Part IB

Geometry

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Paper 1, Section I

2F Geometry

What is a *topological surface*?

Consider

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1\},$$

which you may assume is a topological surface. For the equivalence relation $\sim$ on S^2 generated by $(x, y, z) \sim (-x, -y, -z)$, show that $S^2/\sim$ is a topological surface. For the equivalence relation $\approx$ on S^2 generated by $(x, y, z) \approx (-x, -y, z)$, show that $S^2/\approx$ is homeomorphic to S^2.

Paper 3, Section I

2E Geometry

Let $\mathbb{H}$ be the hyperbolic upper half plane. Explain how the Riemannian metric $\frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ can be used to compute lengths, angles and areas.

Consider the triangle in $\mathbb{H}$ with vertices at $e^{i\alpha}$, $e^{i\beta}$ and ∞, where $0 < \alpha < \beta < \pi$. Compute its area, and deduce the Gauss–Bonnet theorem for a hyperbolic polygon.

Paper 1, Section II

11F Geometry

Define in terms of allowable parametrisations what it means to say that a subset $S \subset \mathbb{R}^3$ is a *smooth surface*.

Let $\phi : \mathbb{R} \to (0, \infty)$ be a smooth function. Show that

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 = \phi(z)^2\}$$

is a smooth surface in $\mathbb{R}^3$.

Suppose $a < b$ and $r > 0$ are such that for all $a \leqslant a' < b' \leqslant b$ we have

$$\mathrm{Area}(\{(x, y, z) \in \Sigma \,:\, a' \leqslant z \leqslant b'\}) = 2\pi r \cdot (b' - a').$$

Show that ϕ must satisfy $r^2 = \phi(t)^2 + \phi(t)^2\phi'(t)^2$ for $a \leqslant t \leqslant b$. Assuming that $\phi(t) < r$ for $a \leqslant t \leqslant b$, show that the graph of the function $\phi|_{[a,b]}$ lies on a circle of radius r.

Paper 2, Section II

11F Geometry

Let $U \subset \mathbb{R}^2$ and $f : U \to \mathbb{R}$ be a smooth function. Derive a formula for the first and second fundamental forms of the surface in $\mathbb{R}^3$ parametrised by

$$\begin{aligned} \sigma : U &\longrightarrow \mathbb{R}^3 \\ (u, v) &\longmapsto (u, v, f(u, v)) \end{aligned}$$

in terms of f. State a formula for the Gaussian curvature in terms of the first and second fundamental forms, and hence give a formula for the Gaussian curvature of this surface.

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface and $P \subset \mathbb{R}^3$ be a plane. Supposing that Σ is tangent to P along a smooth curve $\gamma \subset \mathbb{R}^3$ and otherwise lies on one side of P, show that the Gaussian curvature of Σ is zero at all points on γ.

Paper 3, Section II

12E Geometry

Let $\sigma : V \to \Sigma$ be a smooth parametrisation of an embedded surface $\Sigma \subset \mathbb{R}^3$, and let $\gamma : (a, b) \to \Sigma$; $t \mapsto \sigma(u(t), v(t))$ be a smooth curve. Show by differentiating $\sigma_u \cdot \gamma'$ and $\sigma_v \cdot \gamma'$ that γ satisfies the geodesic equations if and only if $\gamma''(t)$ is normal to the surface. Deduce that geodesics are parametrised at constant speed.

Now assume in addition that Σ is a surface of revolution. Let $\rho(t)$ be the distance from $\gamma(t)$ to the axis of revolution, and let $\theta(t)$ be the angle between γ and the parallel at $\gamma(t)$. Prove that if γ is a geodesic then it satisfies the Clairaut relation

$$\rho(t) \cos \theta(t) = \text{constant}.$$

On the hyperboloid $\Sigma = \{x^2 + y^2 = z^2 + 1\}$ give examples of

(i) a curve parametrised at constant speed, which satisfies the Clairaut relation, but is *not* a geodesic,

(ii) a plane that meets Σ in a pair of disjoint geodesics,

(iii) a plane that meets Σ in a pair of geodesics that intersect at right angles.

Are there any geodesics entirely contained in the region $z > 0$? Are there any geodesics $\gamma \subset \Sigma$ with $\phi(\gamma) = \gamma$ for every isometry $\phi : \Sigma \to \Sigma$? Justify your answers.

Paper 4, Section II

11E Geometry

(a) Show that the Möbius maps commuting with $z \mapsto 1/\overline{z}$ are of the form

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$$

where $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 \neq 0$. Which of these maps preserve the unit disc?

(b) Write down the Riemannian metric on the disc model $\mathbb{D}$ of the hyperbolic plane. Describe the geodesics passing through O and prove that they are length minimising curves. Deduce that every geodesic is part of a circle or line preserved by the transformation $z \mapsto 1/\overline{z}$. [You may assume that the maps in part (a) that preserve the unit disc are isometries.]

(c) Let $P \in \mathbb{D}$ be a point at a hyperbolic distance $\rho > 0$ from O. Let ℓ be the hyperbolic line passing through P at right angles to OP. Show that ℓ has Euclidean radius $1/\sinh\rho$ and centre at a distance $1/\tanh\rho$ from O.

(d) Consider a hyperbolic quadrilateral with three right angles, and angle θ at the remaining vertex v. Show that

$$\cos\theta = \tanh a \tanh b$$

where a and b are the hyperbolic lengths of the sides incident with v.

Paper 1, Section I

2E Geometry

Give a characterisation of the geodesics on a smooth embedded surface in $\mathbb{R}^3$.

Write down all the geodesics on the cylinder $x^2 + y^2 = 1$ passing through the point $(x, y, z) = (1, 0, 0)$. Verify that these satisfy your characterisation of a geodesic. Which of these geodesics are closed?

Can $\mathbb{R}^2 \setminus \{(0, 0)\}$ be equipped with an abstract Riemannian metric such that every point lies on a unique closed geodesic? Briefly justify your answer.

Paper 3, Section I

2F Geometry

Consider the space $S_{a,b} \subset \mathbb{R}^3$ defined by

$$x^2 + y^2 + z^3 + az + b = 0$$

for unknown real constants a, b with $(a, b) \neq (0, 0)$.

(a) Stating any result you use, show that $S_{a,b}$ is a smooth surface in $\mathbb{R}^3$ whenever $4a^3 + 27b^2 \neq 0$.

