Part IB

Methods

Paper 2, Section I

3A Methods

Expand $f(x) = x^3 - \pi^2 x$ as a Fourier series on $-\pi < x < \pi$.

Use the series and Parseval's theorem for Fourier series (which you may quote without proof) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Paper 3, Section I

5A Methods

Calculate the Green's function $G(x;\xi)$ given by the solution to

$$\frac{d^2G}{dx^2} - G = \delta(x-\xi);\; G(0;\xi) = 0 \text{ and } G(x;\xi) \to 0 \text{ as } x \to \infty,$$

where $\xi \in (0,\infty)$, $x \in (0,\infty)$ and $\delta(x)$ is the Dirac δ-function.

Use this Green's function to calculate an explicit solution $y(x)$ to the boundary value problem

$$\frac{d^2y}{dx^2} - y = e^{-2x},$$

where $x \in (0,\infty)$, $y(0) = 0$ and $y(x) \to 0$ as $x \to \infty$.

Paper 1, Section II

13A Methods

(a) Let $y_0(x)$ be a non-trivial solution of the Sturm–Liouville problem

$$\mathcal{L}(y_0; \lambda_0) = 0; \;\; y_0(0) = y_0(1) = 0,$$

where

$$\mathcal{L}(y; \lambda) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda w(x)]\, y.$$

Show that, if $y(x)$ and $f(x)$ are related by

$$\mathcal{L}(y; \lambda_0) = f,$$

with $y(x)$ satisfying the same boundary conditions as $y_0(x)$, then

$$\int_0^1 y_0 f \, dx = 0. \qquad (\star)$$

(b) Now assume that y_0 is normalised so that

$$\int_0^1 w y_0^2 \, dx = 1,$$

and consider the problem

$$\mathcal{L}(y; \lambda) = y^{m+1}; \;\; y(0) = y(1) = 0,$$

where m is a positive integer. By choosing f appropriately in $(\star)$ deduce that, if

$$\lambda - \lambda_0 = \epsilon^m \mu \text{ and } y(x) = \epsilon y_0(x) + \epsilon^2 y_1(x),$$

where $0 < \epsilon \ll 1$ and $\mu = O(1)$, then

$$\mu = \int_0^1 y_0^{m+2} \, dx + O(\epsilon).$$

Paper 2, Section II
14A Methods

(a) Laplace's equation in plane polar coordinates has the form

$$\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0.$$

Using separation of variables, show that the general solution is:

$$\phi(r,\theta) = a_0 + c_0 \ln r + \sum_{n=1}^{\infty}\left(a_n r^n + c_n r^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(b_n r^n + d_n r^{-n}\right)\sin n\theta,$$

for arbitrary real constants a_i, b_i, c_i and d_i.

Which (if any) constants must be zero for the solution to be regular in:

(i) the interior of a disc centred at the origin?

(ii) the exterior of a disc centred at the origin?

(iii) an annular region centred at the origin?

(b) Consider 2π-periodic functions $f(\theta)$ such that

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos n\theta,$$

for some coefficients A_n.

(i) Solve Laplace's equation $\nabla^2\phi = 0$ in the annulus $1 < r < e^2$ with boundary conditions:

$$\phi(r,\theta) = \begin{cases} f(\theta) - 1, & r = 1 \\ f(\theta) + 1, & r = e^2, \end{cases}$$

for general $f(\theta)$.

(ii) Calculate the explicit solution for the specific choice:

$$f(\theta) = \begin{cases} \frac{\pi}{2} - \theta, & 0 \leqslant \theta < \pi \\ -\frac{3\pi}{2} + \theta, & \pi \leqslant \theta < 2\pi. \end{cases}$$

Paper 3, Section II

14A Methods

(a) You are given that $f(x)$, $g(x)$ and $h(x)$ are all absolutely integrable functions with absolutely integrable Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$ and $\tilde{h}(k)$ such that

$$\tilde{h}(k) = [\tilde{f}(k)][\tilde{g}(k)],$$

i.e. that $\tilde{h}(k)$ is the product of $\tilde{f}(k)$ and $\tilde{g}(k)$. Express $h(x)$ in terms of an integral expression involving $f(x)$ and $g(x)$.

(b) If $p'(x) = g(x)$, express $\tilde{p}(k)$ in terms of $\tilde{g}(k)$. [You may assume that the transforms are well-defined.]

(c) Express the inverse transforms of $\cos ka$ and $\sin ka$ in terms of the δ-function, where a is a positive constant.

(d) Consider the following wave problem for $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}; \ u(x,0) = f(x), \ \frac{\partial}{\partial t}u(x,0) = g(x).$$

Use parts (a)-(c) to construct d'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi. \qquad (\star)$$

[No credit will be given for using any other approach to derive $(\star)$. You may assume the expression derived in part (a) applies.]

(e) Consider the specific case

$$f(x) = 0; \ g(x) = \begin{cases} x & \text{for } |x| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $t > 1$, identify a region of the x-t plane including the line $x = 0$ where $u(x,t) = 0$. Briefly interpret this result physically. [*Hint: You may find it useful to consider the lines $x = 1 - t$ and $x = -1 + t$.*]

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx \ \textit{and} \ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\, dk.]$$

Paper 4, Section II

14A Methods

(a) Using Fourier transforms with respect to x, express in integral form the general solution $\theta(x,t)$ to the (unforced) heat equation with initial data $\Theta(x)$ and diffusivity $D > 0$:

$$\frac{\partial\theta}{\partial t} = D\frac{\partial^2\theta}{\partial x^2};\ \theta(x,0) = \Theta(x).$$

[You may quote the convolution theorem for Fourier transforms without proof.]

(b) By constructing an appropriate Green's function, express in integral form the general solution $\theta_f(x,t)$ to the forced heat equation with homogeneous initial data:

$$\frac{\partial\theta_f}{\partial t} - D\frac{\partial^2\theta_f}{\partial x^2} = f(x,t);\ \theta_f(x,0) = 0,$$

for some function $f(x,t)$.

(c) Now consider the combined problem:

$$\frac{\partial\theta_c}{\partial t} - D\frac{\partial^2\theta_c}{\partial x^2} = -A\delta\left(x + 2\sqrt{D}\right)\delta(t-1);\ \theta_c(x,0) = \delta\left(x - 2\sqrt{D}\right),$$

where A is a positive real constant. Determine $\theta_c(x,t)$, and hence deduce that $\theta_c(0,2) = 0$ if

$$A = \sqrt{\frac{e}{2}}.$$

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \ \text{ and } \ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,dk.$$

You may also quote the transform pair

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right);\ \tilde{g}(k,t) = e^{-Dk^2t},$$

as well as any relevant properties of the δ-function without proof.]

Paper 2, Section I
3B Methods

The function $u(x,y)$ satisfies

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = 0\,,$$

with boundary data $u(x,0) = f(x^2)$. Find and sketch the characteristic curves. Hence determine $u(x,y)$.

Paper 3, Section I
5A Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0,1,\ldots, \text{ for } -1 \leqslant x \leqslant 1.$$

Show that $Q_n(x) = P_n'(x)$ satisfies an equation which can be recast in self-adjoint form with eigenvalue $(n-1)(n+2)$. Write down the orthogonality relation for $Q_n(x)$, $Q_m(x)$ for $n \neq m$.

Paper 1, Section II
13B Methods

A uniform string of length l and mass per unit length μ is stretched horizontally under tension $T = \mu c^2$ and fixed at both ends. The string is subject to the gravitational force μg per unit length and a resistive force with value

$$-2k\mu \frac{\partial y}{\partial t}$$

per unit length, where $y(x,t)$ is the transverse, vertical displacement of the string and k is a positive constant.

(a) Derive the equation of motion of the string assuming that $y(x,t)$ remains small.

[In the remaining parts of the question you should assume that gravity is negligible.]

(b) Find $y(x,t)$ for $t > 0$, given that

$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = A \sin\left(\frac{\pi x}{l}\right) \qquad (\star)$$

with A constant, and $k = \pi c/l$.

(c) An extra transverse force

$$\alpha\mu \sin\left(\frac{3\pi x}{l}\right) \cos kt$$

per unit length is applied to the string, where α is a constant. With the initial conditions $(\star)$, find $y(x,t)$ for $t > 0$ and comment on the behaviour of the string as $t \to \infty$.

Compute the total energy E of the string as $t \to \infty$.

Paper 2, Section II
14A Methods

(a) Verify that $y = e^{-x}$ is a solution of the differential equation

$$(x+\lambda+1)y'' + (x+\lambda)y' - y = 0,$$

where λ is a constant. Find a second solution of the form $y = ax+b$.

(b) Let $\mathcal{L}$ be the operator

$$\mathcal{L}[y] = y'' + \frac{(x+\lambda)}{(x+\lambda+1)}y' - \frac{1}{(x+\lambda+1)}y$$

acting on functions $y(x)$ satisfying

$$y(0) = \lambda y'(0) \quad \text{and} \quad \lim_{x\to\infty} y(x) = 0. \qquad (\star)$$

The Green's function $G(x;\xi)$ for $\mathcal{L}$ satisfies

$$\mathcal{L}[G] = \delta(x-\xi),$$

with $\xi > 0$. Show that

$$G(x;\xi) = -\frac{(x+\lambda)}{(\xi+\lambda+1)}$$

for $0 \leqslant x < \xi$, and find $G(x;\xi)$ for $x > \xi$.

(c) Hence or otherwise find the solution when $\lambda = 2$ for the problem

$$\mathcal{L}[y] = -(x+3)e^{-x},$$

for $x \geqslant 0$ and $y(x)$ satisfying the boundary conditions given in $(\star)$.

Paper 3, Section II
14A Methods

(a) Prove Green's third identity for functions $u(\boldsymbol{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\boldsymbol{r}_0) = \int_S \left(u \frac{\partial G_{fs}}{\partial n} - \frac{\partial u}{\partial n} G_{fs} \right) dS,$$

where $G_{fs}(\boldsymbol{r}; \boldsymbol{r}_0) = -1/(4\pi|\boldsymbol{r} - \boldsymbol{r}_0|)$ is the free space Green's function.

(b) A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half-space $z > 0$ with boundary condition

$$\left. \frac{\partial u}{\partial z} \right|_{z=0} = p(x, y),$$

where both u and its spatial derivatives decay sufficiently rapidly as $|\boldsymbol{r}| \to \infty$.

(i) Explain why it is necessary to assume that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) dx\, dy = 0.$$

(ii) Using the method of images or otherwise, construct an appropriate Green's function $G(\boldsymbol{r}; \boldsymbol{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$.

(iii) Hence find the solution in the form

$$u(x_0, y_0, z_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) f(x - x_0, y - y_0, z_0) dx\, dy,$$

where f is to be determined.

(iv) Now let

$$p(x, y) = \begin{cases} \sin(x) & \text{for } |x|, |y| < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, determine the leading order term for u (proportional to z_0^{-3}) as $z_0 \to \infty$.

Paper 4, Section II
14B Methods

(a) Let $h(x) = m'(x)$. Express the Fourier transform $\tilde{h}(k)$ of $h(x)$ in terms of the Fourier transform $\tilde{m}(k)$ of $m(x)$, given that $m \to 0$ as $|x| \to \infty$. [You need to show an explicit calculation.]

(b) Calculate the inverse Fourier transform of

$$\tilde{m}(k) = -i\pi \, \mathrm{sgn}(k) e^{-\alpha|k|} \,,$$

with $\mathrm{Re}\,\alpha > 0$.

(c) The function $u(x, y)$ obeys Laplace's equation $\nabla^2 u = 0$ in the region defined by $-\infty < x < \infty$ and $0 < y < a$, with real positive a, where $u(x, 0) = f(x)$, $u(x, a) = g(x)$ and $u \to 0$ as $|x| \to \infty$.

(i) By performing a suitable Fourier transform of Laplace's equation, determine the ordinary differential equation satisfied by $\tilde{u}(k, y)$. Hence express $\tilde{u}(k, y)$ in terms of the Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$ of $f(x)$ and $g(x)$.

(ii) Find $\tilde{u}(k, y)$ for

$$f(x) = 0, \qquad g(x) = \frac{x}{x^2 + a^2} - \frac{x}{x^2 + 9a^2}.$$

Hence, determine $u(x, y)$.

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad \text{and} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \,.]$$

Paper 2, Section I

3C Methods

Consider the differential operator

$$\mathcal{L}y = \frac{d^2y}{dx^2} + 2\frac{dy}{dx}$$

acting on real functions $y(x)$ with $0 \leqslant x \leqslant 1$.

(i) Recast the eigenvalue equation $\mathcal{L}y = -\lambda y$ in Sturm-Liouville form $\tilde{\mathcal{L}}y = -\lambda w y$, identifying $\tilde{\mathcal{L}}$ and w.

(ii) If boundary conditions $y(0) = y(1) = 0$ are imposed, show that the eigenvalues form an infinite discrete set $\lambda_1 < \lambda_2 < \ldots$ and find the corresponding eigenfunctions $y_n(x)$ for $n = 1, 2, \ldots$. If $f(x) = x - x^2$ on $0 \leqslant x \leqslant 1$ is expanded in terms of your eigenfunctions i.e. $f(x) = \sum_{n=1}^{\infty} A_n\, y_n(x)$, give an expression for A_n. The expression can be given in terms of integrals that you need not evaluate.

Paper 3, Section I

5A Methods

Let $f(\theta)$ be a 2π-periodic function with Fourier expansion

$$f(\theta) \;=\; \frac{1}{2}a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \cos n\theta \;+\; b_n \sin n\theta \right) \;.$$

Find the Fourier coefficients a_n and b_n for

$$f(\theta) \;=\; \begin{cases} 1\,, & 0 < \theta < \pi \\ -1\,, & \pi < \theta < 2\pi\,. \end{cases}$$

Hence, or otherwise, find the Fourier coefficients A_n and B_n for the 2π-periodic function F defined by

$$F(\theta) \;=\; \begin{cases} \theta\,, & 0 < \theta < \pi \\ 2\pi - \theta\,, & \pi < \theta < 2\pi\,. \end{cases}$$

Use your answers to evaluate

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \qquad \text{and} \qquad \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2}\;.$$

Paper 1, Section II

13C Methods

(a) By introducing the variables $\xi = x + ct$ and $\eta = x - ct$ (where c is a constant), derive d'Alembert's solution of the initial value problem for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

where $-\infty < x < \infty$, $t \geqslant 0$ and ϕ and ψ are given functions (and subscripts denote partial derivatives).

(b) Consider the forced wave equation with homogeneous initial conditions:

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0$$

where $-\infty < x < \infty$, $t \geqslant 0$ and f is a given function. You may assume that the solution is given by

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds.$$

For the forced wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$, now in the half space $x \geqslant 0$ (and with $t \geqslant 0$ as before), find (in terms of f) the solution for $u(x,t)$ that satisfies the (inhomogeneous) initial conditions

$$u(x,0) = \sin x, \quad u_t(x,0) = 0, \qquad \text{for } x \geqslant 0$$

and the boundary condition $u(0,t) = 0$ for $t \geqslant 0$.

Paper 2, Section II
14A Methods

The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ and its inverse are given by

$$\tilde{f}(k) \quad = \quad \int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk.$$

(a) Calculate the Fourier transform of the function $f(x)$ defined by:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ -1 & \text{for } -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that the inverse Fourier transform of $\tilde{g}(k) = e^{-\lambda|k|}$, for λ a positive real constant, is given by

$$g(x) = \frac{\lambda}{\pi(x^2+\lambda^2)}.$$

(c) Consider the problem in the quarter plane $0 \leqslant x$, $0 \leqslant y$:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0; \\ u(x,0) &= \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases} \\ u(0,y) = \lim_{x\to\infty} u(x,y) = \lim_{y\to\infty} u(x,y) &= 0. \end{aligned}$$

Use the answers from parts (a) and (b) to show that

$$u(x,y) \quad = \quad \frac{4xy}{\pi}\int_0^1 \frac{v dv}{[(x-v)^2+y^2][(x+v)^2+y^2]}.$$

(d) Hence solve the problem in the quarter plane $0 \leqslant x$, $0 \leqslant y$:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0; \\ w(x,0) &= \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases} \\ w(0,y) &= \begin{cases} 1 & \text{for } 0 < y < 1, \\ 0 & \text{otherwise;} \end{cases} \\ \lim_{x\to\infty} w(x,y) = \lim_{y\to\infty} w(x,y) &= 0. \end{aligned}$$

[*You may quote without proof any property of Fourier transforms.*]

Paper 3, Section II

14A Methods

Let $P(x)$ be a solution of Legendre's equation with eigenvalue λ,

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0\,,$$

such that P and its derivatives $P^{(k)}(x) = d^kP/dx^k$, $k = 0, 1, 2, \ldots$, are regular at all points x with $-1 \leqslant x \leqslant 1$.

