Part IB

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Linear Algebra

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Paper 1, Section I

1F   Linear Algebra

Let V and W be finite-dimensional real vector spaces, and $\mathcal{L}(V, W)$ denote the vector space of linear maps from V to W. Prove that the dimensions of these vector spaces satisfy

$$\dim(\mathcal{L}(V, W)) = \dim(V) \cdot \dim(W).$$

If $A \leqslant V$ and $B \leqslant W$ are vector subspaces, let

$$X = \{\phi \in \mathcal{L}(V, W) \,:\, \phi(A) \leqslant B\},$$

which you may assume is a vector subspace of $\mathcal{L}(V, W)$. Prove a formula for the dimension of X in terms of the dimensions of V, W, A and B.

If S and T are vector subspaces of V such that $V = S + T$, let

$$Y = \{\phi \in \mathcal{L}(V, V) \,:\, \phi(S) \leqslant S \text{ and } \phi(T) \leqslant T\},$$

which you may assume is a vector subspace of $\mathcal{L}(V, V)$. Prove a formula for the dimension of Y in terms of the dimensions of V, S, and T.

Paper 4, Section I

1F   Linear Algebra

Let V be a finite-dimensional real vector space. What is a *non-degenerate bilinear form* on V?

If $B_1(-, -)$ is a non-degenerate bilinear form on V and $B_2(-, -)$ is a bilinear form on V, which may be degenerate, show that there is a linear map $\alpha : V \to V$ such that

$$B_2(v, w) = B_1(v, \alpha(w)) \text{ for all } v, w \in V.$$

Show that

$$\{w \in V \,:\, B_2(v, w) = 0 \text{ for all } v \in V\} = \operatorname{Ker}(\alpha).$$

[*You may use any results on dual vector spaces provided they are clearly stated.*]

Paper 1, Section II

8F   Linear Algebra

For each of the following statements give a proof or counterexample.

(a) If A and B are 3×3 complex matrices with the same characteristic polynomial and the same minimal polynomial, then they are conjugate.

(b) There are three mutually non-conjugate complex matrices with characteristic polynomial $(2 - t)^2(1 - t)^5$ and minimal polynomial $(2 - t)^2(1 - t)^2$.

(c) If $\alpha : V \to V$ is a linear isomorphism from a finite-dimensional complex vector space to itself such that some iterate α^N with $N > 0$ is diagonalisable, then α is diagonalisable.

(d) A real matrix which is diagonalisable when considered as a complex matrix is also diagonalisable as a real matrix.

(e) Two real matrices which are conjugate when considered as complex matrices are also conjugate as real matrices.

Paper 2, Section II

8F   Linear Algebra

What is the *characteristic polynomial* of a square matrix A?

State and prove the Cayley–Hamilton theorem for square complex matrices.

For square matrices X and Y let us write $[X, Y] = XY - YX$. Given another square matrix Z, show that $[X, YZ] = [X, Y]Z + Y[X, Z]$.

Suppose now that A and B are square complex matrices such that $[B, A]$ commutes with A, i.e. $[[B, A], A] = 0$. Show that for any polynomial $\varphi(t)$ we have

$$[B, \varphi(A)] = \varphi'(A)[B, A],$$

where $\varphi'(t)$ denotes the derivative of φ. For a polynomial $f(t)$, whose kth derivative is denoted by $f^{(k)}(t)$, satisfying $f(A) = 0$, show by induction that $f^{(k)}(A)[B, A]^{2^k - 1} = 0$. Deduce that some power of the matrix $[B, A]$ is zero.

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Paper 3, Section II

9F   Linear Algebra

Let V be a finite-dimensional real inner product space, and $\alpha : V \to V$ be a linear map. What does it mean to say that α is *self-adjoint*?

If $\alpha : V \to V$ is self-adjoint, prove that there is an orthonormal basis for V consisting of eigenvectors of α.

Let P_n denote the vector space of real polynomials of degree at most n. Show that

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) e^{-x} \, dx$$

defines an inner product on this vector space, and that the linear map $\alpha : P_n \to P_n$ given by

$$\alpha(f) = x f'' + (1 - x) f'$$

is self-adjoint with respect to this inner product.

Show that α has eigenvalues $0, -1, -2, -3, \ldots, -n$. When $n = 2$ determine corresponding eigenvectors.

[*Hint: You may use the identity $\int_0^\infty x^n e^{-x} \, dx = n!$.*]

Paper 4, Section II

8F   Linear Algebra

If V and W are finite-dimensional vector spaces and $\gamma : V \to W$ is a linear map, what is the *matrix representation* of γ with respect to bases $\mathcal{B}$ of V and $\mathcal{C}$ of W?

If $\alpha, \beta : V \to V$ are linear maps, what does it mean to say that they are *conjugate*? How is this interpreted in terms of matrices representing α and β with respect to a basis $\mathcal{B}$ of V?

Let V be a vector space and $\beta : V \to V$ be a linear isomorphism. Write $\mathcal{L}(V, V)$ for the vector space of linear maps from V to V, and define a function by

$$\phi_\beta : \mathcal{L}(V, V) \longrightarrow \mathcal{L}(V, V)$$
$$\alpha \longmapsto \beta^{-1} \alpha \beta.$$

Show that ϕ_β is a linear isomorphism, and that if β is conjugate to β' then ϕ_β is conjugate to $\phi_{\beta'}$.

Assuming that V is a 2-dimensional complex vector space, determine the Jordan Normal Form of ϕ_β in terms of that of β.

Paper 1, Section I
1F   Linear Algebra

Define the determinant of a matrix $A \in M_n(\mathbb{C})$.

(a) Assume A is a block matrix of the form $\begin{pmatrix} M & X \\ 0 & N \end{pmatrix}$, where M and N are square matrices. Show that $\det A = \det M \det N$.

(b) Assume A is a block matrix of the form $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$, where M and N are square matrices of sizes k and $n - k$. Express $\det A$ in terms of $\det M$ and $\det N$.

[You may assume properties of column operations if clearly stated.]

Paper 4, Section I
1F   Linear Algebra

What is a *Hermitian form* on a complex vector space V? If φ and ψ are two Hermitian forms and $\varphi(v, v) = \psi(v, v)$ for all $v \in V$, prove that $\varphi(v, w) = \psi(v, w)$ for all $v, w \in V$.

Determine whether the Hermitian form on $\mathbb{C}^2$ defined by the matrix

$$A = \begin{pmatrix} 4 & 2i \\ -2i & 3 \end{pmatrix}$$

is positive definite.

Paper 1, Section II
8F   Linear Algebra

(a) Let V be a finite dimensional complex inner product space, and let α be an endomorphism of V. Define its adjoint α^*.

Assume that α is normal, i.e. α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$.

  (i) Show that α and α^* have a common eigenvector $\mathbf{v}$. What is the relation between the corresponding eigenvalues?

  (ii) Deduce that V has an orthonormal basis of eigenvectors of α.

(b) Now consider a real matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ which is skew-symmetric, i.e. $A^T = -A$.

  (i) Can A have a non-zero real eigenvalue?

  (ii) Use the results of part (a) to show that there exists an orthogonal matrix $R \in O(n)$ such that $R^T A R$ is block-diagonal with the non-zero blocks of the form $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Paper 2, Section II
8F   Linear Algebra

Let V be a real vector space (not necessarily finite-dimensional). Define the *dual space* V^*. Prove that if f_1, $f_2 \in V^*$ are such that $f_1(v)f_2(v) = 0$ for all $v \in V$, then f_1 or f_2 is the zero element in V^*.

Now suppose that V is a finite-dimensional real vector space.

Let ϕ be a symmetric bilinear form on V. State Sylvester's law of inertia for ϕ.

Let q be a quadratic form on V, let r denote its rank and σ its signature. Show that q can be factorised as $q(v) = f_1(v)f_2(v)$ with $f_1, f_2 \in V^*$ for all $v \in V$ if and only if $r + |\sigma| \leqslant 2$.

A vector $v_0 \in V$ is called isotropic if $q(v_0) = 0$. Show that if there exist v_1 and v_2 in V such that $q(v_1) > 0$ and $q(v_2) < 0$, then one can construct a basis of V consisting of isotropic vectors.

Paper 3, Section II
9F   Linear Algebra

Suppose that α is an endomorphism of an n-dimensional complex vector space. Define the *minimal polynomial m_α* of α. State the Cayley–Hamilton theorem, and explain why m_α exists and is unique.

(a) If α has minimal polynomial $m_\alpha(x) = x^m$, what is the minimal polynomial of α^3?

(b) If $\lambda \neq 0$ is an eigenvalue for α, show that λ^3 is an eigenvalue for α^3. Describe the λ^3–eigenspace of α^3 in terms of eigenspaces of α.

(c) Assume α is invertible with minimal polynomial $m_\alpha(x) = \prod_{i=1}^k (x - \lambda_i)^{c_i}$.

   (i) Show that the minimal polynomial m_{α^3} of α^3 must divide $\prod_{i=1}^k (x - \lambda_i^3)^{c_i}$.

   (ii) Prove that equality holds if in addition all λ_i are real (in other words, we have $m_{\alpha^3}(x) = \prod_{i=1}^k (x - \lambda_i^3)^{c_i}$).

2022

Paper 4, Section II
8F   Linear Algebra

Let V and W be finite dimensional vector spaces, and α a linear map from V to W. Define the *rank $r(\alpha)$* and *nullity $n(\alpha)$* of α. State and prove the rank-nullity theorem.

Assume now that α and β are linear maps from V to itself, and let $n = \dim V$. Prove the following inequalities for the linear maps $\alpha + \beta$ and $\alpha\beta$:

$$|r(\alpha) - r(\beta)| \leqslant r(\alpha + \beta) \leqslant \min\{r(\alpha) + r(\beta), n\}$$

and

$$\max\{r(\alpha) + r(\beta) - n, 0\} \leqslant r(\alpha\beta) \leqslant \min\{r(\alpha), r(\beta)\}.$$

For arbitrary values of n and $0 \leqslant r(\alpha), r(\beta) \leqslant n$, show that each of the four bounds can be attained for some (α, β). Can both upper bounds always be attained simultaneously?

Paper 1, Section I
1E Linear Algebra

Let V be a vector space over $\mathbb{R}$, $\dim V = n$, and let $\langle,\rangle$ be a non-degenerate anti-symmetric bilinear form on V.

Let $v \in V$, $v \neq 0$. Show that $v^\perp$ is of dimension $n-1$ and $v \in v^\perp$. Show that if $W \subseteq v^\perp$ is a subspace with $W \oplus \mathbb{R}v = v^\perp$, then the restriction of $\langle,\rangle$ to W is non-degenerate.

Conclude that the dimension of V is even.

Paper 4, Section I
1E Linear Algebra

Let $\mathrm{Mat}_n(\mathbb{C})$ be the vector space of n by n complex matrices.

Given $A \in \mathrm{Mat}_n(\mathbb{C})$, define the linear map $\varphi_A : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$,

$$X \mapsto AX - XA.$$

(i) Compute a basis of eigenvectors, and their associated eigenvalues, when A is the diagonal matrix

$$A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}.$$

What is the rank of φ_A?

(ii) Now let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Write down the matrix of the linear transformation φ_A with respect to the standard basis of $\mathrm{Mat}_2(\mathbb{C})$.

What is its Jordan normal form?

Paper 1, Section II
8E   Linear Algebra

Let $d \geqslant 1$, and let $J_d = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \ldots & \ldots & \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{C})$.

(a) (i) Compute J_d^n, for all $n \geqslant 0$.

(ii) Hence, or otherwise, compute $(\lambda I + J_d)^n$, for all $n \geqslant 0$.

(b) Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $\varphi \in \mathrm{End}(V)$. Suppose $\varphi^n = 0$ for some $n > 1$.

(i) Determine the possible eigenvalues of φ.

(ii) What are the possible Jordan blocks of φ?

(iii) Show that if $\varphi^2 = 0$, there exists a decomposition

$$V = U \oplus W_1 \oplus W_2,$$

where $\varphi(U) = \varphi(W_1) = 0$, $\varphi(W_2) = W_1$, and $\dim W_2 = \dim W_1$.

Paper 2, Section II
8E   Linear Algebra

(a) Compute the characteristic polynomial and minimal polynomial of

$$A = \begin{pmatrix} -2 & -6 & -9 \\ 3 & 7 & 9 \\ -1 & -2 & -2 \end{pmatrix}.$$

Write down the Jordan normal form for A.

(b) Let V be a finite-dimensional vector space over $\mathbb{C}$, $f : V \to V$ be a linear map, and for $\alpha \in \mathbb{C}$, $n \geqslant 1$, write

$$W_{\alpha,n} := \{ v \in V \mid (f - \alpha I)^n v = 0 \}.$$

(i) Given $v \in W_{\alpha,n}$, $v \neq 0$, construct a non-zero eigenvector for f in terms of v.

(ii) Show that if $w_1, \ldots, w_d$ are non-zero eigenvectors for f with eigenvalues $\alpha_1, \ldots, \alpha_d$, and $\alpha_i \neq \alpha_j$ for all $i \neq j$, then $w_1, \ldots, w_d$ are linearly independent.

(iii) Show that if $v_1 \in W_{\alpha_1,n}, \ldots, v_d \in W_{\alpha_d,n}$ are all non-zero, and $\alpha_i \neq \alpha_j$ for all $i \neq j$, then $v_1, \ldots, v_d$ are linearly independent.

Paper 3, Section II

9E   Linear Algebra

(a)(i) State the rank-nullity theorem.

Let U and W be vector spaces. Write down the definition of their direct sum $U \oplus W$ and the inclusions $i : U \to U \oplus W$, $j : W \to U \oplus W$.

Now let U and W be subspaces of a vector space V. Define $l : U \cap W \to U \oplus W$ by $l(x) = ix - jx$.

Describe the quotient space $(U \oplus W)/\text{Im}(l)$ as a subspace of V.

(ii) Let $V = \mathbb{R}^5$, and let U be the subspace of V spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \\ 1 \\ -2 \end{pmatrix},$$

and W the subspace of V spanned by the vectors

$$\begin{pmatrix} 3 \\ 2 \\ -3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

Determine the dimension of $U \cap W$.

(b) Let A, B be complex n by n matrices with $\text{rank}(B) = k$.

Show that $\det(A + tB)$ is a polynomial in t of degree at most k.

Show that if $k = n$ the polynomial is of degree precisely n.

