Part IB

Markov Chains

Paper 3, Section I

8H Markov Chains

A gang of thieves decides to commit a robbery every week. The gang only robs one of three possible targets: Art museums, Banks, or Casinos, which they conveniently denote by $\{A, B, C\}$. The places they rob follows a Markov chain with the following transition probability matrix:

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 3/4 & 0 & 1/4 \\ 3/8 & 1/8 & 1/2 \end{pmatrix}.$$

- (a) Find the stationary distribution of this Markov chain.
- (b) Is the Markov chain reversible?

(c) Since this spate of robberies had been going on for a long time (i.e., the Markov chain is in stationarity), the police approach Detective Holmes for assistance. Detective Holmes arrives at the crime scene, which happens to be a bank. Detective Holmes asks the police, "What is the probability that these thieves robbed a bank two weeks ago, as well?" The police, not having taken Part IB Markov Chains, are stumped. Please help the police by finding this probability.

Paper 4, Section I

7H Markov Chains

Consider the Markov chain in the figure below.



- (a) Let $g(i) = \mathbb{E}_i[T_0]$ be the expected time to get absorbed in state 0 starting from state *i*. Find g(1), g(2) and g(3).
- (b) Suppose the Markov chain is initialised in state 1. What is the probability it will visit 3 before getting absorbed in 0?
- (c) Suppose the Markov chain is initialised in state 1. What is the expected number of visits to state 3 before the chain gets absorbed in 0?

Paper 1, Section II

19H Markov Chains

Label the vertices of a binary tree by all binary vectors, with the exception of the "root" node, which is labeled \emptyset . Let $p_0, p_1 > 0$ such that $p_0 + p_1 < 1$, and let $p = 1 - p_0 - p_1$. Consider a Markov chain X_n on the binary tree with transition probabilities as follows:

$$\mathbb{P}(X_{n+1} = (b_1, b_2, \dots, b_k, i) | X_n = (b_1, b_2, \dots, b_k)) = p_i \quad \text{for } i = 0, 1,$$

$$\mathbb{P}(X_{n+1} = (b_1, b_2, \dots, b_{k-1}) | X_n = (b_1, b_2, \dots, b_k)) = p$$

for any non-root vertex $(b_1, b_2, \ldots, b_k) \in \{0, 1\}^k$, and

$$\mathbb{P}(X_{n+1} = i | X_n = \emptyset) = p_i \quad \text{for } i = 0, 1,$$
$$\mathbb{P}(X_{n+1} = \emptyset | X_n = \emptyset) = p$$

for the root vertex. The figure below shows the states and the transition probabilities for the first two levels of the tree.



- (a) Prove that the Markov chain is irreducible and find its period. Justify your answers.
- (b) What are the conditions on p_0, p_1 so that the chain is transient/null recurrent/positive recurrent? Justify your answer.
- (c) Assume that the p_0, p_1 are chosen such that the chain is positive recurrent. Let $\ell(X_n)$ denote the length of the string representing state X_n . For example, $\ell(\emptyset) = 0$ and $\ell(0010) = 4$. Prove that the following limit exists

$$\lim_{n \to \infty} \mathbb{P}(\ell(X_n) = k | X_0 = \emptyset),$$

and determine its value.

Paper 2, Section II

18H Markov Chains

Let $(X_n)_{n\geq 0}$ be a Markov chain with finite state space S. Let $(Y_n)_{n\geq 0}$ denote another Markov chain on the same state space S. Let P denote the transition probability matrix of $(X_n)_{n\geq 0}$ and Q denote the transition probability matrix of $(Y_n)_{n\geq 0}$. You are given that for each $i, j \in S$,

$$P_{ij} > 0 \implies Q_{ij} > 0.$$

For each of the following statements provide a proof or counterexample:

(a) If $(X_n)_{n \ge 0}$ is irreducible, then $(Y_n)_{n \ge 0}$ is also irreducible.

(b) If every state in $(X_n)_{n \ge 0}$ is aperiodic, then every state in $(Y_n)_{n \ge 0}$ is also aperiodic.

(c) If $(X_n)_{n\geq 0}$ has no transient states, then $(Y_n)_{n\geq 0}$ also has no transient states.

(d) For $i \in S$, let μ_i denote the mean of the first return time to i starting from i, in the Markov chain $(X_n)_{n \ge 0}$, and let η_i denote the mean of the first return time to i starting from i, in the Markov chain $(Y_n)_{n \ge 0}$. Then $\eta_i \le \mu_i$.

Paper 3, Section I

8H Markov Chains

Let X be an irreducible, positive recurrent and reversible Markov chain taking values in S and let π be its invariant distribution. For $A \subseteq S$, we write

$$T_A = \min\{n \ge 0 : X_n \in A\} \quad \text{and} \quad T_A^+ = \min\{n \ge 1 : X_n \in A\}.$$

(a) Prove that for all $A \subseteq S$ and $z \in A$, we have

$$\mathbb{P}_{\pi}(X_{T_A} = z) = \pi(z)\mathbb{E}_{z}[T_A^+].$$

(b) Let π_A be the probability measure defined by $\pi_A(x) = \pi(x)/\pi(A)$ for $x \in A$. Prove that

$$\mathbb{E}_{\pi_A}[T_A^+] = \frac{1}{\pi(A)}$$

Paper 4, Section I

7H Markov Chains

Let X be an irreducible Markov chain with transition matrix P and values in the set S. For $i \in S$, let $T_i = \min\{n \ge 1 : X_n = i\}$ and $V_i = \sum_{n=0}^{\infty} \mathbf{1}(X_n = i)$.

- (a) Suppose $X_0 = i$. Show that V_i has a geometric distribution.
- (b) Suppose X is transient. Prove that for all $i, j \in S$, we have

$$P^n(i,j) \to 0$$
 as $n \to \infty$.

Paper 1, Section II 19H Markov Chains

The *n*-th iteration of the Sierpinski triangle is constructed as follows: start with an equilateral triangle, subdivide it into 4 congruent equilateral triangles, and remove the central one. Repeat the same procedure n - 1 times on each smaller triangle that is not removed. We call G_n the graph whose vertices are the corners of the triangles and edges the segments joining them, as shown in the figure:



Let A, B, and C be the corners of the original triangle. Let X be a simple random walk on G_n , i.e., from every vertex, it jumps to a neighbour chosen uniformly at random. Let

$$T_{BC} = \min\{i \ge 0 : X_i \in \{B, C\}\}.$$

- (a) Suppose n = 1. Show that $\mathbb{E}_A[T_{BC}] = 5$.
- (b) Suppose n = 2. Show that $\mathbb{E}_A[T_{BC}] = 5^2$.
- (c) Show that $\mathbb{E}_A[T_{BC}] = 5^n$ when X is a simple random walk on G_n , for $n \in \mathbb{N}$.

Paper 2, Section II 18H Markov Chains

Let X be a random walk on $\mathbb{N} = \{0, 1, 2, ...\}$ with $X_0 = 0$ and transition matrix given by

$$P(i, i+1) = \frac{1}{3} = 1 - P(i, i-1), \text{ for } i \ge 1, \text{ and } P(0, 0) = \frac{2}{3} = 1 - P(0, 1).$$

- (a) Prove that X is positive recurrent.
- (b) Let Y be an independent walk with matrix P and suppose that $Y_0 = 0$. Find the limit

$$\lim_{n \to \infty} \mathbb{P}(X_n = 0, Y_n = 1) \,,$$

stating clearly any theorems you use.

(c) Let $T = \min\{n \ge 1 : (X_n, Y_n) = (0, 0)\}$. Find the expected number of times that Y visits 1 by time T.