(a) Show by induction that

$$(1-x^2)\frac{d^2}{dx^2}\left[P^{(k)}\right] - 2(k+1)x\frac{d}{dx}\left[P^{(k)}\right] + \lambda_k P^{(k)} = 0$$

for some constant λ_k. Find λ_k explicitly and show that its value is negative when k is sufficiently large, for a fixed value of λ.

(b) Write the equation for $P^{(k)}(x)$ in part (a) in self-adjoint form. Hence deduce that if $P^{(k)}(x)$ is not identically zero, then $\lambda_k \geqslant 0$.

[*Hint: Establish a relation between integrals of the form* $\int_{-1}^{1}[P^{(k+1)}(x)]^2 f(x)\,dx$ *and* $\int_{-1}^{1}[P^{(k)}(x)]^2 g(x)\,dx$ *for certain functions* $f(x)$ *and* $g(x)$*.*]

(c) Use the results of parts (a) and (b) to show that if $P(x)$ is a non-zero, regular solution of Legendre's equation on $-1 \leqslant x \leqslant 1$, then $P(x)$ is a polynomial of degree n and $\lambda = n(n+1)$ for some integer $n = 0, 1, 2, \ldots$.

Paper 4, Section II
14C Methods

The function $\theta(x,t)$ obeys the diffusion equation

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}\,. \qquad (*)$$

Verify that

$$\theta(x,t) = \frac{1}{\sqrt{t}}\, e^{-x^2/4Dt}$$

is a solution of $(*)$, and by considering $\int_{-\infty}^{\infty} \theta(x,t)\,dx$, find the solution having the initial form $\theta(x,0) = \delta(x)$ at $t=0$.

Find, in terms of the error function, the solution of $(*)$ having the initial form

$$\theta(x,0) = \begin{cases} 1\,, & |x| \leqslant 1\,, \\ 0\,, & |x| > 1\,. \end{cases}$$

Sketch a graph of this solution at various times $t \geqslant 0$.

[*The error function is*

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy\,.]$$

Paper 2, Section I

4B Methods

Find the Fourier transform of the function

$$f(x) = \begin{cases} A\,, & |x| \leqslant 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

Determine the convolution of the function $f(x)$ with itself.

State the convolution theorem for Fourier transforms. Using it, or otherwise, determine the Fourier transform of the function

$$g(x) = \begin{cases} B(2-|x|)\,, & |x| \leqslant 2 \\ 0\,, & |x| > 2\,. \end{cases}$$

Paper 1, Section II

14B Methods

Consider the equation

$$\nabla^2 \phi = \delta(x) g(y) \qquad (*)$$

on the two-dimensional strip $-\infty < x < \infty$, $0 \leqslant y \leqslant a$, where $\delta(x)$ is the delta function and $g(y)$ is a smooth function satisfying $g(0) = g(a) = 0$. $\phi(x,y)$ satisfies the boundary conditions $\phi(x,0) = \phi(x,a) = 0$ and $\lim_{x\to\pm\infty} \phi(x,y) = 0$. By using solutions of Laplace's equation for $x < 0$ and $x > 0$, matched suitably at $x = 0$, find the solution of $(*)$ in terms of Fourier coefficients of $g(y)$.

Find the solution of $(*)$ in the limiting case $g(y) = \delta(y-c)$, where $0 < c < a$, and hence determine the Green's function $\phi(x,y)$ in the strip, satisfying

$$\nabla^2 \phi = \delta(x-b)\delta(y-c)$$

and the same boundary conditions as before.

Paper 2, Section II

13A Methods

(i) The solution to the equation

$$\frac{d}{dx}\Big(x\frac{dF}{dx}\Big) \;+\; \alpha^2 xF \;=\; 0$$

that is regular at the origin is $F(x) = CJ_0(\alpha x)$, where α is a real, positive parameter, J_0 is a Bessel function, and C is an arbitrary constant. The Bessel function has infinitely many zeros: $J_0(\gamma_k) = 0$ with $\gamma_k > 0$, for $k = 1, 2, \ldots$. Show that

$$\int_0^1 J_0(\alpha x)\, J_0(\beta x)\, x\, dx \;=\; \frac{\beta J_0(\alpha) J_0'(\beta) - \alpha J_0(\beta) J_0'(\alpha)}{\alpha^2 - \beta^2}\;, \quad \alpha \neq \beta\,,$$

(where α and β are real and positive) and deduce that

$$\int_0^1 J_0(\gamma_k x)\, J_0(\gamma_\ell x)\, x\, dx \;=\; 0\,, \quad k \neq \ell\,; \qquad \int_0^1 (J_0(\gamma_k x))^2\, x\, dx \;=\; \frac{1}{2}(J_0'(\gamma_k))^2\,.$$

[*Hint: For the second identity, consider* $\alpha = \gamma_k$ *and* $\beta = \gamma_k + \epsilon$ *with* ϵ *small.*]

(ii) The displacement $z(r,t)$ of the membrane of a circular drum of unit radius obeys

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial z}{\partial r}\Big) \;=\; \frac{\partial^2 z}{\partial t^2}\;, \qquad z(1,t) = 0\,,$$

where r is the radial coordinate on the membrane surface, t is time (in certain units), and the displacement is assumed to have no angular dependence. At $t = 0$ the drum is struck, so that

$$z(r,0) = 0\;, \qquad \frac{\partial z}{\partial t}(r,0) \;=\; \begin{cases} U\,, & r < b \\ 0\,, & r > b \end{cases}$$

where U and $b < 1$ are constants. Show that the subsequent motion is given by

$$z(r,t) \;=\; \sum_{k=1}^{\infty} C_k\, J_0(\gamma_k r)\, \sin(\gamma_k t) \qquad \text{where} \qquad C_k = -2bU\frac{J_0'(\gamma_k b)}{\gamma_k^2 (J_0'(\gamma_k))^2}\;.$$

Paper 2, Section I

5B Methods

Let r, θ, ϕ be spherical polar coordinates, and let P_n denote the nth Legendre polynomial. Write down the most general solution for $r > 0$ of Laplace's equation $\nabla^2 \Phi = 0$ that takes the form $\Phi(r, \theta, \phi) = f(r) P_n(\cos\theta)$.

Solve Laplace's equation in the spherical shell $1 \leqslant r \leqslant 2$ subject to the boundary conditions

$$\begin{aligned} \Phi &= 3\cos 2\theta \quad \text{at} \quad r = 1\,, \\ \Phi &= 0 \quad\quad\quad\;\; \text{at} \quad r = 2\,. \end{aligned}$$

[*The first three Legendre polynomials are*

$$P_0(x) = 1, \quad P_1(x) = x \quad \textit{and} \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}\,.]$$

Paper 4, Section I

5D Methods

Let

$$g_\epsilon(x) = \frac{-2\epsilon x}{\pi(\epsilon^2 + x^2)^2}\,.$$

By considering the integral $\int_{-\infty}^{\infty} \phi(x)\, g_\epsilon(x)\, dx$, where ϕ is a smooth, bounded function that vanishes sufficiently rapidly as $|x| \to \infty$, identify $\lim_{\epsilon \to 0} g_\epsilon(x)$ in terms of a generalized function.

Paper 3, Section I

7D Methods

Define the *discrete Fourier transform* of a sequence $\{x_0, x_1, \ldots, x_{N-1}\}$ of N complex numbers.

Compute the discrete Fourier transform of the sequence

$$x_n = \frac{1}{N}\,(1 + e^{2\pi i n/N})^{N-1} \qquad \text{for} \;\; n = 0, \ldots, N-1\,.$$

Paper 1, Section II

14B Methods

The Bessel functions $J_n(r)$ $(n \geqslant 0)$ can be defined by the expansion

$$e^{ir\cos\theta} = J_0(r) + 2\sum_{n=1}^{\infty} i^n J_n(r)\cos n\theta\,. \qquad (*)$$

By using Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$, or otherwise, show that

$$(\nabla^2 + 1)e^{ir\cos\theta} = 0\,.$$

Deduce that $J_n(r)$ satisfies Bessel's equation

$$\left(r^2\frac{d^2}{dr^2} + r\frac{d}{dr} - (n^2 - r^2)\right) J_n(r) = 0\,.$$

By expanding the left-hand side of $(*)$ up to cubic order in r, derive the series expansions of $J_0(r)$, $J_1(r)$, $J_2(r)$ and $J_3(r)$ up to this order.

Paper 3, Section II

15D Methods

By differentiating the expression $\psi(t) = H(t)\,\sin(\alpha t)/\alpha$, where α is a constant and $H(t)$ is the Heaviside step function, show that

$$\frac{d^2\psi}{dt^2} + \alpha^2\psi = \delta(t)\,,$$

where $\delta(t)$ is the Dirac δ-function.

Hence, by taking a Fourier transform with respect to the spatial variables only, derive the retarded Green's function for the wave operator $\partial_t^2 - c^2\nabla^2$ in three spatial dimensions.

[*You may use that*

$$\frac{1}{2\pi}\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\sin(kct)}{kc}\,d^3k = -\frac{i}{c|\mathbf{x}-\mathbf{y}|}\int_{-\infty}^{\infty} e^{ik|\mathbf{x}-\mathbf{y}|}\,\sin(kct)\,dk$$

without proof.]

Thus show that the solution to the homogeneous wave equation $\partial_t^2 u - c^2\nabla^2 u = 0$, subject to the initial conditions $u(\mathbf{x},0) = 0$ and $\partial_t u(\mathbf{x},0) = f(\mathbf{x})$, may be expressed as

$$u(\mathbf{x},t) = \langle f\rangle\,t\,,$$

where $\langle f\rangle$ is the average value of f on a sphere of radius ct centred on $\mathbf{x}$. Interpret this result.

Paper 2, Section II

16D Methods

For $n = 0, 1, 2, \ldots$, the degree n polynomial $C_n^\alpha(x)$ satisfies the differential equation

$$(1 - x^2)y'' - (2\alpha + 1)xy' + n(n + 2\alpha)y = 0$$

where α is a real, positive parameter. Show that, when $m \neq n$,

$$\int_a^b C_m^\alpha(x)\, C_n^\alpha(x)\, w(x)\, dx = 0$$

for a weight function $w(x)$ and values $a < b$ that you should determine.

Suppose that the roots of $C_n^\alpha(x)$ that lie inside the domain (a, b) are $\{x_1, x_2, \ldots, x_k\}$, with $k \leqslant n$. By considering the integral

$$\int_a^b C_n^\alpha(x) \prod_{i=1}^{k} (x - x_i)\, w(x)\, dx\,,$$

show that in fact all n roots of $C_n^\alpha(x)$ lie in (a, b).

Paper 4, Section II
17B Methods

(a) Show that the operator

$$\frac{d^4}{dx^4} + p\frac{d^2}{dx^2} + q\frac{d}{dx} + r\,,$$

where $p(x)$, $q(x)$ and $r(x)$ are real functions, is self-adjoint (for suitable boundary conditions which you need not state) if and only if

$$q = \frac{dp}{dx}\,.$$

(b) Consider the eigenvalue problem

$$\frac{d^4y}{dx^4} + p\frac{d^2y}{dx^2} + \frac{dp}{dx}\frac{dy}{dx} = \lambda y \qquad (*)$$

on the interval $[a,b]$ with boundary conditions

$$y(a) = \frac{dy}{dx}(a) = y(b) = \frac{dy}{dx}(b) = 0\,.$$

Assuming that $p(x)$ is everywhere negative, show that all eigenvalues λ are positive.

(c) Assume now that $p \equiv 0$ and that the eigenvalue problem $(*)$ is on the interval $[-c,c]$ with $c>0$. Show that $\lambda = 1$ is an eigenvalue provided that

$$\cos c \sinh c \pm \sin c \cosh c = 0$$

and show graphically that this condition has just one solution in the range $0 < c < \pi$.

[*You may assume that all eigenfunctions are either symmetric or antisymmetric about* $x = 0$.]

Paper 2, Section I

5C Methods

Show that

$$a(x,y)\left(\frac{dy}{ds}\right)^2 - 2b(x,y)\frac{dx}{ds}\frac{dy}{ds} + c(x,y)\left(\frac{dx}{ds}\right)^2 = 0$$

along a characteristic curve $(x(s), y(s))$ of the 2^{nd}-order pde

$$a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy} = f(x,y)\,.$$

Paper 4, Section I

5A Methods

By using separation of variables, solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 1, \quad 0 < y < 1,$$

subject to

$$\begin{aligned} u(0,y) &= 0 && 0 \leqslant y \leqslant 1, \\ u(1,y) &= 0 && 0 \leqslant y \leqslant 1, \\ u(x,0) &= 0 && 0 \leqslant x \leqslant 1, \\ u(x,1) &= 2\sin(3\pi x) && 0 \leqslant x \leqslant 1. \end{aligned}$$

Paper 3, Section I

7A Methods

(a) Determine the Green's function $G(x;\xi)$ satisfying

$$G'' - 4G' + 4G = \delta(x-\xi),$$

with $G(0;\xi) = G(1;\xi) = 0$. Here $'$ denotes differentiation with respect to x.

(b) Using the Green's function, solve

$$y'' - 4y' + 4y = e^{2x},$$

with $y(0) = y(1) = 0$.

Paper 1, Section II

14C Methods

Define the *convolution* $f*g$ of two functions f and g. Defining the *Fourier transform* $\tilde{f}$ of f by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx\,,$$

show that

$$\widetilde{f*g}\,(k) = \tilde{f}(k)\,\tilde{g}(k)\,.$$

Given that the Fourier transform of $f(x) = 1/x$ is

$$\tilde{f}(k) = -i\pi\,\mathrm{sgn}(k)\,,$$

find the Fourier transform of $\sin(x)/x^2$.

Paper 3, Section II

15A Methods

Consider the Dirac delta function, $\delta(x)$, defined by the sampling property

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0),$$

for any suitable function $f(x)$ and real constant x_0.

(a) Show that $\delta(\alpha x) = |\alpha|^{-1}\delta(x)$ for any non-zero $\alpha \in \mathbb{R}$.

(b) Show that $x\delta'(x) = -\delta(x)$, where $'$ denotes differentiation with respect to x.

(c) Calculate

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(m)}(x)\,dx,$$

where $\delta^{(m)}(x)$ is the m^{th} derivative of the delta function.

(d) For

$$\gamma_n(x) = \frac{1}{\pi}\frac{n}{(nx)^2+1},$$

show that $\lim_{n\to\infty} \gamma_n(x) = \delta(x)$.

(e) Find expressions in terms of the delta function and its derivatives for

(i)

$$\lim_{n\to\infty} n^3 x\, e^{-x^2 n^2}.$$

(ii)

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^n \cos(kx)\,dk.$$

(f) Hence deduce that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}\,dk = \delta(x).$$

[*You may assume that*

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi} \quad \textit{and} \quad \int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy = \pi.]$$

Paper 2, Section II
16A Methods

(a) Let $f(x)$ be a 2π-periodic function (i.e. $f(x) = f(x+2\pi)$ for all x) defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} x & x \in [0, \pi] \\ -x & x \in [-\pi, 0] \end{cases}$$

Find the Fourier series of $f(x)$ in the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

(b) Find the general solution to

$$y'' + 2y' + y = f(x)$$

where $f(x)$ is as given in part (a) and $y(x)$ is 2π-periodic.

Paper 4, Section II

17C Methods

Let Ω be a bounded region in the plane, with smooth boundary $\partial\Omega$. Green's second identity states that for any smooth functions u, v on Ω

$$\int_\Omega \left(u\,\nabla^2 v - v\,\nabla^2 u\right)\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} u\,(\mathbf{n}\cdot\boldsymbol{\nabla} v) - v\,(\mathbf{n}\cdot\boldsymbol{\nabla} u)\;\mathrm{d}s\,,$$

where $\mathbf{n}$ is the outward pointing normal to $\partial\Omega$. Using this identity with v replaced by

$$G_0(\mathbf{x};\mathbf{x}_0) = \frac{1}{2\pi}\ln\left(\|\mathbf{x}-\mathbf{x}_0\|\right) = \frac{1}{4\pi}\ln\left((x-x_0)^2+(y-y_0)^2\right)$$

and taking care of the singular point $(x,y)=(x_0,y_0)$, show that if u solves the Poisson equation $\nabla^2 u = -\rho$ then

$$\begin{aligned} u(\mathbf{x}) = &-\int_\Omega G_0(\mathbf{x};\mathbf{x}_0)\,\rho(\mathbf{x}_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 \\ &+\oint_{\partial\Omega}\Big(u(\mathbf{x}_0)\,\mathbf{n}\cdot\boldsymbol{\nabla}G_0(\mathbf{x};\mathbf{x}_0) - G_0(\mathbf{x};\mathbf{x}_0)\,\mathbf{n}\cdot\boldsymbol{\nabla}u(\mathbf{x}_0)\Big)\,\mathrm{d}s \end{aligned}$$

at any $\mathbf{x}=(x,y)\in\Omega$, where all derivatives are taken with respect to $\mathbf{x}_0=(x_0,y_0)$.