(b) What about the cases where $4a^3 + 27b^2 = 0$? Briefly justify your answer.

Paper 1, Section II

11E Geometry

(a) Let $\mathbb{H}$ be the upper half plane model of the hyperbolic plane. Let G be the group of orientation preserving isometries of $\mathbb{H}$. Write down the general form of an element of G. Show that G acts transitively on (i) the points in $\mathbb{H}$, (ii) the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$, and (iii) the set of hyperbolic lines in $\mathbb{H}$.

(b) Show that if $P \in \mathbb{H}$ then $\{g \in G \,|\, g(P) = P\}$ is isomorphic to $\mathrm{SO}(2)$.

(c) Show that for any two distinct points $P, Q \in \mathbb{H}$ there exists a unique $g \in G$ with $g(P) = Q$ and $g(Q) = P$.

(d) Show that if ℓ, m are hyperbolic lines meeting at $P \in \mathbb{H}$ with angle θ then the points of intersection of ℓ, m with the boundary of $\mathbb{H}$, when taken in a suitable order, have cross ratio $\cos^2(\theta/2)$.

Paper 2, Section II
11F Geometry

Consider the surface $S \subset \mathbb{R}^3$ given by

$$(\sinh u \cos v, \sinh u \sin v, v) \qquad \text{for } u, v > 0.$$

Sketch S. Calculate its first fundamental form.

(a) Find a surface of revolution S' such that there is a local isometry between S and S'. Do they have the same Gauss curvature?

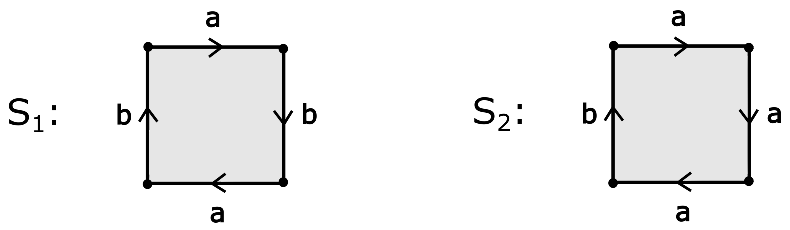
(b) Given an oriented surface $R \subset \mathbb{R}^3$, define the *Gauss map* of R. Describe the image of the Gauss map for S' equipped with the orientation associated to the outward-pointing normal. Use this to calculate the total Gaussian curvature of S'.

(c) By considering the total Gaussian curvature of S, or otherwise, show that there does not exist a global isometry between S and S'.

You should carefully state any result(s) you use.

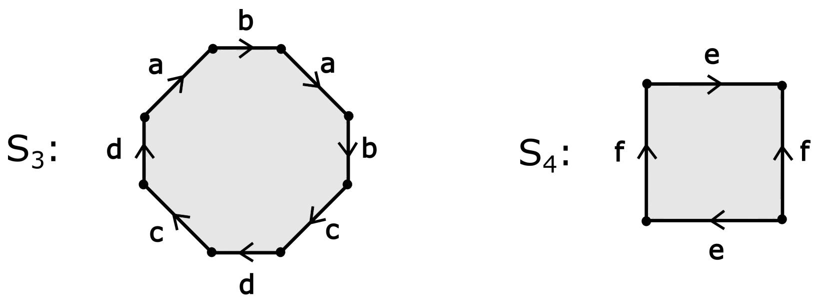
Paper 3, Section II
12F Geometry

(a) Define a *topological surface.* Consider the topological spaces S_1 and S_2 given by identifying the sides of a square as drawn. Show that S_1 is a topological surface. [*Hint: It may help to find a finite group G acting on the 2-sphere S^2 such that S^2/G is homeomorphic to S_1.*]



Is S_2 a topological surface? Briefly justify your answer.

(b) By cutting each along a suitable diagonal, show that the two topological surfaces S_3 and S_4 defined by gluing edges of polygons as shown are homeomorphic.



If you delete an open disc from S_4, can the resulting surface be embedded in $\mathbb{R}^3$? Briefly justify your answer. Can S_4 itself be embedded in $\mathbb{R}^3$? State any result you use.

Paper 4, Section II
11E Geometry

(a) Write down the metric on the unit disc model $\mathbb{D}$ of the hyperbolic plane. Let C be the Euclidean circle centred at the origin with Euclidean radius r. Show that C is a hyperbolic circle and compute its hyperbolic radius.

(b) Let Δ be a hyperbolic triangle with angles α, β, γ, and side lengths (opposite the corresponding angles) a, b, c. State the hyperbolic sine formula. The hyperbolic cosine formula is $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$. Show that if $\gamma = \pi/2$ then

$$\tan \alpha = \frac{\sinh a}{\cosh a \sinh b} \qquad \text{and} \qquad \tan \alpha \tan \beta \cosh c = 1.$$

(c) Write down the Gauss–Bonnet formula for a hyperbolic triangle. Show that the hyperbolic polygon in $\mathbb{D}$ with vertices at $re^{2\pi ik/n}$ for $k = 0, 1, 2, \ldots, n-1$ has hyperbolic area

$$A_n(r) = 2n \left[\cot^{-1} \left(\frac{1-r^2}{1+r^2} \cot \left(\frac{\pi}{n} \right) \right) - \frac{\pi}{n} \right].$$

(d) Show that there exists a hyperbolic hexagon with all interior angles a right angle. Draw pictures illustrating how such hexagons may be used to construct a closed hyperbolic surface of any genus at least 2.

Paper 1, Section I

2F Geometry

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let $\Sigma = f^{-1}(0)$ (assumed not empty). Show that if the differential $Df_p \neq 0$ for all $p \in \Sigma$, then Σ is a smooth surface in $\mathbb{R}^3$.

Is $\{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 = \cosh(z^2)\}$ a smooth surface? Is every surface $\Sigma \subset \mathbb{R}^3$ of the form $f^{-1}(0)$ for some smooth $f : \mathbb{R}^3 \to \mathbb{R}$? Justify your answers.

Paper 3, Section I

2E Geometry

State the local Gauss–Bonnet theorem for geodesic triangles on a surface. Deduce the Gauss–Bonnet theorem for closed surfaces. [Existence of a geodesic triangulation can be assumed.]