[TURN OVER]

Paper 3, Section I

8H Markov Chains

Consider a Markov chain $(X_n)_{n \ge 0}$ on a state space I.

(a) Define the notion of a *communicating class*. What does it mean for a communicating class to be *closed*?

(b) Taking $I=\{1,\ldots,6\},$ find the communicating classes associated with the transition matrix P given by

	$\int 0$	0	0	0	$\frac{1}{4}$	$\left(\frac{3}{4}\right)$
P =	$\frac{1}{4}$	0	0	0	$\frac{1}{2}$	$\frac{1}{4}$
	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
	$\frac{1}{4}$	$\frac{1}{2}$	0	0	0	$\frac{1}{4}$
	\backslash_1	0	0	0	0	0/

and identify which are closed.

(c) Find the expected time for the Markov chain with transition matrix P above to reach 6 starting from 1.

Paper 4, Section I

7H Markov Chains

Show that the simple symmetric random walk on $\mathbb Z$ is recurrent.

Three particles perform independent simple symmetric random walks on \mathbb{Z} . What is the probability that they are all simultaneously at 0 infinitely often? Justify your answer.

[You may assume without proof that there exist constants A, B > 0 such that $A\sqrt{n}(n/e)^n \leq n! \leq B\sqrt{n}(n/e)^n$ for all positive integers n.]

Paper 1, Section II 19H Markov Chains

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. What is a stopping time of $(X_n)_{n\geq 0}$? What is the strong Markov property?

The exciting game of 'Unopoly' is played by a single player on a board of 4 squares. The player starts with $\pounds m$ (where $m \in \mathbb{N}$). During each turn, the player tosses a fair coin and moves one or two places in a clockwise direction $(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1)$ according to whether the coin lands heads or tails respectively. The player collects $\pounds 2$ each time they pass (or land on) square 1. If the player lands on square 3 however, they immediately lose $\pounds 1$ and go back to square 2. The game continues indefinitely unless the player is on square 2 with $\pounds 0$, in which case the player loses the game and the game ends.



(a) By setting up an appropriate Markov chain, show that if the player is at square 2 with $\pounds m$, where $m \ge 1$, the probability that they are ever at square 2 with $\pounds (m-1)$ is 2/3.

(b) Find the probability of losing the game when the player starts on square 1 with $\pounds m$, where $m \ge 1$.

[*Hint: Take the state space of your Markov chain to be* $\{1, 2, 4\} \times \{0, 1, ...\}$.]

Paper 2, Section II

18H Markov Chains

Let P be a transition matrix on state space I. What does it mean for a distribution π to be an *invariant distribution*? What does it mean for π and P to be in *detailed balance*? Show that if π and P are in detailed balance, then π is an invariant distribution.

(a) Assuming that an invariant distribution exists, state the relationship between this and

- (i) the expected return time to a state i;
- (ii) the expected time spent in a state i between visits to a state k.

(b) Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix $P = (p_{ij})_{i,j\in I}$ where $I = \{0, 1, 2, \ldots\}$. The transition probabilities are given for $i \geq 1$ by

$$p_{ij} = \begin{cases} q^{-(i+2)} & \text{if } j = i+1, \\ q^{-i} & \text{if } j = i-1, \\ 1 - q^{-(i+2)} - q^{-i} & \text{if } j = i, \end{cases}$$

where $q \ge 2$. For $p \in (0, 1]$ let $p_{01} = p = 1 - p_{00}$. Compute the following, justifying your answers:

- (i) The expected time spent in states $\{2, 4, 6, \ldots\}$ between visits to state 1;
- (ii) The expected time taken to return to state 1, starting from 1;
- (iii) The expected time taken to hit state 0 starting from 1.

Paper 2, Section I

7H Markov Chains

Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $\{1,2\}$ and transition matrix

$$P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}$$

where $\alpha, \beta \in (0, 1]$. Compute $\mathbb{P}(X_n = 1 | X_0 = 1)$. Find the value of $\mathbb{P}(X_n = 1 | X_0 = 2)$.

Paper 1, Section II

20H Markov Chains

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. What is a stopping time of $(X_n)_{n\geq 0}$? What is the strong Markov property?

A porter is trying to apprehend a student who is walking along a long narrow path at night. Being unaware of the porter, the student's location Y_n at time $n \ge 0$ evolves as a simple symmetric random walk on \mathbb{Z} . The porter's initial location Z_0 is 2m units to the right of the student, so $Z_0 - Y_0 = 2m$ where $m \ge 1$. The future locations Z_{n+1} of the porter evolve as follows: The porter moves to the left (so $Z_{n+1} = Z_n - 1$) with probability $q \in (\frac{1}{2}, 1)$, and to the right with probability 1 - q whenever $Z_n - Y_n > 2$. When $Z_n - Y_n = 2$, the porter's probability of moving left changes to $r \in (0, 1)$, and the probability of moving right is 1 - r.

(a) By setting up an appropriate Markov chain, show that for $m \ge 2$, the expected time for the porter to be a distance 2(m-1) away from the student is 2/(2q-1).

(b) Show that the expected time for the porter to catch the student, i.e. for their locations to coincide, is

$$\frac{2}{r} + \left(m + \frac{1}{r} - 2\right)\frac{2}{2q-1}.$$

[You may use without proof the fact that the time for the porter to catch the student is finite with probability 1 for any $m \ge 1$.]

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Paper 4, Section I

9H Markov Chains

For a Markov chain X on a state space S with $u, v \in S$, we let $p_{uv}(n)$ for $n \in \{0, 1, ...\}$ be the probability that $X_n = v$ when $X_0 = u$.

(a) Let X be a Markov chain. Prove that if X is recurrent at a state v, then $\sum_{n=0}^{\infty} p_{vv}(n) = \infty$. [You may use without proof that the number of returns of a Markov chain to a state v when starting from v has the geometric distribution.]

(b) Let X and Y be independent simple symmetric random walks on \mathbb{Z}^2 starting from the origin 0. Let $Z = \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = Y_n\}}$. Prove that $\mathbb{E}[Z] = \sum_{n=0}^{\infty} p_{00}(2n)$ and deduce that $\mathbb{E}[Z] = \infty$. [You may use without proof that $p_{xy}(n) = p_{yx}(n)$ for all $x, y \in \mathbb{Z}^2$ and $n \in \mathbb{N}$, and that X is recurrent at 0.]

Paper 3, Section I

9H Markov Chains

Suppose that (X_n) is a Markov chain with state space S.

- (a) Give the definition of a *communicating class*.
- (b) Give the definition of the *period* of a state $a \in S$.
- (c) Show that if two states communicate then they have the same period.

Paper 2, Section II 20H Markov Chains

Fix $n \ge 1$ and let G be the graph consisting of a copy of $\{0, \ldots, n\}$ joining vertices A and B, a copy of $\{0, \ldots, n\}$ joining vertices B and C, and a copy of $\{0, \ldots, n\}$ joining vertices B and D. Let E be the vertex adjacent to B on the segment from B to C. Shown below is an illustration of G in the case n = 5. The vertices are solid squares and edges are indicated by straight lines.



Let (X_k) be a simple random walk on G. In other words, in each time step, X moves to one of its neighbours with equal probability. Assume that $X_0 = A$.

(a) Compute the expected amount of time for X to hit B.

(b) Compute the expected amount of time for X to hit E. [Hint: first show that the expected amount of time x for X to go from B to E satisfies $x = \frac{1}{3} + \frac{2}{3}(L+x)$ where L is the expected return time of X to B when starting from B.]

(c) Compute the expected amount of time for X to hit C. [Hint: for each i, let v_i be the vertex which is i places to the right of B on the segment from B to C. Derive an equation for the expected amount of time x_i for X to go from v_i to v_{i+1} .]

