0, 1] \times [0, 1]$ connected? Justify your answers.

(*a*) $\{(x, y) : x \neq 0, y \neq 0, \text{ and } x/y \in \mathbb{Q}\}$.

(*b*) $\{(x, y) : (x = 0) \text{ or } (x \neq 0 \text{ and } y \in \mathbb{Q})\}$.

4/II/14F **Metric and Topological Spaces**

Explain what is meant by a base for a topology. Illustrate your definition by describing bases for the topology induced by a metric on a set, and for the product topology on the cartesian product of two topological spaces.

A topological space $(X, \mathcal{T})$ is said to be *separable* if there is a countable subset $C \subseteq X$ which is dense, i.e. such that $C \cap U \neq \emptyset$ for every nonempty $U \in \mathcal{T}$. Show that a product of two separable spaces is separable. Show also that a metric space is separable if and only if its topology has a countable base, and deduce that every subspace of a separable metric space is separable.

Now let $X = \mathbb{R}$ with the topology $\mathcal{T}$ having as a base the set of all half-open intervals

$$[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}$$

with $a < b$. Show that X is separable, but that the subspace $Y = \{(x, -x) : x \in \mathbb{R}\}$ of $X \times X$ is not separable.

[*You may assume standard results on countability.*]

1/II/12A **Metric and Topological Spaces**

Let X and Y be topological spaces. Define the product topology on $X \times Y$ and show that if X and Y are Hausdorff then so is $X \times Y$.

Show that the following statements are equivalent.

(i) X is a Hausdorff space.

(ii) The diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of $X \times X$, in the product topology.

(iii) For any topological space Y and any continuous maps $f, g : Y \to X$, the set $\{y \in Y : f(y) = g(y)\}$ is closed in Y.

2/I/4A **Metric and Topological Spaces**

Are the following statements true or false? Give a proof or a counterexample as appropriate.

(i) If $f : X \to Y$ is a continuous map of topological spaces and $S \subseteq X$ is compact then $f(S)$ is compact.

(ii) If $f : X \to Y$ is a continuous map of topological spaces and $K \subseteq Y$ is compact then $f^{-1}(K) = \{x \in X : f(x) \in K\}$ is compact.

(iii) If a metric space M is complete and a metric space T is homeomorphic to M then T is complete.

3/I/4A **Metric and Topological Spaces**

(a) Let X be a connected topological space such that each point x of X has a neighbourhood homeomorphic to $\mathbb{R}^n$. Prove that X is path-connected.

(b) Let τ denote the topology on $\mathbb{N} = \{1, 2, \ldots\}$, such that the open sets are $\mathbb{N}$, the empty set, and all the sets $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$. Prove that any continuous map from the topological space $(\mathbb{N}, \tau)$ to the Euclidean $\mathbb{R}$ is constant.

4/II/14A **Metric and Topological Spaces**

(a) For a subset A of a topological space X, define the *closure* $cl(A)$ of A. Let $f : X \to Y$ be a map to a topological space Y. Prove that f is continuous if and only if $f(cl(A)) \subseteq cl(f(A))$, for each $A \subseteq X$.

(b) Let M be a metric space. A subset S of M is called *dense* in M if the closure of S is equal to M.

Prove that if a metric space M is compact then it has a countable subset which is dense in M.

1/II/12F **Metric and Topological Spaces**

(i) Define the product topology on $X \times Y$ for topological spaces X and Y, proving that your definition does define a topology.

(ii) Let X be the logarithmic spiral defined in polar coordinates by $r = e^{\theta}$, where $-\infty < \theta < \infty$. Show that X (with the subspace topology from $\mathbf{R}^2$) is homeomorphic to the real line.

2/I/4F **Metric and Topological Spaces**

Which of the following subspaces of Euclidean space are connected? Justify your answers.

(i) $\{(x, y, z) \in \mathbf{R}^3 : z^2 - x^2 - y^2 = 1\}$;

(ii) $\{(x, y) \in \mathbf{R}^2 : x^2 = y^2\}$;

(iii) $\{(x, y, z) \in \mathbf{R}^3 : z = x^2 + y^2\}$.

3/I/4F **Metric and Topological Spaces**

Which of the following are topological spaces? Justify your answers.

(i) The set $X = \mathbf{Z}$ of the integers, with a subset A of X called "open" when A is either finite or the whole set X;

(ii) The set $X = \mathbf{Z}$ of the integers, with a subset A of X called "open" when, for each element $x \in A$ and every even integer n, $x + n$ is also in A.

4/II/14F **Metric and Topological Spaces**

(a) Show that every compact subset of a Hausdorff topological space is closed.

(b) Let X be a compact metric space. For F a closed subset of X and p any point of X, show that there is a point q in F with

$$d(p, q) = \inf_{q' \in F} d(p, q').$$

Suppose that for every x and y in X there is a point m in X with $d(x, m) = (1/2)d(x, y)$ and $d(y, m) = (1/2)d(x, y)$. Show that X is connected.

1/II/12A **Metric and Topological Spaces**

Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Show that the definition

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

defines a metric on the product $X \times Y$, under which the projection map $\pi : X \times Y \to Y$ is continuous.

If (X, d_X) is compact, show that every sequence in X has a subsequence converging to a point of X. Deduce that the projection map π then has the property that, for any closed subset $F \subset X \times Y$, the image $\pi(F)$ is closed in Y. Give an example to show that this fails if (X, d_X) is not assumed compact.

2/I/4A **Metric and Topological Spaces**

Let X be a topological space. Suppose that $U_1, U_2, \ldots$ are connected subsets of X with $U_j \cap U_1$ non-empty for all $j > 0$. Prove that

$$W = \bigcup_{j>0} U_j$$

is connected. If each U_j is path-connected, prove that W is path-connected.

3/I/4A **Metric and Topological Spaces**

Show that a topology τ_1 is determined on the real line $\mathbf{R}$ by specifying that a non-empty subset is open if and only if it is a union of half-open intervals $\{a \leq x < b\}$, where $a < b$ are real numbers. Determine whether $(\mathbf{R}, \tau_1)$ is Hausdorff.

Let τ_2 denote the *cofinite* topology on $\mathbf{R}$ (that is, a non-empty subset is open if and only if its complement is finite). Prove that the identity map induces a continuous map $(\mathbf{R}, \tau_1) \to (\mathbf{R}, \tau_2)$.

4/II/14A **Metric and Topological Spaces**

Let (M, d) be a metric space, and F a non-empty closed subset of M. For $x \in M$, set

$$d(x, F) = \inf_{z \in F} d(x, z).$$

Prove that $d(x, F)$ is a continuous function of x, and that it is strictly positive for $x \notin F$.

A topological space is called *normal* if for any pair of disjoint closed subsets F_1, F_2, there exist disjoint open subsets $U_1 \supset F_1$, $U_2 \supset F_2$. By considering the function

$$d(x, F_1) - d(x, F_2),$$

or otherwise, deduce that any metric space is normal.

Suppose now that X is a normal topological space, and that F_1, F_2 are disjoint closed subsets in X. Prove that there exist open subsets $W_1 \supset F_1, W_2 \supset F_2$, whose *closures* are disjoint. In the case when $X = \mathbf{R}^2$ with the standard metric topology, $F_1 = \{(x, -1/x) : x < 0\}$ and $F_2 = \{(x, 1/x) : x > 0\}$, find explicit open subsets W_1, W_2 with the above property.