Let $S_r \subset \mathbb{R}^3$ denote the sphere with radius r centred at the origin. Show that the Gauss curvature of S_r is $1/r^2$. An octant is any of the eight regions in S_r bounded by arcs of great circles arising from the planes $x = 0$, $y = 0$, $z = 0$. Verify directly that the local Gauss–Bonnet theorem holds for an octant. [You may assume that the great circles on S_r are geodesics.]

Paper 1, Section II

11F Geometry

Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the *Gauss map* N and show that the differential DN_p of the Gauss map at any point $p \in S$ is a self-adjoint linear map. Define the *Gauss curvature* κ and compute κ in a given parametrisation.

A point $p \in S$ is called umbilic if DN_p has a repeated eigenvalue. Let $S \subset \mathbb{R}^3$ be a surface such that every point is umbilic and there is a parametrisation $\phi : \mathbb{R}^2 \to S$ such that $S = \phi(\mathbb{R}^2)$. Prove that S is part of a plane or part of a sphere. [*Hint: consider the symmetry of the mixed partial derivatives* $n_{uv} = n_{vu}$, *where* $n(u, v) = N(\phi(u, v))$ *for* $(u, v) \in \mathbb{R}^2$.]

Paper 2, Section II

11E Geometry

Define $\mathbb{H}$, the *upper half plane model* for the hyperbolic plane, and show that $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by isometries, and that these isometries preserve the orientation of $\mathbb{H}$.

Show that every orientation preserving isometry of $\mathbb{H}$ is in $PSL_2(\mathbb{R})$, and hence the full group of isometries of $\mathbb{H}$ is $G = PSL_2(\mathbb{R}) \; \cup \; PSL_2(\mathbb{R})\tau$, where $\tau z = -\bar{z}$.

Let ℓ be a hyperbolic line. Define the reflection σ_ℓ in ℓ. Now let ℓ, ℓ' be two hyperbolic lines which meet at a point $A \in \mathbb{H}$ at an angle θ. What are the possibilities for the group G generated by σ_ℓ and $\sigma_{\ell'}$? Carefully justify your answer.

Paper 3, Section II
12E Geometry

Let $S \subset \mathbb{R}^3$ be an embedded smooth surface and $\gamma : [0,1] \to S$ a parameterised smooth curve on S. What is the *energy* of γ? By applying the Euler–Lagrange equations for stationary curves to the energy function, determine the differential equations for geodesics on S explicitly in terms of a parameterisation of S.

If S contains a straight line ℓ, prove from first principles that each segment $[P,Q] \subset \ell$ (with some parameterisation) is a geodesic on S.

Let $H \subset \mathbb{R}^3$ be the hyperboloid defined by the equation $x^2 + y^2 - z^2 = 1$ and let $P = (x_0, y_0, z_0) \in H$. By considering appropriate isometries, or otherwise, display explicitly *three* distinct (as subsets of H) geodesics $\gamma : \mathbb{R} \to H$ through P in the case when $z_0 \neq 0$ and *four* distinct geodesics through P in the case when $z_0 = 0$. Justify your answer.

Let $\gamma : \mathbb{R} \to H$ be a geodesic, with coordinates $\gamma(t) = (x(t), y(t), z(t))$. Clairaut's relation asserts $\rho(t) \sin \psi(t)$ is constant, where $\rho(t) = \sqrt{x(t)^2 + y(t)^2}$ and $\psi(t)$ is the angle between $\dot{\gamma}(t)$ and the plane through the point $\gamma(t)$ and the z-axis. Deduce from Clairaut's relation that there exist infinitely many geodesics $\gamma(t)$ on H which stay in the half-space $\{z > 0\}$ for all $t \in \mathbb{R}$.

[*You may assume that if $\gamma(t)$ satisfies the geodesic equations on H then γ is defined for all $t \in \mathbb{R}$ and the Euclidean norm $\|\dot{\gamma}(t)\|$ is constant. If you use a version of the geodesic equations for a surface of revolution, then that should be proved.*]

Paper 4, Section II
11F Geometry

Define an *abstract smooth surface* and explain what it means for the surface to be *orientable*. Given two smooth surfaces S_1 and S_2 and a map $f : S_1 \to S_2$, explain what it means for f to be *smooth*.

For the cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 = 1\},$$

let $a : C \to C$ be the orientation reversing diffeomorphism $a(x,y,z) = (-x,-y,-z)$. Let S be the quotient of C by the equivalence relation $p \sim a(p)$ and let $\pi : C \to S$ be the canonical projection map. Show that S can be made into an abstract smooth surface so that π is smooth. Is S orientable? Justify your answer.

Paper 1, Section I

2E Geometry

Define the *Gauss map* of a smooth embedded surface. Consider the surface of revolution S with points

$$\begin{pmatrix}(2+\cos v)\cos u\\(2+\cos v)\sin u\\\sin v\end{pmatrix}\in\mathbb{R}^3$$

for $u, v \in [0, 2\pi]$. Let f be the Gauss map of S. Describe f on the $\{y = 0\}$ cross-section of S, and use this to write down an explicit formula for f.

Let U be the upper hemisphere of the 2-sphere S^2, and K the Gauss curvature of S. Calculate $\int_{f^{-1}(U)} K\,dA$.

Paper 1, Section II

11E Geometry

Let $\mathcal{C}$ be the curve in the (x, z)-plane defined by the equation

$$(x^2-1)^2+(z^2-1)^2=5\,.$$

Sketch $\mathcal{C}$, taking care with inflection points.

Let S be the surface of revolution in $\mathbb{R}^3$ given by spinning $\mathcal{C}$ about the z-axis. Write down an equation defining S. Stating any result you use, show that S is a smooth embedded surface.

Let r be the radial coordinate on the (x, y)-plane. Show that the Gauss curvature of S vanishes when $r = 1$. Are these the only points at which the Gauss curvature of S vanishes? Briefly justify your answer.

Paper 2, Section II

11F Geometry

Let $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the hyperbolic half-plane with the metric $g_H = (dx^2 + dy^2)/y^2$. Define the *length* of a continuously differentiable curve in H with respect to g_H.

What are the *hyperbolic lines* in H? Show that for any two distinct points z, w in H, the infimum $\rho(z, w)$ of the lengths (with respect to g_H) of curves from z to w is attained by the segment $[z, w]$ of the hyperbolic line with an appropriate parameterisation.