Give an example where $k \geqslant 1$ but this polynomial is zero.

Paper 4, Section II

8E   Linear Algebra

(a) Let V be a complex vector space of dimension n.

What is a *Hermitian form* on V?

Given a Hermitian form, define the matrix A of the form with respect to the basis $v_1, \ldots, v_n$ of V, and describe in terms of A the value of the Hermitian form on two elements of V.

Now let $w_1, \ldots, w_n$ be another basis of V. Suppose $w_i = \sum_j p_{ij} v_j$, and let $P = (p_{ij})$. Write down the matrix of the form with respect to this new basis in terms of A and P.

Let $N = V^\perp$. Describe the dimension of N in terms of the matrix A.

(b) Write down the matrix of the real quadratic form

$$x^2 + y^2 + 2z^2 + 2xy + 2xz - 2yz.$$

Using the Gram–Schmidt algorithm, find a basis which diagonalises the form. What are its rank and signature?

(c) Let V be a real vector space, and $\langle, \rangle$ a symmetric bilinear form on it. Let A be the matrix of this form in some basis.

Prove that the signature of $\langle, \rangle$ is the number of positive eigenvalues of A minus the number of negative eigenvalues.

Explain, using an example, why the eigenvalues themselves depend on the choice of a basis.

Paper 1, Section I

1F   Linear Algebra

Define what it means for two $n \times n$ matrices A and B to be *similar*. Define the *Jordan normal form* of a matrix.

Determine whether the matrices

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

are similar, carefully stating any theorem you use.

Paper 1, Section II

8F   Linear Algebra

Let $\mathcal{M}_n$ denote the vector space of $n \times n$ matrices over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What is the *rank* $r(A)$ of a matrix $A \in \mathcal{M}_n$?

Show, stating accurately any preliminary results that you require, that $r(A) = n$ if and only if A is non-singular, i.e. $\det A \neq 0$.

Does $\mathcal{M}_n$ have a basis consisting of non-singular matrices? Justify your answer.

Suppose that an $n \times n$ matrix A is non-singular and every entry of A is either 0 or 1. Let c_n be the largest possible number of 1's in such an A. Show that $c_n \leqslant n^2 - n + 1$. Is this bound attained? Justify your answer.

[*Standard properties of the adjugate matrix can be assumed, if accurately stated.*]

Paper 2, Section II

8F   Linear Algebra

Let V be a finite-dimensional vector space over a field. Show that an endomorphism α of V is idempotent, i.e. $\alpha^2 = \alpha$, if and only if α is a projection onto its image.

Determine whether the following statements are true or false, giving a proof or counterexample as appropriate:

(i) If $\alpha^3 = \alpha^2$, then α is idempotent.

(ii) The condition $\alpha(1 - \alpha)^2 = 0$ is equivalent to α being idempotent.

(iii) If α and β are idempotent and such that $\alpha + \beta$ is also idempotent, then $\alpha\beta = 0$.

(iv) If α and β are idempotent and $\alpha\beta = 0$, then $\alpha + \beta$ is also idempotent.

Paper 4, Section I

1F   Linear Algebra

What is an *eigenvalue* of a matrix A? What is the *eigenspace* corresponding to an eigenvalue λ of A?

Consider the matrix

$$A = \begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}$$

for $(a, b, c, d) \in \mathbb{R}^4$ a non-zero vector. Show that A has rank 1. Find the eigenvalues of A and describe the corresponding eigenspaces. Is A diagonalisable?

Paper 2, Section I

1F   Linear Algebra

If U and W are finite-dimensional subspaces of a vector space V, prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Let

$$\begin{aligned} U &= \{\mathbf{x} \in \mathbb{R}^4 \,|\, x_1 = 7x_3 + 8x_4, \ x_2 + 5x_3 + 6x_4 = 0\}, \\ W &= \{\mathbf{x} \in \mathbb{R}^4 \,|\, x_1 + 2x_2 + 3x_3 = 0, \ x_4 = 0\}. \end{aligned}$$

Show that $U + W$ is 3-dimensional and find a linear map $\ell : \mathbb{R}^4 \to \mathbb{R}$ such that

$$U + W = \{\mathbf{x} \in \mathbb{R}^4 \,|\, \ell(\mathbf{x}) = 0\}.$$

Paper 1, Section I

1F   Linear Algebra

Define a *basis* of a vector space V.

If V has a finite basis $\mathcal{B}$, show using only the definition that any other basis $\mathcal{B}'$ has the same cardinality as $\mathcal{B}$.

Paper 1, Section II

9F   Linear Algebra

What is the *adjugate* $\mathrm{adj}(A)$ of an $n \times n$ matrix A? How is it related to $\det(A)$?

(a) Define matrices $B_0, B_1, \ldots, B_{n-1}$ by

$$\mathrm{adj}(tI - A) = \sum_{i=0}^{n-1} B_i t^{n-1-i}$$

and scalars $c_0, c_1, \ldots, c_n$ by

$$\det(tI - A) = \sum_{j=0}^{n} c_j t^{n-j} .$$

Find a recursion for the matrices B_i in terms of A and the c_j's.

(b) By considering the partial derivatives of the multivariable polynomial

$$p(t_1, t_2, \ldots, t_n) = \det\left(\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} - A \right),$$

show that

$$\frac{d}{dt}(\det(tI - A)) = \mathrm{Tr}(\mathrm{adj}(tI - A)) .$$

(c) Hence show that the c_j's may be expressed in terms of $\mathrm{Tr}(A), \mathrm{Tr}(A^2), \ldots, \mathrm{Tr}(A^n)$.

Paper 4, Section II
10F Linear Algebra

If U is a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$, prove that the linear map $\phi : U \to U^*$ given by $\phi(u)(u') = \langle u, u' \rangle$ is an isomorphism. [You do not need to show that it is linear.]

If V and W are inner product spaces and $\alpha : V \to W$ is a linear map, what is meant by the *adjoint* α^* of α? If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for V, $\{f_1, f_2, \ldots, f_m\}$ is an orthonormal basis for W, and A is the matrix representing α in these bases, derive a formula for the matrix representing α^* in these bases.

Prove that $\mathrm{Im}(\alpha) = \mathrm{Ker}(\alpha^*)^\perp$.

If $w_0 \notin \mathrm{Im}(\alpha)$ then the linear equation $\alpha(v) = w_0$ has no solution, but we may instead search for a $v_0 \in V$ minimising $||\alpha(v) - w_0||^2$, known as a least-squares solution. Show that v_0 is such a least-squares solution if and only if it satisfies $\alpha^* \alpha(v_0) = \alpha^*(w_0)$. Hence find a least-squares solution to the linear equation

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Paper 3, Section II
10F Linear Algebra

If q is a quadratic form on a finite-dimensional real vector space V, what is the associated *symmetric bilinear form* $\varphi(\cdot, \cdot)$? Prove that there is a basis for V with respect to which the matrix for φ is diagonal. What is the *signature* of q?

If $R \leqslant V$ is a subspace such that $\varphi(r, v) = 0$ for all $r \in R$ and all $v \in V$, show that $q'(v + R) = q(v)$ defines a quadratic form on the quotient vector space V/R. Show that the signature of q' is the same as that of q.

If $e, f \in V$ are vectors such that $\varphi(e, e) = 0$ and $\varphi(e, f) = 1$, show that there is a direct sum decomposition $V = \mathrm{span}(e, f) \oplus U$ such that the signature of $q|_U$ is the same as that of q.

Paper 2, Section II

10F Linear Algebra

Let A and B be $n \times n$ matrices over $\mathbb{C}$.

(a) Assuming that A is invertible, show that AB and BA have the same characteristic polynomial.

(b) By considering the matrices $A - sI$, show that AB and BA have the same characteristic polynomial even when A is singular.

(c) Give an example to show that the minimal polynomials $m_{AB}(t)$ and $m_{BA}(t)$ of AB and BA may be different.

(d) Show that $m_{AB}(t)$ and $m_{BA}(t)$ differ at most by a factor of t. Stating carefully any results which you use, deduce that if AB is diagonalisable then so is $(BA)^2$.

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Paper 1, Section I

1E   Linear Algebra

State the Rank-Nullity Theorem.

If $\alpha : V \to W$ and $\beta : W \to X$ are linear maps and W is finite dimensional, show that

$$\dim \operatorname{Im}(\alpha) = \dim \operatorname{Im}(\beta\alpha) + \dim(\operatorname{Im}(\alpha) \cap \operatorname{Ker}(\beta)).$$

If $\gamma : U \to V$ is another linear map, show that

$$\dim \operatorname{Im}(\beta\alpha) + \dim \operatorname{Im}(\alpha\gamma) \leqslant \dim \operatorname{Im}(\alpha) + \dim \operatorname{Im}(\beta\alpha\gamma).$$

Paper 2, Section I

1E   Linear Algebra

Let V be a real vector space. Define the *dual* vector space V^* of V. If U is a subspace of V, define the *annihilator* U^0 of U. If $x_1, x_2, \ldots, x_n$ is a basis for V, define its dual $x_1^*, x_2^*, \ldots, x_n^*$ and prove that it is a basis for V^*.

If V has basis x_1, x_2, x_3, x_4 and U is the subspace spanned by

$$x_1 + 2x_2 + 3x_3 + 4x_4 \quad \text{and} \quad 5x_1 + 6x_2 + 7x_3 + 8x_4,$$

give a basis for U^0 in terms of the dual basis $x_1^*, x_2^*, x_3^*, x_4^*$.

Paper 4, Section I

1E   Linear Algebra

Define a *quadratic form* on a finite dimensional real vector space. What does it mean for a quadratic form to be *positive definite*?

Find a basis with respect to which the quadratic form

$$x^2 + 2xy + 2y^2 + 2yz + 3z^2$$

is diagonal. Is this quadratic form positive definite?

Paper 1, Section II

9E   Linear Algebra

Define a *Jordan block* $J_m(\lambda)$. What does it mean for a complex $n \times n$ matrix to be in *Jordan normal form*?

If A is a matrix in Jordan normal form for an endomorphism $\alpha : V \to V$, prove that

$$\dim \mathrm{Ker}((\alpha - \lambda I)^r) - \dim \mathrm{Ker}((\alpha - \lambda I)^{r-1})$$

is the number of Jordan blocks $J_m(\lambda)$ of A with $m \geqslant r$.

Find a matrix in Jordan normal form for $J_m(\lambda)^2$. *[Consider all possible values of λ.]*

Find a matrix in Jordan normal form for the complex matrix

$$\begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix}$$

assuming it is invertible.

Paper 2, Section II

10E   Linear Algebra

If X is an $n \times m$ matrix over a field, show that there are invertible matrices P and Q such that

$$Q^{-1}XP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for some $0 \leqslant r \leqslant \min(m, n)$, where I_r is the identity matrix of dimension r.

For a square matrix of the form $A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix}$ with B and C square matrices, prove that $\det(A) = \det(B)\det(C)$.

If $A \in M_{n \times n}(\mathbb{C})$ and $B \in M_{m \times m}(\mathbb{C})$ have no common eigenvalue, show that the linear map

$$L : M_{n \times m}(\mathbb{C}) \longrightarrow M_{n \times m}(\mathbb{C})$$
$$X \longmapsto AX - XB$$

is injective.

Paper 4, Section II

10E Linear Algebra

Let V be a finite dimensional inner-product space over $\mathbb{C}$. What does it mean to say that an endomorphism of V is *self-adjoint*? Prove that a self-adjoint endomorphism has real eigenvalues and may be diagonalised.

An endomorphism $\alpha : V \to V$ is called *positive definite* if it is self-adjoint and satisfies $\langle \alpha(x), x \rangle > 0$ for all non-zero $x \in V$; it is called *negative definite* if $-\alpha$ is positive definite. Characterise the property of being positive definite in terms of eigenvalues, and show that the sum of two positive definite endomorphisms is positive definite.

Show that a self-adjoint endomorphism $\alpha : V \to V$ has all eigenvalues in the interval $[a, b]$ if and only if $\alpha - \lambda I$ is positive definite for all $\lambda < a$ and negative definite for all $\lambda > b$.

Let $\alpha, \beta : V \to V$ be self-adjoint endomorphisms whose eigenvalues lie in the intervals $[a, b]$ and $[c, d]$ respectively. Show that all of the eigenvalues of $\alpha + \beta$ lie in the interval $[a + c, b + d]$.

Paper 3, Section II

10E Linear Algebra

State and prove the Cayley–Hamilton Theorem.

Let A be an $n \times n$ complex matrix. Using division of polynomials, show that if $p(x)$ is a polynomial then there is another polynomial $r(x)$ of degree at most $(n-1)$ such that $p(\lambda) = r(\lambda)$ for each eigenvalue λ of A and such that $p(A) = r(A)$.

Hence compute the $(1, 1)$ entry of the matrix A^{1000} when

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Paper 2, Section I

1F Linear Algebra

State and prove the Rank–Nullity theorem.

Let α be a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$ of rank 2. Give an example to show that $\mathbb{R}^3$ may be the direct sum of the kernel of α and the image of α, and also an example where this is not the case.

Paper 1, Section I

1F Linear Algebra

State and prove the Steinitz Exchange Lemma.

Deduce that, for a subset S of $\mathbb{R}^n$, any two of the following imply the third:

    (i) S is linearly independent

    (ii) S is spanning

    (iii) S has exactly n elements

Let e_1, e_2 be a basis of $\mathbb{R}^2$. For which values of λ do $\lambda e_1 + e_2,\ e_1 + \lambda e_2$ form a basis of $\mathbb{R}^2$?

Paper 4, Section I

1F Linear Algebra

Briefly explain the Gram–Schmidt orthogonalisation process in a real finite-dimensional inner product space V.

For a subspace U of V, define $U^\perp$, and show that $V = U \oplus U^\perp$.

For which positive integers n does

$$(f, g) = f(1)g(1) + f(2)g(2) + f(3)g(3)$$

define an inner product on the space of all real polynomials of degree at most n?

Paper 1, Section II

9F   Linear Algebra

Let U and V be finite-dimensional real vector spaces, and let $\alpha : U \to V$ be a surjective linear map. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) There is a linear map $\beta : V \to U$ such that $\beta\alpha$ is the identity map on U.

(ii) There is a linear map $\beta : V \to U$ such that $\alpha\beta$ is the identity map on V.

(iii) There is a subspace W of U such that the restriction of α to W is an isomorphism from W to V.