In the case that Ω is the unit disc $\|\mathbf{x}\|\leqslant 1$, use the method of images to show that the solution to Laplace's equation $\nabla^2 u = 0$ inside Ω, subject to the boundary condition

$$u(1,\theta) = \delta(\theta-\alpha),$$

is

$$u(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)}\,,$$

where (r,θ) are polar coordinates in the disc and α is a constant.

[*Hint: The image of a point* $\mathbf{x}_0\in\Omega$ *is the point* $\mathbf{y}_0 = \mathbf{x}_0/\|\mathbf{x}_0\|^2$*, and then*

$$\|\mathbf{x}-\mathbf{x}_0\| = \|\mathbf{x}_0\|\;\|\mathbf{x}-\mathbf{y}_0\|$$

for all $\mathbf{x}\in\partial\Omega$.]

Paper 2, Section I

5B Methods

Expand $f(x) = x$ as a Fourier series on $-\pi < x < \pi$.

By integrating the series show that x^2 on $-\pi < x < \pi$ can be written as

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ ,$$

where a_n, $n = 1, 2, \ldots$, should be determined and

$$a_0 = 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

By evaluating a_0 another way show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \, .$$

Paper 4, Section I

5A Methods

The Legendre polynomials, $P_n(x)$ for integers $n \geqslant 0$, satisfy the Sturm–Liouville equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

and the recursion formula

$$(n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x), \qquad P_0(x) = 1, \quad P_1(x) = x.$$

(i) For all $n \geqslant 0$, show that $P_n(x)$ is a polynomial of degree n with $P_n(1) = 1$.

(ii) For all $m, n \geqslant 0$, show that $P_n(x)$ and $P_m(x)$ are orthogonal over the range $x \in [-1, 1]$ when $m \neq n$.

(iii) For each $n \geqslant 0$ let

$$R_n(x) = \frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right].$$

Assume that for each n there is a constant α_n such that $P_n(x) = \alpha_n R_n(x)$ for all x. Determine α_n for each n.

Paper 3, Section I

7A Methods

Using the substitution $u(x,y) = v(x,y)e^{-x^2}$, find $u(x,y)$ that satisfies

$$u_x + x\,u_y + 2\,x\,u = e^{-x^2}$$

with boundary data $u(0,y) = y\,e^{-y^2}$.

Paper 1, Section II

14B Methods

(a)

(i) Compute the Fourier transform $\tilde{h}(k)$ of $h(x) = e^{-a|x|}$, where a is a real positive constant.

(ii) Consider the boundary value problem

$$-\frac{d^2u}{dx^2} + \omega^2 u = e^{-|x|} \quad \text{on } -\infty < x < \infty$$

with real constant $\omega \neq \pm 1$ and boundary condition $u(x) \to 0$ as $|x| \to \infty$.
Find the Fourier transform $\tilde{u}(k)$ of $u(x)$ and hence solve the boundary value problem. You should clearly state any properties of the Fourier transform that you use.

(b) Consider the wave equation

$$v_{tt} = v_{xx} \quad \text{on} \quad -\infty < x < \infty \text{ and } t > 0$$

with initial conditions

$$v(x,0) = f(x) \qquad v_t(x,0) = g(x).$$

Show that the Fourier transform $\tilde{v}(k,t)$ of the solution $v(x,t)$ with respect to the variable x is given by

$$\tilde{v}(k,t) = \tilde{f}(k)\cos kt + \frac{\tilde{g}(k)}{k}\sin kt$$

where $\tilde{f}(k)$ and $\tilde{g}(k)$ are the Fourier transforms of the initial conditions.
Starting from $\tilde{v}(k,t)$ derive d'Alembert's solution for the wave equation:

$$v(x,t) = \frac{1}{2}\Big(f(x-t) + f(x+t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi\,.$$

You should state clearly any properties of the Fourier transform that you use.

Paper 3, Section II

15A Methods

Let $\mathcal{L}$ be the linear differential operator

$$\mathcal{L}\, y \;= y''' - y'' - 2y'$$

where $'$ denotes differentiation with respect to x.

Find the Green's function, $G(x;\xi)$, for $\mathcal{L}$ satisfying the homogeneous boundary conditions $G(0;\xi) = 0$, $G'(0;\xi) = 0$, $G''(0;\xi) = 0$.

Using the Green's function, solve

$$\mathcal{L}y = e^{x}\Theta(x-1)$$

with boundary conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$. Here $\Theta(x)$ is the Heaviside step function having value 0 for $x < 0$ and 1 for $x > 0$.

Paper 2, Section II

16A Methods

Laplace's equation for ϕ in cylindrical coordinates (r, θ, z), is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Use separation of variables to find an expression for the general solution to Laplace's equation in cylindrical coordinates that is 2π-periodic in θ.

Find the bounded solution $\phi(r,\theta,z)$ that satisfies

$$\nabla^2\phi = 0 \qquad z \geqslant 0, \quad 0 \leqslant r \leqslant 1,$$
$$\phi(1,\theta,z) = e^{-4z}(\cos\theta + \sin 2\theta) + 2\,e^{-z}\,\sin 2\theta.$$

Paper 4, Section II
17B Methods
(a)

(i) For the diffusion equation

$$\frac{\partial\phi}{\partial t} - K\frac{\partial^2\phi}{\partial x^2} = 0 \qquad \text{on } -\infty < x < \infty \text{ and } t \geqslant 0,$$

with diffusion constant K, state the properties (in terms of the Dirac delta function) that define the *fundamental solution* $F(x,t)$ and the *Green's function* $G(x,t;y,\tau)$.
You are not required to give expressions for these functions.

(ii) Consider the initial value problem for the homogeneous equation:

$$\frac{\partial\phi}{\partial t} - K\frac{\partial^2\phi}{\partial x^2} = 0, \quad \phi(x,t_0) = \alpha(x) \quad \text{on } -\infty < x < \infty \text{ and } t \geqslant t_0, \qquad \text{(A)}$$

and the forced equation with homogeneous initial condition (and given forcing term $h(x,t)$):

$$\frac{\partial\phi}{\partial t} - K\frac{\partial^2\phi}{\partial x^2} = h(x,t), \quad \phi(x,0) = 0 \quad \text{on } -\infty < x < \infty \text{ and } t \geqslant 0. \qquad \text{(B)}$$

Given that F and G in part (i) are related by

$$G(x,t;y,\tau) = \Theta(t-\tau)F(x-y,t-\tau)$$

(where $\Theta(t)$ is the Heaviside step function having value 0 for $t<0$ and 1 for $t>0$), show how the solution of (B) can be expressed in terms of solutions of (A) with suitable initial conditions. Briefly interpret your expression.

(b) A semi-infinite conducting plate lies in the (x_1,x_2) plane in the region $x_1 \geqslant 0$. The boundary along the x_2 axis is perfectly insulated. Let (r,θ) denote standard polar co-ordinates on the plane. At time $t=0$ the entire plate is at temperature zero except for the region defined by $-\pi/4 < \theta < \pi/4$ and $1 < r < 2$ which has constant initial temperature $T_0 > 0$. Subsequently the temperature of the plate obeys the two-dimensional heat equation with diffusion constant K. Given that the fundamental solution of the two-dimensional heat equation on $\mathbb{R}^2$ is

$$F(x_1,x_2,t) = \frac{1}{4\pi Kt}e^{-(x_1^2+x_2^2)/(4Kt)},$$

show that the origin $(0,0)$ on the plate reaches its maximum temperature at time $t = 3/(8K\log 2)$.

Paper 2, Section I

5A Methods

Use the method of characteristics to find $u(x,y)$ in the first quadrant $x \geqslant 0$, $y \geqslant 0$, where $u(x,y)$ satisfies

$$\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} = \cos x,$$

with boundary data $u(x,0) = \cos x$.

Paper 4, Section I

5A Methods

Consider the function $f(x)$ defined by

$$f(x) = x^2, \quad \text{for} \quad -\pi < x < \pi.$$

Calculate the Fourier series representation for the 2π-periodic extension of this function. Hence establish that

$$\frac{\pi^2}{6} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and that

$$\frac{\pi^2}{12} \quad = \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Paper 3, Section I

7A Methods

Calculate the Green's function $G(x;\xi)$ given by the solution to

$$\frac{d^2G}{dx^2} = \delta(x-\xi); \quad G(0;\xi) = \frac{dG}{dx}(1;\xi) = 0,$$

where $\xi \in (0,1)$, $x \in (0,1)$ and $\delta(x)$ is the Dirac δ-function. Use this Green's function to calculate an explicit solution $y(x)$ to the boundary value problem

$$\frac{d^2y}{dx^2} = xe^{-x},$$

where $x \in (0,1)$, and $y(0) = y'(1) = 0$.

Paper 1, Section II

14A Methods

(a) Consider the general self-adjoint problem for $y(x)$ on $[a, b]$:

$$-\frac{d}{dx}\left[p(x)\frac{d}{dx}y\right] + q(x)y = \lambda w(x)y; \quad y(a) = y(b) = 0,$$

where λ is the eigenvalue, and $w(x) > 0$. Prove that eigenfunctions associated with distinct eigenvalues are orthogonal with respect to a particular inner product which you should define carefully.

(b) Consider the problem for $y(x)$ given by

$$xy'' + 3y' + \left(\frac{1+\lambda}{x}\right) y = 0; \quad y(1) = y(e) = 0.$$

(i) Recast this problem into self-adjoint form.

(ii) Calculate the complete set of eigenfunctions and associated eigenvalues for this problem. [*Hint: You may find it useful to make the substitution* $x = e^s$.]

(iii) Verify that the eigenfunctions associated with distinct eigenvalues are indeed orthogonal.

Paper 3, Section II

15B Methods

(a) Show that the Fourier transform of $f(x) = e^{-a^2x^2}$, for $a > 0$, is

$$\tilde{f}(k) = \frac{\sqrt{\pi}}{a}\, e^{-\frac{k^2}{4a^2}}\,,$$

stating clearly any properties of the Fourier transform that you use.
[*Hint: You may assume that* $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$.]

(b) Consider now the Cauchy problem for the diffusion equation in one space dimension, i.e. solving for $\theta(x,t)$ satisfying:

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2} \quad \text{with } \theta(x,0) = g(x),$$

where D is a positive constant and $g(x)$ is specified. Consider the following property of a solution:
Property P: If the initial data $g(x)$ is positive and it is non-zero only within a bounded region (i.e. there is a constant α such that $\theta(x,0) = 0$ for all $|x| > \alpha$), then for any $\epsilon > 0$ (however small) and β (however large) the solution $\theta(\beta,\epsilon)$ can be non-zero, i.e. the solution can become non-zero arbitrarily far away after an arbitrarily short time.

Does Property P hold for solutions of the diffusion equation? Justify your answer (deriving any expression for the solution $\theta(x,t)$ that you use).

(c) Consider now the wave equation in one space dimension:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

with given initial data $u(x,0) = \phi(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ (and c is a constant).
Does Property P (with $g(x)$ and $\theta(\beta,\epsilon)$ now replaced by $\phi(x)$ and $u(\beta,\epsilon)$ respectively) hold for solutions of the wave equation? Justify your answer again as above.

Paper 2, Section II

16A Methods

Consider a bar of length π with free ends, subject to longitudinal vibrations. You may assume that the longitudinal displacement $y(x,t)$ of the bar satisfies the wave equation with some wave speed c:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

for $x \in (0,\pi)$ and $t > 0$ with boundary conditions:

$$\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(\pi,t) = 0,$$

for $t > 0$. The bar is initially at rest so that

$$\frac{\partial y}{\partial t}(x,0) = 0$$

for $x \in (0,\pi)$, with a spatially varying initial longitudinal displacement given by

$$y(x,0) = bx$$

for $x \in (0,\pi)$, where b is a real constant.

(a) Using separation of variables, show that

$$y(x,t) = \frac{b\pi}{2} - \frac{4b}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]\cos[(2n-1)ct]}{(2n-1)^2}.$$

(b) Determine a periodic function $P(x)$ such that this solution may be expressed as

$$y(x,t) = \frac{1}{2}[P(x+ct) + P(x-ct)].$$

Paper 4, Section II

17B Methods

Let $\mathcal{D}$ be a 2-dimensional region in $\mathbb{R}^2$ with boundary $\partial\mathcal{D}$.
In this question you may assume Green's second identity:

$$\int_{\mathcal{D}} (f\,\nabla^2 g - g\,\nabla^2 f)\,dA = \int_{\partial\mathcal{D}} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dl,$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\mathcal{D}$, and f and g are suitably regular functions that include the free space Green's function in two dimensions. You may also assume that the free space Green's function for the Laplace equation in two dimensions is given by

$$G_0(\boldsymbol{r}, \boldsymbol{r_0}) = \frac{1}{2\pi}\log|\boldsymbol{r} - \boldsymbol{r_0}|.$$

(a) State the conditions required on a function $G(\boldsymbol{r}, \boldsymbol{r_0})$ for it to be a Dirichlet Green's function for the Laplace operator on $\mathcal{D}$. Suppose that $\nabla^2\psi = 0$ on $\mathcal{D}$. Show that if G is a Dirichlet Green's function for $\mathcal{D}$ then

$$\psi(\boldsymbol{r_0}) = \int_{\partial\mathcal{D}} \psi(\boldsymbol{r})\frac{\partial}{\partial n}G(\boldsymbol{r}, \boldsymbol{r_0})\,dl \qquad \text{for } \boldsymbol{r_0} \in \mathcal{D}.$$

(b) Consider the Laplace equation $\nabla^2\phi = 0$ in the quarter space

$$\mathcal{D} = \{(x, y) : x \geqslant 0 \text{ and } y \geqslant 0\},$$

with boundary conditions

$$\phi(x, 0) = e^{-x^2},\ \ \phi(0, y) = e^{-y^2} \text{ and } \phi(x, y) \to 0 \text{ as } \sqrt{x^2 + y^2} \to \infty.$$

Using the method of images, show that the solution is given by

$$\phi(x_0, y_0) = F(x_0, y_0) + F(y_0, x_0),$$

where

$$F(x_0, y_0) = \frac{4x_0y_0}{\pi}\int_0^\infty \frac{t\,e^{-t^2}}{\left[(t - x_0)^2 + y_0^2\right]\left[(t + x_0)^2 + y_0^2\right]}\,dt.$$

Paper 4, Section I

5C Methods

(a) The convolution $f*g$ of two functions $f,\ g:\mathbb{R}\to\mathbb{C}$ is related to their Fourier transforms $\tilde{f},\ \tilde{g}$ by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}(k)e^{ikx}\,dk=\int_{-\infty}^{\infty}f(u)g(x-u)\,du\,.$$

Derive Parseval's theorem for Fourier transforms from this relation.

(b) Let $a>0$ and

$$f(x)=\begin{cases}\cos x & \text{for}\quad x\in[-a,a]\\ 0 & \text{elsewhere.}\end{cases}$$

(i) Calculate the Fourier transform $\tilde{f}(k)$ of $f(x)$.

(ii) Determine how the behaviour of $\tilde{f}(k)$ in the limit $|k|\to\infty$ depends on the value of a. Briefly interpret the result.

Paper 2, Section I

5C Methods

(i) Write down the trigonometric form for the Fourier series and its coefficients for a function $f:[-L,L)\to\mathbb{R}$ extended to a $2L$-periodic function on $\mathbb{R}$.

(ii) Calculate the Fourier series on $[-\pi,\pi)$ of the function $f(x)=\sin(\lambda x)$ where λ is a real constant. Take the limit $\lambda\to k$ with $k\in\mathbb{Z}$ in the coefficients of this series and briefly interpret the resulting expression.

Paper 3, Section I

7C Methods

(a) From the defining property of the δ function,

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)\,,$$

for any function f, prove that

(i) $\delta(-x) = \delta(x)$,

(ii) $\delta(ax) = |a|^{-1}\delta(x)$ for $a \in \mathbb{R}$, $a \neq 0$,

(iii) If $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto g(x)$ is smooth and has isolated zeros x_i where the derivative $g'(x_i) \neq 0$, then

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}\,.$$

(b) Show that the function $\gamma(x)$ defined by

$$\gamma(x) = \lim_{s \to 0} \frac{e^{x/s}}{s\left(1 + e^{x/s}\right)^2}\,,$$

is the $\delta(x)$ function.

Paper 1, Section II
14C Methods

(i) Briefly describe the Sturm–Liouville form of an eigenfunction equation for real valued functions with a linear, second-order ordinary differential operator. Briefly summarize the properties of the solutions.

(ii) Derive the condition for self-adjointness of the differential operator in (i) in terms of the boundary conditions of solutions y_1, y_2 to the Sturm–Liouville equation. Give at least three types of boundary conditions for which the condition for self-adjointness is satisfied.