Justify all of your answers.

Paper 1, Section II

20H Markov Chains

Let P be a transition matrix for a Markov chain (X_n) on a state space with N elements with $N < \infty$. Assume that the Markov chain is aperiodic and irreducible and let π be its unique invariant distribution. Assume that $X_0 \sim \pi$.

(a) Let $P^*(x,y) = \mathbb{P}[X_0 = y | X_1 = x]$. Show that $P^*(x,y) = \pi(y)P(y,x)/\pi(x)$.

(b) Let $T = \min\{n \ge 1 : X_n = X_0\}$. Compute $\mathbb{E}[T]$ in terms of an explicit function of N.

(c) Suppose that a cop and a robber start from a common state chosen from π . The robber then takes one step according to P^* and stops. The cop then moves according to P independently of the robber until the cop catches the robber (i.e., the cop visits the state occupied by the robber). Compute the expected amount of time for the cop to catch the robber.

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Paper 3, Section I

9H Markov Chains

The mathematics course at the University of Barchester is a three-year one. After the end-of-year examinations there are three possibilities:

- (i) failing and leaving (probability p);
- (ii) taking that year again (probability q);
- (iii) going on to the next year (or graduating, if the current year is the third one) (probability r).

Thus there are five states for a student (1st year, 2nd year, 3rd year, left without a degree, graduated).

Write down the 5×5 transition matrix. Classify the states, assuming $p, q, r \in (0, 1)$. Find the probability that a student will eventually graduate.

Paper 4, Section I

9H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S.

- (a) What does it mean to say that a distribution π is the *invariant distribution* for the chain?
- (b) What does it mean to say that the chain is *in detailed balance* with respect to a distribution π ? Show that if the chain is in detailed balance with respect to a distribution π then π is the invariant distribution for the chain.
- (c) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise} \end{cases}$$

where D_i is the number of vertices adjacent to vertex *i*. Show that the random walk is in detailed balance with respect to its invariant distribution.

Paper 1, Section II 20H Markov Chains

A coin-tossing game is played by two players, A_1 and A_2 . Each player has a coin and the probability that the coin tossed by player A_i comes up heads is p_i , where $0 < p_i < 1, i = 1, 2$. The players toss their coins according to the following scheme: A_1 tosses first and then after each head, A_2 pays A_1 one pound and A_1 has the next toss, while after each tail, A_1 pays A_2 one pound and A_2 has the next toss.

Define a Markov chain to describe the state of the game. Find the probability that the game ever returns to a state where neither player has lost money.

Paper 2, Section II 20H Markov Chains

For a finite irreducible Markov chain, what is the relationship between the invariant probability distribution and the mean recurrence times of states?

A particle moves on the 2^n vertices of the hypercube, $\{0,1\}^n$, in the following way: at each step the particle is equally likely to move to each of the *n* adjacent vertices, independently of its past motion. (Two vertices are *adjacent* if the Euclidean distance between them is one.) The initial vertex occupied by the particle is (0, 0, ..., 0). Calculate the expected number of steps until the particle

- (i) first returns to (0, 0, ..., 0),
- (ii) first visits $(0, 0, \dots, 0, 1)$,
- (iii) first visits $(0, 0, \dots, 0, 1, 1)$.

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Paper 3, Section I

9H Markov Chains

(a) What does it mean to say that a Markov chain is *reversible*?

(b) Let G be a finite connected graph on n vertices. What does it mean to say that X is a *simple random walk* on G?

Find the unique invariant distribution π of X.

Show that X is reversible when $X_0 \sim \pi$.

[You may use, without proof, results about detailed balance equations, but you should state them clearly.]

Paper 4, Section I

9H Markov Chains

Prove that the simple symmetric random walk on \mathbb{Z}^3 is transient.

[Any combinatorial inequality can be used without proof.]

Paper 1, Section II 20H Markov Chains

A rich and generous man possesses n pounds. Some poor cousins arrive at his mansion. Being generous he decides to give them money. On day 1, he chooses uniformly at random an integer between n-1 and 1 inclusive and gives it to the first cousin. Then he is left with x pounds. On day 2, he chooses uniformly at random an integer between x-1 and 1 inclusive and gives it to the second cousin and so on. If x = 1 then he does not give the next cousin any money. His choices of the uniform numbers are independent. Let X_i be his fortune at the end of day i.

Show that X is a Markov chain and find its transition probabilities.

Let τ be the first time he has 1 pound left, i.e. $\tau = \min\{i \ge 1 : X_i = 1\}$. Show that

$$\mathbb{E}[\tau] = \sum_{i=1}^{n-1} \frac{1}{i}.$$

Paper 2, Section II

20H Markov Chains

Let Y_1, Y_2, \ldots be i.i.d. random variables with values in $\{1, 2, \ldots\}$ and $\mathbb{E}[Y_1] = \mu < \infty$. Moreover, suppose that the greatest common divisor of $\{n : \mathbb{P}(Y_1 = n) > 0\}$ is 1. Consider the following process

 $X_n = \inf\{m \ge n : Y_1 + \ldots + Y_k = m, \text{ for some } k \ge 0\} - n.$

(a) Show that X is a Markov chain and find its transition probabilities.

(b) Let $T_0 = \inf\{n \ge 1 : X_n = 0\}$. Find $\mathbb{E}_0[T_0]$.

(c) Find the limit as $n \to \infty$ of $\mathbb{P}(X_n = 0)$. State carefully any theorems from the course that you are using.

Paper 4, Section I

9H Markov Chains

Consider two boxes, labelled A and B. Initially, there are no balls in box A and k balls in box B. Each minute later, one of the k balls is chosen uniformly at random and is moved to the opposite box. Let X_n denote the number of balls in box A at time n, so that $X_0 = 0$.

(a) Find the transition probabilities of the Markov chain $(X_n)_{n\geq 0}$ and show that it is reversible in equilibrium.

(b) Find $\mathbb{E}(T)$, where $T = \inf\{n \ge 1 : X_n = 0\}$ is the next time that all k balls are again in box B.

Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n \ge 0}$ be a Markov chain such that $X_0 = i$. Prove that

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \frac{1}{\mathbb{P}_i(X_n \neq i \text{ for all } n \ge 1)}$$

where $1/0 = +\infty$. [You may use the strong Markov property without proof.]

Paper 2, Section II 20H Markov Chains

(a) Prove that every open communicating class of a Markov chain is transient. Prove that every finite transient communicating class is open. Give an example of a Markov chain with an infinite transient closed communicating class.

(b) Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $\{a, b, c, d\}$ and transition probabilities given by the matrix

$$P = \begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/4 & 0 & 3/4 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}.$$

- (i) Compute $\mathbb{P}(X_n = b | X_0 = d)$ for a fixed $n \ge 0$.
- (ii) Compute $\mathbb{P}(X_n = c \text{ for some } n \ge 1 | X_0 = a)$.
- (iii) Show that P^n converges as $n \to \infty$, and determine the limit. [Results from lectures can be used without proof if stated carefully.]

Paper 1, Section II 20H Markov Chains

Let $(X_n)_{n \ge 0}$ be a simple symmetric random walk on the integers, starting at $X_0 = 0$.

(a) What does it mean to say that a Markov chain is *irreducible*? What does it mean to say that an irreducible Markov chain is *recurrent*? Show that $(X_n)_{n \ge 0}$ is irreducible and recurrent.

[Hint: You may find it helpful to use the limit

$$\lim_{k \to \infty} \sqrt{k} 2^{-2k} \binom{2k}{k} = \sqrt{\pi}.$$

You may also use without proof standard necessary and sufficient conditions for recurrence.]