(iv) If X and Y are subspaces of U with $U = X \oplus Y$ then $V = \alpha(X) \oplus \alpha(Y)$.

(v) If X and Y are subspaces of U with $V = \alpha(X) \oplus \alpha(Y)$ then $U = X \oplus Y$.

Paper 2, Section II

10F   Linear Algebra

Let $\alpha : U \to V$ and $\beta : V \to W$ be linear maps between finite-dimensional real vector spaces.

Show that the rank $r(\beta\alpha)$ satisfies $r(\beta\alpha) \leqslant \min(r(\beta), r(\alpha))$. Show also that $r(\beta\alpha) \geqslant r(\alpha) + r(\beta) - \dim V$. For each of these two inequalities, give examples to show that we may or may not have equality.

Now let V have dimension $2n$ and let $\alpha : V \to V$ be a linear map of rank $2n - 2$ such that $\alpha^n = 0$. Find the rank of α^k for each $1 \leqslant k \leqslant n - 1$.

Paper 4, Section II

10F Linear Algebra

What is the *dual* X^* of a finite-dimensional real vector space X? If X has a basis $e_1, \ldots, e_n$, define the dual basis, and prove that it is indeed a basis of X^*.

[No results on the dimension of duals may be assumed without proof.]

Write down (without making a choice of basis) an isomorphism from X to X^{**}. Prove that your map is indeed an isomorphism.

Does every basis of X^* arise as the dual basis of some basis of X? Justify your answer.

A subspace W of X^* is called *separating* if for every non-zero $x \in X$ there is a $T \in W$ with $T(x) \neq 0$. Show that the only separating subspace of X^* is X^* itself.

Now let X be the (infinite-dimensional) space of all real polynomials. Explain briefly how we may identify X^* with the space of all real sequences. Give an example of a proper subspace of X^* that is separating.

Paper 3, Section II

10F Linear Algebra

Let f be a quadratic form on a finite-dimensional real vector space V. Prove that there exists a diagonal basis for f, meaning a basis with respect to which the matrix of f is diagonal.

Define the rank r and signature s of f in terms of this matrix. Prove that r and s are independent of the choice of diagonal basis.

In terms of r, s, and the dimension n of V, what is the greatest dimension of a subspace on which f is zero?

Now let f be the quadratic form on $\mathbb{R}^3$ given by $f(x, y, z) = x^2 - y^2$. For which points v in $\mathbb{R}^3$ is it the case that there is some diagonal basis for f containing v?

Paper 4, Section I
1F   Linear Algebra
For which real numbers x do the vectors

$$(x, 1, 1, 1), \quad (1, x, 1, 1), \quad (1, 1, x, 1), \quad (1, 1, 1, x),$$

not form a basis of $\mathbb{R}^4$? For each such value of x, what is the dimension of the subspace of $\mathbb{R}^4$ that they span? For each such value of x, provide a basis for the spanned subspace, and extend this basis to a basis of $\mathbb{R}^4$.

Paper 2, Section I
1F   Linear Algebra
Find a linear change of coordinates such that the quadratic form

$$2x^2 + 8xy - 6xz + y^2 - 4yz + 2z^2$$

takes the form

$$\alpha x^2 + \beta y^2 + \gamma z^2,$$

for real numbers α, β and γ.

Paper 1, Section I
1F   Linear Algebra
(a) Consider the linear transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}.$$

Find a basis of $\mathbb{R}^3$ in which α is represented by a diagonal matrix.

(b) Give a list of 6×6 matrices such that any linear transformation $\beta : \mathbb{R}^6 \to \mathbb{R}^6$ with characteristic polynomial

$$(x - 2)^4 (x + 7)^2$$

and minimal polynomial

$$(x - 2)^2 (x + 7)$$

is similar to one of the matrices on your list. No two distinct matrices on your list should be similar. [No proof is required.]

Paper 1, Section II

9F   Linear Algebra

Let $M_{n,n}$ denote the vector space over $F = \mathbb{R}$ or $\mathbb{C}$ of $n \times n$ matrices with entries in F. Let $\mathrm{Tr} : M_{n,n} \to F$ denote the trace functional, i.e., if $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in M_{n,n}$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

(a) Show that Tr is a linear functional.

(b) Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for $A, B \in M_{n,n}$.

(c) Show that Tr is unique in the following sense: If $f : M_{n,n} \to F$ is a linear functional such that $f(AB) = f(BA)$ for each $A, B \in M_{n,n}$, then f is a scalar multiple of the trace functional. If, in addition, $f(I) = n$, then $f = \mathrm{Tr}$.

(d) Let $W \subseteq M_{n,n}$ be the subspace spanned by matrices C of the form $C = AB - BA$ for $A, B \in M_{n,n}$. Show that W is the kernel of Tr.

Paper 4, Section II

10F   Linear Algebra

(a) Let $\alpha : V \to W$ be a linear transformation between finite dimensional vector spaces over a field $F = \mathbb{R}$ or $\mathbb{C}$.

Define the *dual map* of α. Let δ be the dual map of α. Given a subspace $U \subseteq V$, define the annihilator U° of U. Show that $(\ker \alpha)^\circ$ and the image of δ coincide. Conclude that the dimension of the image of α is equal to the dimension of the image of δ. Show that $\dim \ker(\alpha) - \dim \ker(\delta) = \dim V - \dim W$.

(b) Now suppose in addition that V, W are inner product spaces. Define the *adjoint* α^* of α. Let $\beta : U \to V$, $\gamma : V \to W$ be linear transformations between finite dimensional inner product spaces. Suppose that the image of β is equal to the kernel of γ. Then show that $\beta\beta^* + \gamma^*\gamma$ is an isomorphism.

Paper 3, Section II
10F Linear Algebra

Let $\alpha : V \to V$ be a linear transformation defined on a finite dimensional inner product space V over $\mathbb{C}$. Recall that α is normal if α and its adjoint α^* commute. Show that α being normal is equivalent to each of the following statements:

(i) $\alpha = \alpha_1 + i\alpha_2$ where α_1, α_2 are self-adjoint operators and $\alpha_1\alpha_2 = \alpha_2\alpha_1$;

(ii) there is an orthonormal basis for V consisting of eigenvectors of α;

(iii) there is a polynomial g with complex coefficients such that $\alpha^* = g(\alpha)$.

Paper 2, Section II
10F Linear Algebra

Let $M_{n,n}$ denote the vector space over a field $F = \mathbb{R}$ or $\mathbb{C}$ of $n \times n$ matrices with entries in F. Given $B \in M_{n,n}$, consider the two linear transformations $R_B, L_B : M_{n,n} \to M_{n,n}$ defined by
$$L_B(A) = BA, \qquad R_B(A) = AB.$$

(a) Show that $\det L_B = (\det B)^n$.

[For parts (b) and (c), you may assume the analogous result $\det R_B = (\det B)^n$ without proof.]

(b) Now let $F = \mathbb{C}$. For $B \in M_{n,n}$, write B^* for the conjugate transpose of B, i.e., $B^* := \overline{B}^T$. For $B \in M_{n,n}$, define the linear transformation $M_B : M_{n,n} \to M_{n,n}$ by

$$M_B(A) = BAB^*.$$

Show that $\det M_B = |\det B|^{2n}$.

(c) Again let $F = \mathbb{C}$. Let $W \subseteq M_{n,n}$ be the set of Hermitian matrices. [Note that W is not a vector space over $\mathbb{C}$ but only over $\mathbb{R}$.] For $B \in M_{n,n}$ and $A \in W$, define $T_B(A) = BAB^*$. Show that T_B is an $\mathbb{R}$-linear operator on W, and show that as such,

$$\det T_B = |\det B|^{2n}.$$

Paper 4, Section I

1E   Linear Algebra

Define the *dual space* V^* of a vector space V. Given a basis $\{x_1, \ldots, x_n\}$ of V define its *dual* and show it is a basis of V^*.

Let V be a 3-dimensional vector space over $\mathbb{R}$ and let $\{\zeta_1, \zeta_2, \zeta_3\}$ be the basis of V^* dual to the basis $\{x_1, x_2, x_3\}$ for V. Determine, in terms of the ζ_i, the bases dual to each of the following:

(a) $\{x_1 + x_2, x_2 + x_3, x_3\}$,

(b) $\{x_1 + x_2, x_2 + x_3, x_3 + x_1\}$.

Paper 2, Section I

1E   Linear Algebra

Let q denote a quadratic form on a real vector space V. Define the *rank* and *signature* of q.

Find the rank and signature of the following quadratic forms.

(a) $q(x, y, z) = x^2 + y^2 + z^2 - 2xz - 2yz$.

(b) $q(x, y, z) = xy - xz$.

(c) $q(x, y, z) = xy - 2z^2$.

Paper 1, Section I

1E   Linear Algebra

Let U and V be finite dimensional vector spaces and $\alpha : U \to V$ a linear map. Suppose W is a subspace of U. Prove that

$$r(\alpha) \geqslant r(\alpha|_W) \geqslant r(\alpha) - \dim(U) + \dim(W)$$

where $r(\alpha)$ denotes the rank of α and $\alpha|_W$ denotes the restriction of α to W. Give examples showing that each inequality can be both a strict inequality and an equality.

Paper 1, Section II

9E   Linear Algebra

Determine the characteristic polynomial of the matrix

$$M = \begin{pmatrix} x & 1 & 1 & 0 \\ 1-x & 0 & -1 & 0 \\ 2 & 2x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For which values of $x \in \mathbb{C}$ is M invertible? When M is not invertible determine (i) the Jordan normal form J of M, (ii) the minimal polynomial of M.

Find a basis of $\mathbb{C}^4$ such that J is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix P such that $P^{-1}MP = J$.

Paper 4, Section II

10E Linear Algebra

Suppose U and W are subspaces of a vector space V. Explain what is meant by $U \cap W$ and $U + W$ and show that both of these are subspaces of V.

Show that if U and W are subspaces of a finite dimensional space V then

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 5 \\ -1 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel and with image W.

What is the dimension of the space spanned by all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$
(i) with $(1,1,1,1,1)^T$ in the kernel and with image contained in W,
(ii) with $(1,1,1,1,1)^T$ in the kernel or with image contained in W?

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20

Paper 3, Section II
10E Linear Algebra

Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices over a field $\mathbb{F}$. We say $A_1, A_2, \ldots, A_k$ are simultaneously diagonalisable if there exists an invertible matrix P such that $P^{-1}A_iP$ is diagonal for all $1 \leqslant i \leqslant k$. We say the matrices are commuting if $A_iA_j = A_jA_i$ for all i, j.

(i) Suppose $A_1, A_2, \ldots, A_k$ are simultaneously diagonalisable. Prove that they are commuting.

(ii) Define an *eigenspace* of a matrix. Suppose $B_1, B_2, \ldots, B_k$ are commuting $n \times n$ matrices over a field $\mathbb{F}$. Let E denote an eigenspace of B_1. Prove that $B_i(E) \leqslant E$ for all i.

(iii) Suppose $B_1, B_2, \ldots, B_k$ are commuting diagonalisable matrices. Prove that they are simultaneously diagonalisable.

(iv) Are the 2×2 diagonalisable matrices over $\mathbb{C}$ simultaneously diagonalisable? Explain your answer.

Paper 2, Section II
10E Linear Algebra

(i) Suppose A is a matrix that does not have -1 as an eigenvalue. Show that $A + I$ is non-singular. Further, show that A commutes with $(A + I)^{-1}$.

(ii) A matrix A is called skew-symmetric if $A^T = -A$. Show that a real skew-symmetric matrix does not have -1 as an eigenvalue.

(iii) Suppose A is a real skew-symmetric matrix. Show that $U = (I - A)(I + A)^{-1}$ is orthogonal with determinant 1.

(iv) Verify that every orthogonal matrix U with determinant 1 which does not have -1 as an eigenvalue can be expressed as $(I - A)(I + A)^{-1}$ where A is a real skew-symmetric matrix.

Paper 4, Section I

1G   Linear Algebra

Let V denote the vector space of all real polynomials of degree at most 2. Show that

$$(f, g) \;=\; \int_{-1}^{1} f(x)g(x) \, dx$$

defines an inner product on V.

Find an orthonormal basis for V.

Paper 2, Section I

1G   Linear Algebra

State and prove the Rank–Nullity Theorem.

Let α be a linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$. What are the possible dimensions of the kernel of α? Justify your answer.

Paper 1, Section I

1G   Linear Algebra

State and prove the Steinitz Exchange Lemma. Use it to prove that, in a finite-dimensional vector space: any two bases have the same size, and every linearly independent set extends to a basis.

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Is $e_1 + e_2$, $e_2 + e_3$, $e_3 + e_1$ a basis for $\mathbb{R}^3$? Is $e_1 + e_2$, $e_2 + e_3$, $e_3 + e_4$, $e_4 + e_1$ a basis for $\mathbb{R}^4$? Justify your answers.

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Paper 1, Section II
9G   Linear Algebra

Let V be an n-dimensional real vector space, and let T be an endomorphism of V. We say that T *acts* on a subspace W if $T(W) \subset W$.

(i) For any $x \in V$, show that T acts on the linear span of $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$.

(ii) If $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$ spans V, show directly (i.e. without using the Cayley–Hamilton Theorem) that T satisfies its own characteristic equation.

(iii) Suppose that T acts on a subspace W with $W \neq \{0\}$ and $W \neq V$. Let $e_1, \ldots, e_k$ be a basis for W, and extend to a basis $e_1, \ldots, e_n$ for V. Describe the matrix of T with respect to this basis.

(iv) Using (i), (ii) and (iii) and induction, give a proof of the Cayley–Hamilton Theorem.

[Simple properties of determinants may be assumed without proof.]

Paper 4, Section II
10G Linear Algebra

Let V be a real vector space. What is the *dual* V^* of V? If $e_1, \ldots, e_n$ is a basis for V, define the *dual basis* $e_1^*, \ldots, e_n^*$ for V^*, and show that it is indeed a basis for V^*.

[No result about dimensions of dual spaces may be assumed.]

For a subspace U of V, what is the *annihilator* of U? If V is n-dimensional, how does the dimension of the annihilator of U relate to the dimension of U?

Let $\alpha : \ V \to W$ be a linear map between finite-dimensional real vector spaces. What is the *dual map* α^*? Explain why the rank of α^* is equal to the rank of α. Prove that the kernel of α^* is the annihilator of the image of α, and also that the image of α^* is the annihilator of the kernel of α.