(iii) Consider the inhomogeneous Sturm–Liouville equation with weighted linear term

$$\frac{1}{w(x)}\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)-\frac{q(x)}{w(x)}y-\lambda y=f(x)\,,$$

on the interval $a\leqslant x\leqslant b$, where p and q are real functions on $[a,b]$ and w is the weighting function. Let $G(x,\xi)$ be a Green's function satisfying

$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right)-q(x)\,G(x,\xi)=\delta(x-\xi)\,.$$

Let solutions y and the Green's function G satisfy the same boundary conditions of the form $\alpha y'+\beta y=0$ at $x=a$, $\mu y'+\nu y=0$ at $x=b$ (α, β are not both zero and μ, ν are not both zero) and likewise for G for the same constants α, β, μ and ν. Show that the Sturm–Liouville equation can be written as a so-called *Fredholm* integral equation of the form

$$\psi(\xi)=U(\xi)+\lambda\int_a^b K(x,\xi)\psi(x)dx\,,$$

where $K(x,\xi)=\sqrt{w(\xi)\,w(x)}G(x,\xi)$, $\psi=\sqrt{w}y$ and U depends on K, w and the forcing term f. Write down U in terms of an integral involving f, K and w.

(iv) Derive the Fredholm integral equation for the Sturm–Liouville equation on the interval $[0,1]$

$$\frac{d^2y}{dx^2}-\lambda y=0\,,$$

with $y(0)=y(1)=0$.

Paper 3, Section II

15C Methods

(i) Consider the Poisson equation $\nabla^2\psi(\mathbf{r}) = f(\mathbf{r})$ with forcing term f on the infinite domain $\mathbb{R}^3$ with $\lim_{|\mathbf{r}|\to\infty}\psi = 0$. Derive the Green's function $G(\mathbf{r},\mathbf{r}') = -1/(4\pi|\mathbf{r}-\mathbf{r}'|)$ for this equation using the divergence theorem. [You may assume without proof that the divergence theorem is valid for the Green's function.]

(ii) Consider the *Helmholtz equation*

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = f(\mathbf{r})\,, \qquad (\dagger)$$

where k is a real constant. A Green's function $g(\mathbf{r},\mathbf{r}')$ for this equation can be constructed from $G(\mathbf{r},\mathbf{r}')$ of (i) by assuming $g(\mathbf{r},\mathbf{r}') = U(r)G(\mathbf{r},\mathbf{r}')$ where $r = |\mathbf{r}-\mathbf{r}'|$ and $U(r)$ is a regular function. Show that $\lim_{r\to 0} U(r) = 1$ and that U satisfies the equation

$$\frac{d^2U}{dr^2} + k^2U(r) = 0\,. \qquad (\ddagger)$$

(iii) Take the Green's function with the specific solution $U(r) = e^{ikr}$ to Eq. ($\ddagger$) and consider the Helmholtz equation ($\dagger$) on the semi-infinite domain $z > 0$, $x, y \in \mathbb{R}$. Use the method of images to construct a Green's function for this problem that satisfies the boundary conditions

$$\frac{\partial g}{\partial z'} = 0 \;\text{ on }\; z' = 0 \quad\text{and}\quad \lim_{|\mathbf{r}|\to\infty} g(\mathbf{r},\mathbf{r}') = 0\,.$$

(iv) A solution to the Helmholtz equation on a bounded domain can be constructed in complete analogy to that of the Poisson equation using the Green's function in Green's 3rd identity

$$\psi(\mathbf{r}) = \int_{\partial V}\left[\psi(\mathbf{r}')\frac{\partial g(\mathbf{r},\mathbf{r}')}{\partial n'} - g(\mathbf{r},\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial n'}\right]dS' + \int_V f(\mathbf{r}')g(\mathbf{r},\mathbf{r}')dV'\,,$$

where V denotes the volume of the domain, ∂V its boundary and $\partial/\partial n'$ the outgoing normal derivative on the boundary. Now consider the homogeneous Helmholtz equation $\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = 0$ on the domain $z > 0$, $x, y \in \mathbb{R}$ with boundary conditions $\psi(\mathbf{r}) = 0$ at $|\mathbf{r}| \to \infty$ and

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=0} = \begin{cases} 0 & \text{for } \rho > a \\ A & \text{for } \rho \leqslant a \end{cases}$$

where $\rho = \sqrt{x^2+y^2}$ and A and a are real constants. Construct a solution in integral form to this equation using cylindrical coordinates (z, ρ, φ) with $x = \rho\cos\varphi$, $y = \rho\sin\varphi$.

Paper 2, Section II

16C Methods

(i) The Laplace operator in spherical coordinates is

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

Show that general, regular axisymmetric solutions $\psi(r,\theta)$ to the equation $\vec{\nabla}^2\psi = 0$ are given by

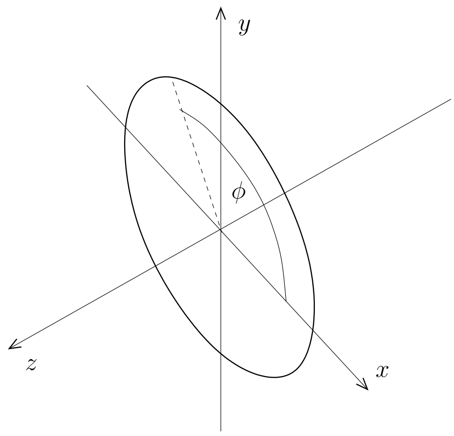
$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right)P_n(\cos\theta)\,,$$

where A_n, B_n are constants and P_n are the Legendre polynomials. [You may use without proof that regular solutions to Legendre's equation $-\frac{d}{dx}[(1-x^2)\frac{d}{dx}y(x)] = \lambda\, y(x)$ are given by $P_n(x)$ with $\lambda = n(n+1)$ and non-negative integer n.]

(ii) Consider a uniformly charged wire in the form of a ring of infinitesimal width with radius $r_0 = 1$ and a constant charge per unit length σ. By Coulomb's law, the electric potential due to a point charge q at a point a distance d from the charge is

$$U = \frac{q}{4\pi\epsilon_0 d}\,,$$

where ϵ_0 is a constant. Let the z-axis be perpendicular to the circle and pass through the circle's centre (see figure). Show that the potential due to the charged ring at a point on the z-axis at location z is given by



$$V = \frac{\sigma}{2\epsilon_0\sqrt{1+z^2}}\,.$$

(iii) The potential V generated by the charged ring of (ii) at arbitrary points (excluding points directly on the ring which can be ignored for this question) is determined by Laplace's equation $\vec{\nabla}^2 V = 0$. Calculate this potential with the boundary condition $\lim_{r\to\infty} V = 0$, where $r = \sqrt{x^2+y^2+z^2}$. [You may use without proof that

$$\frac{1}{\sqrt{1+x^2}} = \sum_{m=0}^{\infty} x^{2m}\,(-1)^m\,\frac{(2m)!}{2^{2m}\,(m!)^2}\,,$$

for $|x| < 1$. Furthermore, the Legendre polynomials are normalized such that $P_n(1) = 1$.]

Paper 4, Section II
17C Methods
Describe the method of characteristics to construct solutions for 1st-order, homogeneous, linear partial differential equations

$$\alpha(x,y)\frac{\partial u}{\partial x} + \beta(x,y)\frac{\partial u}{\partial y} = 0\,,$$

with initial data prescribed on a curve $x_0(\sigma)$, $y_0(\sigma)$: $u(x_0(\sigma), y_0(\sigma)) = h(\sigma)$.

Consider the partial differential equation (here the two independent variables are time t and spatial direction x)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,,$$

with initial data $u(t=0,x) = e^{-x^2}$.

(i) Calculate the characteristic curves of this equation and show that u remains constant along these curves. Qualitatively sketch the characteristics in the (x,t) diagram, i.e. the x axis is the horizontal and the t axis is the vertical axis.

(ii) Let $\tilde{x}_0$ denote the x value of a characteristic at time $t = 0$ and thus label the characteristic curves. Let $\tilde{x}$ denote the x value at time t of a characteristic with given $\tilde{x}_0$. Show that $\partial\tilde{x}/\partial\tilde{x}_0$ becomes a non-monotonic function of $\tilde{x}_0$ (at fixed t) at times $t > \sqrt{e/2}$, i.e. $\tilde{x}(\tilde{x}_0)$ has a local minimum or maximum. Qualitatively sketch snapshots of the solution $u(t,x)$ for a few fixed values of $t \in [0, \sqrt{e/2}]$ and briefly interpret the onset of the non-monotonic behaviour of $\tilde{x}(\tilde{x}_0)$ at $t = \sqrt{e/2}$.

Paper 4, Section I
5D Methods
Consider the ordinary differential equation

$$\frac{d^2\psi}{dz^2} - \left[\frac{15k^2}{4(k|z|+1)^2} - 3k\delta(z)\right]\psi = 0\,, \qquad (\dagger)$$

where k is a positive constant and δ denotes the Dirac delta function. Physically relevant solutions for ψ are bounded over the entire range $z \in \mathbb{R}$.

(i) Find piecewise bounded solutions to this differential equations in the ranges $z > 0$ and $z < 0$, respectively. [*Hint: The equation* $\frac{d^2y}{dx^2} - \frac{c}{x^2}y = 0$ *for a constant c may be solved using the Ansatz* $y = x^\alpha$.]

(ii) Derive a matching condition at $z = 0$ by integrating ($\dagger$) over the interval $(-\epsilon, \epsilon)$ with $\epsilon \to 0$ and use this condition together with the requirement that ψ be continuous at $z = 0$ to determine the solution over the entire range $z \in \mathbb{R}$.

Paper 2, Section I
5D Methods
(i) Calculate the Fourier series for the periodic extension on $\mathbb{R}$ of the function

$$f(x) = x(1-x)\,,$$

defined on the interval $[0, 1)$.

(ii) Explain why the Fourier series for the periodic extension of $f'(x)$ can be obtained by term-by-term differentiation of the series for $f(x)$.

(iii) Let $G(x)$ be the Fourier series for the periodic extension of $f'(x)$. Determine the value of $G(0)$ and explain briefly how it is related to the values of f'.

Paper 3, Section I
7D Methods
Using the method of characteristics, solve the differential equation

$$-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0\,,$$

where x, $y \in \mathbb{R}$ and $u = \cos y^2$ on $x = 0$, $y \geqslant 0$.

Paper 1, Section II

14D Methods

(a) Legendre's differential equation may be written

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0\,, \qquad y(1) = 1\,.$$

Show that for non-negative integer n, this equation has a solution $P_n(x)$ that is a polynomial of degree n. Find P_0, P_1 and P_2 explicitly.

(b) Laplace's equation in spherical coordinates for an axisymmetric function $U(r,\theta)$ (i.e. no ϕ dependence) is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) = 0\,.$$

Use separation of variables to find the general solution for $U(r,\theta)$.

Find the solution $U(r,\theta)$ that satisfies the boundary conditions

$$U(r,\theta) \to v_0\, r\, \cos\theta \quad \text{ as } r\to\infty\,,$$

$$\frac{\partial U}{\partial r} = 0 \quad \text{ at } r = r_0\,,$$

where v_0 and r_0 are constants.

Paper 3, Section II

15D Methods

Let $\mathcal{L}$ be a linear second-order differential operator on the interval $[0, \pi/2]$. Consider the problem

$$\mathcal{L}y(x) = f(x)\,; \qquad y(0) = y(\pi/2) = 0\,,$$

with $f(x)$ bounded in $[0, \pi/2]$.

(i) How is a Green's function for this problem defined?

(ii) How is a solution $y(x)$ for this problem constructed from the Green's function?

(iii) Describe the continuity and jump conditions used in the construction of the Green's function.

(iv) Use the continuity and jump conditions to construct the Green's function for the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{5}{4}y = f(x)$$

on the interval $[0, \pi/2]$ with the boundary conditions $y(0) = 0$, $y(\pi/2)$=0 and an arbitrary bounded function $f(x)$. Use the Green's function to construct a solution $y(x)$ for the particular case $f(x) = e^{x/2}$.

Paper 2, Section II
16D Methods
The Fourier transform $\tilde{f}$ of a function f is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx\,, \quad \text{so that} \;\; f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk\,.$$

A Green's function $G(t, t', x, x')$ for the diffusion equation in one spatial dimension satisfies

$$\frac{\partial G}{\partial t} - D \frac{\partial^2 G}{\partial x^2} = \delta(t - t')\, \delta(x - x')\,.$$

(a) By applying a Fourier transform, show that the Fourier transform $\tilde{G}$ of this Green's function and the Green's function G are

$$\begin{aligned} \tilde{G}(t, t', k, x') &= H(t - t')\, e^{-ikx'}\, e^{-Dk^2(t-t')}\,, \\ G(t, t', x, x') &= \frac{H(t - t')}{\sqrt{4\pi D(t - t')}} e^{-\frac{(x-x')^2}{4D(t-t')}}\,, \end{aligned}$$

where H is the Heaviside function. [*Hint: The Fourier transform $\tilde{F}$ of a Gaussian* $F(x) = \dfrac{1}{\sqrt{4\pi a}} e^{-\frac{x^2}{4a}}$, $a = \text{const}$, *is given by* $\tilde{F}(k) = e^{-ak^2}$.]

(b) The analogous result for the Green's function for the diffusion equation in two spatial dimensions is

$$G(t, t', x, x', y, y') = \frac{H(t - t')}{4\pi D(t - t')} e^{-\frac{1}{4D(t-t')}\left[(x-x')^2 + (y-y')^2\right]}\,.$$

Use this Green's function to construct a solution for $t \geqslant 0$ to the diffusion equation

$$\frac{\partial \Psi}{\partial t} - D \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) = p(t)\, \delta(x)\, \delta(y)\,,$$

with the initial condition $\Psi(0, x, y) = 0$.

Now set

$$p(t) = \begin{cases} p_0 = \text{const} & \text{for} \quad 0 \leqslant t \leqslant t_0 \\ 0 & \text{for} \quad t > t_0 \end{cases}$$

Find the solution $\Psi(t, x, y)$ for $t > t_0$ in terms of the exponential integral defined by

$$Ei(-\eta) = -\int_{\eta}^{\infty} \frac{e^{-\lambda}}{\lambda} d\lambda\,.$$

Use the approximation $Ei(-\eta) \approx \ln \eta + C$, valid for $\eta \ll 1$, to simplify this solution $\Psi(t, x, y)$. Here $C \approx 0.577$ is Euler's constant.

Paper 4, Section II
17D Methods
Let $f(x)$ be a complex-valued function defined on the interval $[-L, L]$ and periodically extended to $x \in \mathbb{R}$.

(i) Express $f(x)$ as a complex Fourier series with coefficients c_n, $n \in \mathbb{Z}$. How are the coefficients c_n obtained from $f(x)$?

(ii) State Parseval's theorem for complex Fourier series.

(iii) Consider the function $f(x) = \cos(\alpha x)$ on the interval $[-\pi, \pi]$ and periodically extended to $x \in \mathbb{R}$ for a complex but non-integer constant α. Calculate the complex Fourier series of $f(x)$.

(iv) Prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)} \, .$$

(v) Now consider the case where α is a real, non-integer constant. Use Parseval's theorem to obtain a formula for

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2 - \alpha^2)^2} \, .$$

What value do you obtain for this series for $\alpha = 5/2$?

Paper 2, Section I

5B Methods

Consider the equation

$$xu_x + (x+y)u_y = 1$$

subject to the Cauchy data $u(1, y) = y$. Using the method of characteristics, obtain a solution to this equation.

Paper 4, Section I

5C Methods

Show that the general solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

can be written in the form

$$y(x,t) = f(ct - x) + g(ct + x)\,.$$

For the boundary conditions

$$y(0,t) = y(L,t) = 0, \qquad t > 0\,,$$

find the relation between f and g and show that they are $2L$-periodic. Hence show that

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2\right) dx$$

is independent of t.

Paper 3, Section I
7C Methods

The solution to the Dirichlet problem on the half-space $D = \{\mathbf{x} = (x, y, z) : z > 0\}$:

$$\nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in D\,, \qquad u(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \qquad u(x, y, 0) = h(x, y),$$

is given by the formula

$$u(\mathbf{x}_0) = u(x_0, y_0, z_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \frac{\partial}{\partial n} G(\mathbf{x}, \mathbf{x}_0)\, dx\, dy\,,$$

where n is the outward normal to ∂D.

State the boundary conditions on G and explain how G is related to G_3, where

$$G_3(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0|}$$

is the fundamental solution to the Laplace equation in three dimensions.

Using the method of images find an explicit expression for the function $\frac{\partial}{\partial n} G(\mathbf{x}, \mathbf{x}_0)$ in the formula.