(b) What does it mean to say that an irreducible Markov chain is *positive recurrent*? Determine, with proof, whether $(X_n)_{n \ge 0}$ is positive recurrent.

(c) Let

$$T = \inf\{n \ge 1 : X_n = 0\}$$

be the first time the chain returns to the origin. Compute $\mathbb{E}[s^T]$ for a fixed number 0 < s < 1.

Part IB, 2016 List of Questions

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Paper 4, Section I

9H Markov Chains

Let X_0, X_1, X_2, \ldots be independent identically distributed random variables with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p, 0 . Let <math>Z_n = X_{n-1} + cX_n, n = 1, 2, \ldots$, where c is a constant. For each of the following cases, determine whether or not $(Z_n : n \ge 1)$ is a Markov chain:

- (a) c = 0;
- (b) c = 1;
- (c) c = 2.

In each case, if $(Z_n : n \ge 1)$ is a Markov chain, explain why, and give its state space and transition matrix; if it is not a Markov chain, give an example to demonstrate that it is not.

Paper 3, Section I

9H Markov Chains

Define what is meant by a *communicating class* and a *closed class* in a Markov chain.

A Markov chain $(X_n : n \ge 0)$ with state space $\{1, 2, 3, 4\}$ has transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Write down the communicating classes for this Markov chain and state whether or not each class is closed.

If $X_0 = 2$, let N be the smallest n such that $X_n \neq 2$. Find $\mathbb{P}(N = n)$ for n = 1, 2, ...and $\mathbb{E}(N)$. Describe the evolution of the chain if $X_0 = 2$.

Paper 2, Section II

20H Markov Chains

(a) What does it mean for a transition matrix P and a distribution λ to be in *detailed balance*? Show that if P and λ are in detailed balance then $\lambda = \lambda P$.

(b) A mathematician owns r bicycles, which she sometimes uses for her journey from the station to College in the morning and for the return journey in the evening. If it is fine weather when she starts a journey, and if there is a bicycle available at the current location, then she cycles; otherwise she takes the bus. Assume that with probability p, 0 , it is fine when she starts a journey, independently of all other journeys. Let $<math>X_n$ denote the number of bicycles at the current location, just before the mathematician starts the *n*th journey.

- (i) Show that $(X_n; n \ge 0)$ is a Markov chain and write down its transition matrix.
- (ii) Find the invariant distribution of the Markov chain.
- (iii) Show that the Markov chain satisfies the necessary conditions for the convergence theorem for Markov chains and find the limiting probability that the mathematician's nth journey is by bicycle.

[Results from the course may be used without proof provided that they are clearly stated.]

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Paper 1, Section II

20H Markov Chains

Consider a particle moving between the vertices of the graph below, taking steps along the edges. Let X_n be the position of the particle at time n. At time n + 1 the particle moves to one of the vertices adjoining X_n , with each of the adjoining vertices being equally likely, independently of previous moves. Explain briefly why $(X_n; n \ge 0)$ is a Markov chain on the vertices. Is this chain irreducible? Find an invariant distribution for this chain.

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Suppose that the particle starts at B. By adapting the transition matrix, or otherwise, find the probability that the particle hits vertex A before vertex F.

Find the expected first passage time from B to F given no intermediate visit to A.

[Results from the course may be used without proof provided that they are clearly stated.]

Paper 4, Section I

9H Markov Chains

Let $(X_n : n \ge 0)$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$.

(a) Let $W_n = X_{2n}$, n = 0, 1, 2, ... Show that $(W_n : n \ge 0)$ is a Markov chain and give its transition matrix. If $\lambda_i = \mathbb{P}(X_0 = i)$, $i \in S$, find $\mathbb{P}(W_1 = 0)$ in terms of the λ_i and the $p_{i,j}$.

[Results from the course may be quoted without proof, provided they are clearly stated.]

- (b) Suppose that $S = \{-1, 0, 1\}$, $p_{0,1} = p_{-1,-1} = 0$ and $p_{-1,0} \neq p_{1,0}$. Let $Y_n = |X_n|$, $n = 0, 1, 2, \ldots$ In terms of the $p_{i,j}$, find
 - (i) $\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 1, Y_{n-1} = 0)$ and
 - (ii) $\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 1, Y_{n-1} = 1, Y_{n-2} = 0).$

What can you conclude about whether or not $(Y_n : n \ge 0)$ is a Markov chain?

Paper 3, Section I

9H Markov Chains

Let $(X_n : n \ge 0)$ be a homogeneous Markov chain with state space S. For i, j in S let $p_{i,j}(n)$ denote the *n*-step transition probability $\mathbb{P}(X_n = j \mid X_0 = i)$.

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- (i) Express the (m + n)-step transition probability $p_{i,j}(m + n)$ in terms of the *n*-step and *m*-step transition probabilities.
- (ii) Write $i \to j$ if there exists $n \ge 0$ such that $p_{i,j}(n) > 0$, and $i \leftrightarrow j$ if $i \to j$ and $j \to i$. Prove that if $i \leftrightarrow j$ and $i \ne j$ then either both i and j are recurrent or both i and j are transient. [You may assume that a state i is recurrent if and only if $\sum_{n=0}^{\infty} p_{i,i}(n) = \infty$, and otherwise i is transient.]
- (iii) A Markov chain has state space $\{0, 1, 2, 3\}$ and transition matrix

$$\left(\begin{array}{ccccc} \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right),$$

For each state i, determine whether i is recurrent or transient. [Results from the course may be quoted without proof, provided they are clearly stated.]

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Paper 1, Section II

20H Markov Chains

Consider a homogeneous Markov chain $(X_n : n \ge 0)$ with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$. For a state *i*, define the terms *aperiodic*, *positive recurrent* and *ergodic*.

Let $S = \{0, 1, 2, ...\}$ and suppose that for $i \ge 1$ we have $p_{i,i-1} = 1$ and

$$p_{0,0} = 0, \, p_{0,j} = pq^{j-1}, \, j = 1, 2, \dots,$$

where $p = 1 - q \in (0, 1)$. Show that this Markov chain is irreducible.

Let $T_0 = \inf\{n \ge 1 : X_n = 0\}$ be the first passage time to 0. Find $\mathbb{P}(T_0 = n \mid X_0 = 0)$ and show that state 0 is ergodic.

Find the invariant distribution π for this Markov chain. Write down:

- (i) the mean recurrence time for state $i, i \ge 1$;
- (ii) $\lim_{n\to\infty} \mathbb{P}(X_n \neq 0 \mid X_0 = 0).$

[Results from the course may be quoted without proof, provided they are clearly stated.]

Paper 2, Section II

20H Markov Chains

Let $(X_n : n \ge 0)$ be a homogeneous Markov chain with state space S and transition matrix $P = (p_{i,j} : i, j \in S)$. For $A \subseteq S$, let

$$H^A = \inf\{n \ge 0 : X_n \in A\}$$
 and $h_i^A = \mathbb{P}(H^A < \infty \mid X_0 = i), i \in S.$

Prove that $h^A = (h_i^A : i \in S)$ is the minimal non-negative solution to the equations

$$h_i^A = \begin{cases} 1 & \text{for } i \in A\\ \sum_{j \in S} p_{i,j} h_j^A & \text{otherwise.} \end{cases}$$

Three people A, B and C play a series of two-player games. In the first game, two people play and the third person sits out. Any subsequent game is played between the winner of the previous game and the person sitting out the previous game. The overall winner of the series is the first person to win two consecutive games. The players are evenly matched so that in any game each of the two players has probability $\frac{1}{2}$ of winning the game, independently of all other games. For n = 1, 2, ..., let X_n be the ordered pair consisting of the winners of games n and n + 1. Thus the state space is $\{AA, AB, AC, BA, BB, BC, CA, CB, CC\}$, and, for example, $X_1 = AC$ if A wins the first game and C wins the second.