[Results about the matrices representing a map and its dual may be used without proof, provided they are stated clearly.]

Now let V be the vector space of all real polynomials, and define elements $L_0, L_1, \ldots$ of V^* by setting $L_i(p)$ to be the coefficient of X^i in p (for each $p \in V$). Do the L_i form a basis for V^*?

Paper 3, Section II

10G Linear Algebra

Let q be a nonsingular quadratic form on a finite-dimensional real vector space V. Prove that we may write $V = P \bigoplus N$, where the restriction of q to P is positive definite, the restriction of q to N is negative definite, and $q(x + y) = q(x) + q(y)$ for all $x \in P$ and $y \in N$. [No result on diagonalisability may be assumed.]

Show that the dimensions of P and N are independent of the choice of P and N. Give an example to show that P and N are not themselves uniquely defined.

Find such a decomposition $V = P \bigoplus N$ when $V = \mathbb{R}^3$ and q is the quadratic form $q((x, y, z)) = x^2 + 2y^2 - 2xy - 2xz$.

Paper 2, Section II

10G Linear Algebra

Define the *determinant* of an $n \times n$ complex matrix A. Explain, with justification, how the determinant of A changes when we perform row and column operations on A.

Let A, B, C be complex $n \times n$ matrices. Prove the following statements.

(i) $\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det A \, \det B$.

(ii) $\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det(A + iB) \, \det(A - iB)$.

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Paper 4, Section I
1E   Linear Algebra

What is a quadratic form on a finite dimensional real vector space V? What does it mean for two quadratic forms to be isomorphic (*i.e.* congruent)? State Sylvester's law of inertia and explain the definition of the quantities which appear in it. Find the signature of the quadratic form on $\mathbb{R}^3$ given by $q(\mathbf{v}) = \mathbf{v}^T A \mathbf{v}$, where

$$A = \begin{pmatrix} -2 & 1 & 6 \\ 1 & -1 & -3 \\ 6 & -3 & 1 \end{pmatrix}.$$

Paper 2, Section I
1E   Linear Algebra

If A is an $n \times n$ invertible Hermitian matrix, let

$$U_A = \{U \in M_{n \times n}(\mathbb{C}) \,|\, \overline{U}^T A U = A\}.$$

Show that U_A with the operation of matrix multiplication is a group, and that $\det U$ has norm 1 for any $U \in U_A$. What is the relation between U_A and the complex Hermitian form defined by A?

If $A = I_n$ is the $n \times n$ identity matrix, show that any element of U_A is diagonalizable.

Paper 1, Section I
1E   Linear Algebra

What is the adjugate of an $n \times n$ matrix A? How is it related to A^{-1}? Suppose all the entries of A are integers. Show that all the entries of A^{-1} are integers if and only if $\det A = \pm 1$.

Paper 1, Section II

9E   Linear Algebra

If V_1 and V_2 are vector spaces, what is meant by $V_1 \oplus V_2$? If V_1 and V_2 are subspaces of a vector space V, what is meant by $V_1 + V_2$?

Stating clearly any theorems you use, show that if V_1 and V_2 are subspaces of a finite dimensional vector space V, then

$$\dim V_1 + \dim V_2 = \dim(V_1 \cap V_2) + \dim(V_1 + V_2).$$

Let $V_1, V_2 \subset \mathbb{R}^4$ be subspaces with bases

$$V_1 = \langle (3,2,4,-1), (1,2,1,-2), (-2,3,3,2) \rangle$$
$$V_2 = \langle (1,4,2,4), (-1,1,-1,-1), (3,1,2,0) \rangle.$$

Find a basis $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ for $V_1 \cap V_2$ such that the first component of $\mathbf{v}_1$ and the second component of $\mathbf{v}_2$ are both 0.

Paper 4, Section II

10E Linear Algebra

What does it mean for an $n \times n$ matrix to be in Jordan form? Show that if $A \in M_{n \times n}(\mathbb{C})$ is in Jordan form, there is a sequence (A_m) of diagonalizable $n \times n$ matrices which converges to A, in the sense that the (ij)th component of A_m converges to the (ij)th component of A for all i and j. [*Hint: A matrix with distinct eigenvalues is diagonalizable.*] Deduce that the same statement holds for all $A \in M_{n \times n}(\mathbb{C})$.

Let $V = M_{2 \times 2}(\mathbb{C})$. Given $A \in V$, define a linear map $T_A : V \to V$ by $T_A(B) = AB + BA$. Express the characteristic polynomial of T_A in terms of the trace and determinant of A. [*Hint: First consider the case where A is diagonalizable.*]

Paper 3, Section II

10E Linear Algebra

Let V and W be finite dimensional real vector spaces and let $T : V \to W$ be a linear map. Define the dual space V^* and the dual map T^*. Show that there is an isomorphism $\iota : V \to (V^*)^*$ which is canonical, in the sense that $\iota \circ S = (S^*)^* \circ \iota$ for any automorphism S of V.

Now let W be an inner product space. Use the inner product to show that there is an injective map from $\operatorname{im} T$ to $\operatorname{im} T^*$. Deduce that the row rank of a matrix is equal to its column rank.

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Paper 2, Section II
10E Linear Algebra

Define what it means for a set of vectors in a vector space V to be linearly dependent. Prove from the definition that any set of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.

Using this or otherwise, prove that if V has a finite basis consisting of n elements, then any basis of V has exactly n elements.

Let V be the vector space of bounded continuous functions on $\mathbb{R}$. Show that V is infinite dimensional.

Paper 4, Section I

1F   Linear Algebra

Let V be a complex vector space with basis $\{e_1, \ldots, e_n\}$. Define $T : V \to V$ by $T(e_i) = e_i - e_{i+1}$ for $i < n$ and $T(e_n) = e_n - e_1$. Show that T is diagonalizable and find its eigenvalues. [You may use any theorems you wish, as long as you state them clearly.]

Paper 2, Section I

1F   Linear Algebra

Define the determinant $\det A$ of an $n \times n$ real matrix A. Suppose that X is a matrix with block form

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A, B and C are matrices of dimensions $n \times n$, $n \times m$ and $m \times m$ respectively. Show that $\det X = (\det A)(\det C)$.

Paper 1, Section I

1F   Linear Algebra

Define the notions of basis and dimension of a vector space. Prove that two finite-dimensional real vector spaces with the same dimension are isomorphic.

In each case below, determine whether the set S is a basis of the real vector space V:

(i) $V = \mathbb{C}$ is the complex numbers; $S = \{1, i\}$.

(ii) $V = \mathbb{R}[x]$ is the vector space of all polynomials in x with real coefficients; $S = \{1, (x-1), (x-1)(x-2), (x-1)(x-2)(x-3), \ldots\}$.

(iii) $V = \{f : [0,1] \to \mathbb{R}\}$; $S = \{\chi_p \,|\, p \in [0,1]\}$, where

$$\chi_p(x) = \begin{cases} 1 & x = p \\ 0 & x \neq p. \end{cases}$$

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Paper 1, Section II
9F   Linear Algebra

Define what it means for two $n \times n$ matrices to be similar to each other. Show that if two $n \times n$ matrices are similar, then the linear transformations they define have isomorphic kernels and images.

If A and B are $n \times n$ real matrices, we define $[A, B] = AB - BA$. Let

$$K_A = \{X \in M_{n \times n}(\mathbb{R}) \,|\, [A, X] = 0\}$$
$$L_A = \{[A, X] \,|\, X \in M_{n \times n}(\mathbb{R})\}\,.$$

Show that K_A and L_A are linear subspaces of $M_{n \times n}(\mathbb{R})$. If A and B are similar, show that $K_A \cong K_B$ and $L_A \cong L_B$.

Suppose that A is diagonalizable and has characteristic polynomial

$$(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2}\,,$$

where $\lambda_1 \neq \lambda_2$. What are $\dim K_A$ and $\dim L_A$?

Paper 4, Section II
10F   Linear Algebra

Let V be a finite-dimensional real vector space of dimension n. A bilinear form $B : V \times V \to \mathbb{R}$ is *nondegenerate* if for all $\mathbf{v} \neq 0$ in V, there is some $\mathbf{w} \in V$ with $B(\mathbf{v}, \mathbf{w}) \neq 0$. For $\mathbf{v} \in V$, define $\langle \mathbf{v} \rangle^\perp = \{\mathbf{w} \in V \,|\, B(\mathbf{v}, \mathbf{w}) = 0\}$. Assuming B is nondegenerate, show that $V = \langle \mathbf{v} \rangle \oplus \langle \mathbf{v} \rangle^\perp$ whenever $B(\mathbf{v}, \mathbf{v}) \neq 0$.

Suppose that B is a nondegenerate, symmetric bilinear form on V. Prove that there is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V with $B(\mathbf{v}_i, \mathbf{v}_j) = 0$ for $i \neq j$. [If you use the fact that symmetric matrices are diagonalizable, you must prove it.]

Define the signature of a quadratic form. Explain how to determine the signature of the quadratic form associated to B from the basis you constructed above.

A linear subspace $V' \subset V$ is said to be *isotropic* if $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{v}, \mathbf{w} \in V'$. Show that if B is nondegenerate, the maximal dimension of an isotropic subspace of V is $(n - |\sigma|)/2$, where σ is the signature of the quadratic form associated to B.

Paper 3, Section II

10F Linear Algebra

What is meant by the Jordan normal form of an $n \times n$ complex matrix?

Find the Jordan normal forms of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 9 & 6 & 3 & 0 \\ 15 & 12 & 9 & 3 \end{pmatrix}.$$

Suppose A is an invertible $n \times n$ complex matrix. Explain how to derive the characteristic and minimal polynomials of A^n from the characteristic and minimal polynomials of A. Justify your answer. [*Hint: write each polynomial as a product of linear factors.*]

Paper 2, Section II

10F Linear Algebra

(i) Define the transpose of a matrix. If V and W are finite-dimensional real vector spaces, define the dual of a linear map $T : V \to W$. How are these two notions related?

Now suppose V and W are finite-dimensional inner product spaces. Use the inner product on V to define a linear map $V \to V^*$ and show that it is an isomorphism. Define the adjoint of a linear map $T : V \to W$. How are the adjoint of T and its dual related? If A is a matrix representing T, under what conditions is the adjoint of T represented by the transpose of A?

(ii) Let $V = C[0,1]$ be the vector space of continuous real-valued functions on $[0,1]$, equipped with the inner product

$$\langle f, g \rangle = \int_0^1 f(t) g(t) \, dt \,.$$

Let $T : V \to V$ be the linear map

$$Tf(t) = \int_0^t f(s) \, ds \,.$$

What is the adjoint of T?

Paper 1, Section I

1G   Linear Algebra

(i) State the rank-nullity theorem for a linear map between finite-dimensional vector spaces.

(ii) Show that a linear transformation $f : V \to V$ of a finite-dimensional vector space V is bijective if it is injective or surjective.

(iii) Let V be the $\mathbb{R}$-vector space $\mathbb{R}[X]$ of all polynomials in X with coefficients in $\mathbb{R}$. Give an example of a linear transformation $f : V \to V$ which is surjective but not bijective.

Paper 2, Section I

1G   Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space with an inner product. Let W be an m-dimensional subspace of V and $W^\perp$ its orthogonal complement, so that every element $v \in V$ can be uniquely written as $v = w + w'$ for $w \in W$ and $w' \in W^\perp$.

The *reflection map* with respect to W is defined as the linear map

$$f_W : V \ni w + w' \longmapsto w - w' \in V.$$

Show that f_W is an orthogonal transformation with respect to the inner product, and find its determinant.

Paper 4, Section I

1G   Linear Algebra

(i) Let V be a vector space over a field F, and W_1, W_2 subspaces of V. Define the subset $W_1 + W_2$ of V, and show that $W_1 + W_2$ and $W_1 \cap W_2$ are subspaces of V.

(ii) When W_1, W_2 are finite-dimensional, state a formula for $\dim(W_1 + W_2)$ in terms of $\dim W_1$, $\dim W_2$ and $\dim(W_1 \cap W_2)$.

(iii) Let V be the $\mathbb{R}$-vector space of all $n \times n$ matrices over $\mathbb{R}$. Let S be the subspace of all symmetric matrices and T the subspace of all upper triangular matrices (the matrices (a_{ij}) such that $a_{ij} = 0$ whenever $i > j$). Find $\dim S$, $\dim T$, $\dim(S \cap T)$ and $\dim(S + T)$. Briefly justify your answer.

Paper 1, Section II

9G   Linear Algebra

Let V, W be finite-dimensional vector spaces over a field F and $f : V \to W$ a linear map.

(i) Show that f is injective if and only if the image of every linearly independent subset of V is linearly independent in W.

(ii) Define the dual space V^* of V and the dual map $f^* : W^* \to V^*$.

(iii) Show that f is surjective if and only if the image under f^* of every linearly independent subset of W^* is linearly independent in V^*.

Paper 2, Section II

10G Linear Algebra

Let n be a positive integer, and let V be a $\mathbb{C}$-vector space of complex-valued functions on $\mathbb{R}$, generated by the set $\{\ \cos kx,\ \sin kx;\ k = 0, 1, \dots, n-1\ \}$.

(i) Let $\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$ for $f, g \in V$. Show that this is a positive definite Hermitian form on V.

(ii) Let $\Delta(f) = \dfrac{d^2}{dx^2}f(x)$. Show that Δ is a self-adjoint linear transformation of V with respect to the form defined in (i).

(iii) Find an orthonormal basis of V with respect to the form defined in (i), which consists of eigenvectors of Δ.

Paper 3, Section II

10G Linear Algebra

(i) Let A be an $n \times n$ complex matrix and $f(X)$ a polynomial with complex coefficients. By considering the Jordan normal form of A or otherwise, show that if the eigenvalues of A are $\lambda_1, \dots, \lambda_n$ then the eigenvalues of $f(A)$ are $f(\lambda_1), \dots, f(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$. Write B as $B = f(A)$ for a polynomial f with

$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and find the eigenvalues of B.

[Hint: compute the powers of A.]

Paper 4, Section II

10G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space and $f, g : V \to V$ linear transformations. Suppose f is invertible and diagonalisable, and $f \circ g = t \cdot (g \circ f)$ for some real number $t > 1$.

(i) Show that g is nilpotent, i.e. some positive power of g is 0.