The 'hyperbolic Pythagoras theorem' asserts that if a hyperbolic triangle ABC has angle $\pi/2$ at C then

$$\cosh c = \cosh a \cosh b\,,$$

where a, b, c are the lengths of the sides BC, AC, AB, respectively.

Let l and m be two hyperbolic lines in H such that

$$\inf\{\rho(z, w) : z \in l,\, w \in m\} = d > 0.$$

Prove that the distance d is attained by the points of intersection with a hyperbolic line h that meets each of l, m orthogonally. Give an example of two hyperbolic lines l and m such that the infimum of $\rho(z, w)$ is *not* attained by any $z \in l$, $w \in m$.

[*You may assume that every Möbius transformation that maps H onto itself is an isometry of g_H.*]

Paper 1, Section I

3E Geometry

Describe the Poincaré disc model D for the hyperbolic plane by giving the appropriate Riemannian metric.

Calculate the distance between two points $z_1, z_2 \in D$. You should carefully state any results about isometries of D that you use.

Paper 3, Section I

5E Geometry

State a formula for the area of a spherical triangle with angles α, β, γ.

Let $n \geqslant 3$. What is the area of a convex spherical n-gon with interior angles $\alpha_1, \ldots, \alpha_n$? Justify your answer.

Find the range of possible values for the interior angle of a regular convex spherical n-gon.

Paper 3, Section II

14E Geometry

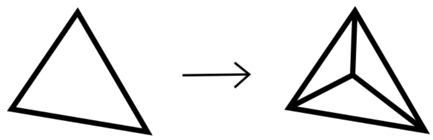
Define a *geodesic triangulation* of an abstract closed smooth surface. Define the *Euler number* of a triangulation, and state the Gauss–Bonnet theorem for closed smooth surfaces. Given a vertex in a triangulation, its valency is defined to be the number of edges incident at that vertex.

(a) Given a triangulation of the torus, show that the average valency of a vertex of the triangulation is 6.

(b) Consider a triangulation of the sphere.

(i) Show that the average valency of a vertex is strictly less than 6.

(ii) A triangulation can be subdivided by replacing one triangle Δ with three sub-triangles, each one with vertices two of the original ones, and a fixed interior point of Δ.



Using this, or otherwise, show that there exist triangulations of the sphere with average vertex valency arbitrarily close to 6.

(c) Suppose S is a closed abstract smooth surface of everywhere negative curvature. Show that the average vertex valency of a triangulation of S is bounded above and below.

Paper 2, Section II
14E Geometry

Define a *smooth embedded surface* in $\mathbb{R}^3$. Sketch the surface C given by

$$\left(\sqrt{2x^2+2y^2}-4\right)^2+2z^2=2$$

and find a smooth parametrisation for it. Use this to calculate the Gaussian curvature of C at every point.

Hence or otherwise, determine which points of the embedded surface

$$\left(\sqrt{x^2+2xz+z^2+2y^2}-4\right)^2+(z-x)^2=2$$

have Gaussian curvature zero. [*Hint: consider a transformation of* $\mathbb{R}^3$.]

[*You should carefully state any result that you use.*]

Paper 4, Section II
15E Geometry

Let $H=\{x+iy\,|\,x,y\in\mathbb{R},y>0\}$ be the upper-half plane with hyperbolic metric $\frac{dx^2+dy^2}{y^2}$. Define the group $PSL(2,\mathbb{R})$, and show that it acts by isometries on H. [If you use a generation statement you must carefully state it.]

(a) Prove that $PSL(2,\mathbb{R})$ acts transitively on the collection of pairs (l,P), where l is a hyperbolic line in H and $P\in l$.

(b) Let $l^+\subset H$ be the imaginary half-axis. Find the isometries of H which fix l^+ pointwise. Hence or otherwise find all isometries of H.

(c) Describe without proof the collection of all hyperbolic lines which meet l^+ with (signed) angle α, $0<\alpha<\pi$. Explain why there exists a hyperbolic triangle with angles α,β and γ whenever $\alpha+\beta+\gamma<\pi$.

(d) Is this triangle unique up to isometry? Justify your answer. [You may use without proof the fact that Möbius maps preserve angles.]

Paper 1, Section I
3G Geometry

(a) State the Gauss–Bonnet theorem for spherical triangles.

(b) Prove that any geodesic triangulation of the sphere has Euler number equal to 2.

(c) Prove that there is no geodesic triangulation of the sphere in which every vertex is adjacent to exactly 6 triangles.

Paper 3, Section I
5G Geometry

Consider a quadrilateral $ABCD$ in the hyperbolic plane whose sides are hyperbolic line segments. Suppose angles ABC, BCD and CDA are right-angles. Prove that AD is longer than BC.

[You may use without proof the distance formula in the upper-half-plane model

$$\rho(z_1, z_2) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right| \ .]$$

Paper 3, Section II
14G Geometry

Let U be an open subset of the plane $\mathbb{R}^2$, and let $\sigma : U \to S$ be a smooth parametrization of a surface S. A *coordinate curve* is an arc either of the form

$$\alpha_{v_0}(t) = \sigma(t, v_0)$$

for some constant v_0 and $t \in [u_1, u_2]$, or of the form

$$\beta_{u_0}(t) = \sigma(u_0, t)$$

for some constant u_0 and $t \in [v_1, v_2]$. A *coordinate rectangle* is a rectangle in S whose sides are coordinate curves.

Prove that all coordinate rectangles in S have opposite sides of the same length if and only if $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$ at all points of S, where E and G are the usual components of the first fundamental form, and (u, v) are coordinates in U.

Paper 2, Section II
14G Geometry

For any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),$$

the corresponding Möbius transformation is

$$z \mapsto Az = \frac{az+b}{cz+d},$$

which acts on the upper half-plane $\mathbb{H}$, equipped with the hyperbolic metric ρ.

(a) Assuming that $|\mathrm{tr}\ A| > 2$, prove that A is conjugate in $SL(2, \mathbb{R})$ to a diagonal matrix B. Determine the relationship between $|\mathrm{tr}\ A|$ and $\rho(i, Bi)$.

(b) For a diagonal matrix B with $|\mathrm{tr}\ B| > 2$, prove that

$$\rho(x, Bx) > \rho(i, Bi)$$

for all $x \in \mathbb{H}$ not on the imaginary axis.

(c) Assume now that $|\mathrm{tr}\ A| < 2$. Prove that A fixes a point in $\mathbb{H}$.