The first game is between A and B. Treating AA, BB and CC as absorbing states, or otherwise, find the probability of winning the series for each of the three players.

Paper 4, Section I

9H Markov Chains

Suppose P is the transition matrix of an irreducible recurrent Markov chain with state space I. Show that if x is an invariant measure and $x_k > 0$ for some $k \in I$, then $x_j > 0$ for all $j \in I$.

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Let

$$\gamma_j^k = p_{kj} + \sum_{t=1}^{\infty} \sum_{i_1 \neq k, \dots, i_t \neq k} p_{ki_t} p_{i_t i_{t-1}} \cdots p_{i_1 j}.$$

Give a meaning to γ_j^k and explain why $\gamma_k^k = 1$.

Suppose x is an invariant measure with $x_k = 1$. Prove that $x_j \ge \gamma_j^k$ for all j.

Paper 3, Section I

9H Markov Chains

Prove that if a distribution π is in detailed balance with a transition matrix P then it is an invariant distribution for P.

Consider the following model with 2 urns. At each time, t = 0, 1, ... one of the following happens:

- with probability β a ball is chosen at random and moved to the other urn (but nothing happens if both urns are empty);
- with probability γ a ball is chosen at random and removed (but nothing happens if both urns are empty);
- with probability α a new ball is added to a randomly chosen urn,

where $\alpha + \beta + \gamma = 1$ and $\alpha < \gamma$. State (i, j) denotes that urns 1, 2 contain *i* and *j* balls respectively. Prove that there is an invariant measure

$$\lambda_{i,j} = \frac{(i+j)!}{i!j!} (\alpha/2\gamma)^{i+j}.$$

Find the proportion of time for which there are n balls in the system.

Paper 1, Section II

20H Markov Chains

A Markov chain has state space $\{a, b, c\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 3/5 & 2/5 \\ 3/4 & 0 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix},$$

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where the rows 1,2,3 correspond to a, b, c, respectively. Show that this Markov chain is equivalent to a random walk on some graph with 6 edges.

Let k(i, j) denote the mean first passage time from i to j.

(i) Find k(a, a) and k(a, b).

(ii) Given $X_0 = a$, find the expected number of steps until the walk first completes a step from b to c.

(iii) Suppose the distribution of X_0 is $(\pi_1, \pi_2, \pi_3) = (5, 4, 3)/12$. Let $\tau(a, b)$ be the least *m* such that $\{a, b\}$ appears as a subsequence of $\{X_0, X_1, \ldots, X_m\}$. By comparing the distributions of $\{X_0, X_1, \ldots, X_m\}$ and $\{X_m, \ldots, X_1, X_0\}$ show that $E\tau(a, b) = E\tau(b, a)$ and that

$$k(b,a) - k(a,b) = \sum_{i \in \{a,b,c\}} \pi_i [k(i,a) - k(i,b)].$$

Paper 2, Section II

20H Markov Chains

(i) Suppose $(X_n)_{n\geq 0}$ is an irreducible Markov chain and $f_{ij} = P(X_n = j \text{ for some } n \geq 1 \mid X_0 = i)$. Prove that $f_{ii} \geq f_{ij}f_{ji}$ and that

$$\sum_{n=0}^{\infty} P_i(X_n = i) = \sum_{n=1}^{\infty} f_{ii}^{n-1}.$$

(ii) Let $(X_n)_{n \ge 0}$ be a symmetric random walk on the \mathbb{Z}^2 lattice. Prove that $(X_n)_{n \ge 0}$ is recurrent. You may assume, for $n \ge 1$,

$$1/2 < 2^{-2n} \sqrt{n} \binom{2n}{n} < 1.$$

(iii) A princess and monster perform independent random walks on the \mathbb{Z}^2 lattice. The trajectory of the princess is the symmetric random walk $(X_n)_{n\geq 0}$. The monster's trajectory, denoted $(Z_n)_{n\geq 0}$, is a sleepy version of an independent symmetric random walk $(Y_n)_{n\geq 0}$. Specifically, given an infinite sequence of integers $0 = n_0 < n_1 < \cdots$, the monster sleeps between these times, so $Z_{n_i+1} = \cdots = Z_{n_{i+1}} = Y_{i+1}$. Initially, $X_0 = (100, 0)$ and $Z_0 = Y_0 = (0, 100)$. The princess is captured if and only if at some future time she and the monster are simultaneously at (0, 0).

Compare the capture probabilities for an active monster, who takes $n_{i+1} = n_i + 1$ for all *i*, and a sleepy monster, who takes n_i spaced sufficiently widely so that

$$P(X_k = (0,0) \text{ for some } k \in \{n_i + 1, \dots, n_{i+1}\}) > 1/2.$$

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Paper 3, Section I

9H Markov Chains

A runner owns k pairs of running shoes and runs twice a day. In the morning she leaves her house by the front door, and in the evening she leaves by the back door. On starting each run she looks for shoes by the door through which she exits, and runs barefoot if none are there. At the end of each run she is equally likely to return through the front or back doors. She removes her shoes (if any) and places them by the door. In the morning of day 1 all shoes are by the back door so she must run barefoot.

Let $p_{00}^{(n)}$ be the probability that she runs barefoot on the morning of day n + 1. What conditions are satisfied in this problem which ensure $\lim_{n\to\infty} p_{00}^{(n)}$ exists? Show that its value is $\pi_0 = 1/(2k+1)$.

Find the expected number of days that will pass until the first morning that she finds all k pairs of shoes at her front door.

Paper 4, Section I

9H Markov Chains

Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain with $p_{ij}^{(n)} = P(X_n = j | X_0 = i)$. Define the meaning of the statements:

- (i) state i is transient,
- (ii) state i is aperiodic.

Give a criterion for transience that can be expressed in terms of the probabilities $(p_{ii}^{(n)}, n = 0, 1, ...).$

Prove that if a state i is transient then all states are transient.

Prove that if a state i is aperiodic then all states are aperiodic.

Suppose that $p_{ii}^{(n)} = 0$ unless *n* is divisible by 3. Given any other state *j*, prove that $p_{jj}^{(n)} = 0$ unless *n* is divisible by 3.

Paper 1, Section II

20H Markov Chains

A Markov chain $(X_n)_{n\geq 0}$ has as its state space the integers, with

$$p_{i,i+1} = p, \quad p_{i,i-1} = q = 1 - p,$$

and $p_{ij} = 0$ otherwise. Assume p > q.

Let $T_j = \inf\{n \ge 1 : X_n = j\}$ if this is finite, and $T_j = \infty$ otherwise. Let V_0 be the total number of hits on 0, and let $V_0(n)$ be the total number of hits on 0 within times $0, \ldots, n-1$. Let

$$h_i = P(V_0 > 0 \mid X_0 = i)$$

$$r_i(n) = E[V_0(n) \mid X_0 = i]$$

$$r_i = E[V_0 \mid X_0 = i].$$

- (i) Quoting an appropriate theorem, find, for every i, the value of h_i .
- (ii) Show that if $(x_i, i \in \mathbb{Z})$ is any non-negative solution to the system of equations

$$x_0 = 1 + qx_1 + px_{-1},$$

 $x_i = qx_{i-1} + px_{i+1}, \text{ for all } i \neq 0,$

then $x_i \ge r_i(n)$ for all i and n.

- (iii) Show that $P(V_0(T_1) \ge k \mid X_0 = 1) = q^k$ and $E[V_0(T_1) \mid X_0 = 1] = q/p$.
- (iv) Explain why $r_{i+1} = (q/p)r_i$ for i > 0.
- (v) Find r_i for all i.