(ii) Suppose that there is a non-zero vector $v \in V$ with $f(v) = v$ and $g^{n-1}(v) \neq 0$. Determine the diagonal form of f.

Paper 1, Section I

1F   Linear Algebra

Suppose that V is the complex vector space of polynomials of degree at most $n-1$ in the variable z. Find the Jordan normal form for each of the linear transformations $\dfrac{d}{dz}$ and $z\dfrac{d}{dz}$ acting on V.

Paper 2, Section I

1F   Linear Algebra

Suppose that ϕ is an endomorphism of a finite-dimensional complex vector space.

(i) Show that if λ is an eigenvalue of ϕ, then λ^2 is an eigenvalue of ϕ^2.

(ii) Show conversely that if μ is an eigenvalue of ϕ^2, then there is an eigenvalue λ of ϕ with $\lambda^2 = \mu$.

Paper 4, Section I

1F   Linear Algebra

Define the notion of an inner product on a finite-dimensional real vector space V, and the notion of a self-adjoint linear map $\alpha : V \to V$.

Suppose that V is the space of real polynomials of degree at most n in a variable t. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$$

is an inner product on V, and that the map $\alpha : V \to V$:

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint.

Paper 1, Section II
9F   Linear Algebra

Let V denote the vector space of $n \times n$ real matrices.

(1) Show that if $\psi(A, B) = \operatorname{tr}(AB^T)$, then ψ is a positive-definite symmetric bilinear form on V.

(2) Show that if $q(A) = \operatorname{tr}(A^2)$, then q is a quadratic form on V. Find its rank and signature.

[Hint: Consider symmetric and skew-symmetric matrices.]

Paper 2, Section II
10F   Linear Algebra

(i) Show that two $n \times n$ complex matrices A, B are similar (i.e. there exists invertible P with $A = P^{-1}BP$) if and only if they represent the same linear map $\mathbb{C}^n \to \mathbb{C}^n$ with respect to different bases.

(ii) Explain the notion of Jordan normal form of a square complex matrix.

(iii) Show that any square complex matrix A is similar to its transpose.

(iv) If A is invertible, describe the Jordan normal form of A^{-1} in terms of that of A.

Justify your answers.

Paper 3, Section II
10F   Linear Algebra

Suppose that V is a finite-dimensional vector space over $\mathbb{C}$, and that $\alpha : V \to V$ is a $\mathbb{C}$-linear map such that $\alpha^n = 1$ for some $n > 1$. Show that if V_1 is a subspace of V such that $\alpha(V_1) \subset V_1$, then there is a subspace V_2 of V such that $V = V_1 \oplus V_2$ and $\alpha(V_2) \subset V_2$.

[Hint: Show, for example by picking bases, that there is a linear map $\pi : V \to V_1$ with $\pi(x) = x$ for all $x \in V_1$. Then consider $\rho : V \to V_1$ with $\rho(y) = \frac{1}{n} \sum_{i=0}^{n-1} \alpha^i \pi \alpha^{-i}(y)$.]

Paper 4, Section II

10F   Linear Algebra

(i) Show that the group $O_n(\mathbb{R})$ of orthogonal $n \times n$ real matrices has a normal subgroup $SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) \mid \det A = 1\}$.

(ii) Show that $O_n(\mathbb{R}) = SO_n(\mathbb{R}) \times \{\pm I_n\}$ if and only if n is odd.

(iii) Show that if n is even, then $O_n(\mathbb{R})$ is not the direct product of $SO_n(\mathbb{R})$ with any normal subgroup.

[*You may assume that the only elements of $O_n(\mathbb{R})$ that commute with all elements of $O_n(\mathbb{R})$ are $\pm I_n$.*]

Paper 1, Section I
1G   Linear Algebra

(1) Let V be a finite-dimensional vector space and let $T : V \to V$ be a non-zero endomorphism of V. If $\ker(T) = \operatorname{im}(T)$ show that the dimension of V is an even integer. Find the minimal polynomial of T. [*You may assume the rank-nullity theorem.*]

(2) Let A_i, $1 \leqslant i \leqslant 3$, be non-zero subspaces of a vector space V with the property that

$$V = A_1 \oplus A_2 = A_2 \oplus A_3 = A_1 \oplus A_3.$$

Show that there is a 2-dimensional subspace $W \subset V$ for which all the $W \cap A_i$ are one-dimensional.

Paper 2, Section I
1G   Linear Algebra

Let V denote the vector space of polynomials $f(x, y)$ in two variables of total degree at most n. Find the dimension of V.

If $S : V \to V$ is defined by

$$(Sf)(x, y) = x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2},$$

find the kernel of S and the image of S. Compute the trace of S for each n with $1 \leqslant n \leqslant 4$.

Paper 4, Section I
1G   Linear Algebra

Show that every endomorphism of a finite-dimensional vector space satisfies some polynomial, and define the *minimal polynomial* of such an endomorphism.

Give a linear transformation of an eight-dimensional complex vector space which has minimal polynomial $x^2(x-1)^3$.

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Paper 1, Section II
9G   Linear Algebra

Define the *dual* of a vector space V. State and prove a formula for its dimension.

Let V be the vector space of real polynomials of degree at most n. If $\{a_0, \ldots, a_n\}$ are distinct real numbers, prove that there are unique real numbers $\{\lambda_0, \ldots, \lambda_n\}$ with

$$\frac{dp}{dx}(0) \;=\; \sum_{j=0}^{n} \lambda_j p(a_j)$$

for every $p(x) \in V$.

Paper 2, Section II
10G   Linear Algebra

Let V be a finite-dimensional vector space and let $T : V \to V$ be an endomorphism of V. Show that there is a positive integer l such that $V = \ker(T^l) \oplus \operatorname{im}(T^l)$. Hence, or otherwise, show that if T has zero determinant there is some non-zero endomorphism S with $TS = 0 = ST$.

Suppose T_1 and T_2 are endomorphisms of V for which $T_i^2 = T_i$, $i = 1, 2$. Show that T_1 is similar to T_2 if and only if they have the same rank.

Paper 3, Section II
10G   Linear Algebra

For each of the following, provide a proof or counterexample.

(1) If A, B are complex $n \times n$ matrices and $AB = BA$, then A and B have a common eigenvector.

(2) If A, B are complex $n \times n$ matrices and $AB = BA$, then A and B have a common eigenvalue.

(3) If A, B are complex $n \times n$ matrices and $(AB)^n = 0$ then $(BA)^n = 0$.

(4) If $T : V \to V$ is an endomorphism of a finite-dimensional vector space V and λ is an eigenvalue of T, then the dimension of $\{v \in V \mid (T - \lambda I)v = 0\}$ equals the multiplicity of λ as a root of the minimal polynomial of T.

(5) If $T : V \to V$ is an endomorphism of a finite-dimensional complex vector space V, λ is an eigenvalue of T, and $W_i = \{v \in V \mid (T - \lambda I)^i(v) = 0\}$, then $W_c = W_{c+1}$ where c is the multiplicity of λ as a root of the minimal polynomial of T.

Paper 4, Section II

10G Linear Algebra

What does it mean to say two real symmetric bilinear forms A and B on a vector space V are *congruent* ?

State and prove Sylvester's law of inertia, and deduce that the rank and signature determine the congruence class of a real symmetric bilinear form. [*You may use without proof a result on diagonalisability of real symmetric matrices, provided it is clearly stated.*]

How many congruence classes of symmetric bilinear forms on a real n-dimensional vector space are there? Such a form ψ defines a family of subsets $\{x \in \mathbb{R}^n \mid \psi(x,x) = t\}$, for $t \in \mathbb{R}$. For how many of the congruence classes are these associated subsets all bounded subsets of $\mathbb{R}^n$? Is the quadric surface

$$\{3x^2 + 6y^2 + 5z^2 + 4xy + 2xz + 8yz = 1\}$$

a bounded or unbounded subset of $\mathbb{R}^3$? Justify your answers.

1/I/1E   **Linear Algebra**

Let A be an $n \times n$ matrix over $\mathbb{C}$. What does it mean to say that λ is an eigenvalue of A? Show that A has at least one eigenvalue. For each of the following statements, provide a proof or a counterexample as appropriate.

(i) If A is Hermitian, all eigenvalues of A are real.

(ii) If all eigenvalues of A are real, A is Hermitian.

(iii) If all entries of A are real and positive, all eigenvalues of A have positive real part.

(iv) If A and B have the same trace and determinant then they have the same eigenvalues.

1/II/9E   **Linear Algebra**

Let A be an $m \times n$ matrix of real numbers. Define the row rank and column rank of A and show that they are equal.

Show that if a matrix A' is obtained from A by elementary row and column operations then $\mathrm{rank}(A') = \mathrm{rank}(A)$.

Let P, Q and R be $n \times n$ matrices. Show that the $2n \times 2n$ matrices $\begin{pmatrix} PQ & 0 \\ Q & QR \end{pmatrix}$ and $\begin{pmatrix} 0 & PQR \\ Q & 0 \end{pmatrix}$ have the same rank.

Hence, or otherwise, prove that

$$\mathrm{rank}(PQ) + \mathrm{rank}(QR) \leqslant \mathrm{rank}(Q) + \mathrm{rank}(PQR).$$

2/I/1E   **Linear Algebra**

Suppose that V and W are finite-dimensional vector spaces over $\mathbb{R}$. What does it mean to say that $\psi : V \to W$ is a linear map? State the rank-nullity formula. Using it, or otherwise, prove that a linear map $\psi : V \to V$ is surjective if, and only if, it is injective.

Suppose that $\psi : V \to V$ is a linear map which has a right inverse, that is to say there is a linear map $\phi : V \to V$ such that $\psi\phi = \mathrm{id}_V$, the identity map. Show that $\phi\psi = \mathrm{id}_V$.

Suppose that A and B are two $n \times n$ matrices over $\mathbb{R}$ such that $AB = I$. Prove that $BA = I$.

2/II/10E   Linear Algebra

Define the determinant $\det(A)$ of an $n \times n$ square matrix A over the complex numbers. If A and B are two such matrices, show that $\det(AB) = \det(A)\det(B)$.

Write $p_M(\lambda) = \det(M - \lambda I)$ for the characteristic polynomial of a matrix M. Let A, B, C be $n \times n$ matrices and suppose that C is nonsingular. Show that $p_{BC} = p_{CB}$. Taking $C = A + tI$ for appropriate values of t, or otherwise, deduce that $p_{BA} = p_{AB}$.

Show that if $p_A = p_B$ then $\operatorname{tr}(A) = \operatorname{tr}(B)$. Which of the following statements is true for all $n \times n$ matrices A, B, C? Justify your answers.

(i) $p_{ABC} = p_{ACB}$;

(ii) $p_{ABC} = p_{BCA}$.

3/II/10E   Linear Algebra

Let $k = \mathbb{R}$ or $\mathbb{C}$. What is meant by a quadratic form $q : k^n \to k$? Show that there is a basis $\{v_1, \ldots, v_n\}$ for k^n such that, writing $x = x_1 v_1 + \ldots + x_n v_n$, we have $q(x) = a_1 x_1^2 + \ldots + a_n x_n^2$ for some scalars $a_1, \ldots, a_n \in \{-1, 0, 1\}$.

Suppose that $k = \mathbb{R}$. Define the rank and signature of q and compute these quantities for the form $q : \mathbb{R}^3 \to \mathbb{R}$ given by $q(x) = -3x_1^2 + x_2^2 + 2x_1 x_2 - 2x_1 x_3 + 2x_2 x_3$.

Suppose now that $k = \mathbb{C}$ and that $q_1, \ldots, q_d : \mathbb{C}^n \to \mathbb{C}$ are quadratic forms. If $n \geqslant 2^d$, show that there is some nonzero $x \in \mathbb{C}^n$ such that $q_1(x) = \ldots = q_d(x) = 0$.

4/I/1E   Linear Algebra

Describe (without proof) what it means to put an $n \times n$ matrix of complex numbers into Jordan normal form. Explain (without proof) the sense in which the Jordan normal form is unique.

Put the following matrix in Jordan normal form:

$$\begin{pmatrix} -7 & 3 & -5 \\ 7 & -1 & 5 \\ 17 & -6 & 12 \end{pmatrix}.$$

4

4/II/10E   **Linear Algebra**

What is meant by a Hermitian matrix? Show that if A is Hermitian then all its eigenvalues are real and that there is an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A.

A Hermitian matrix is said to be *positive definite* if $\langle Ax, x \rangle > 0$ for all $x \neq 0$. We write $A > 0$ in this case. Show that A is positive definite if, and only if, all of its eigenvalues are positive. Show that if $A > 0$ then A has a unique positive definite square root $\sqrt{A}$.

Let A, B be two positive definite Hermitian matrices with $A - B > 0$. Writing $C = \sqrt{A}$ and $X = \sqrt{A} - \sqrt{B}$, show that $CX + XC > 0$. By considering eigenvalues of X, or otherwise, show that $X > 0$.

1/I/1G   **Linear Algebra**

Suppose that $\{e_1, \ldots, e_3\}$ is a basis of the complex vector space $\mathbb{C}^3$ and that $A : \mathbb{C}^3 \to \mathbb{C}^3$ is the linear operator defined by $A(e_1) = e_2$, $A(e_2) = e_3$, and $A(e_3) = e_1$.

By considering the action of A on column vectors of the form $(1, \xi, \xi^2)^T$, where $\xi^3 = 1$, or otherwise, find the diagonalization of A and its characteristic polynomial.

1/II/9G   **Linear Algebra**

State and prove Sylvester's law of inertia for a real quadratic form.

[*You may assume that for each real symmetric matrix A there is an orthogonal matrix U, such that $U^{-1}AU$ is diagonal.*]

Suppose that V is a real vector space of even dimension $2m$, that Q is a non-singular quadratic form on V and that U is an m-dimensional subspace of V on which Q vanishes. What is the signature of Q?

2/I/1G   **Linear Algebra**

Suppose that S, T are endomorphisms of the 3-dimensional complex vector space $\mathbb{C}^3$ and that the eigenvalues of each of them are 1, 2, 3. What are their characteristic and minimal polynomials? Are they conjugate?

2/II/10G   **Linear Algebra**

Suppose that P is the complex vector space of complex polynomials in one variable, z.

(i) Show that the form $\langle \ , \ \rangle$ defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \cdot \overline{g\left(e^{i\theta}\right)} \, d\theta$$

is a positive definite Hermitian form on P.

(ii) Find an orthonormal basis of P for this form, in terms of the powers of z.