Paper 1, Section II
14B Methods

(i) Let $f(x) = x$, $0 < x \leqslant \pi$. Obtain the Fourier sine series and sketch the odd and even periodic extensions of $f(x)$ over the interval $-2\pi \leqslant x \leqslant 2\pi$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

(ii) Consider the eigenvalue problem

$$\mathcal{L}y = -\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} = \lambda y, \qquad \lambda \in \mathbb{R}$$

with boundary conditions $y(0) = y(\pi) = 0$. Find the eigenvalues and corresponding eigenfunctions. Recast $\mathcal{L}$ in Sturm-Liouville form and give the orthogonality condition for the eigenfunctions. Using the Fourier sine series obtained in part (i), or otherwise, and assuming completeness of the eigenfunctions, find a series for y that satisfies

$$\mathcal{L}y = xe^{-x}$$

for the given boundary conditions.

Paper 3, Section II

15C Methods

The Laplace equation in plane polar coordinates has the form

$$\nabla^2\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0\,.$$

Using separation of variables, derive the general solution to the equation that is single-valued in the domain $1 < r < 2$.

For

$$f(\theta) = \sum_{n=1}^{\infty} A_n \sin n\theta\,,$$

solve the Laplace equation in the annulus with the boundary conditions:

$$\nabla^2\phi = 0, \quad 1 < r < 2, \qquad \phi(r,\theta) = \begin{cases} f(\theta), & r = 1 \\ f(\theta) + 1, & r = 2. \end{cases}$$

Paper 2, Section II

16B Methods

The steady-state temperature distribution $u(x)$ in a uniform rod of finite length satisfies the boundary value problem

$$-D\frac{d^2}{dx^2}u(x) = f(x)\,, \qquad 0 < x < l$$

$$u(0) = 0\,, \qquad u(l) = 0$$

where $D > 0$ is the (constant) diffusion coefficient. Determine the Green's function $G(x,\xi)$ for this problem. Now replace the above homogeneous boundary conditions with the inhomogeneous boundary conditions $u(0) = \alpha, \quad u(l) = \beta$ and give a solution to the new boundary value problem. Hence, obtain the steady-state solution for the following problem with the specified boundary conditions:

$$-D\frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial t}u(x,t) = x\,, \quad 0 < x < 1\,,$$

$$u(0,t) = 1/D\,, \qquad u(1,t) = 2/D\,, \qquad t > 0\,.$$

[You may assume that a steady-state solution exists.]

Paper 4, Section II

17C Methods

Find the inverse Fourier transform $G(x)$ of the function

$$g(k) = e^{-a|k|}, \qquad a > 0, \qquad -\infty < k < \infty\,.$$

Assuming that appropriate Fourier transforms exist, determine the solution $\psi(x, y)$ of

$$\nabla^2 \psi = 0, \qquad -\infty < x < \infty, \qquad 0 < y < 1,$$

with the following boundary conditions

$$\psi(x, 0) = \delta(x), \qquad \psi(x, 1) = \frac{1}{\pi}\frac{1}{x^2+1}\,.$$

Here $\delta(x)$ is the Dirac delta-function.

Paper 2, Section I

5C Methods

Using the method of characteristics, obtain a solution to the equation

$$u_x + 2xu_y = y$$

subject to the Cauchy data $u(0, y) = 1 + y^2$ for $-\frac{1}{2} < y < \frac{1}{2}$.

Sketch the characteristics and specify the greatest region of the plane in which a unique solution exists.

Paper 4, Section I

5D Methods

Show that the general solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

can be written in the form

$$y(x, t) = f(x - ct) + g(x + ct)\,.$$

Hence derive the solution $y(x, t)$ subject to the initial conditions

$$y(x, 0) = 0, \qquad \frac{\partial y}{\partial t}(x, 0) = \psi(x)\,.$$

Paper 3, Section I

7D Methods

For the step-function

$$F(x) = \begin{cases} 1\,, & |x| \leqslant 1/2 \\ 0\,, & \text{otherwise,} \end{cases}$$

its convolution with itself is the hat-function

$$G(x) = [F * F](x) = \begin{cases} 1 - |x|\,, & |x| \leqslant 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Find the Fourier transforms of F and G, and hence find the values of the integrals

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2}\, dy\,, \qquad I_2 = \int_{-\infty}^{\infty} \frac{\sin^4 y}{y^4}\, dy\,.$$

Paper 1, Section II

14C Methods

Consider the regular Sturm-Liouville (S-L) system

$$(\mathcal{L}y)(x) - \lambda\omega(x)y(x) = 0\,, \quad a \leqslant x \leqslant b\,,$$

where

$$(\mathcal{L}y)(x) := -[p(x)y'(x)]' + q(x)y(x)$$

with $\omega(x) > 0$ and $p(x) > 0$ for all x in $[a,b]$, and the boundary conditions on y are

$$\begin{cases} A_1\, y(a) + A_2\, y'(a) = 0 \\ B_1\, y(b) + B_2\, y'(b) \ = 0\,. \end{cases}$$

Show that with these boundary conditions, $\mathcal{L}$ is self-adjoint. By considering $y\mathcal{L}y$, or otherwise, show that the eigenvalue λ can be written as

$$\lambda = \frac{\int_a^b (py'^2 + qy^2)\, dx - [pyy']_a^b}{\int_a^b \omega y^2\, dx}\,.$$

Now suppose that $a = 0$ and $b = \ell$, that $p(x) = 1$, $q(x) \geqslant 0$ and $\omega(x) = 1$ for all $x \in [0,\ell]$, and that $A_1 = 1$, $A_2 = 0$, $B_1 = k \in \mathbb{R}^+$ and $B_2 = 1$. Show that the eigenvalues of this regular S-L system are strictly positive. Assuming further that $q(x) = 0$, solve the system explicitly, and with the aid of a graph, show that there exist infinitely many eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$. Describe the behaviour of λ_n as $n \to \infty$.

Paper 3, Section II
15D Methods

Consider Legendre's equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0\,.$$

Show that if $\lambda = n(n+1)$, with n a non-negative integer, this equation has a solution $y = P_n(x)$, a polynomial of degree n. Find P_0, P_1 and P_2 explicitly, subject to the condition $P_n(1) = 1$.

The general solution of Laplace's equation $\nabla^2\psi = 0$ in spherical polar coordinates, in the axisymmetric case, has the form

$$\psi(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-(n+1)})P_n(\cos\theta)\,.$$

Hence, find the solution of Laplace's equation in the region $a \leqslant r \leqslant b$ satisfying the boundary conditions

$$\begin{cases} \psi(r,\theta) = 1\,, & r = a \\ \psi(r,\theta) = 3\cos^2\theta\,, & r = b\,. \end{cases}$$

Paper 2, Section II
16C Methods

Consider the linear differential operator $\mathcal{L}$ defined by

$$\mathcal{L}y := -\frac{d^2y}{dx^2} + y$$

on the interval $0 \leqslant x < \infty$. Given the boundary conditions $y(0) = 0$ and $\lim_{x\to\infty} y(x) = 0$, find the Green's function $G(x,\xi)$ for $\mathcal{L}$ with these boundary conditions. Hence, or otherwise, obtain the solution of

$$\mathcal{L}y = \begin{cases} 1, & 0 \leqslant x \leqslant \mu \\ 0, & \mu < x < \infty \end{cases}$$

subject to the above boundary conditions, where μ is a positive constant. Show that your piecewise solution is continuous at $x = \mu$ and has the value

$$y(\mu) = \frac{1}{2}(1 + e^{-2\mu} - 2e^{-\mu})\,.$$

Paper 4, Section II

17D Methods

Let $D \subset \mathbb{R}^2$ be a two-dimensional domain with boundary $S = \partial D$, and let

$$G_2 = G_2(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \log |\mathbf{r} - \mathbf{r}_0| \,,$$

where $\mathbf{r}_0$ is a point in the interior of D. From Green's second identity,

$$\int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\ell = \int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, da \,,$$

derive Green's third identity

$$u(\mathbf{r}_0) = \int_D G_2 \nabla^2 u \, da + \int_S \left(u \frac{\partial G_2}{\partial n} - G_2 \frac{\partial u}{\partial n} \right) d\ell \,.$$

[Here $\frac{\partial}{\partial n}$ denotes the normal derivative on S.]

Consider the Dirichlet problem on the unit disc $D_1 = \{\mathbf{r} \in \mathbb{R}^2 : |\mathbf{r}| \leqslant 1\}$:

$$\begin{aligned} \nabla^2 u = 0, \quad & \mathbf{r} \in D_1 \,, \\ u(\mathbf{r}) = f(\mathbf{r}), \quad & \mathbf{r} \in S_1 = \partial D_1 \,. \end{aligned}$$

Show that, with an appropriate function $G(\mathbf{r}, \mathbf{r}_0)$, the solution can be obtained by the formula

$$u(\mathbf{r}_0) = \int_{S_1} f(\mathbf{r}) \frac{\partial}{\partial n} G(\mathbf{r} \,, \mathbf{r}_0) \, d\ell \,.$$

State the boundary conditions on G and explain how G is related to G_2.

For $\mathbf{r}, \mathbf{r}_0 \in \mathbb{R}^2$, prove the identity

$$\left| \frac{\mathbf{r}}{|\mathbf{r}|} - \mathbf{r}_0 |\mathbf{r}| \right| = \left| \frac{\mathbf{r}_0}{|\mathbf{r}_0|} - \mathbf{r} |\mathbf{r}_0| \right| ,$$

and deduce that if the point $\mathbf{r}$ lies on the unit circle, then

$$|\mathbf{r} - \mathbf{r}_0| = |\mathbf{r}_0| |\mathbf{r} - \mathbf{r}_0^*| \,, \text{ where } \mathbf{r}_0^* = \frac{\mathbf{r}_0}{|\mathbf{r}_0|^2} \,.$$

Hence, using the method of images, or otherwise, find an expression for the function $G(\mathbf{r} \,, \mathbf{r}_0)$. [An expression for $\frac{\partial}{\partial n} G$ is not required.]

Paper 2, Section I

5A Methods

The Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

for $-1 \leqslant x \leqslant 1$ and non-negative integers n.

Write the Legendre equation as an eigenvalue equation for an operator L in Sturm-Liouville form. Show that L is self-adjoint and find the orthogonality relation between the eigenfunctions.

Paper 3, Section I

7A Methods

The Fourier transform $\widetilde{h}(k)$ of the function $h(x)$ is defined by

$$\widetilde{h}(k) = \int_{-\infty}^{\infty} h(x) e^{-ikx} dx.$$

(i) State the inverse Fourier transform formula expressing $h(x)$ in terms of $\widetilde{h}(k)$.

(ii) State the convolution theorem for Fourier transforms.

(iii) Find the Fourier transform of the function $f(x) = e^{-|x|}$. Hence show that the convolution of the function $f(x) = e^{-|x|}$ with itself is given by the integral expression

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{(1+k^2)^2} dk.$$

Paper 4, Section I

5A Methods

Use the method of characteristics to find a continuous solution $u(x,y)$ of the equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0,$$

subject to the condition $u(0,y) = y^4$.

In which region of the plane is the solution uniquely determined?

Paper 1, Section II

14A Methods

Let $f(t)$ be a real function defined on an interval $(-T, T)$ with Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right).$$

State and prove Parseval's theorem for $f(t)$ and its Fourier series. Write down the formulae for a_0, a_n and b_n in terms of $f(t)$, $\cos \dfrac{n\pi t}{T}$ and $\sin \dfrac{n\pi t}{T}$.

Find the Fourier series of the square wave function defined on $(-\pi, \pi)$ by

$$g(t) = \begin{cases} 0 & -\pi < t \leqslant 0 \\ 1 & \quad 0 < t < \pi. \end{cases}$$

Hence evaluate

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)}.$$

Using some of the above results evaluate

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

What is the sum of the Fourier series for $g(t)$ at $t = 0$? Comment on your answer.

Paper 2, Section II

16A Methods

Use a Green's function to find an integral expression for the solution of the equation

$$\frac{d^2\theta}{dt^2} + 4\frac{d\theta}{dt} + 29\,\theta = f(t)$$

for $t \geqslant 0$ subject to the initial conditions

$$\theta(0) = 0 \quad \text{and} \quad \frac{d\theta}{dt}(0) = 0.$$

Paper 3, Section II

15A Methods

A uniform stretched string of length L, density per unit length μ and tension $T = \mu c^2$ is fixed at both ends. Its transverse displacement is given by $y(x,t)$ for $0 \leqslant x \leqslant L$. The motion of the string is resisted by the surrounding medium with a resistive force per unit length of $-2k\mu \dfrac{\partial y}{\partial t}$.

(i) Show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} - c^2 \frac{\partial^2 y}{\partial x^2} = 0$$

provided that the transverse motion can be regarded as small.

(ii) Suppose now that $k = \dfrac{\pi c}{L}$. Find the displacement of the string for $t \geqslant 0$ given the initial conditions

$$y(x,0) = A \sin\left(\frac{\pi x}{L}\right) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = 0.$$

(iii) Sketch the transverse displacement at $x = \dfrac{L}{2}$ as a function of time for $t \geqslant 0$.

Paper 4, Section II

17A Methods

Let D be a two dimensional domain with boundary ∂D. Establish Green's second identity

$$\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dA = \int_{\partial D} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds$$

where $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on ∂D.

State the differential equation and boundary conditions which are satisfied by a Dirichlet Green's function $G(\mathbf{r}, \mathbf{r}_0)$ for the Laplace operator on the domain D, where $\mathbf{r}_0$ is a fixed point in the interior of D.

Suppose that $\nabla^2 \psi = 0$ on D. Show that

$$\psi(\mathbf{r}_0) = \int_{\partial D} \psi(\mathbf{r}) \frac{\partial}{\partial n} G(\mathbf{r}, \mathbf{r_0}) ds.$$

Consider Laplace's equation in the upper half plane,

$$\nabla^2 \psi(x, y) = 0, \quad -\infty < x < \infty \quad \text{and} \quad y > 0,$$

with boundary conditions $\psi(x, 0) = f(x)$ where $f(x) \to 0$ as $|x| \to \infty$, and $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$. Show that the solution is given by the integral formula

$$\psi(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{(x - x_0)^2 + y_0^2} \, dx.$$

[*Hint: It might be useful to consider*

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} (\log |\mathbf{r} - \mathbf{r}_0| - \log |\mathbf{r} - \tilde{\mathbf{r}}_0|)$$

for suitable $\tilde{\mathbf{r}}_\mathbf{0}$. *You may assume* $\nabla^2 \log |\mathbf{r} - \mathbf{r}_0| = 2\pi \delta(\mathbf{r} - \mathbf{r}_0)$.]

Paper 2, Section I

5A Methods

Consider the initial value problem

$$\mathcal{L}x(t) = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0, \quad t \geqslant 0,$$

where $\mathcal{L}$ is a second-order linear operator involving differentiation with respect to t. Explain briefly how to solve this by using a Green's function.

Now consider

$$\ddot{x}(t) = \begin{cases} a & 0 \leqslant t \leqslant T, \\ 0 & T < t < \infty, \end{cases}$$

where a is a constant, subject to the same initial conditions. Solve this using the Green's function, and explain how your answer is related to a problem in Newtonian dynamics.

Paper 3, Section I

7B Methods

Show that Laplace's equation $\nabla^2\phi = 0$ in polar coordinates (r, θ) has solutions proportional to $r^{\pm\alpha} \sin \alpha\theta$, $r^{\pm\alpha} \cos \alpha\theta$ for any constant α.

Find the function ϕ satisfying Laplace's equation in the region $a < r < b$, $0 < \theta < \pi$, where $\phi(a, \theta) = \sin^3 \theta$, $\phi(b, \theta) = \phi(r, 0) = \phi(r, \pi) = 0$.

[*The Laplacian ∇^2 in polar coordinates is*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.\Big]$$

Paper 4, Section I

5A Methods

(a) By considering strictly monotonic differentiable functions $\varphi(x)$, such that the zeros satisfy $\varphi(c) = 0$ but $\varphi'(c) \neq 0$, establish the formula

$$\int_{-\infty}^{\infty} f(x)\delta(\varphi(x))dx = \frac{f(c)}{|\varphi'(c)|} .$$

Hence show that for a general differentiable function with only such zeros, labelled by c,

$$\int_{-\infty}^{\infty} f(x)\delta(\varphi(x))\, dx = \sum_c \frac{f(c)}{|\varphi'(c)|} .$$

(b) Hence by changing to plane polar coordinates, or otherwise, evaluate,

$$I = \int_0^{\infty} \int_0^{\infty} (x^3 + y^2 x)\delta(x^2 + y^2 - 1)\, dy dx.$$

Paper 1, Section II

14A Methods

(a) A function $f(t)$ is periodic with period 2π and has continuous derivatives up to and including the kth derivative. Show by integrating by parts that the Fourier coefficients of $f(t)$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt\, dt,$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt\, dt,$$

decay at least as fast as $1/n^k$ as $n \to \infty$.

(b) Calculate the Fourier series of $f(t) = |\sin t|$ on $[0, 2\pi]$.

(c) Comment on the decay rate of your Fourier series.