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Paper 2, Section II

20H Markov Chains

Let $(X_n)_{n\geq 0}$ be the symmetric random walk on vertices of a connected graph. At each step this walk jumps from the current vertex to a neighbouring vertex, choosing uniformly amongst them. Let $T_i = \inf\{n \geq 1 : X_n = i\}$. For each $i \neq j$ let $q_{ij} = P(T_j < T_i \mid X_0 = i)$ and $m_{ij} = E(T_j \mid X_0 = i)$. Stating any theorems that you use:

- (i) Prove that the invariant distribution π satisfies detailed balance.
- (ii) Use reversibility to explain why $\pi_i q_{ij} = \pi_j q_{ji}$ for all i, j.

Consider a symmetric random walk on the graph shown below.



- (iii) Find m_{33} .
- (iv) The removal of any edge (i, j) leaves two disjoint components, one which includes i and one which includes j. Prove that $m_{ij} = 1 + 2e_{ij}(i)$, where $e_{ij}(i)$ is the number of edges in the component that contains i.
- (v) Show that $m_{ij} + m_{ji} \in \{18, 36, 54, 72\}$ for all $i \neq j$.

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Paper 3, Section I 9H Markov Chains

Let $(X_n)_{n \ge 0}$ be a Markov chain with state space S.

(i) What does it mean to say that $(X_n)_{n\geq 0}$ has the strong Markov property? Your answer should include the definition of the term *stopping time*.

(ii) Show that

 $\mathbb{P}(X_n=i \text{ at least } k \text{ times } |X_0=i) = \left[\mathbb{P}(X_n=i \text{ at least once } |X_0=i)\right]^k$

for a state $i \in S$. You may use without proof the fact that $(X_n)_{n \ge 0}$ has the strong Markov property.

Paper 4, Section I

9H Markov Chains

Let $(X_n)_{n \ge 0}$ be a Markov chain on a state space S, and let $p_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i).$

(i) What does the term *communicating class* mean in terms of this chain?

(ii) Show that $p_{ii}(m+n) \ge p_{ij}(m)p_{ji}(n)$.

(iii) The period d_i of a state *i* is defined to be

$$d_i = \gcd\{n \ge 1 : p_{ii}(n) > 0\}.$$

Show that if i and j are in the same communicating class and $p_{jj}(r) > 0$, then d_i divides r.

Paper 1, Section II

20H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S.

(i) What does it mean to say π is the invariant distribution for the chain?

(ii) What does it mean to say the chain is in detailed balance with respect to π ?

(iii) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise,} \end{cases}$$

where D_i is the number of vertices adjacent to vertex *i*. Show that the random walk is in detailed balance with respect to its invariant distribution.

(iv) Let π be the invariant distribution for the transition matrix P, and define an inner product for vectors $x, y \in \mathbb{R}^S$ by the formula

$$\langle x, y \rangle = \sum_{i \in S} x_i \pi_i y_i.$$

Show that the equation

$$\langle x, Py \rangle = \langle Px, y \rangle$$

holds for all vectors $x, y \in \mathbb{R}^S$ if and only if the chain is in detailed balance with respect to π . [Here $z \in \mathbb{R}^S$ means $z = (z_i)_{i \in S}$.]

Paper 2, Section II

20H Markov Chains

(i) Let $(X_n)_{n \ge 0}$ be a Markov chain on the finite state space S with transition matrix P. Fix a subset $A \subseteq S$, and let

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$$H = \inf\{n \ge 0 : X_n \in A\}.$$

Fix a function g on S such that $0 < g(i) \leq 1$ for all $i \in S$, and let

$$V_i = \mathbb{E}\left[\prod_{n=0}^{H-1} g(X_n) | X_0 = i\right]$$

where $\prod_{n=0}^{-1} a_n = 1$ by convention. Show that

$$V_i = \begin{cases} 1 & \text{if } i \in A\\ g(i) \sum_{j \in S} P_{ij} V_j & \text{otherwise.} \end{cases}$$

(ii) A flea lives on a polyhedron with N vertices, labelled $1, \ldots, N$. It hops from vertex to vertex in the following manner: if one day it is on vertex i > 1, the next day it hops to one of the vertices labelled $1, \ldots, i-1$ with equal probability, and it dies upon reaching vertex 1. Let X_n be the position of the flea on day n. What are the transition probabilities for the Markov chain $(X_n)_{n \ge 0}$?

(iii) Let H be the number of days the flea is alive, and let

$$V_i = \mathbb{E}(s^H | X_0 = i)$$

where s is a real number such that $0 < s \leq 1$. Show that $V_1 = 1$ and

$$\frac{i}{s}V_{i+1} = V_i + \frac{i-1}{s}V_i$$

for $i \ge 1$. Conclude that

$$\mathbb{E}(s^{H}|X_{0} = N) = \prod_{i=1}^{N-1} \left(1 + \frac{s-1}{i}\right).$$

[Hint. Use part (i) with $A = \{1\}$ and a well-chosen function g.]

(iv) Show that

$$\mathbb{E}(H|X_0 = N) = \sum_{i=1}^{N-1} \frac{1}{i}.$$

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Paper 3, Section I

9E Markov Chains

An intrepid tourist tries to ascend Springfield's famous infinite staircase on an icy day. When he takes a step with his right foot, he reaches the next stair with probability 1/2, otherwise he falls down and instantly slides back to the bottom with probability 1/2. Similarly, when he steps with his left foot, he reaches the next stair with probability 1/3, or slides to the bottom with probability 2/3. Assume that he always steps first with his right foot when he is at the bottom, and alternates feet as he ascends. Let X_n be his position after his *n*th step, so that $X_n = i$ when he is on the stair $i, i = 0, 1, 2, \ldots$, where 0 is the bottom stair.

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(a) Specify the transition probabilities p_{ij} for the Markov chain $(X_n)_{n \ge 0}$ for any $i, j \ge 0$.

(b) Find the equilibrium probabilities π_i , for $i \ge 0$. [Hint: $\pi_0 = 5/9$.]

(c) Argue that the chain is irreducible and aperiodic and evaluate the limit

$$\lim_{n \to \infty} \mathbb{P}\left(X_n = i\right)$$

for each $i \ge 0$.

Paper 4, Section I

9E Markov Chains

Consider a Markov chain $(X_n)_{n \ge 0}$ with state space $\{a, b, c, d\}$ and transition probabilities given by the following table.

	a	b	c	d
a	1/4	1/4	1/2	0
b	0	1/4	0	3/4
c	1/2	0	1/4	1/4
d	0	1/2	0	1/2

By drawing an appropriate diagram, determine the communicating classes of the chain, and classify them as either open or closed. Compute the following transition and hitting probabilities:

- $\mathbb{P}(X_n = b | X_0 = d)$ for a fixed $n \ge 0$,
- $\mathbb{P}(X_n = c \text{ for some } n \ge 1 | X_0 = a).$

Paper 1, Section II

20E Markov Chains

Let $(X_n)_{n \ge 0}$ be a Markov chain.

(a) What does it mean to say that a state *i* is positive recurrent? How is this property related to the equilibrium probability π_i ? You do not need to give a full proof, but you should carefully state any theorems you use.

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(b) What is a communicating class? Prove that if states i and j are in the same communicating class and i is positive recurrent then j is positive recurrent also.

A frog is in a pond with an infinite number of lily pads, numbered $1, 2, 3, \ldots$ She hops from pad to pad in the following manner: if she happens to be on pad *i* at a given time, she hops to one of pads $(1, 2, \ldots, i, i + 1)$ with equal probability.