(iii) Generalize this construction to complex vector spaces of complex polynomials in any finite number of variables.

3/II/10G   **Linear Algebra**

(i) Define the terms *row-rank*, *column-rank* and *rank* of a matrix, and state a relation between them.

(ii) Fix positive integers m, n, p with $m, n \geqslant p$. Suppose that A is an $m \times p$ matrix and B a $p \times n$ matrix. State and prove the best possible upper bound on the rank of the product AB.

4/I/1G   **Linear Algebra**

Suppose that $\alpha : V \to W$ is a linear map of finite-dimensional complex vector spaces. What is the dual map α^* of the dual vector spaces?

Suppose that we choose bases of V, W and take the corresponding dual bases of the dual vector spaces. What is the relation between the matrices that represent α and α^* with respect to these bases? Justify your answer.

4/II/10G   **Linear Algebra**

(i) State and prove the Cayley–Hamilton theorem for square complex matrices.

(ii) A square matrix A is *of order n* for a strictly positive integer n if $A^n = I$ and no smaller positive power of A is equal to I.

Determine the order of a complex 2×2 matrix A of trace zero and determinant 1.

1/I/1H   Linear Algebra

Define what is meant by the *minimal polynomial* of a complex $n \times n$ matrix, and show that it is unique. Deduce that the minimal polynomial of a real $n \times n$ matrix has real coefficients.

For $n > 2$, find an $n \times n$ matrix with minimal polynomial $(t-1)^2(t+1)$.

1/II/9H   Linear Algebra

Let U, V be finite-dimensional vector spaces, and let θ be a linear map of U into V. Define the *rank* $r(\theta)$ and the *nullity* $n(\theta)$ of θ, and prove that

$$r(\theta) + n(\theta) = \dim U.$$

Now let θ, ϕ be endomorphisms of a vector space U. Define the endomorphisms $\theta + \phi$ and $\theta\phi$, and prove that

$$r(\theta + \phi) \leqslant r(\theta) + r(\phi)$$
$$n(\theta\phi) \leqslant n(\theta) + n(\phi).$$

Prove that equality holds in **both** inequalities if and only if $\theta + \phi$ is an isomorphism and $\theta\phi$ is zero.

2/I/1E   Linear Algebra

State Sylvester's law of inertia.

Find the rank and signature of the quadratic form q on $\mathbf{R}^n$ given by

$$q(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2.$$

2/II/10E   **Linear Algebra**

Suppose that V is the set of complex polynomials of degree at most n in the variable x. Find the dimension of V as a complex vector space.

Define

$$e_k : V \to \mathbf{C} \quad \text{by} \quad e_k(\phi) = \frac{d^k \phi}{dx^k}(0).$$

Find a subset of $\{e_k \,|\, k \in \mathbf{N}\}$ that is a basis of the dual vector space V^*. Find the corresponding dual basis of V.

Define

$$D : V \to V \quad \text{by} \quad D(\phi) = \frac{d\phi}{dx}.$$

Write down the matrix of D with respect to the basis of V that you have just found, and the matrix of the map dual to D with respect to the dual basis.

3/II/10H   **Linear Algebra**

(a) Define what is meant by the *trace* of a complex $n \times n$ matrix A. If T denotes an $n \times n$ invertible matrix, show that A and TAT^{-1} have the same trace.

(b) If $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, show that the endomorphism of $\mathbf{C}^r$ defined by the matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \cdots & \lambda_1^r \\ \vdots & \cdots & \vdots \\ \lambda_r & \cdots & \lambda_r^r \end{pmatrix}$$

has trivial kernel, and hence that the same is true for the transposed matrix Λ^t.

For arbitrary complex numbers $\lambda_1, \ldots, \lambda_n$, show that the vector $(1, \ldots, 1)^t$ is not in the kernel of the endomorphism of $\mathbf{C}^n$ defined by the matrix

$$\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ \vdots & \cdots & \vdots \\ \lambda_1^n & \cdots & \lambda_n^n \end{pmatrix},$$

unless all the λ_i are zero.

[*Hint: reduce to the case when $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, with $r \leqslant n$, and each λ_j for $j > r$ is either zero or equal to some λ_i with $i \leqslant r$. If the kernel of the endomorphism contains $(1, \ldots, 1)^t$, show that it also contains a vector of the form $(m_1, \ldots, m_r, 0, \ldots, 0)^t$ with the m_i strictly positive integers.*]

(c) Assuming the fact that any complex $n \times n$ matrix is conjugate to an upper-triangular one, prove that if A is an $n \times n$ matrix such that A^k has zero trace for all $1 \leqslant k \leqslant n$, then $A^n = 0$.

4

4/I/1H   Linear Algebra

Suppose V is a vector space over a field k. A finite set of vectors is said to be a *basis* for V if it is both linearly independent and spanning. Prove that any two finite bases for V have the same number of elements.

4/II/10E   Linear Algebra

Suppose that α is an orthogonal endomorphism of the finite-dimensional real inner product space V. Suppose that V is decomposed as a direct sum of mutually orthogonal α-invariant subspaces. How small can these subspaces be made, and how does α act on them? Justify your answer.

Describe the possible matrices for α with respect to a suitably chosen orthonormal basis of V when $\dim V = 3$.

1/I/1C   Linear Algebra

Let V be an n-dimensional vector space over $\mathbf{R}$, and let $\beta : V \to V$ be a linear map. Define the minimal polynomial of β. Prove that β is invertible if and only if the constant term of the minimal polynomial of β is non-zero.

1/II/9C   Linear Algebra

Let V be a finite dimensional vector space over $\mathbf{R}$, and V^* be the dual space of V. If W is a subspace of V, we define the subspace $\alpha(W)$ of V^* by

$$\alpha(W) = \{f \in V^* : f(w) = 0 \text{ for all } w \text{ in } W\}.$$

Prove that $\dim(\alpha(W)) = \dim(V) - \dim(W)$. Deduce that, if $A = (a_{ij})$ is any real $m \times n$-matrix of rank r, the equations

$$\sum_{j=1}^{n} a_{ij}\, x_j = 0 \quad (i = 1, \ldots, m)$$

have $n - r$ linearly independent solutions in $\mathbf{R}^n$.

2/I/1C   Linear Algebra

Let Ω be the set of all 2×2 matrices of the form $\alpha = aI + bJ + cK + dL$, where a, b, c, d are in $\mathbf{R}$, and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (i^2 = -1).$$

Prove that Ω is closed under multiplication and determine its dimension as a vector space over $\mathbf{R}$. Prove that

$$(aI + bJ + cK + dL)\,(aI - bJ - cK - dL) = (a^2 + b^2 + c^2 + d^2)I\,,$$

and deduce that each non-zero element of Ω is invertible.

2/II/10C   Linear Algebra

(i) Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in $\mathbf{C}$. Define the determinant of A, the cofactor of each a_{ij}, and the adjugate matrix $\mathrm{adj}(A)$. Assuming the expansion of the determinant of a matrix in terms of its cofactors, prove that

$$\mathrm{adj}(A)\, A = \det(A) I_n \,,$$

where I_n is the $n \times n$ identity matrix.

(ii) Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \,.$$

Show the eigenvalues of A are $\pm 1, \pm i$, where $i^2 = -1$, and determine the diagonal matrix to which A is similar. For each eigenvalue, determine a non-zero eigenvector.

3/II/10B   Linear Algebra

Let S be the vector space of functions $f : \mathbf{R} \to \mathbf{R}$ such that the nth derivative of f is defined and continuous for every $n \geqslant 0$. Define linear maps $A, B : S \to S$ by $A(f) = df/dx$ and $B(f)(x) = xf(x)$. Show that

$$[A, B] = 1_S,$$

where in this question $[A, B]$ means $AB - BA$ and 1_S is the identity map on S.

Now let V be any real vector space with linear maps $A, B : V \to V$ such that $[A, B] = 1_V$. Suppose that there is a nonzero element $y \in V$ with $Ay = 0$. Let W be the subspace of V spanned by y, By, $B^2 y$, and so on. Show that $A(By)$ is in W and give a formula for it. More generally, show that $A(B^i y)$ is in W for each $i \geqslant 0$, and give a formula for it.

Show, using your formula or otherwise, that $\{y, By, B^2 y, \ldots\}$ are linearly independent. (Or, equivalently: show that $y, By, B^2 y, \ldots, B^n y$ are linearly independent for every $n \geqslant 0$.)

4/I/1B   Linear Algebra

Define what it means for an $n \times n$ complex matrix to be unitary or Hermitian. Show that every eigenvalue of a Hermitian matrix is real. Show that every eigenvalue of a unitary matrix has absolute value 1.

Show that two eigenvectors of a Hermitian matrix that correspond to different eigenvalues are orthogonal, using the standard inner product on $\mathbf{C}^n$.

4/II/10B   **Linear Algebra**

(i) Let V be a finite-dimensional real vector space with an inner product. Let $e_1, \ldots, e_n$ be a basis for V. Prove by an explicit construction that there is an orthonormal basis $f_1, \ldots, f_n$ for V such that the span of $e_1, \ldots, e_i$ is equal to the span of $f_1, \ldots, f_i$ for every $1 \leqslant i \leqslant n$.

(ii) For any real number a, consider the quadratic form

$$q_a(x, y, z) = xy + yz + zx + ax^2$$

on $\mathbf{R}^3$. For which values of a is q_a nondegenerate? When q_a is nondegenerate, compute its signature in terms of a.

1/I/1H   **Linear Algebra**

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_{r+1}\}$ is a linearly independent set of distinct elements of a vector space V and $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{f}_{r+1}, \ldots, \mathbf{f}_m\}$ spans V. Prove that $\mathbf{f}_{r+1}, \ldots, \mathbf{f}_m$ may be reordered, as necessary, so that $\{\mathbf{e}_1, \ldots \mathbf{e}_{r+1}, \mathbf{f}_{r+2}, \ldots, \mathbf{f}_m\}$ spans V.

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a linearly independent set of distinct elements of V and that $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ spans V. Show that $n \leqslant m$.

1/II/12H   **Linear Algebra**

Let U and W be subspaces of the finite-dimensional vector space V. Prove that both the sum $U + W$ and the intersection $U \cap W$ are subspaces of V. Prove further that

$$\dim U + \dim W = \dim (U + W) + \dim (U \cap W).$$

Let U, W be the kernels of the maps $A, B : \mathbb{R}^4 \to \mathbb{R}^2$ given by the matrices A and B respectively, where

$$A = \begin{pmatrix} 1 & 2 & -1 & -3 \\ -1 & 1 & 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & -4 \end{pmatrix}.$$

Find a basis for the intersection $U \cap W$, and extend this first to a basis of U, and then to a basis of $U + W$.

2/I/1E   **Linear Algebra**

For each n let A_n be the $n \times n$ matrix defined by

$$(A_n)_{ij} = \begin{cases} i & i \leqslant j, \\ j & i > j. \end{cases}$$

What is $\det A_n$? Justify your answer.

[*It may be helpful to look at the cases $n = 1, 2, 3$ before tackling the general case.*]

2/II/12E   **Linear Algebra**

Let Q be a quadratic form on a real vector space V of dimension n. Prove that there is a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with respect to which Q is given by the formula

$$Q\Big(\sum_{i=1}^{n} x_i \mathbf{e}_i\Big) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \,.$$

Prove that the numbers p and q are uniquely determined by the form Q. By means of an example, show that the subspaces $\langle \mathbf{e}_1, \ldots, \mathbf{e}_p \rangle$ and $\langle \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \rangle$ need not be uniquely determined by Q.

3/I/1E   **Linear Algebra**

Let V be a finite-dimensional vector space over $\mathbb{R}$. What is the *dual space* of V? Prove that the dimension of the dual space is the same as that of V.

3/II/13E   **Linear Algebra**

(i) Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\alpha : V \to V$ be an endomorphism. Suppose that the characteristic polynomial of α is $\Pi_{i=1}^{k}(x - \lambda_i)^{n_i}$, where the λ_i are distinct and $n_i > 0$ for every i.

Describe all possibilities for the minimal polynomial and prove that there are no further ones.

(ii) Give an example of a matrix for which both the characteristic and the minimal polynomial are $(x-1)^3(x-3)$.

(iii) Give an example of two matrices A, B with the same rank and the same minimal and characteristic polynomials such that there is no invertible matrix P with $PAP^{-1} = B$.

4/I/1E   **Linear Algebra**

Let V be a real n-dimensional inner-product space and let $W \subset V$ be a k-dimensional subspace. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be an orthonormal basis for W. In terms of this basis, give a formula for the orthogonal projection $\pi : V \to W$.

Let $v \in V$. Prove that πv is the closest point in W to v.

[*You may assume that the sequence* $\mathbf{e}_1, \ldots, \mathbf{e}_k$ *can be extended to an orthonormal basis* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *of* V.]

4/II/11E   **Linear Algebra**

(i) Let V be an n-dimensional inner-product space over $\mathbb{C}$ and let $\alpha : V \to V$ be a Hermitian linear map. Prove that V has an orthonormal basis consisting of eigenvectors of α.

(ii) Let $\beta : V \to V$ be another Hermitian map. Prove that $\alpha\beta$ is Hermitian if and only if $\alpha\beta = \beta\alpha$.

(iii) A Hermitian map α is *positive-definite* if $\langle \alpha v, v \rangle > 0$ for every non-zero vector v. If α is a positive-definite Hermitian map, prove that there is a unique positive-definite Hermitian map β such that $\beta^2 = \alpha$.

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1/I/5E **Linear Mathematics**

Let V be the subset of $\mathbb{R}^5$ consisting of all quintuples $(a_1, a_2, a_3, a_4, a_5)$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and
$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 .$$

Prove that V is a subspace of $\mathbb{R}^5$. Solve the above equations for a_1 and a_2 in terms of a_3, a_4 and a_5. Hence, exhibit a basis for V, explaining carefully why the vectors you give form a basis.

1/II/14E **Linear Mathematics**

(a) Let U, U' be subspaces of a finite-dimensional vector space V. Prove that $\dim(U + U') = \dim U + \dim U' - \dim(U \cap U')$.