(d) Give an example of a matrix A in $SL(2, \mathbb{R})$ that does not preserve any point or hyperbolic line in $\mathbb{H}$. Justify your answer.

Paper 4, Section II
15G Geometry

A Möbius strip in $\mathbb{R}^3$ is parametrized by

$$\sigma(u, v) = \big(Q(u, v)\sin u, Q(u, v)\cos u, v\cos(u/2)\big)$$

for $(u, v) \in U = (0, 2\pi) \times \mathbb{R}$, where $Q \equiv Q(u, v) = 2 - v\sin(u/2)$. Show that the Gaussian curvature is

$$K = \frac{-1}{(v^2/4 + Q^2)^2}$$

at $(u, v) \in U$.

Paper 1, Section I

3G Geometry

Give the definition for the *area* of a hyperbolic triangle with interior angles α, β, γ.

Let $n \geqslant 3$. Show that the area of a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$ is $(n-2)\pi - \sum \alpha_i$.

Show that for every $n \geqslant 3$ and for every A with $0 < A < (n-2)\pi$ there exists a regular hyperbolic n-gon with area A.

Paper 3, Section I

5G Geometry

Let

$$\pi(x,y,z) = \frac{x+iy}{1-z}$$

be stereographic projection from the unit sphere S^2 in $\mathbb{R}^3$ to the Riemann sphere $\mathbb{C}_\infty$. Show that if r is a rotation of S^2, then $\pi r \pi^{-1}$ is a Möbius transformation of $\mathbb{C}_\infty$ which can be represented by an element of $SU(2)$. (You may assume without proof any result about generation of $SO(3)$ by a particular set of rotations, but should state it carefully.)

Paper 2, Section II

14G Geometry

Let $H = \{\, \mathbf{x} \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{x} = c \,\}$ be a hyperplane in $\mathbb{R}^n$, where $\mathbf{u}$ is a unit vector and c is a constant. Show that the reflection map

$$\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{u} \cdot \mathbf{x} - c)\mathbf{u}$$

is an isometry of $\mathbb{R}^n$ which fixes H pointwise.

Let $\mathbf{p}$, $\mathbf{q}$ be distinct points in $\mathbb{R}^n$. Show that there is a unique reflection R mapping $\mathbf{p}$ to $\mathbf{q}$, and that $R \in O(n)$ if and only if $\mathbf{p}$ and $\mathbf{q}$ are equidistant from the origin.

Show that every isometry of $\mathbb{R}^n$ can be written as a product of at most $n+1$ reflections. Give an example of an isometry of $\mathbb{R}^2$ which cannot be written as a product of fewer than 3 reflections.

Paper 3, Section II
14G Geometry

Let $\sigma \colon U \to \mathbb{R}^3$ be a parametrised surface, where $U \subset \mathbb{R}^2$ is an open set.

(a) Explain what are the *first and second fundamental forms* of the surface, and what is its *Gaussian curvature*. Compute the Gaussian curvature of the hyperboloid $\sigma(x, y) = (x, y, xy)$.

(b) Let $\mathbf{a}(x)$ and $\mathbf{b}(x)$ be parametrised curves in $\mathbb{R}^3$, and assume that

$$\sigma(x, y) = \mathbf{a}(x) + y\mathbf{b}(x).$$

Find a formula for the first fundamental form, and show that the Gaussian curvature vanishes if and only if

$$\mathbf{a}' \cdot (\mathbf{b} \times \mathbf{b}') = 0\,.$$

Paper 4, Section II
15G Geometry

What is a *hyperbolic line* in (a) the disc model (b) the upper half-plane model of the hyperbolic plane? What is the *hyperbolic distance* $d(P, Q)$ between two points P, Q in the hyperbolic plane? Show that if γ is any continuously differentiable curve with endpoints P and Q then its length is at least $d(P, Q)$, with equality if and only if γ is a monotonic reparametrisation of the hyperbolic line segment joining P and Q.

What does it mean to say that two hyperbolic lines L, L' are (a) *parallel* (b) *ultraparallel*? Show that L and L' are ultraparallel if and only if they have a common perpendicular, and if so, then it is unique.

A *horocycle* is a curve in the hyperbolic plane which in the disc model is a Euclidean circle with exactly one point on the boundary of the disc. Describe the horocycles in the upper half-plane model. Show that for any pair of horocycles there exists a hyperbolic line which meets both orthogonally. For which pairs of horocycles is this line unique?

Paper 1, Section I

3F Geometry

(a) Describe the Poincaré disc model D for the hyperbolic plane by giving the appropriate Riemannian metric.

(b) Let $a \in D$ be some point. Write down an isometry $f : D \to D$ with $f(a) = 0$.

(c) Using the Poincaré disc model, calculate the distance from 0 to $re^{i\theta}$ with $0 \leqslant r < 1$.

(d) Using the Poincaré disc model, calculate the area of a disc centred at a point $a \in D$ and of hyperbolic radius $\rho > 0$.

Paper 3, Section I

5F Geometry

(a) State Euler's formula for a triangulation of a sphere.

(b) A sphere is decomposed into hexagons and pentagons with precisely three edges at each vertex. Determine the number of pentagons.

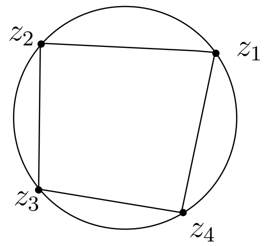
Paper 3, Section II

14F Geometry

(a) Define the *cross-ratio* $[z_1, z_2, z_3, z_4]$ of four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$. Show that the cross-ratio is invariant under Möbius transformations. Express $[z_2, z_1, z_3, z_4]$ in terms of $[z_1, z_2, z_3, z_4]$.

(b) Show that $[z_1, z_2, z_3, z_4]$ is real if and only if z_1, z_2, z_3, z_4 lie on a line or circle in $\mathbb{C} \cup \{\infty\}$.

(c) Let z_1, z_2, z_3, z_4 lie on a circle in $\mathbb{C}$, given in anti-clockwise order as depicted.