Paper 2, Section II
16B Methods

Explain briefly the use of the method of characteristics to solve linear first-order partial differential equations.

Use the method to solve the problem

$$(x-y)\frac{\partial u}{\partial x}+(x+y)\frac{\partial u}{\partial y}=\alpha u,$$

where α is a constant, with initial condition $u(x,0)=x^2$, $x \geqslant 0$.

By considering your solution explain:

(i) why initial conditions cannot be specified on the whole x-axis;

(ii) why a single-valued solution in the entire plane is not possible if $\alpha \neq 2$.

Paper 3, Section II
15A Methods

(a) Put the equation

$$x\frac{d^2u}{dx^2}+\frac{du}{dx}+\lambda xu=0, \qquad 0\leqslant x\leqslant 1,$$

into Sturm–Liouville form.

(b) Suppose $u_n(x)$ are eigenfunctions such that $u_n(x)$ are bounded as x tends to zero and

$$x\frac{d^2u_n}{dx^2}+\frac{du_n}{dx}+\lambda_n xu_n=0, \qquad 0\leqslant x\leqslant 1.$$

Identify the weight function $w(x)$ and the most general boundary conditions on $u_n(x)$ which give the orthogonality relation

$$(\lambda_m-\lambda_n)\int_0^1 u_m(x)w(x)u_n(x)\,dx=0.$$

(c) The equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0, \qquad x>0,$$

has a solution $J_0(x)$ and a second solution which is not bounded at the origin. The zeros of $J_0(x)$ arranged in ascending order are $j_n, n=1,2,....$ Given that $u_n(1)=0$, show that the eigenvalues of the Sturm–Liouville problem in (b) are $\lambda={j_n}^2, n=1,2,....$

(d) Using the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$ and integration by parts, show that

$$\int_0^1 J_0(\alpha x)J_0(\beta x)x\,dx=\frac{\beta J_0(\alpha)J_0'(\beta)-\alpha J_0(\beta)J_0'(\alpha)}{\alpha^2-\beta^2} \qquad (\alpha\neq\beta).$$

Paper 4, Section II
17B Methods

Defining the function $G_{f3}(\mathbf{r};\mathbf{r}_0) = -1/(4\pi|\mathbf{r}-\mathbf{r}_0|)$, prove Green's third identity for functions $u(\mathbf{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\mathbf{r}_0) = \int_S \left(u\frac{\partial G_{f_3}}{\partial n} - \frac{\partial u}{\partial n} G_{f_3} \right) dS.$$

A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half plane $z > 0$:

$$u = O(|\mathbf{x}|^{-a}), \quad \frac{\partial u}{\partial r} = O(|\mathbf{x}|^{-a-1}) \text{ as } |\mathbf{x}| \to \infty, \qquad \frac{\partial u}{\partial z} = p(x,y) \text{ on } z = 0,$$

where $a > 0$. It is assumed that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 0$. Explain why this condition is necessary.

Construct an appropriate Green's function $G(\mathbf{r};\mathbf{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$, using the method of images or otherwise. Hence find the solution in the form

$$u(x_0, y_0, z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) f(x - x_0, y - y_0, z_0)\,dx\,dy,$$

where f is to be determined.

Now let

$$p(x,y) = \begin{cases} x & |x|, |y| < a, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, show that

$$u \to \frac{-2a^4 x_0}{3\pi z_0^3} \qquad \text{as} \quad z_0 \to \infty\ .$$

Paper 2, Section I

5B Mathematical Methods

Expand $f(x) = x,\ 0 < x < \pi$, as a half-range sine series.

By integrating the series show that a Fourier cosine series for x^2, $0 < x < \pi$, can be written as

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\ ,$$

where $a_n,\ n = 1, 2, \ldots,$ should be determined and

$$a_0 = 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}\ .$$

By evaluating a_0 another way show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}\ .$$

Paper 4, Section I

5B Mathematical Methods

Describe briefly the method of Lagrange multipliers for finding the stationary points of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

Show that at a stationary point (a, b)

$$\begin{vmatrix} \dfrac{\partial f}{\partial x}(a,b) & \dfrac{\partial g}{\partial x}(a,b) \\[2ex] \dfrac{\partial f}{\partial y}(a,b) & \dfrac{\partial g}{\partial y}(a,b) \end{vmatrix} = 0\ .$$

Find the maximum distance from the origin to the curve

$$x^2 + y^2 + xy - 4 = 0\ .$$

Paper 1, Section II

14B Mathematical Methods

Find a power series solution about $x = 0$ of the equation

$$xy'' + (1 - x)y' + \lambda y = 0,$$

with $y(0) = 1$, and show that y is a polynomial if and only if λ is a non-negative integer n. Let y_n be the solution for $\lambda = n$. Establish an orthogonality relation between y_m and y_n $(m \neq n)$.

Show that $y_m y_n$ is a polynomial of degree $m + n$, and hence that

$$y_m y_n = \sum_{p=0}^{m+n} a_p y_p$$

for appropriate choices of the coefficients a_p and with $a_{m+n} \neq 0$.

For given $n > 0$, show that the functions

$$\{y_m,\ y_m y_n : m = 0, 1, 2, \ldots, n - 1\}$$

are linearly independent.

Let $f(x)$ be a polynomial of degree 3. Explain why the expansion

$$f(x) = a_0 y_0(x) + a_1 y_1(x) + a_2 y_2(x) + a_3 y_1(x) y_2(x)$$

holds for appropriate choices of $a_p,\ p = 0, 1, 2, 3$. Hence show that

$$\int_0^\infty e^{-x} f(x)\, dx = w_1 f(\alpha_1) + w_2 f(\alpha_2)\ ,$$

where

$$w_1 = \frac{y_1(\alpha_2)}{y_1(\alpha_2) - y_1(\alpha_1)}\ , \qquad w_2 = \frac{-y_1(\alpha_1)}{y_1(\alpha_2) - y_1(\alpha_1)}\ ,$$

and α_1, α_2 are the zeros of y_2. You need not construct the polynomials $y_1(x), y_2(x)$ explicitly.

Paper 2, Section II

15B Mathematical Methods

A string of uniform density ρ is stretched under tension along the x-axis and undergoes small transverse oscillations in the (x, y) plane with amplitude $y(x, t)$. Given that waves in the string travel at velocity c, write down the equation of motion satisfied by $y(x, t)$.

The string is now fixed at $x = 0$ and $x = L$. Derive the general separable solution for the amplitude $y(x, t)$.

For $t < 0$ the string is at rest. At time $t = 0$ the string is struck by a hammer in the interval $[l - a/2, l + a/2]$, distance being measured from one end. The effect of the hammer is to impart a constant velocity v to the string inside the interval and zero velocity outside it. Calculate the proportion of the total energy given to the string in each mode.

If $l = L/3$ and $a = L/10$, find all the modes of the string which are not excited in the motion.

Paper 3, Section I

6A Methods

The Fourier transform $\tilde{f}(\omega)$ of a suitable function $f(t)$ is defined as $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$. Consider the function $h(t) = e^{\alpha t}$ for $t > 0$, and zero otherwise. Show that

$$\tilde{h}(\omega) = \frac{1}{i\omega - \alpha},$$

provided $\Re(\alpha) < 0$.

The angle $\theta(t)$ of a forced, damped pendulum satisfies

$$\ddot{\theta} + 2\dot{\theta} + 5\theta = e^{-4t},$$

with initial conditions $\theta(0) = \dot{\theta}(0) = 0$. Show that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{4i}\left[\frac{1}{(i\omega + 1 - 2i)} - \frac{1}{(i\omega + 1 + 2i)}\right].$$

Paper 3, Section II

15A Methods

A function $g(r)$ is chosen to make the integral

$$\int_a^b f(r, g, g')dr$$

stationary, subject to given values of $g(a)$ and $g(b)$. Find the Euler–Lagrange equation for $g(r)$.

In a certain three-dimensional electrostatics problem the potential ϕ depends only on the radial coordinate r, and the energy functional of ϕ is

$$\mathcal{E}[\phi] = 2\pi \int_{R_1}^{R_2} \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + \frac{1}{2\lambda^2}\phi^2 \right] r^2 dr \ ,$$

where λ is a parameter. Show that the Euler–Lagrange equation associated with minimizing the energy $\mathcal{E}$ is equivalent to

$$\frac{1}{r}\frac{d^2(r\phi)}{dr^2} - \frac{1}{\lambda^2}\phi = 0 \ . \tag{1}$$

Find the general solution of this equation, and the solution for the region $R_1 \leqslant r \leqslant R_2$ which satisfies $\phi(R_1) = \phi_1$ and $\phi(R_2) = 0$.

Consider an annular region in two dimensions, where the potential is a function of the radial coordinate r only. Write down the equivalent expression for the energy functional $\mathcal{E}$ above, in cylindrical polar coordinates, and derive the equivalent of (1).

Paper 4, Section II

16A Methods

Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 ,$$

with $y_1(0) = 0$ and $y_2(1) = 0$. Show that the Green's function $G(x, \xi)$ for the interval $0 \leqslant x, \xi \leqslant 1$ and with $G(0, \xi) = G(1, \xi) = 0$ can be written in the form

$$G(x,\xi) = \begin{cases} y_1(x)y_2(\xi)/W(\xi); & 0 < x < \xi, \\ y_2(x)y_1(\xi)/W(\xi); & \xi < x < 1, \end{cases}$$

where $W(x) = W[y_1(x), y_2(x)]$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

Use this result to find the Green's function $G(x, \xi)$ that satisfies

$$\frac{d^2G}{dx^2} + 3\frac{dG}{dx} + 2G = \delta(x - \xi) ,$$

in the interval $0 \leqslant x, \xi \leqslant 1$ and with $G(0, \xi) = G(1, \xi) = 0$. Hence obtain an integral expression for the solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \begin{cases} 0; & 0 < x < x_0, \\ 2; & x_0 < x < 1, \end{cases}$$

for the case $x < x_0$.

1/II/14D **Methods**

Write down the Euler–Lagrange equation for the variational problem for $y(x)$ that extremizes the integral I defined as

$$I = \int_{x_1}^{x_2} f(x, y, y')dx,$$

with boundary conditions $y(x_1) = y_1, y(x_2) = y_2$, where y_1 and y_2 are positive constants such that $y_2 > y_1$, with $x_2 > x_1$. Find a first integral of the equation when f is independent of y, i.e. $f = f(x, y')$.

A light ray moves in the (x, y) plane from (x_1, y_1) to (x_2, y_2) with speed $c(x)$ taking a time T. Show that the equation of the path that makes T an extremum satisfies

$$\frac{dy}{dx} = \frac{c(x)}{\sqrt{k^2 - c^2(x)}},$$

where k is a constant and write down an integral relating k, x_1, x_2, y_1 and y_2.

When $c(x) = ax$ where a is a constant and $k = ax_2$, show that the path is given by

$$(y_2 - y)^2 = x_2^2 - x^2.$$

2/I/5D **Methods**

Describe briefly the method of Lagrange multipliers for finding the stationary values of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$.

Use the method to find the largest possible volume of a circular cylinder that has surface area A (including both ends).

2/II/15D **Methods**

(a) Legendre's equation may be written in the form

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right)+\lambda y=0.$$

Show that there is a series solution for y of the form

$$y=\sum_{k=0}^{\infty}a_k x^k,$$

where the a_k satisfy the recurrence relation

$$\frac{a_{k+2}}{a_k}=-\frac{(\lambda-k(k+1))}{(k+1)(k+2)}.$$

Hence deduce that there are solutions for $y(x)=P_n(x)$ that are polynomials of degree n, provided that $\lambda=n(n+1)$. Given that a_0 is then chosen so that $P_n(1)=1$, find the explicit form for $P_2(x)$.

(b) Laplace's equation for $\Phi(r,\theta)$ in spherical polar coordinates (r,θ,ϕ) may be written in the axisymmetric case as

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial\Phi}{\partial x}\right)=0,$$

where $x=\cos\theta$.

Write down without proof the general form of the solution obtained by the method of separation of variables. Use it to find the form of Φ exterior to the sphere $r=a$ that satisfies the boundary conditions, $\Phi(a,x)=1+x^2$, and $\lim_{r\to\infty}\ \Phi(r,x)=0$.

3/I/6D **Methods**

Let $\mathcal{L}$ be the operator

$$\mathcal{L}y = \frac{d^2y}{dx^2} - k^2 y$$

on functions $y(x)$ satisfying $\lim_{x\to-\infty}\ y(x) = 0$ and $\lim_{x\to\infty}\ y(x) = 0$.

Given that the Green's function $G(x;\xi)$ for $\mathcal{L}$ satisfies

$$\mathcal{L}G = \delta(x-\xi),$$

show that a solution of

$$\mathcal{L}y = S(x),$$

for a given function $S(x)$, is given by

$$y(x) = \int_{-\infty}^{\infty} G(x;\xi)S(\xi)d\xi.$$

Indicate why this solution is unique.

Show further that the Green's function is given by

$$G(x;\xi) = -\frac{1}{2|k|}\exp(-|k||x-\xi|).$$

3/II/15D **Methods**

Let $\lambda_1 < \lambda_2 < \ldots \lambda_n \ldots$ and $y_1(x)$, $y_2(x)$, $\ldots y_n(x) \ldots$ be the eigenvalues and corresponding eigenfunctions for the Sturm–Liouville system

$$\mathcal{L}y_n = \lambda_n w(x) y_n,$$

where

$$\mathcal{L}y \equiv \frac{d}{dx}\left(-p(x)\frac{dy}{dx}\right) + q(x)y,$$

with $p(x) > 0$ and $w(x) > 0$. The boundary conditions on y are that $y(0) = y(1) = 0$.

Show that two distinct eigenfunctions are orthogonal in the sense that

$$\int_0^1 w y_n y_m \, dx = \delta_{nm} \int_0^1 w y_n^2 \, dx.$$

Show also that if y has the form

$$y = \sum_{n=1}^{\infty} a_n y_n,$$

with a_n being independent of x, then

$$\frac{\int_0^1 y\mathcal{L}y \, dx}{\int_0^1 w y^2 \, dx} \geq \lambda_1.$$

Assuming that the eigenfunctions are complete, deduce that a solution of the diffusion equation,

$$\frac{\partial y}{\partial t} = -\frac{1}{w}\mathcal{L}y,$$

that satisfies the boundary conditions given above is such that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^1 w y^2 \, dx\right) \leq -\lambda_1 \int_0^1 w y^2 \, dx.$$

4/I/5A **Methods**

Find the half-range Fourier cosine series for $f(x) = x^2$, $0 < x < 1$. Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4/II/16A **Methods**

Assume $F(x)$ satisfies

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \,,$$

and that the series

$$g(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly in $[0 \leqslant \tau \leqslant 2\pi]$.

If $\tilde{F}$ is the Fourier transform of F, prove that

$$g(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \tilde{F}(n) e^{in\tau} \,.$$

[*Hint: prove that g is periodic and express its Fourier expansion coefficients in terms of* $\tilde{F}$].

In the case that $F(x) = e^{-|x|}$, evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \,.$$

1/II/14D **Methods**

Define the Fourier transform $\tilde{f}(k)$ of a function $f(x)$ that tends to zero as $|x| \to \infty$, and state the inversion theorem. State and prove the convolution theorem.

Calculate the Fourier transforms of

$$\begin{array}{llll} & (i) & f(x) & = e^{-a|x|}, \\ \text{and} & (ii) & g(x) & = \begin{cases} 1\,, & |x| \leqslant b \\ 0\,, & |x| > b\,. \end{cases} \end{array}$$

Hence show that

$$\int_{-\infty}^{\infty} \frac{\sin(bk)\, e^{ikx}}{k\,(a^2+k^2)}\, dk \;=\; \frac{\pi \sinh(ab)}{a^2}\, e^{-ax} \qquad \text{for} \quad x > b\,,$$

and evaluate this integral for all other (real) values of x.

2/I/5D **Methods**

Show that a smooth function $y(x)$ that satisfies $y(0) = y'(1) = 0$ can be written as a Fourier series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \sin \lambda_n x, \quad 0 \leqslant x \leqslant 1\,,$$

where the λ_n should be specified. Write down an integral expression for a_n.

Hence solve the following differential equation

$$y'' - \alpha^2 y = x \cos \pi x\,,$$

with boundary conditions $y(0) = y'(1) = 0$, in the form of an infinite series.