(c) Find the equilibrium distribution of the corresponding Markov chain.

(d) Now suppose the frog starts on pad k and stops when she returns to it. Show that the expected number of times the frog hops is e(k-1)! where $e = 2.718 \ldots$ What is the expected number of times she will visit the lily pad k + 1?

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Paper 2, Section II

20E Markov Chains

Let $(X_n)_{n \ge 0}$ be a simple, symmetric random walk on the integers $\{\ldots, -1, 0, 1, \ldots\}$, with $X_0 = 0$ and $\mathbb{P}(X_{n+1} = i \pm 1 | X_n = i) = 1/2$. For each integer $a \ge 1$, let $T_a = \inf\{n \ge 0 : X_n = a\}$. Show that T_a is a stopping time.

Define a random variable Y_n by the rule

$$Y_n = \begin{cases} X_n & \text{if } n < T_a, \\ 2a - X_n & \text{if } n \ge T_a. \end{cases}$$

Show that $(Y_n)_{n \ge 0}$ is also a simple, symmetric random walk.

Let $M_n = \max_{0 \le i \le n} X_n$. Explain why $\{M_n \ge a\} = \{T_a \le n\}$ for $a \ge 0$. By using the process $(Y_n)_{n \ge 0}$ constructed above, show that, for $a \ge 0$,

$$\mathbb{P}(M_n \ge a, X_n \le a-1) = \mathbb{P}(X_n \ge a+1),$$

and thus

$$\mathbb{P}(M_n \ge a) = \mathbb{P}(X_n \ge a) + \mathbb{P}(X_n \ge a+1).$$

Hence compute

 $\mathbb{P}\left(M_n=a\right)$

when a and n are positive integers with $n \ge a$. [Hint: if n is even, then X_n must be even, and if n is odd, then X_n must be odd.]

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Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n\geq 0}$ be a simple random walk on the integers: the random variables $\xi_n \equiv X_n - X_{n-1}$ are independent, with distribution

$$P(\xi = 1) = p, \quad P(\xi = -1) = q,$$

where 0 , and <math>q = 1-p. Consider the hitting time $\tau = \inf\{n : X_n = 0 \text{ or } X_n = N\}$, where N > 1 is a given integer. For fixed $s \in (0,1)$ define $\xi_k = E[s^{\tau} : X_{\tau} = 0|X_0 = k]$ for $k = 0, \ldots, N$. Show that the ξ_k satisfy a second-order difference equation, and hence find them.

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Paper 4, Section I

9H Markov Chains

In chess, a bishop is allowed to move only in straight diagonal lines. Thus if the bishop stands on the square marked A in the diagram, it is able in one move to reach any of the squares marked with an asterisk. Suppose that the bishop moves at random around the chess board, choosing at each move with equal probability from the squares it can reach, the square chosen being independent of all previous choices. The bishop starts at the bottom left-hand corner of the board.

If X_n is the position of the bishop at time n, show that $(X_n)_{n\geq 0}$ is a reversible Markov chain, whose statespace you should specify. Find the invariant distribution of this Markov chain.

What is the expected number of moves the bishop will make before first returning to its starting square?

				*		
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*		*				
	A					
*		*				
			*			
				*		
					*	

Paper 1, Section II

19H Markov Chains

A gerbil is introduced into a maze at the node labelled 0 in the diagram. It roams at random through the maze until it reaches the node labelled 1. At each vertex, it chooses to move to one of the neighbouring nodes with equal probability, independently of all other choices. Find the mean number of moves required for the gerbil to reach node 1.

Suppose now that the gerbil is intelligent, in that when it reaches a node it will not immediately return to the node from which it has just come, choosing with equal probability from all other neighbouring nodes. Express the movement of the gerbil in terms of a Markov chain whose states and transition probabilities you should specify. Find the mean number of moves until the intelligent gerbil reaches node 1. Compare with your answer to the first part, and comment briefly.



Paper 2, Section II

20H Markov Chains

Suppose that B is a non-empty subset of the statespace I of a Markov chain X with transition matrix P, and let $\tau \equiv \inf\{n \ge 0 : X_n \in B\}$, with the convention that $\inf \emptyset = \infty$. If $h_i = P(\tau < \infty | X_0 = i)$, show that the equations

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(a)
$$g_i \ge (Pg)_i \equiv \sum_{j \in I} p_{ij}g_j \ge 0 \quad \forall i,$$

$$(b) g_i = 1 \forall i \in B$$

are satisfied by g = h.

If g satisfies (a), prove that g also satisfies

$$(c) g_i \geqslant (\tilde{P}g)_i \quad \forall i,$$

where

$$\tilde{p}_{ij} = \begin{cases} p_{ij} & (i \notin B), \\ \delta_{ij} & (i \in B). \end{cases}$$

By interpreting the transition matrix \tilde{P} , prove that h is the minimal solution to the equations (a), (b).

Now suppose that P is irreducible. Prove that P is recurrent if and only if the only solutions to (a) are constant functions.

1/II/19H Markov Chains

The village green is ringed by a fence with N fenceposts, labelled $0, 1, \ldots, N-1$. The village idiot is given a pot of paint and a brush, and started at post 0 with instructions to paint all the posts. He paints post 0, and then chooses one of the two nearest neighbours, 1 or N-1, with equal probability, moving to the chosen post and painting it. After painting a post, he chooses with equal probability one of the two nearest neighbours, moves there and paints it (regardless of whether it is already painted). Find the distribution of the last post unpainted.

2/II/20H Markov Chains

A Markov chain with state–space $I=\mathbb{Z}^+$ has non-zero transition probabilities $p_{00}=q_0$ and

 $p_{i,i+1} = p_i, \quad p_{i+1,i} = q_{i+1} \qquad (i \in I).$

Prove that this chain is recurrent if and only if

$$\sum_{n \ge 1} \prod_{r=1}^n \frac{q_r}{p_r} = \infty.$$

Prove that this chain is positive-recurrent if and only if

$$\sum_{n \ge 1} \prod_{r=1}^n \frac{p_{r-1}}{q_r} < \infty.$$

3/I/9H Markov Chains

What does it mean to say that a Markov chain is recurrent?

Stating clearly any general results to which you appeal, prove that the symmetric simple random walk on $\mathbb Z$ is recurrent.

4/I/9H Markov Chains

A Markov chain on the state–space $I = \{1, 2, 3, 4, 5, 6, 7\}$ has transition matrix

	/ 0	1/2	1/4	0	1/4	0	0 \	
	1/3	0	1/2	0	0	1/6	0	
	0	0	0	1	0	0	0	
P =	0	0	1	0	0	0	0	
	0	0	0	0	0	1	0	
	0	0	0	0	0	0	1	
	0 /	0	0	0	1/2	0	1/2/	

Classify the chain into its communicating classes, deciding for each what the period is, and whether the class is recurrent.

For each $i, j \in I$ say whether the limit $\lim_{n\to\infty} p_{ij}^{(n)}$ exists, and evaluate the limit when it does exist.

1/II/19C Markov Chains

Consider a Markov chain $(X_n)_{n \ge 0}$ on states $\{0, 1, \ldots, r\}$ with transition matrix (P_{ij}) , where $P_{0,0} = 1 = P_{r,r}$, so that 0 and r are absorbing states. Let

 $A = (X_n = 0, \text{ for some } n \ge 0),$

be the event that the chain is absorbed in 0. Assume that $h_i = \mathbb{P}(A \mid X_0 = i) > 0$ for $1 \leq i < r$.

Show carefully that, conditional on the event A, $(X_n)_{n \ge 0}$ is a Markov chain and determine its transition matrix.