Show that $[z_1, z_2, z_3, z_4]$ is a negative real number, and that $[z_2, z_1, z_3, z_4]$ is a positive real number greater than 1. Show that $|[z_1, z_2, z_3, z_4]| + 1 = |[z_2, z_1, z_3, z_4]|$. Use this to deduce Ptolemy's relation on lengths of edges and diagonals of the inscribed 4-gon:

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|.$$

Paper 2, Section II

14F Geometry

(a) Let ABC be a hyperbolic triangle, with the angle at A at least $\pi/2$. Show that the side BC has maximal length amongst the three sides of ABC.
[You may use the hyperbolic cosine formula without proof. This states that if a, b and c are the lengths of BC, AC, and AB respectively, and α, β and γ are the angles of the triangle at A, B and C respectively, then

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha.]$$

(b) Given points z_1, z_2 in the hyperbolic plane, let w be any point on the hyperbolic line segment joining z_1 to z_2, and let w' be any point not on the hyperbolic line passing through z_1, z_2, w. Show that

$$\rho(w', w) \leqslant \max\{\rho(w', z_1), \rho(w', z_2)\},$$

where ρ denotes hyperbolic distance.

(c) The diameter of a hyperbolic triangle Δ is defined to be

$$\sup\{\rho(P, Q) \mid P, Q \in \Delta\}.$$

Show that the diameter of Δ is equal to the length of its longest side.

Paper 4, Section II

15F Geometry

Let $\alpha(s) = (f(s), g(s))$ be a simple curve in $\mathbb{R}^2$ parameterised by arc length with $f(s) > 0$ for all s, and consider the surface of revolution S in $\mathbb{R}^3$ defined by the parameterisation

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

(a) Calculate the first and second fundamental forms for S. Show that the Gaussian curvature of S is given by

$$K = -\frac{f''(u)}{f(u)}.$$

(b) Now take $f(s) = \cos s + 2$, $g(s) = \sin s$, $0 \leqslant s < 2\pi$. What is the integral of the Gaussian curvature over the surface of revolution S determined by f and g?
[You may use the Gauss-Bonnet theorem without proof.]

(c) Now suppose S has constant curvature $K \equiv 1$, and suppose there are two points $P_1, P_2 \in \mathbb{R}^3$ such that $S \cup \{P_1, P_2\}$ is a smooth closed embedded surface. Show that S is a unit sphere, minus two antipodal points.
[Do not attempt to integrate an expression of the form $\sqrt{1 - C^2 \sin^2 u}$ when $C \neq 1$. Study the behaviour of the surface at the largest and smallest possible values of u.]

Paper 1, Section I

3F Geometry

(i) Give a model for the hyperbolic plane. In this choice of model, describe hyperbolic lines.

Show that if ℓ_1, ℓ_2 are two hyperbolic lines and $p_1 \in \ell_1$, $p_2 \in \ell_2$ are points, then there exists an isometry g of the hyperbolic plane such that $g(\ell_1) = \ell_2$ and $g(p_1) = p_2$.

(ii) Let T be a triangle in the hyperbolic plane with angles $30°$, $30°$ and $45°$. What is the area of T?

Paper 3, Section I

5F Geometry

State the sine rule for spherical triangles.

Let Δ be a spherical triangle with vertices A, B, and C, with angles α, β and γ at the respective vertices. Let a, b, and c be the lengths of the edges BC, AC and AB respectively. Show that $b = c$ if and only if $\beta = \gamma$. [You may use the cosine rule for spherical triangles.] Show that this holds if and only if there exists a reflection M such that $M(A) = A$, $M(B) = C$ and $M(C) = B$.

Are there equilateral triangles on the sphere? Justify your answer.

Paper 3, Section II

14F Geometry

Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation on the Riemann sphere $\mathbb{C}_\infty$.

(i) Show that T has either one or two fixed points.

(ii) Show that if T is a Möbius transformation corresponding to (under stereographic projection) a rotation of S^2 through some fixed non-zero angle, then T has two fixed points, z_1, z_2, with $z_2 = -1/\bar{z}_1$.

(iii) Suppose T has two fixed points z_1, z_2 with $z_2 = -1/\bar{z}_1$. Show that either T corresponds to a rotation as in (ii), or one of the fixed points, say z_1, is attractive, i.e. $T^n z \to z_1$ as $n \to \infty$ for any $z \neq z_2$.

Paper 2, Section II

14F Geometry

(a) For each of the following subsets of $\mathbb{R}^3$, explain briefly why it is a smooth embedded surface or why it is not.

$$S_1 = \{(x,y,z)\,|\, x = y, z = 3\} \cup \{(2,3,0)\}$$
$$S_2 = \{(x,y,z)\,|\, x^2 + y^2 - z^2 = 1\}$$
$$S_3 = \{(x,y,z)\,|\, x^2 + y^2 - z^2 = 0\}$$

(b) Let $f : U = \{(u,v)|v > 0\} \to \mathbb{R}^3$ be given by

$$f(u,v) = (u^2, uv, v),$$

and let $S = f(U) \subseteq \mathbb{R}^3$. You may assume that S is a smooth embedded surface.

Find the first fundamental form of this surface.

Find the second fundamental form of this surface.

Compute the Gaussian curvature of this surface.

Paper 4, Section II

15F Geometry

Let $\alpha(s) = \big(f(s), g(s)\big)$ be a curve in $\mathbb{R}^2$ parameterized by arc length, and consider the surface of revolution S in $\mathbb{R}^3$ defined by the parameterization

$$\sigma(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big).$$

In what follows, you may use that a curve $\sigma \circ \gamma$ in S, with $\gamma(t) = \big(u(t), v(t)\big)$, is a geodesic if and only if

$$\ddot{u} = f(u)\frac{df}{du}\dot{v}^2, \quad \frac{d}{dt}\big(f(u)^2\dot{v}\big) = 0.$$

(i) Write down the first fundamental form for S, and use this to write down a formula which is equivalent to $\sigma \circ \gamma$ being a unit speed curve.

(ii) Show that for a given u_0, the circle on S determined by $u = u_0$ is a geodesic if and only if $\frac{df}{du}(u_0) = 0$.

(iii) Let $\gamma(t) = \big(u(t), v(t)\big)$ be a curve in $\mathbb{R}^2$ such that $\sigma \circ \gamma$ parameterizes a unit speed curve that is a geodesic in S. For a given time t_0, let $\theta(t_0)$ denote the angle between the curve $\sigma \circ \gamma$ and the circle on S determined by $u = u(t_0)$. Derive *Clairault's relation* that

$$f\big(u(t)\big)\cos\big(\theta(t)\big)$$

is independent of t.