(b) Let V and W be finite-dimensional vector spaces and let α and β be linear maps from V to W. Prove that

$$\operatorname{rank}(\alpha + \beta) \leqslant \operatorname{rank} \alpha + \operatorname{rank} \beta .$$

(c) Deduce from this result that

$$\operatorname{rank}(\alpha + \beta) \geqslant |\operatorname{rank} \alpha - \operatorname{rank} \beta| .$$

(d) Let $V = W = \mathbb{R}^n$ and suppose that $1 \leqslant r \leqslant s \leqslant n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\operatorname{rank} \alpha = r$, $\operatorname{rank} \beta = s$ and $\operatorname{rank}(\alpha + \beta) = s - r$. Suppose that $r + s \geqslant n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\operatorname{rank} \alpha = r$, $\operatorname{rank} \beta = s$ and $\operatorname{rank}(\alpha + \beta) = n$.

2/I/6E **Linear Mathematics**

Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. For each i let $\mathbf{v}_i$ be the vector $(1, a_i, a_i^2, \ldots, a_i^{n-1})$. Let A be the $n \times n$ matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and let $\mathbf{c}$ be a column vector of size n. Prove that $A\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c} = \mathbf{0}$. Deduce that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.

[*You may use general facts about matrices if you state them clearly.*]

2/II/15E   Linear Mathematics

(a) Let $A = (a_{ij})$ be an $m \times n$ matrix and for each $k \leqslant n$ let A_k be the $m \times k$ matrix formed by the first k columns of A. Suppose that $n > m$. Explain why the nullity of A is non-zero. Prove that if k is minimal such that A_k has non-zero nullity, then the nullity of A_k is 1.

(b) Suppose that no column of A consists entirely of zeros. Deduce from (a) that there exist scalars $b_1, \ldots, b_k$ (where k is defined as in (a)) such that $\sum_{j=1}^{k} a_{ij} b_j = 0$ for every $i \leqslant m$, but whenever $\lambda_1, \ldots, \lambda_k$ are distinct real numbers there is some $i \leqslant m$ such that $\sum_{j=1}^{k} a_{ij} \lambda_j b_j \neq 0$.

(c) Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be bases for the same real m-dimensional vector space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers such that for every j the vectors $\mathbf{v}_1 + \lambda_j \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda_j \mathbf{w}_m$ are linearly dependent. For each j, let $a_{1j}, \ldots, a_{mj}$ be scalars, not all zero, such that $\sum_{i=1}^{m} a_{ij}(\mathbf{v}_i + \lambda_j \mathbf{w}_i) = \mathbf{0}$. By applying the result of (b) to the matrix (a_{ij}), deduce that $n \leqslant m$.

(d) It follows that the vectors $\mathbf{v}_1 + \lambda \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda \mathbf{w}_m$ are linearly dependent for at most m values of λ. Explain briefly how this result can also be proved using determinants.

3/I/7G   Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U and let β be another endomorphism of U that commutes with α. If λ is an eigenvalue of α, show that β maps the kernel of $\alpha - \lambda \iota$ into itself, where ι is the identity map. Suppose now that α is diagonalizable with n distinct real eigenvalues where $n = \dim U$. Prove that if there exists an endomorphism β of U such that $\alpha = \beta^2$, then $\lambda \geqslant 0$ for all eigenvalues λ of α.

3/II/17G   Linear Mathematics

Define the *determinant* $\det(A)$ of an $n \times n$ complex matrix A. Let $A_1, \ldots, A_n$ be the columns of A, let σ be a permutation of $\{1, \ldots, n\}$ and let A^σ be the matrix whose columns are $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$. Prove from your definition of determinant that $\det(A^\sigma) = \epsilon(\sigma) \det(A)$, where $\epsilon(\sigma)$ is the sign of the permutation σ. Prove also that $\det(A) = \det(A^t)$.

Define the *adjugate* matrix $\mathrm{adj}(A)$ and prove from your definitions that $A\,\mathrm{adj}(A) = \mathrm{adj}(A)\,A = \det(A)\,I$, where I is the identity matrix. Hence or otherwise, prove that if $\det(A) \neq 0$, then A is invertible.

Let C and D be real $n \times n$ matrices such that the complex matrix $C + iD$ is invertible. By considering $\det(C + \lambda D)$ as a function of λ or otherwise, prove that there exists a real number λ such that $C + \lambda D$ is invertible. [*You may assume that if a matrix A is invertible, then $\det(A) \neq 0$.*]

Deduce that if two real matrices A and B are such that there exists an invertible complex matrix P with $P^{-1} A P = B$, then there exists an invertible **real** matrix Q such that $Q^{-1} A Q = B$.

4/I/6G   Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U such that $\alpha^2 = \alpha$. Show that U can be written as the direct sum of the kernel of α and the image of α. Hence or otherwise, find the characteristic polynomial of α in terms of the dimension of U and the rank of α. Is α diagonalizable? Justify your answer.

4/II/15G   Linear Mathematics

Let $\alpha \in L(U, V)$ be a linear map between finite-dimensional vector spaces. Let

$$M^l(\alpha) = \{\beta \in L(V, U) : \ \beta\,\alpha = 0\} \qquad \text{and}$$

$$M^r(\alpha) = \{\beta \in L(V, U) : \ \alpha\,\beta = 0\} \ .$$

(a) Prove that $M^l(\alpha)$ and $M^r(\alpha)$ are subspaces of $L(V, U)$ of dimensions

$$\dim M^l(\alpha) = (\dim V - \operatorname{rank}\alpha)\dim U \qquad \text{and}$$

$$\dim M^r(\alpha) = \dim\ker(\alpha)\dim V \ .$$

[*You may use the result that there exist bases in U and V so that α is represented by*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and r is the rank of α.]

(b) Let $\Phi \colon L(U, V) \to L(V^*, U^*)$ be given by $\Phi(\alpha) = \alpha^*$, where α^* is the dual map induced by α. Prove that Φ is an isomorphism. [*You may assume that Φ is linear, and you may use the result that a finite-dimensional vector space and its dual have the same dimension.*]

(c) Prove that

$$\Phi(M^l(\alpha)) = M^r(\alpha^*) \quad \text{and} \quad \Phi(M^r(\alpha)) = M^l(\alpha^*).$$

[*You may use the results that $(\beta\,\alpha)^* = \alpha^*\,\beta^*$ and that β^{**} can be identified with β under the canonical isomorphism between a vector space and its double dual.*]

(d) Conclude that $\operatorname{rank}(\alpha) = \operatorname{rank}(\alpha^*)$.

1/I/8G Quadratic Mathematics

Let U and V be finite-dimensional vector spaces. Suppose that b and c are bilinear forms on $U \times V$ and that b is non-degenerate. Show that there exist linear endomorphisms S of U and T of V such that $c(x, y) = b(S(x), y) = b(x, T(y))$ for all $(x, y) \in U \times V$.

1/II/17G Quadratic Mathematics

(a) Suppose p is an odd prime and a an integer coprime to p. Define the *Legendre symbol* $\left(\frac{a}{p}\right)$ and state Euler's criterion.

(b) Compute $\left(\frac{-1}{p}\right)$ and prove that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

whenever a and b are coprime to p.

(c) Let n be any integer such that $1 \leqslant n \leqslant p-2$. Let m be the unique integer such that $1 \leqslant m \leqslant p-2$ and $mn \equiv 1 \pmod{p}$. Prove that

$$\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+m}{p}\right).$$

(d) Find

$$\sum_{n=1}^{p-2} \left(\frac{n(n+1)}{p}\right).$$

2/I/8G Quadratic Mathematics

Let U be a finite-dimensional real vector space and b a positive definite symmetric bilinear form on $U \times U$. Let $\psi \colon U \to U$ be a linear map such that $b(\psi(x), y) + b(x, \psi(y)) = 0$ for all x and y in U. Prove that if ψ is invertible, then the dimension of U must be even. By considering the restriction of ψ to its image or otherwise, prove that the rank of ψ is always even.

2/II/17G   Quadratic Mathematics

Let S be the set of all 2×2 complex matrices A which are *hermitian*, that is, $A^* = A$, where $A^* = \overline{A}^t$.

(a) Show that S is a real 4-dimensional vector space. Consider the real symmetric bilinear form b on this space defined by

$$b(A, B) = \tfrac{1}{2} \left(\operatorname{tr}(A\,B) - \operatorname{tr}(A)\operatorname{tr}(B) \right).$$

Prove that $b(A, A) = -\det A$ and $b(A, I) = -\tfrac{1}{2}\operatorname{tr}(A)$, where I denotes the identity matrix.

(b) Consider the three matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Prove that the basis I, A_1, A_2, A_3 of S diagonalizes b. Hence or otherwise find the rank and signature of b.

(c) Let Q be the set of all 2×2 complex matrices C which satisfy $C + C^* = \operatorname{tr}(C)\,I$. Show that Q is a real 4-dimensional vector space. Given $C \in Q$, put

$$\Phi(C) = \frac{1 - i}{2}\operatorname{tr}(C)\,I + i\,C.$$

Show that Φ takes values in S and is a linear isomorphism between Q and S.

(d) Define a real symmetric bilinear form on Q by setting $c(C, D) = -\tfrac{1}{2}\operatorname{tr}(C\,D)$, $C, D \in Q$. Show that $b(\Phi(C), \Phi(D)) = c(C, D)$ for all $C, D \in Q$. Find the rank and signature of the symmetric bilinear form c defined on Q.

3/I/9G   Quadratic Mathematics

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Explain what is meant by the *discriminant* d of f. State a necessary and sufficient condition for some form of discriminant d to represent an odd prime number p. Using this result or otherwise, find the primes p which can be represented by the form $x^2 + 3y^2$.

3/II/19G   **Quadratic Mathematics**

Let U be a finite-dimensional real vector space endowed with a positive definite inner product. A linear map $\tau : U \to U$ is said to be an *orthogonal projection* if τ is self-adjoint and $\tau^2 = \tau$.

(a) Prove that for every orthogonal projection τ there is an orthogonal decomposition

$$U = \ker(\tau) \oplus \operatorname{im}(\tau).$$

(b) Let $\phi \colon U \to U$ be a linear map. Show that if $\phi^2 = \phi$ and $\phi\phi^* = \phi^*\phi$, where ϕ^* is the adjoint of ϕ, then ϕ is an orthogonal projection. [*You may find it useful to prove first that if $\phi\phi^* = \phi^*\phi$, then ϕ and ϕ^* have the same kernel.*]

(c) Show that given a subspace W of U there exists a unique orthogonal projection τ such that $\operatorname{im}(\tau) = W$. If W_1 and W_2 are two subspaces with corresponding orthogonal projections τ_1 and τ_2, show that $\tau_2 \circ \tau_1 = 0$ if and only if W_1 is orthogonal to W_2.

(d) Let $\phi \colon U \to U$ be a linear map satisfying $\phi^2 = \phi$. Prove that one can define a positive definite inner product on U such that ϕ becomes an orthogonal projection.

1/I/5G   **Linear Mathematics**

Define $f : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$f(a, b, c) = (a + 3b - c, 2b + c, -4b - c).$$

Find the characteristic polynomial and the minimal polynomial of f. Is f diagonalisable? Are f and f^2 linearly independent endomorphisms of $\mathbb{C}^3$? Justify your answers.

1/II/14G   **Linear Mathematics**

Let α be an endomorphism of a vector space V of finite dimension n .

(a) What is the dimension of the vector space of linear endomorphisms of V? Show that there exists a non-trivial polynomial $p(X)$ such that $p(\alpha) = 0$. Define what is meant by the minimal polynomial m_α of α.

(b) Show that the eigenvalues of α are precisely the roots of the minimal polynomial of α.

(c) Let W be a subspace of V such that $\alpha(W) \subseteq W$ and let β be the restriction of α to W. Show that m_β divides m_α.

(d) Give an example of an endomorphism α and a subspace W as in (c) not equal to V for which $m_\alpha = m_\beta$, and $\deg(m_\alpha) > 1$.

2/I/6G   **Linear Mathematics**

Let A be a complex 4×4 matrix such that $A^3 = A^2$. What are the possible minimal polynomials of A? If A is not diagonalisable and $A^2 \neq 0$, list all possible Jordan normal forms of A.

2/II/15G   **Linear Mathematics**

(a) A complex $n \times n$ matrix is said to be unipotent if $U - I$ is nilpotent, where I is the identity matrix. Show that U is unipotent if and only if 1 is the only eigenvalue of U.

(b) Let T be an invertible complex matrix. By considering the Jordan normal form of T show that there exists an invertible matrix P such that

$$PTP^{-1} = D_0 + N,$$

where D_0 is an invertible diagonal matrix, N is an upper triangular matrix with zeros in the diagonal and $D_0 N = N D_0$.

(c) Set $D = P^{-1} D_0 P$ and show that $U = D^{-1} T$ is unipotent.

(d) Conclude that any invertible matrix T can be written as $T = DU$ where D is diagonalisable, U is unipotent and $DU = UD$.

3/I/7F   Linear Mathematics

Which of the following statements are true, and which false? Give brief justifications for your answers.

(a) If U and W are subspaces of a vector space V, then $U \cap W$ is always a subspace of V.

(b) If U and W are distinct subspaces of a vector space V, then $U \cup W$ is never a subspace of V.

(c) If U, W and X are subspaces of a vector space V, then $U \cap (W + X) = (U \cap W) + (U \cap X)$.

(d) If U is a subspace of a finite-dimensional space V, then there exists a subspace W such that $U \cap W = \{0\}$ and $U + W = V$.

3/II/17F   Linear Mathematics

Define the *determinant* of an $n \times n$ matrix A, and prove from your definition that if A' is obtained from A by an elementary row operation (i.e. by adding a scalar multiple of the ith row of A to the jth row, for some $j \neq i$), then $\det A' = \det A$.

Prove also that if X is a $2n \times 2n$ matrix of the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where O denotes the $n \times n$ zero matrix, then $\det X = \det A \det C$. Explain briefly how the $2n \times 2n$ matrix

$$\begin{pmatrix} B & I \\ O & A \end{pmatrix}$$

can be transformed into the matrix

$$\begin{pmatrix} B & I \\ -AB & O \end{pmatrix}$$

by a sequence of elementary row operations. Hence or otherwise prove that $\det AB = \det A \det B$.

4/I/6F   Linear Mathematics

Define the *rank* and *nullity* of a linear map between finite-dimensional vector spaces. State the rank–nullity formula.