2/II/15D Methods

Let $y_0(x)$ be a non-zero solution of the Sturm-Liouville equation

$$L(y_0;\lambda_0) \equiv \frac{d}{dx}\left(p(x)\frac{dy_0}{dx}\right) + (q(x)+\lambda_0 w(x))\, y_0 = 0$$

with boundary conditions $y_0(0) = y_0(1) = 0$. Show that, if $y(x)$ and $f(x)$ are related by

$$L(y;\lambda_0) = f\,,$$

with $y(x)$ satisfying the same boundary conditions as $y_0(x)$, then

$$\int_0^1 y_0 f\, dx = 0\,. \qquad (*)$$

Suppose that y_0 is normalised so that

$$\int_0^1 w y_0^2 dx = 1\,,$$

and consider the problem

$$L(y;\lambda) = y^3\,; \quad y(0) = y(1) = 0\,.$$

By choosing f appropriately in $(*)$ deduce that, if

$$\lambda - \lambda_0 = \epsilon^2\mu \;\; [\mu = O(1), \epsilon \ll 1]\,, \quad \text{and} \quad y(x) = \epsilon y_0(x) + \epsilon^2 y_1(x)$$

then

$$\mu = \int_0^1 y_0^4 dx + O(\epsilon)\,.$$

3/I/6E Methods

Describe the method of Lagrange multipliers for finding extrema of a function $f(x, y, z)$ subject to the constraint that $g(x, y, z) = c$.

Illustrate the method by finding the maximum and minimum values of xy for points (x, y, z) lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,,$$

with a, b and c all positive.

3/II/15E **Methods**

Legendre's equation may be written

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0 \quad \text{with} \quad y(1)=1\,.$$

Show that if n is a positive integer, this equation has a solution $y=P_n(x)$ that is a polynomial of degree n. Find P_0, P_1 and P_2 explicitly.

Write down a general separable solution of Laplace's equation, $\nabla^2\phi=0$, in spherical polar coordinates (r,θ). (A derivation of this result is *not* required.)

Hence or otherwise find ϕ when

$$\nabla^2\phi=0, \quad a<r<b\,,$$

with $\phi=\sin^2\theta$ both when $r=a$ and when $r=b$.

4/I/5B **Methods**

Show that the general solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\,,$$

where c is a constant, is

$$y=f(x+ct)+g(x-ct)\,,$$

where f and g are twice differentiable functions. Briefly discuss the physical interpretation of this solution.

Calculate $y(x,t)$ subject to the initial conditions

$$y(x,0)=0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0)=\psi(x)\,.$$

4/II/16E **Methods**

Write down the Euler-Lagrange equation for extrema of the functional

$$I = \int_a^b F(y, y')\, dx\,.$$

Show that a first integral of this equation is given by

$$F - y' \frac{\partial F}{\partial y'} = C\,.$$

A road is built between two points A and B in the plane $z = 0$ whose polar coordinates are $r = a$, $\theta = 0$ and $r = a$, $\theta = \pi/2$ respectively. Owing to congestion, the traffic speed at points along the road is kr^2 with k a positive constant. If the equation describing the road is $r = r(\theta)$, obtain an integral expression for the total travel time T from A to B.

[*Arc length in polar coordinates is given by* $ds^2 = dr^2 + r^2 d\theta^2$.]

Calculate T for the circular road $r = a$.

Find the equation for the road that minimises T and determine this minimum value.

1/II/14A Methods

Define a *second rank tensor*. Show from your definition that if M_{ij} is a second rank tensor then M_{ii} is a scalar.

A rigid body consists of a thin flat plate of material having density $\rho(\mathbf{x})$ per unit area, where $\mathbf{x}$ is the position vector. The body occupies a region D of the (x, y)-plane; its thickness in the z-direction is negligible. The moment of inertia tensor of the body is given as

$$M_{ij} = \int_D (x_k x_k \delta_{ij} - x_i x_j)\rho \, dS.$$

Show that the z-direction is an eigenvector of M_{ij} and write down an integral expression for the corresponding eigenvalue $M_\perp$.

Hence or otherwise show that if the remaining eigenvalues of M_{ij} are M_1 and M_2 then

$$M_\perp = M_1 + M_2.$$

Find M_{ij} for a circular disc of radius a and uniform density having its centre at the origin.

2/I/5A Methods

Describe briefly the method of Lagrange multipliers for finding the stationary values of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$.

Use the method to find the smallest possible surface area (including both ends) of a circular cylinder that has volume V.

2/II/15G **Methods**

Verify that $y = e^{-x}$ is a solution of the differential equation

$$(x+2)y'' + (x+1)y' - y = 0,$$

and find a second solution of the form $ax + b$.

Let L be the operator

$$L[y] = y'' + \frac{(x+1)}{(x+2)}y' - \frac{1}{(x+2)}y$$

on functions $y(x)$ satisfying

$$y'(0) = y(0) \quad \text{and} \quad \lim_{x \to \infty} y(x) = 0.$$

The Green's function $G(x, \xi)$ for L satisfies

$$L[G] = \delta(x - \xi),$$

with $\xi > 0$. Show that

$$G(x, \xi) = -\frac{(\xi+1)}{(\xi+2)}e^{\xi - x}$$

for $x > \xi$, and find $G(x, \xi)$ for $x < \xi$.

Hence or otherwise find the solution of

$$L[y] = -(x+2)e^{-x},$$

for $x \geqslant 0$, with $y(x)$ satisfying the boundary conditions above.

3/I/6A **Methods**

If T_{ij} is a second rank tensor such that $b_i T_{ij} c_j = 0$ for every vector $\mathbf{b}$ and every vector $\mathbf{c}$, show that $T_{ij} = 0$.

Let S be a closed surface with outward normal $\mathbf{n}$ that encloses a three-dimensional region having volume V. The position vector is $\mathbf{x}$. Use the divergence theorem to find

$$\int_S (\mathbf{b}\cdot\mathbf{x})(\mathbf{c}\cdot\mathbf{n})\, dS$$

for constant vectors $\mathbf{b}$ and $\mathbf{c}$. Hence find

$$\int_S x_i n_j \, dS,$$

and deduce the values of

$$\int_S \mathbf{x}\cdot\mathbf{n}\, dS \quad \text{and} \quad \int_S \mathbf{x}\times\mathbf{n}\, dS.$$

3/II/15G **Methods**

(a) Find the Fourier sine series of the function

$$f(x) = x$$

for $0 \leqslant x \leqslant 1$.

(b) The differential operator L acting on y is given by

$$L[y] = y'' + y'.$$

Show that the eigenvalues λ in the eigenvalue problem

$$L[y] = \lambda y, \quad y(0) = y(1) = 0,$$

are given by $\lambda = -n^2\pi^2 - \frac{1}{4}, \quad n = 1, 2, \ldots$, and find the corresponding eigenfunctions $y_n(x)$.

By expressing the equation $L[y] = \lambda y$ in Sturm-Liouville form or otherwise, write down the orthogonality relation for the y_n. Assuming the completeness of the eigenfunctions and using the result of part (a), find, in the form of a series, a function $y(x)$ which satisfies

$$L[y] = xe^{-x/2}$$

and $y(0) = y(1) = 0$.

4/I/5G **Methods**

A finite-valued function $f(r,\theta,\phi)$, where r,θ,ϕ are spherical polar coordinates, satisfies Laplace's equation in the regions $r<1$ and $r>1$, and $f\to 0$ as $r\to\infty$. At $r=1$, f is continuous and its derivative with respect to r is discontinuous by $A\sin^2\theta$, where A is a constant. Write down the general axisymmetric solution for f in the two regions and use the boundary conditions to find f.

$$\left[\mathit{Hint}:\quad P_2(\cos\theta)=\frac{1}{2}\left(3\cos^2\theta-1\right)\,.\right]$$

4/II/16B **Methods**

The integral

$$I=\int_a^b F(y(x),y'(x))dx\,,$$

where F is some functional, is defined for the class of functions $y(x)$ for which $y(a)=y_0$, with the value $y(b)$ at the upper endpoint unconstrained. Suppose that $y(x)$ extremises the integral among the functions in this class. By considering perturbed paths of the form $y(x)+\epsilon\eta(x)$, with $\epsilon\ll 1$, show that

$$\frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big)-\frac{\partial F}{\partial y}=0$$

and that

$$\frac{\partial F}{\partial y'}\Big|_{x=b}=0\,.$$

Show further that

$$F-y'\frac{\partial F}{\partial y'}=k$$

for some constant k.

A bead slides along a frictionless wire under gravity. The wire lies in a vertical plane with coordinates (x,y) and connects the point A with coordinates $(0,0)$ to the point B with coordinates $(x_0,y(x_0))$, where x_0 is given and $y(x_0)$ can take any value less than zero. The bead is released from rest at A and slides to B in a time T. For a prescribed x_0 find both the shape of the wire, and the value of $y(x_0)$, for which T is as small as possible.

1/II/14E **Methods**

Find the Fourier Series of the function

$$f(\theta) = \begin{cases} 1 & 0 \le \theta < \pi, \\ -1 & \pi \le \theta < 2\pi. \end{cases}$$

Find the solution $\phi(r,\theta)$ of the Poisson equation in two dimensions inside the unit disk $r \le 1$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = f(\theta),$$

subject to the boundary condition $\phi(1,\theta)=0$.

[*Hint: The general solution of* $r^2R'' + rR' - n^2R = r^2$ *is* $R = ar^n + br^{-n} - r^2/(n^2-4)$.]

From the solution, show that

$$\int_{r\le 1} f\phi \, dA = -\frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2(n+2)^2}.$$

2/I/5E **Methods**

Consider the differential equation for $x(t)$ in $t>0$

$$\ddot{x} - k^2 x = f(t),$$

subject to boundary conditions $x(0)=0$, and $\dot{x}(0)=0$. Find the Green function $G(t,t')$ such that the solution for $x(t)$ is given by

$$x(t) = \int_0^t G(t,t') f(t') \, dt'.$$

2/II/15E Methods

Write down the Euler–Lagrange equation for the variational problem for $r(z)$

$$\delta \int_{-h}^{h} F(z, r, r')\, dz = 0,$$

with boundary conditions $r(-h) = r(h) = R$, where R is a given positive constant. Show that if F does not depend explicitly on z, i.e. $F = F(r, r')$, then the equation has a first integral

$$F - r' \frac{\partial F}{\partial r'} = \frac{1}{k},$$

where k is a constant.

An axisymmetric soap film $r(z)$ is formed between two circular rings $r = R$ at $z = \pm H$. Find the equation governing the shape which minimizes the surface area. Show that the shape takes the form

$$r(z) = k^{-1} \cosh kz.$$

Show that there exist no solution if $R/H < \sinh A$, where A is the unique positive solution of $A = \coth A$.

3/I/6E Methods

Describe briefly the method of Lagrangian multipliers for finding the stationary points of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$.

Use the method to find the stationary values of xy subject to the constraint $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3/II/15H **Methods**

Obtain the power series solution about $t = 0$ of

$$(1-t^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2}y - 2t\,\frac{\mathrm{d}}{\mathrm{d}t}y + \lambda\, y = 0\,,$$

and show that regular solutions $y(t) = P_n(t)$, which are polynomials of degree n, are obtained only if $\lambda = n(n+1)$, $n = 0, 1, 2, \ldots$. Show that the polynomial must be even or odd according to the value of n.

Show that

$$\int_{-1}^{1} P_n(t)P_m(t)\,\mathrm{d}t = k_n\delta_{nm}\,,$$

for some $k_n > 0$.

Using the identity

$$\left(x\frac{\partial^2}{\partial x^2}x + \frac{\partial}{\partial t}(1-t^2)\frac{\partial}{\partial t}\right)\frac{1}{(1-2xt+x^2)^{\frac{1}{2}}} = 0\,,$$

and considering an expansion $\sum_n a_n(x)P_n(t)$ show that

$$\frac{1}{(1-2xt+x^2)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} x^n P_n(t)\,, \qquad 0 < x < 1\,,$$

if we assume $P_n(1) = 1$.

By considering

$$\int_{-1}^{1} \frac{1}{1-2xt+x^2}\,\mathrm{d}t\,,$$

determine the coefficient k_n.

4/I/5H **Methods**

Show how the general solution of the wave equation for $y(x,t)$,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}y(x,t)-\frac{\partial^2}{\partial x^2}y(x,t)=0\,,$$

can be expressed as

$$y(x,t)=f(ct-x)+g(ct+x)\,.$$

Show that the boundary conditions $y(0,t)=y(L,t)=0$ relate the functions f and g and require them to be periodic with period $2L$.

Show that, with these boundary conditions,

$$\frac{1}{2}\int_0^L\left(\frac{1}{c^2}\Big(\frac{\partial y}{\partial t}\Big)^2+\Big(\frac{\partial y}{\partial x}\Big)^2\right)\mathrm{d}x=\int_{-L}^{L}g'(ct+x)^2\,\mathrm{d}x\,,$$

and that this is a constant independent of t.

4/II/16H **Methods**

Define an isotropic tensor and show that δ_{ij}, ϵ_{ijk} are isotropic tensors.

For $\hat{\mathbf{x}}$ a unit vector and $\mathrm{d}S(\hat{\mathbf{x}})$ the area element on the unit sphere show that

$$\int \mathrm{d}S(\hat{\mathbf{x}})\,\hat{x}_{i_1}\ldots\hat{x}_{i_n}$$

is an isotropic tensor for any n. Hence show that

$$\int \mathrm{d}S(\hat{\mathbf{x}})\,\hat{x}_i\hat{x}_j=a\delta_{ij}\,,\qquad \int \mathrm{d}S(\hat{\mathbf{x}})\,\hat{x}_i\hat{x}_j\hat{x}_k=0\,,$$
$$\int \mathrm{d}S(\hat{\mathbf{x}})\,\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l=b\big(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\big)\,,$$

for some a,b which should be determined.

Explain why

$$\int_V \mathrm{d}^3x\,\big(x_1+\sqrt{-1}\,x_2\big)^n f(|\mathbf{x}|)=0\,,\quad n=2,3,4\,,$$

where V is the region inside the unit sphere.

[*The general isotropic tensor of rank 4 has the form* $a\,\delta_{ij}\delta_{kl}+b\,\delta_{ik}\delta_{jl}+c\,\delta_{il}\delta_{jk}$.]

1/I/6B **Methods**

Write down the general isotropic tensors of rank 2 and 3.

According to a theory of magnetostriction, the mechanical stress described by a second-rank symmetric tensor σ_{ij} is induced by the magnetic field vector B_i. The stress is linear in the magnetic field,

$$\sigma_{ij} = A_{ijk} B_k,$$

where A_{ijk} is a third-rank tensor which depends only on the material. Show that σ_{ij} can be non-zero only in anisotropic materials.

1/II/17B **Methods**

The equation governing small amplitude waves on a string can be written as

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

The end points $x = 0$ and $x = 1$ are fixed at $y = 0$. At $t = 0$, the string is held stationary in the waveform,

$$y(x, 0) = x(1 - x) \qquad \text{in} \quad 0 \le x \le 1.$$

The string is then released. Find $y(x, t)$ in the subsequent motion.

Given that the energy

$$\int_0^1 \left[\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

is constant in time, show that

$$\sum_{\substack{n \text{ odd} \\ n \geqslant 1}} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

2/I/6B **Methods**

Write down the general form of the solution in polar coordinates (r, θ) to Laplace's equation in two dimensions.

Solve Laplace's equation for $\phi(r, \theta)$ in $0 < r < 1$ and in $1 < r < \infty$, subject to the conditions

$$\phi \to 0 \qquad \text{as} \qquad r \to 0 \text{ and } r \to \infty,$$

$$\phi|_{r=1+} = \phi|_{r=1-} \qquad \text{and} \qquad \left.\frac{\partial \phi}{\partial r}\right|_{r=1+} - \left.\frac{\partial \phi}{\partial r}\right|_{r=1-} = \cos 2\theta + \cos 4\theta.$$

2/II/17B Methods

Let $I_{ij}(P)$ be the moment-of-inertia tensor of a rigid body relative to the point P. If G is the centre of mass of the body and the vector GP has components X_i, show that

$$I_{ij}(P) = I_{ij}(G) + M\left(X_k X_k \delta_{ij} - X_i X_j\right),$$

where M is the mass of the body.

Consider a cube of uniform density and side $2a$, with centre at the origin. Find the inertia tensor about the centre of mass, and thence about the corner $P = (a, a, a)$.

Find the eigenvectors and eigenvalues of $I_{ij}(P)$.

3/I/6D Methods

Let

$$S[x] = \int_0^T \tfrac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \mathrm{d}t\,, \qquad x(0) = a\,, \quad x(T) = b\,.$$

For any variation $\delta x(t)$ with $\delta x(0) = \delta x(T) = 0$, show that $\delta S = 0$ when $x = x_c$ with

$$x_c(t) = \frac{1}{\sin\omega T}\Big[a\,\sin\omega(T-t) + b\,\sin\omega t\Big]\,.$$

By using integration by parts, show that

$$S[x_c] = \left[\tfrac{1}{2}x_c\dot{x}_c\right]_0^T = \frac{\omega}{2\sin\omega T}\Big[(a^2+b^2)\cos\omega T - 2ab\Big]\,.$$

3/II/18D **Methods**

Starting from the Euler–Lagrange equations, show that the condition for the variation of the integral $\int I(y, y')\,\mathrm{d}x$ to be stationary is

$$I - y'\frac{\partial I}{\partial y'} = \text{constant}\,.$$

In a medium with speed of light $c(y)$ the ray path taken by a light signal between two points satisfies the condition that the time taken is stationary. Consider the region $0 < y < \infty$ and suppose $c(y) = e^{\lambda y}$. Derive the equation for the light ray path $y(x)$. Obtain the solution of this equation and show that the light ray between $(-a, 0)$ and $(a, 0)$ is given by

$$e^{\lambda y} = \frac{\cos \lambda x}{\cos \lambda a}\,,$$

if $\lambda a < \frac{\pi}{2}$.