Paper 1, Section I

3F Geometry

Determine the second fundamental form of a surface in $\mathbb{R}^3$ defined by the parametrisation

$$\sigma(u,v) = \Big((a+b\cos u)\cos v,\ (a+b\cos u)\sin v,\ b\sin u\Big),$$

for $0<u<2\pi,\ 0<v<2\pi$, with some fixed $a>b>0$. Show that the Gaussian curvature $K(u,v)$ of this surface takes both positive and negative values.

Paper 3, Section I

5F Geometry

Let $f(x)=Ax+b$ be an isometry $\mathbb{R}^n \to \mathbb{R}^n$, where A is an $n\times n$ matrix and $b\in\mathbb{R}^n$. What are the possible values of $\det A$?

Let I denote the $n\times n$ identity matrix. Show that if $n=2$ and $\det A>0$, but $A\neq I$, then f has a fixed point. Must f have a fixed point if $n=3$ and $\det A>0$, but $A\neq I$? Justify your answer.

Paper 3, Section II

14F Geometry

Let $\mathcal{T}$ be a decomposition of the two-dimensional sphere into polygonal domains, with every polygon having at least three edges. Let V, E, and F denote the numbers of vertices, edges and faces of $\mathcal{T}$, respectively. State Euler's formula. Prove that $2E \geqslant 3F$.

Suppose that at least three edges meet at every vertex of $\mathcal{T}$. Let F_n be the number of faces of $\mathcal{T}$ that have exactly n edges ($n\geqslant 3$) and let V_m be the number of vertices at which exactly m edges meet ($m\geqslant 3$). Is it possible for $\mathcal{T}$ to have $V_3=F_3=0$? Justify your answer.

By expressing $6F-\sum_n nF_n$ in terms of the V_j, or otherwise, show that $\mathcal{T}$ has at least four faces that are triangles, quadrilaterals and/or pentagons.

Paper 2, Section II

14F Geometry

Let $H = \{x + iy : x, y \in \mathbb{R},\ y > 0\} \subset \mathbb{C}$ be the upper half-plane with a hyperbolic metric $g = \frac{dx^2+dy^2}{y^2}$. Prove that every hyperbolic circle C in H is also a Euclidean circle. Is the centre of C as a hyperbolic circle always the same point as the centre of C as a Euclidean circle? Give a proof or counterexample as appropriate.

Let ABC and $A'B'C'$ be two hyperbolic triangles and denote the hyperbolic lengths of their sides by a, b, c and a', b', c', respectively. Show that if $a = a'$, $b = b'$ and $c = c'$, then there is a hyperbolic isometry taking ABC to $A'B'C'$. Is there always such an isometry if instead the triangles have one angle the same and $a = a'$, $b = b'$? Justify your answer.

[Standard results on hyperbolic isometries may be assumed, provided they are clearly stated.]

Paper 4, Section II

15F Geometry

Define an embedded parametrised surface in $\mathbb{R}^3$. What is the Riemannian metric induced by a parametrisation? State, in terms of the Riemannian metric, the equations defining a geodesic curve $\gamma : (0, 1) \to S$, assuming that γ is parametrised by arc-length.

Let S be a conical surface

$$S = \{(x, y, z) \in \mathbb{R}^3 :\ 3(x^2 + y^2) = z^2,\ \ z > 0\}.$$

Using an appropriate smooth parametrisation, or otherwise, prove that S is locally isometric to the Euclidean plane. Show that any two points on S can be joined by a geodesic. Is this geodesic always unique (up to a reparametrisation)? Justify your answer.

[The expression for the Euclidean metric in polar coordinates on $\mathbb{R}^2$ may be used without proof.]

Paper 1, Section I

3F Geometry

Let l_1 and l_2 be ultraparallel geodesics in the hyperbolic plane. Prove that the l_i have a unique common perpendicular.

Suppose now l_1, l_2, l_3 are pairwise ultraparallel geodesics in the hyperbolic plane. Can the three common perpendiculars be pairwise disjoint? Must they be pairwise disjoint? Briefly justify your answers.

Paper 3, Section I

5F Geometry

Let S be a surface with Riemannian metric having first fundamental form $du^2 + G(u,v)dv^2$. State a formula for the Gauss curvature K of S.

Suppose that S is flat, so K vanishes identically, and that $u = 0$ is a geodesic on S when parametrised by arc-length. Using the geodesic equations, or otherwise, prove that $G(u,v) \equiv 1$, i.e. S is locally isometric to a plane.

Paper 2, Section II

14F Geometry

Let A and B be disjoint circles in $\mathbb{C}$. Prove that there is a Möbius transformation which takes A and B to two concentric circles.

A collection of circles $X_i \subset \mathbb{C}$, $0 \leqslant i \leqslant n-1$, for which

1. X_i is tangent to A, B and X_{i+1}, where indices are *mod* n;
2. the circles are disjoint away from tangency points;

is called a *constellation* on (A,B). Prove that for any $n \geqslant 2$ there is some pair (A,B) and a constellation on (A,B) made up of precisely n circles. Draw a picture illustrating your answer.

Given a constellation on (A,B), prove that the tangency points $X_i \cap X_{i+1}$ for $0 \leqslant i \leqslant n-1$ all lie on a circle. Moreover, prove that if we take any other circle Y_0 tangent to A and B, and then construct Y_i for $i \geqslant 1$ inductively so that Y_i is tangent to A, B and Y_{i-1}, then we will have $Y_n = Y_0$, i.e. the chain of circles will again close up to form a constellation.

Paper 3, Section II
14F Geometry

Show that the set of all straight lines in $\mathbb{R}^2$ admits the structure of an abstract smooth surface S. Show that S is an open Möbius band (i.e. the Möbius band without its boundary circle), and deduce that S admits a Riemannian metric with vanishing Gauss curvature.

Show that there is no metric $d : S \times S \to \mathbb{R}_{\geqslant 0}$, in the sense of metric spaces, which

1. induces the locally Euclidean topology on S constructed above;
2. is invariant under the natural action on S of the group of translations of $\mathbb{R}^2$.

Show that the set of great circles on the two-dimensional sphere admits the structure of a smooth surface S'. Is S' homeomorphic to S ? Does S' admit a Riemannian metric with vanishing Gauss curvature? Briefly justify your answers.