Now consider the case where $P_{i,i+1} = \frac{1}{2} = P_{i,i-1}$, for $1 \leq i < r$. Suppose that $X_0 = i, 1 \leq i < r$, and that the event A occurs; calculate the expected number of transitions until the chain is first in the state 0.

2/II/20C Markov Chains

Consider a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition matrix given by

$$P_{i,j} = \begin{cases} qp^{j-i+1} & \text{ for } i \ge 1 \text{ and } j \ge i-1, \\ qp^j & \text{ for } i=0 \text{ and } j \ge 0, \end{cases}$$
where $0 < n-1$, $q < 1$

and $P_{i,j} = 0$ otherwise, where 0 .

For each value of p, 0 , determine whether the chain is transient, null recurrent or positive recurrent, and in the last case find the invariant distribution.

3/I/9C Markov Chains

Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \beta & \beta \end{pmatrix},$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

Calculate $\mathbb{P}(X_n = 0 \mid X_0 = 0)$ for each $n \ge 0$.

4/I/9C Markov Chains

For a Markov chain with state space S, define what is meant by the following:

- (i) states $i, j \in S$ communicate;
- (ii) state $i \in S$ is recurrent.

Prove that communication is an equivalence relation on S and that if two states i, j communicate and i is recurrent then j is recurrent.



1/II/19C Markov Chains

Explain what is meant by a stopping time of a Markov chain $(X_n)_{n\geq 0}$. State the strong Markov property.

Show that, for any state *i*, the probability, starting from *i*, that $(X_n)_{n\geq 0}$ makes infinitely many visits to *i* can take only the values 0 or 1.

Show moreover that, if

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \infty,$$

then $(X_n)_{n\geq 0}$ makes infinitely many visits to *i* with probability 1.

2/II/20C Markov Chains

Consider the Markov chain $(X_n)_{n\geq 0}$ on the integers \mathbb{Z} whose non-zero transition probabilities are given by $p_{0,1} = p_{0,-1} = 1/2$ and

$$p_{n,n-1} = 1/3, \quad p_{n,n+1} = 2/3, \quad \text{for } n \ge 1,$$

 $p_{n,n-1} = 3/4, \quad p_{n,n+1} = 1/4, \quad \text{for } n \le -1.$

(a) Show that, if $X_0 = 1$, then $(X_n)_{n \ge 0}$ hits 0 with probability 1/2.

(b) Suppose now that $X_0 = 0$. Show that, with probability 1, as $n \to \infty$ either $X_n \to \infty$ or $X_n \to -\infty$.

(c) In the case $X_0 = 0$ compute $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

3/I/9C Markov Chains

A hungry student always chooses one of three places to get his lunch, basing his choice for one day on his gastronomic experience the day before. He sometimes tries a sandwich from Natasha's Patisserie: with probability 1/2 this is delicious so he returns the next day; if the sandwich is less than delicious, he chooses with equal probability 1/4 either to eat in Hall or to cook for himself. Food in Hall leaves no strong impression, so he chooses the next day each of the options with equal probability 1/3. However, since he is a hopeless cook, he never tries his own cooking two days running, always preferring to buy a sandwich the next day. On the first day of term the student has lunch in Hall. What is the probability that 60 days later he is again having lunch in Hall?

[*Note* $0^0 = 1$.]

4/I/9C Markov Chains

A game of chance is played as follows. At each turn the player tosses a coin, which lands heads or tails with equal probability 1/2. The outcome determines a score for that turn, which depends also on the cumulative score so far. Write S_n for the cumulative score after n turns. In particular $S_0 = 0$. When S_n is odd, a head scores 1 but a tail scores 0. When S_n is a multiple of 4, a head scores 4 and a tail scores 1. When S_n is even but is not a multiple of 4, a head scores 2 and a tail scores 1. By considering a suitable four-state Markov chain, determine the long run proportion of turns for which S_n is a multiple of 4. State clearly any general theorems to which you appeal.

1/II/19D Markov Chains

Every night Lancelot and Guinevere sit down with four guests for a meal at a circular dining table. The six diners are equally spaced around the table and just before each meal two individuals are chosen at random and they exchange places from the previous night while the other four diners stay in the same places they occupied at the last meal; the choices on successive nights are made independently. On the first night Lancelot and Guinevere are seated next to each other.

Find the probability that they are seated diametrically opposite each other on the (n+1)th night at the round table, $n \ge 1$.

2/II/20D Markov Chains

Consider a Markov chain $(X_n)_{n \ge 0}$ with state space $\{0, 1, 2, \ldots\}$ and transition probabilities given by

 $P_{i,j} = pq^{i-j+1}, \quad 0 < j \le i+1, \text{ and } P_{i,0} = q^{i+1} \text{ for } i \ge 0,$

with $P_{i,j} = 0$, otherwise, where 0 and <math>q = 1 - p.

For each $i \ge 1$, let

$$h_i = \mathbb{P}(X_n = 0, \text{ for some } n \ge 0 \mid X_0 = i),$$

that is, the probability that the chain ever hits the state 0 given that it starts in state *i*. Write down the equations satisfied by the probabilities $\{h_i, i \ge 1\}$ and hence, or otherwise, show that they satisfy a second-order recurrence relation with constant coefficients. Calculate h_i for each $i \ge 1$.

Determine for each value of p, 0 , whether the chain is transient, null recurrent or positive recurrent and in the last case calculate the stationary distribution.

[*Hint:* When the chain is positive recurrent, the stationary distribution is geometric.]

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3/I/9D Markov Chains

Prove that if two states of a Markov chain communicate then they have the same period.

Consider a Markov chain with state space $\{1,2,\ldots,7\}$ and transition probabilities determined by the matrix

$(^{0})$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
0	0	0	0	0	0	1
0	0	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	0
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$
0	0	0	0	0	1	0
$\sqrt{0}$	1	0	0	0	0	0/

•

Identify the communicating classes of the chain and for each class state whether it is open or closed and determine its period.

4/I/9D Markov Chains

Prove that the simple symmetric random walk in three dimensions is transient.

[You may wish to recall Stirling's formula: $n! \sim (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}$.]



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1/I/11H Markov Chains

Let $P = (P_{ij})$ be a transition matrix. What does it mean to say that P is (a) irreducible, (b) recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix \tilde{P} by

$$\tilde{P}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ (1 - P_{ii})^{-1} P_{ij} & \text{if } i \neq j. \end{cases}$$

Prove that \tilde{P} is also irreducible and recurrent.

1/II/22H Markov Chains

Consider the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$\int 0$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{\overline{1}}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	
$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	•
0	0	0	0	1	0	
$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	$\left(\frac{1}{4}\right)$	

Determine the communicating classes of the chain, and for each class indicate whether it is open or closed.

Suppose that the chain starts in state 2; determine the probability that it ever reaches state 6.

Suppose that the chain starts in state 3; determine the probability that it is in state 6 after exactly n transitions, $n \ge 1$.

2/I/11H Markov Chains

Let $(X_r)_{r\geq 0}$ be an irreducible, positive-recurrent Markov chain on the state space S with transition matrix (P_{ij}) and initial distribution $P(X_0 = i) = \pi_i, i \in S$, where (π_i) is the unique invariant distribution. What does it mean to say that the Markov chain is reversible?

Prove that the Markov chain is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$.



2/II/22H Markov Chains

Consider a Markov chain on the state space $S = \{0, 1, 2, ...\} \cup \{1', 2', 3', ...\}$ with transition probabilities as illustrated in the diagram below, where 0 < q < 1 and p = 1 - q.



For each value of q, 0 < q < 1, determine whether the chain is transient, null recurrent or positive recurrent.

When the chain is positive recurrent, calculate the invariant distribution.