Sketch the path for λa close to $\frac{\pi}{2}$ and evaluate the time taken for a light signal between these points.

[*The substitution* $u = k\,e^{\lambda y}$, *for some constant* k, *should prove useful in solving the differential equation.*]

4/I/6C **Methods**

Chebyshev polynomials $T_n(x)$ satisfy the differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0 \qquad \text{on} \qquad [-1, 1], \tag{$\dagger$}$$

where n is an integer.

Recast this equation into Sturm–Liouville form and hence write down the orthogonality relationship between $T_n(x)$ and $T_m(x)$ for $n \neq m$.

By writing $x = \cos\theta$, or otherwise, show that the polynomial solutions of $(\dagger)$ are proportional to $\cos\left(n \cos^{-1} x\right)$.

4/II/16C **Methods**

Obtain the Green function $G(x,\xi)$ satisfying

$$G'' + \frac{2}{x}G' + k^2 G = \delta(x-\xi),$$

where k is real, subject to the boundary conditions

$$\begin{aligned} G \text{ is finite} \quad &\text{at} \quad x=0, \\ G=0 \quad &\text{at} \quad x=1. \end{aligned}$$

[*Hint: You may find the substitution* $G = H/x$ *helpful.*]

Use the Green function to determine that the solution of the differential equation

$$y'' + \frac{2}{x}y' + k^2 y = 1,$$

subject to the boundary conditions

$$\begin{aligned} y \text{ is finite} \quad &\text{at} \quad x=0, \\ y=0 \quad &\text{at} \quad x=1, \end{aligned}$$

is

$$y = \frac{1}{k^2}\left[1 - \frac{\sin kx}{x \sin k}\right].$$

1/I/2D **Methods**

Fermat's principle of optics states that the path of a light ray connecting two points will be such that the travel time t is a minimum. If the speed of light varies continuously in a medium and is a function $c(y)$ of the distance from the boundary $y = 0$, show that the path of a light ray is given by the solution to

$$c(y)y'' + c'(y)(1 + y'^2) = 0\,,$$

where $y' = \frac{dy}{dx}$, etc. Show that the path of a light ray in a medium where the speed of light c is a constant is a straight line. Also find the path from $(0,0)$ to $(1,0)$ if $c(y) = y$, and sketch it.

1/II/11D **Methods**

(a) Determine the Green's function $G(x,\xi)$ for the operator $\frac{d^2}{dx^2} + k^2$ on $[0,\pi]$ with Dirichlet boundary conditions by solving the boundary value problem

$$\frac{d^2G}{dx^2} + k^2 G = \delta(x-\xi)\ , \quad G(0) = 0,\ G(\pi) = 0$$

when k is not an integer.

(b) Use the method of Green's functions to solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2 y = f(x)\ , \quad y(0) = a,\ y(\pi) = b$$

when k is not an integer.

2/I/2C **Methods**

Explain briefly why the second-rank tensor

$$\int_S x_i x_j \, dS(\mathbf{x})$$

is isotropic, where S is the surface of the unit sphere centred on the origin.

A second-rank tensor is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j) \, dS(\mathbf{x}) \, ,$$

where S is the surface of the unit sphere centred on the origin. Calculate $T(\mathbf{y})$ in the form

$$T_{ij} = \lambda \delta_{ij} + \mu y_i y_j \, ,$$

where λ and μ are to be determined.

By considering the action of T on $\mathbf{y}$ and on vectors perpendicular to $\mathbf{y}$, determine the eigenvalues and associated eigenvectors of T.

2/II/11C **Methods**

State the transformation law for an nth-rank tensor $T_{ij\cdots k}$.

Show that the fourth-rank tensor

$$c_{ijkl} = \alpha \, \delta_{ij} \, \delta_{kl} + \beta \, \delta_{ik} \, \delta_{jl} + \gamma \, \delta_{il} \, \delta_{jk}$$

is isotropic for arbitrary scalars α, β and γ.

The stress σ_{ij} and strain e_{ij} in a linear elastic medium are related by

$$\sigma_{ij} = c_{ijkl} \, e_{kl}.$$

Given that e_{ij} is symmetric and that the medium is isotropic, show that the stress-strain relationship can be written in the form

$$\sigma_{ij} = \lambda \, e_{kk} \, \delta_{ij} + 2\mu \, e_{ij}.$$

Show that e_{ij} can be written in the form $e_{ij} = p\delta_{ij} + d_{ij}$, where d_{ij} is a traceless tensor and p is a scalar to be determined. Show also that necessary and sufficient conditions for the stored elastic energy density $E = \frac{1}{2}\sigma_{ij} \, e_{ij}$ to be non-negative for any deformation of the solid are that

$$\mu \geq 0 \qquad \text{and} \qquad \lambda \geq -\tfrac{2}{3}\mu.$$

3/I/2D **Methods**

Consider the path between two arbitrary points on a cone of interior angle 2α. Show that the arc-length of the path $r(\theta)$ is given by

$$\int (r^2 + r'^2 \mathrm{cosec}^2\,\alpha)^{1/2}\, d\theta \ ,$$

where $r' = \frac{dr}{d\theta}$. By minimizing the total arc-length between the points, determine the equation for the shortest path connecting them.

3/II/12D **Methods**

The transverse displacement $y(x,t)$ of a stretched string clamped at its ends $x = 0, l$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2k\frac{\partial y}{\partial t} \ , \quad y(x,0) = 0, \ \frac{\partial y}{\partial t}(x,0) = \delta(x-a)\,,$$

where $c > 0$ is the wave velocity, and $k > 0$ is the damping coefficient. The initial conditions correspond to a sharp blow at $x = a$ at time $t = 0$.

(a) Show that the subsequent motion of the string is given by

$$y(x,t) = \frac{1}{\sqrt{\alpha_n^2 - k^2}} \sum_n 2e^{-kt} \sin\frac{\alpha_n a}{c} \sin\frac{\alpha_n x}{c} \sin/(\sqrt{\alpha_n^2 - k^2} \ \ t)$$

where $\alpha_n = \pi c n / l$.

(b) Describe what happens in the limits of small and large damping. What critical parameter separates the two cases?

4/I/2D **Methods**

Consider the wave equation in a spherically symmetric coordinate system

$$\frac{\partial^2 u(r,t)}{\partial t^2} = c^2 \Delta u(r,t)\,,$$

where $\Delta u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$ is the spherically symmetric Laplacian operator.

(a) Show that the general solution to the equation above is

$$u(r,t) = \tfrac{1}{r}[f(r+ct) + g(r-ct)]\,,$$

where $f(x), g(x)$ are arbitrary functions.

(b) Using separation of variables, determine the wave field $u(r,t)$ in response to a pulsating source at the origin $u(0,t) = A\,\sin\omega t$.

4/II/11D **Methods**

The velocity potential $\phi(r,\theta)$ for inviscid flow in two dimensions satisfies the Laplace equation

$$\Delta\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0\ .$$

(a) Using separation of variables, derive the general solution to the equation above that is single-valued and finite in each of the domains (i) $0 \leqslant r \leqslant a$; (ii) $a \leqslant r < \infty$.

(b) Assuming ϕ is single-valued, solve the Laplace equation subject to the boundary conditions $\frac{\partial\phi}{\partial r} = 0$ at $r = a$, and $\frac{\partial\phi}{\partial r} \to U\,\cos\theta$ as $r \to \infty$. Sketch the lines of constant potential.

1/I/2A **Methods**

Find the Fourier sine series for $f(x) = x$, on $0 \leqslant x < L$. To which value does the series converge at $x = \frac{3}{2}L$?

Now consider the corresponding cosine series for $f(x) = x$, on $0 \leqslant x < L$. Sketch the cosine series between $x = -2L$ and $x = 2L$. To which value does the series converge at $x = \frac{3}{2}L$? [*You do not need to determine the cosine series explicitly.*]

1/II/11A **Methods**

The potential $\Phi(r, \vartheta)$, satisfies Laplace's equation everywhere except on a sphere of unit radius and $\Phi \to 0$ as $r \to \infty$. The potential is continuous at $r = 1$, but the derivative of the potential satisfies

$$\lim_{r \to 1^+} \frac{\partial \Phi}{\partial r} - \lim_{r \to 1^-} \frac{\partial \Phi}{\partial r} = V \cos^2 \vartheta,$$

where V is a constant. Use the method of separation of variables to find Φ for both $r > 1$ and $r < 1$.

[*The Laplacian in spherical polar coordinates for axisymmetric systems is*

$$\nabla^2 \equiv \frac{1}{r^2} \left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \left(\frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} \right).$$

You may assume that the equation

$$\left((1 - x^2) y' \right)' + \lambda y = 0$$

has polynomial solutions of degree n, which are regular at $x = \pm 1$, if and only if $\lambda = n(n+1)$.]

2/I/2C **Methods**

Write down the transformation law for the components of a second-rank tensor A_{ij} explaining the meaning of the symbols that you use.

A tensor is said to have *cubic symmetry* if its components are unchanged by rotations of $\pi/2$ about each of the three co-ordinate axes. Find the most general second-rank tensor having cubic symmetry.

2/II/11C **Methods**

If $\mathbf{B}$ is a vector, and

$$T_{ij} = \alpha B_i B_j + \beta B_k B_k \delta_{ij} \ ,$$

show for arbitrary scalars α and β that T_{ij} is a symmetric second-rank tensor.

Find the eigenvalues and eigenvectors of T_{ij}.

Suppose now that $\mathbf{B}$ depends upon position $\mathbf{x}$ and that $\nabla\cdot\mathbf{B} = 0$. Find constants α and β such that

$$\frac{\partial}{\partial x_j} T_{ij} = [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i \ .$$

Hence or otherwise show that if $\mathbf{B}$ vanishes everywhere on a surface S that encloses a volume V then

$$\int_V (\nabla \times \mathbf{B}) \times \mathbf{B} \ dV = 0 \ .$$

3/I/2A **Methods**

Write down the wave equation for the displacement $y(x,t)$ of a stretched string with constant mass density and tension. Obtain the general solution in the form

$$y(x,t) = f(x+ct) + g(x-ct),$$

where c is the wave velocity. For a solution in the region $0 \leqslant x < \infty$, with $y(0,t) = 0$ and $y \to 0$ as $x \to \infty$, show that

$$E = \int_0^\infty \left[\frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} c^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx,$$

is constant in time. Express E in terms of the general solution in this case.

3/II/12A **Methods**

Consider the real Sturm-Liouville problem

$$\mathcal{L}y(x) = -(p(x)y')' + q(x)y = \lambda r(x)y,$$

with the boundary conditions $y(a) = y(b) = 0$, where p, q and r are continuous and positive on $[a, b]$. Show that, with suitable choices of inner product and normalisation, the eigenfunctions $y_n(x)$, $n = 1, 2, 3 \ldots$, form an orthonormal set.

Hence show that the corresponding Green's function $G(x, \xi)$ satisfying

$$(\mathcal{L} - \mu r(x))G(x, \xi) = \delta(x - \xi),$$

where μ is not an eigenvalue, is

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n - \mu},$$

where λ_n is the eigenvalue corresponding to y_n.

Find the Green's function in the case where

$$\mathcal{L}y \equiv y'',$$

with boundary conditions $y(0) = y(\pi) = 0$, and deduce, by suitable choice of μ, that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

4/I/2A **Methods**

Use the method of Lagrange multipliers to find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4/II/11A **Methods**

A function $y(x)$ is chosen to make the integral

$$I = \int_a^b f\left(x, y, y', y''\right) dx$$

stationary, subject to given values of $y(a), y'(a), y(b)$ and $y'(b)$. Derive an analogue of the Euler–Lagrange equation for $y(x)$.

Solve this equation for the case where

$$f = x^4 y''^2 + 4y^2 y',$$

in the interval $[0, 1]$ and

$$x^2 y(x) \to 0, \quad xy(x) \to 1$$

as $x \to 0$, whilst

$$y(1) = 2, \quad y'(1) = 0.$$

1/I/2H **Methods**

The even function $f(x)$ has the Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

in the interval $-\pi \leqslant x \leqslant \pi$. Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2.$$

Find the Fourier cosine series of x^2 in the same interval, and show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

1/II/11H **Methods**

Use the substitution $y = x^p$ to find the general solution of

$$\mathcal{L}_x y \equiv \frac{d^2 y}{dx^2} - \frac{2}{x^2} y = 0.$$

Find the Green's function $G(x,\xi)$, $0 < \xi < \infty$, which satisfies

$$\mathcal{L}_x G(x,\xi) = \delta(x-\xi)$$

for $x > 0$, subject to the boundary conditions $G(x,\xi) \to 0$ as $x \to 0$ and as $x \to \infty$, for each fixed ξ.

Hence, find the solution of the equation

$$\mathcal{L}_x y = \begin{cases} 1, & 0 \leqslant x < 1, \\ 0, & x > 1, \end{cases}$$

subject to the same boundary conditions.

Verify that both forms of your solution satisfy the appropriate equation and boundary conditions, and match at $x = 1$.

2/I/2G **Methods**

Show that the symmetric and antisymmetric parts of a second-rank tensor are themselves tensors, and that the decomposition of a tensor into symmetric and antisymmetric parts is unique.

For the tensor A having components

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix},$$

find the scalar a, vector $\mathbf{p}$ and symmetric traceless tensor B such that

$$A\mathbf{x} = a\mathbf{x} + \mathbf{p} \wedge \mathbf{x} + B\mathbf{x}$$

for every vector $\mathbf{x}$.

2/II/11G **Methods**

Explain what is meant by an *isotropic* tensor.

Show that the fourth-rank tensor

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \qquad (*)$$

is isotropic for arbitrary scalars α, β and γ.

Assuming that the most general isotropic tensor of rank 4 has the form $(*)$, or otherwise, evaluate

$$B_{ijkl} = \int_{r<a} x_i x_j \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV,$$

where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$.

3/I/2G **Methods**

Laplace's equation in the plane is given in terms of plane polar coordinates r and θ in the form

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

In each of the cases

$$\text{(i)} \quad 0 \leqslant r \leqslant 1, \quad \text{and} \quad \text{(ii)} \quad 1 \leqslant r < \infty,$$

find the general solution of Laplace's equation which is single-valued and finite.

Solve also Laplace's equation in the annulus $a \leqslant r \leqslant b$ with the boundary conditions

$$\phi = 1 \quad \text{on} \quad r = a \quad \text{for} \quad \text{all} \quad \theta,$$

$$\phi = 2 \quad \text{on} \quad r = b \quad \text{for} \quad \text{all} \quad \theta.$$

3/II/12H **Methods**

Find the Fourier sine series representation on the interval $0 \leqslant x \leqslant l$ of the function

$$f(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ 1, & a \leqslant x \leqslant b, \\ 0, & b < x \leqslant l. \end{cases}$$

The motion of a struck string is governed by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\,\frac{\partial^2 y}{\partial x^2}, \quad \text{for} \quad 0 \leqslant x \leqslant l \quad \text{and} \quad t \geqslant 0,$$

subject to boundary conditions $y = 0$ at $x = 0$ and $x = l$ for $t \geqslant 0$, and to the initial conditions $y = 0$ and $\dfrac{\partial y}{\partial t} = \delta(x - \dfrac{1}{4}l)$ at $t = 0$.

Obtain the solution $y(x,t)$ for this motion. Evaluate $y(x,t)$ for $t = \frac{1}{2}l/c$, and sketch it clearly.

4/I/2H **Methods**

The Legendre polynomial $P_n(x)$ satisfies

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0, 1, \ldots, \; -1 \leqslant x \leqslant 1.$$

Show that $R_n(x) = P_n'(x)$ obeys an equation which can be recast in Sturm–Liouville form and has the eigenvalue $(n-1)(n+2)$. What is the orthogonality relation for $R_n(x), R_m(x)$ for $n \neq m$?

4/II/11H **Methods**

A curve $y(x)$ in the xy-plane connects the points $(\pm a, 0)$ and has a fixed length l, $2a < l < \pi a$. Find an expression for the area A of the surface of the revolution obtained by rotating $y(x)$ about the x-axis.

Show that the area A has a stationary value for

$$y = \frac{1}{k}(\cosh kx - \cosh ka),$$

where k is a constant such that

$$lk = 2 \sinh ka.$$

Show that the latter equation admits a unique positive solution for k.