Let $\alpha\colon U \to V$ and $\beta\colon V \to W$ be linear maps. Prove that

$$\operatorname{rank}(\alpha) + \operatorname{rank}(\beta) - \dim V \leqslant \operatorname{rank}(\beta\alpha) \leqslant \min\{\operatorname{rank}(\alpha), \operatorname{rank}(\beta)\} .$$

Part IB

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4/II/15F   **Linear Mathematics**

Define the *dual space* V^* of a finite-dimensional real vector space V, and explain what is meant by the basis of V^* dual to a given basis of V. Explain also what is meant by the statement that the second dual V^{**} is naturally isomorphic to V.

Let V_n denote the space of real polynomials of degree at most n. Show that, for any real number x, the function e_x mapping p to $p(x)$ is an element of V_n^*. Show also that, if $x_1, x_2, \ldots, x_{n+1}$ are distinct real numbers, then $\{e_{x_1}, e_{x_2}, \ldots, e_{x_{n+1}}\}$ is a basis of V_n^*, and find the basis of V_n dual to it.

Deduce that, for any $(n+1)$ distinct points $x_1, \ldots, x_{n+1}$ of the interval $[-1, 1]$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n+1} \lambda_i p(x_i)$$

for all $p \in V_n$. For $n = 4$ and $(x_1, x_2, x_3, x_4, x_5) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$, find the corresponding scalars λ_i.

1/I/8F   Quadratic Mathematics

Define the *rank* and *signature* of a symmetric bilinear form ϕ on a finite-dimensional real vector space. (If your definitions involve a matrix representation of ϕ, you should explain why they are independent of the choice of representing matrix.)

Let V be the space of all $n \times n$ real matrices (where $n \geqslant 2$), and let ϕ be the bilinear form on V defined by

$$\phi(A, B) = \operatorname{tr} AB - \operatorname{tr} A \operatorname{tr} B \ .$$

Find the rank and signature of ϕ.

[*Hint: You may find it helpful to consider the subspace of symmetric matrices having trace zero, and a suitable complement for this subspace.*]

1/II/17F   Quadratic Mathematics

Let A and B be $n \times n$ real symmetric matrices, such that the quadratic form $\mathbf{x}^T A \mathbf{x}$ is positive definite. Show that it is possible to find an invertible matrix P such that $P^T A P = I$ and $P^T B P$ is diagonal. Show also that the diagonal entries of the matrix $P^T B P$ may be calculated directly from A and B, without finding the matrix P. If

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

find the diagonal entries of $P^T B P$.

2/I/8F   Quadratic Mathematics

Explain what is meant by a *sesquilinear form* on a complex vector space V. If ϕ and ψ are two such forms, and $\phi(v, v) = \psi(v, v)$ for all $v \in V$, prove that $\phi(v, w) = \psi(v, w)$ for all $v, w \in V$. Deduce that if $\alpha : V \to V$ is a linear map satisfying $\phi(\alpha(v), \alpha(v)) = \phi(v, v)$ for all $v \in V$, then $\phi(\alpha(v), \alpha(w)) = \phi(v, w)$ for all $v, w \in V$.

2/II/17F   **Quadratic Mathematics**

Define the *adjoint* α^* of an endomorphism α of a complex inner-product space V. Show that if W is a subspace of V, then $\alpha(W) \subseteq W$ if and only if $\alpha^*(W^\perp) \subseteq W^\perp$.

An endomorphism of a complex inner-product space is said to be *normal* if it commutes with its adjoint. Prove the following facts about a normal endomorphism α of a finite-dimensional space V.

(i) α and α^* have the same kernel.

(ii) α and α^* have the same eigenvectors, with complex conjugate eigenvalues.

(iii) If $E_\lambda = \{x \in V : \alpha(x) = \lambda x\}$, then $\alpha(E_\lambda^\perp) \subseteq E_\lambda^\perp$.

(iv) There is an orthonormal basis of V consisting of eigenvectors of α.

Deduce that an endomorphism α is normal if and only if it can be written as a product $\beta\gamma$, where β is Hermitian, γ is unitary and β and γ commute with each other. [*Hint: Given α, define β and γ in terms of their effect on the basis constructed in (iv).*]

3/I/9F   **Quadratic Mathematics**

Explain what is meant by a *quadratic residue* modulo an odd prime p, and show that a is a quadratic residue modulo p if and only if $a^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}$. Hence characterize the odd primes p for which -1 is a quadratic residue.

State the law of quadratic reciprocity, and use it to determine whether 73 is a quadratic residue (mod 127).

3/II/19F   **Quadratic Mathematics**

Explain what is meant by saying that a positive definite integral quadratic form $f(x,y) = ax^2 + bxy + cy^2$ is *reduced*, and show that every positive definite form is equivalent to a reduced form.

State a criterion for a prime number p to be representable by some form of discriminant d, and deduce that p is representable by a form of discriminant -32 if and only if $p \equiv 1, 2$ or $3 \pmod 8$. Find the reduced forms of discriminant -32, and hence or otherwise show that a prime p is representable by the form $3x^2 + 2xy + 3y^2$ if and only if $p \equiv 3 \pmod 8$.

[*Standard results on when -1 and 2 are squares* (mod p) *may be assumed.*]

1/I/5C   Linear Mathematics

Determine for which values of $x \in \mathbb{C}$ the matrix

$$M = \begin{pmatrix} x & 1 & 1 \\ 1-x & 0 & -1 \\ 2 & 2x & 1 \end{pmatrix}$$

is invertible. Determine the rank of M as a function of x. Find the adjugate and hence the inverse of M for general x.

1/II/14C   Linear Mathematics

(a) Find a matrix M over $\mathbb{C}$ with both minimal polynomial and characteristic polynomial equal to $(x-2)^3(x+1)^2$. Furthermore find two matrices M_1 and M_2 over $\mathbb{C}$ which have the same characteristic polynomial, $(x-3)^5(x-1)^2$, and the same minimal polynomial, $(x-3)^2(x-1)^2$, but which are not conjugate to one another. Is it possible to find a third such matrix, M_3, neither conjugate to M_1 nor to M_2? Justify your answer.

(b) Suppose A is an $n \times n$ matrix over $\mathbb{R}$ which has minimal polynomial of the form $(x - \lambda_1)(x - \lambda_2)$ for distinct roots $\lambda_1 \neq \lambda_2$ in $\mathbb{R}$. Show that the vector space $V = \mathbb{R}^n$ on which A defines an endomorphism $\alpha : V \to V$ decomposes as a direct sum into $V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota)$, where ι is the identity.

[Hint: Express $v \in V$ in terms of $(\alpha - \lambda_1\iota)(v)$ and $(\alpha - \lambda_2\iota)(v)$.]

Now suppose that A has minimal polynomial $(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_m)$ for distinct $\lambda_1,\dots,\lambda_m \in \mathbb{R}$. By induction or otherwise show that

$$V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota) \oplus \dots \oplus \ker(\alpha - \lambda_m\iota).$$

Use this last statement to prove that an arbitrary matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if all roots of its minimal polynomial lie in $\mathbb{R}$ and have multiplicity 1.

2/I/6C   Linear Mathematics

Show that right multiplication by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$ defines a linear transformation $\rho_A : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$. Find the matrix representing ρ_A with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_{2\times 2}(\mathbb{C})$. Prove that the characteristic polynomial of ρ_A is equal to the square of the characteristic polynomial of A, and that A and ρ_A have the same minimal polynomial.

2/II/15C   Linear Mathematics

Define the dual V^* of a vector space V. Given a basis $\{v_1, \ldots, v_n\}$ of V define its dual and show it is a basis of V^*. For a linear transformation $\alpha : V \to W$ define the dual $\alpha^* : W^* \to V^*$.

Explain (with proof) how the matrix representing $\alpha : V \to W$ with respect to given bases of V and W relates to the matrix representing $\alpha^* : W^* \to V^*$ with respect to the corresponding dual bases of V^* and W^*.

Prove that α and α^* have the same rank.

Suppose that α is an invertible endomorphism. Prove that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

3/I/7C   Linear Mathematics

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 5 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix M which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ whose image is W and which contains $(1, 1, 1, 1, 1)^T$ in its kernel. What is the dimension of the space of all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1, 1, 1, 1, 1)^T$ in the kernel, and image contained in W?

3/II/17C   Linear Mathematics

Let V be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a nilpotent endomorphism of V, i.e. $\alpha^m = 0$ for some positive integer m. Prove that α can be represented by a strictly upper-triangular matrix (with zeros along the diagonal). [*You may wish to consider the subspaces* $\ker(\alpha^j)$ *for* $j = 1, \ldots, m$.]

Show that if α is nilpotent, then $\alpha^n = 0$ where n is the dimension of V. Give an example of a 4×4 matrix M such that $M^4 = 0$ but $M^3 \neq 0$.

Let A be a nilpotent matrix and I the identity matrix. Prove that $I + A$ has all eigenvalues equal to 1. Is the same true of $(I+A)(I+B)$ if A and B are nilpotent? Justify your answer.

Part IB

4/I/6C   **Linear Mathematics**

Find the Jordan normal form J of the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and determine both the characteristic and the minimal polynomial of M.

Find a basis of $\mathbb{C}^4$ such that J (the Jordan normal form of M) is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix P such that $P^{-1}MP = J$.

4/II/15C   **Linear Mathematics**

Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that AB and BA have the same characteristic polynomial. [*Hint: Look at* $\det(CBC - xC)$ *for* $C = A + yI$, *where x and y are scalar variables.*]

Show by example that AB and BA need not have the same minimal polynomial.

Suppose that AB is diagonalizable, and let $p(x)$ be its minimal polynomial. Show that the minimal polynomial of BA must divide $xp(x)$. Using this and the first part of the question prove that $(AB)^2$ and $(BA)^2$ are conjugate.

1/I/8B   Quadratic Mathematics

Let $q(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Define what is meant by the *discriminant* d of q, and show that q is positive-definite if and only if $a > 0 > d$. Define what it means for the form q to be *reduced*. For any integer $d < 0$, we define the class number $h(d)$ to be the number of positive-definite reduced binary quadratic forms (with integer coefficients) with discriminant d. Show that $h(d)$ is always finite (for negative d). Find $h(-39)$, and exhibit the corresponding reduced forms.

1/II/17B   Quadratic Mathematics

Let ϕ be a symmetric bilinear form on a finite dimensional vector space V over a field k of characteristic $\neq 2$. Prove that the form ϕ may be diagonalized, and interpret the rank r of ϕ in terms of the resulting diagonal form.

For ϕ a symmetric bilinear form on a real vector space V of finite dimension n, define the *signature* σ of ϕ, proving that it is well-defined. A subspace U of V is called *null* if $\phi|_U \equiv 0$; show that V has a null subspace of dimension $n - \frac{1}{2}(r + |\sigma|)$, but no null subspace of higher dimension.

Consider now the quadratic form q on $\mathbb{R}^5$ given by

$$2(x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 + x_5 x_1).$$

Write down the matrix A for the corresponding symmetric bilinear form, and calculate $\det A$. Hence, or otherwise, find the rank and signature of q.

2/I/8B   Quadratic Mathematics

Let V be a finite-dimensional vector space over a field k. Describe a bijective correspondence between the set of bilinear forms on V, and the set of linear maps of V to its dual space V^*. If ϕ_1, ϕ_2 are non-degenerate bilinear forms on V, prove that there exists an isomorphism $\alpha : V \to V$ such that $\phi_2(u, v) = \phi_1(u, \alpha v)$ for all $u, v \in V$. If furthermore both ϕ_1, ϕ_2 are symmetric, show that α is self-adjoint (i.e. equals its adjoint) with respect to ϕ_1.

2/II/17B   **Quadratic Mathematics**

Suppose p is an odd prime and a an integer coprime to p. Define the Legendre symbol $\left(\frac{a}{p}\right)$, and state (without proof) Euler's criterion for its calculation.

For j any positive integer, we denote by r_j the (unique) integer with $|r_j| \leq (p-1)/2$ and $r_j \equiv aj \bmod p$. Let l be the number of integers $1 \leq j \leq (p-1)/2$ for which r_j is negative. Prove that

$$\left(\frac{a}{p}\right) = (-1)^l.$$

Hence determine the odd primes for which 2 is a quadratic residue.

Suppose that $p_1, \ldots, p_m$ are primes congruent to 7 modulo 8, and let

$$N \;=\; 8(p_1 \ldots p_m)^2 - 1.$$

Show that 2 is a quadratic residue for any prime dividing N. Prove that N is divisible by some prime $p \equiv 7 \bmod 8$. Hence deduce that there are infinitely many primes congruent to 7 modulo 8.

3/I/9B   **Quadratic Mathematics**

Let A be the Hermitian matrix

$$\begin{pmatrix} 1 & i & 2i \\ -i & 3 & -i \\ -2i & i & 5 \end{pmatrix}.$$

Explaining carefully the method you use, find a diagonal matrix D with **rational** entries, and an invertible (complex) matrix T such that $T^*DT = A$, where T^* here denotes the conjugated transpose of T.

Explain briefly why we cannot find T, D as above with T unitary.

[*You may assume that if a monic polynomial $t^3 + a_2 t^2 + a_1 t + a_0$ with integer coefficients has all its roots rational, then all its roots are in fact integers.*]

3/II/19B   **Quadratic Mathematics**

Let J_1 denote the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that T is a 2×2 upper-triangular real matrix with strictly positive diagonal entries and that $J_1{}^{-1}TJ_1T^{-1}$ is orthogonal. Verify that $J_1T = TJ_1$.

Prove that any real invertible matrix A has a decomposition $A = BC$, where B is an orthogonal matrix and C is an upper-triangular matrix with strictly positive diagonal entries.

Let A now denote a $2n \times 2n$ real matrix, and $A = BC$ be the decomposition of the previous paragraph. Let K denote the $2n \times 2n$ matrix with n copies of J_1 on the diagonal, and zeros elsewhere, and suppose that $KA = AK$. Prove that $K^{-1}CKC^{-1}$ is orthogonal. From this, deduce that the entries of $K^{-1}CKC^{-1}$ are zero, apart from n orthogonal 2×2 blocks $E_1, \ldots, E_n$ along the diagonal. Show that each E_i has the form $J_1{}^{-1}C_iJ_1C_i{}^{-1}$, for some 2×2 upper-triangular matrix C_i with strictly positive diagonal entries. Deduce that $KC = CK$ and $KB = BK$.

[*Hint: The invertible $2n \times 2n$ matrices S with 2×2 blocks $S_1, \ldots, S_n$ along the diagonal, but with all other entries below the diagonal zero, form a group under matrix multiplication.*]