Paper 4, Section II
15F Geometry

Let η be a smooth curve in the xz-plane $\eta(s) = (f(s), 0, g(s))$, with $f(s) > 0$ for every $s \in \mathbb{R}$ and $f'(s)^2 + g'(s)^2 = 1$. Let S be the surface obtained by rotating η around the z-axis. Find the first fundamental form of S.

State the equations for a curve $\gamma : (a, b) \to S$ parametrised by arc-length to be a geodesic.

A parallel on S is the closed circle swept out by rotating a single point of η. Prove that for every $n \in \mathbb{Z}_{>0}$ there is some η for which exactly n parallels are geodesics. Sketch possible such surfaces S in the cases $n = 1$ and $n = 2$.

If *every* parallel is a geodesic, what can you deduce about S ? Briefly justify your answer.

Paper 1, Section I

3G Geometry

Describe a collection of charts which cover a circular cylinder of radius R. Compute the first fundamental form, and deduce that the cylinder is locally isometric to the plane.

Paper 3, Section I

5G Geometry

State a formula for the area of a hyperbolic triangle.

Hence, or otherwise, prove that if l_1 and l_2 are disjoint geodesics in the hyperbolic plane, there is at most one geodesic which is perpendicular to both l_1 and l_2.

Paper 2, Section II

14G Geometry

Let S be a closed surface, equipped with a triangulation. Define the Euler characteristic $\chi(S)$ of S. How does $\chi(S)$ depend on the triangulation?

Let V, E and F denote the number of vertices, edges and faces of the triangulation. Show that $2E = 3F$.

Suppose now the triangulation is *tidy*, meaning that it has the property that no two vertices are joined by more than one edge. Deduce that V satisfies

$$V \geqslant \frac{7 + \sqrt{49 - 24\chi(S)}}{2}.$$

Hence compute the minimal number of vertices of a tidy triangulation of the real projective plane. [*Hint: it may be helpful to consider the icosahedron as a triangulation of the sphere* S^2.]

Paper 3, Section II

14G Geometry

Define the *first* and *second fundamental forms* of a smooth surface $\Sigma \subset \mathbb{R}^3$, and explain their geometrical significance.

Write down the geodesic equations for a smooth curve $\gamma : [0,1] \to \Sigma$. Prove that γ is a geodesic if and only if the derivative of the tangent vector to γ is always orthogonal to Σ.

A plane $\Pi \subset \mathbb{R}^3$ cuts Σ in a smooth curve $C \subset \Sigma$, in such a way that reflection in the plane Π is an isometry of Σ (in particular, preserves Σ). Prove that C is a geodesic.

Paper 4, Section II

15G Geometry

Let $\Sigma \subset \mathbb{R}^3$ be a smooth closed surface. Define the *principal curvatures* $\kappa_{\max}$ and $\kappa_{\min}$ at a point $p \in \Sigma$. Prove that the Gauss curvature at p is the product of the two principal curvatures.

A point $p \in \Sigma$ is called a *parabolic point* if at least one of the two principal curvatures vanishes. Suppose $\Pi \subset \mathbb{R}^3$ is a plane and Σ is tangent to Π along a smooth closed curve $C = \Pi \cap \Sigma \subset \Sigma$. Show that C is composed of parabolic points.

Can both principal curvatures vanish at a point of C? Briefly justify your answer.

Paper 1, Section I

3F Geometry

Suppose that $H \subseteq \mathbb{C}$ is the upper half-plane, $H = \{x + iy \,|\, x, y \in \mathbb{R},\ y > 0\}$. Using the Riemannian metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$, define the length of a curve γ and the area of a region Ω in H.

Find the area of

$$\Omega = \{x + iy \,|\, |x| \leqslant \frac{1}{2},\ x^2 + y^2 \geqslant 1\}\,.$$

Paper 3, Section I

5F Geometry

Let $R(x, \theta)$ denote anti-clockwise rotation of the Euclidean plane $\mathbb{R}^2$ through an angle θ about a point x.

Show that $R(x, \theta)$ is a composite of two reflexions.

Assume $\theta, \phi \in (0, \pi)$. Show that the composite $R(y, \phi) \cdot R(x, \theta)$ is also a rotation $R(z, \psi)$. Find z and ψ.

Paper 2, Section II

14F Geometry

Suppose that $\pi : S^2 \to \mathbb{C}_\infty$ is stereographic projection. Show that, via π, every rotation of S^2 corresponds to a Möbius transformation in $PSU(2)$.

Paper 3, Section II

14F Geometry

Suppose that $\eta(u) = (f(u), 0, g(u))$ is a unit speed curve in $\mathbb{R}^3$. Show that the corresponding surface of revolution S obtained by rotating this curve about the z-axis has Gaussian curvature $K = -(d^2 f/du^2)/f$.

Paper 4, Section II

15F Geometry

Suppose that P is a point on a Riemannian surface S. Explain the notion of geodesic polar co-ordinates on S in a neighbourhood of P, and prove that if C is a geodesic circle centred at P of small positive radius, then the geodesics through P meet C at right angles.

Paper 1, Section I

3F Geometry

(i) Define the notion of curvature for surfaces embedded in $\mathbb{R}^3$.

(ii) Prove that the unit sphere in $\mathbb{R}^3$ has curvature $+1$ at all points.

Paper 3, Section I

5F Geometry

(i) Write down the Poincaré metric on the unit disc model D of the hyperbolic plane. Compute the hyperbolic distance ρ from $(0,0)$ to $(r,0)$, with $0 < r < 1$.

(ii) Given a point P in D and a hyperbolic line L in D with P not on L, describe how the minimum distance from P to L is calculated. Justify your answer.

Paper 2, Section II

14F Geometry

Suppose that $a > 0$ and that $S \subset \mathbb{R}^3$ is the half-cone defined by $z^2 = a(x^2 + y^2)$, $z > 0$. By using an explicit smooth parametrization of S, calculate the curvature of S.

Describe the geodesics on S. Show that for $a = 3$, no geodesic intersects itself, while for $a > 3$ some geodesic does so.

Paper 3, Section II

14F Geometry

Describe the hyperbolic metric on the upper half-plane H. Show that any Möbius transformation that preserves H is an isometry of this metric.

Suppose that $z_1, z_2 \in H$ are distinct and that the hyperbolic line through z_1 and z_2 meets the real axis at w_1, w_2. Show that the hyperbolic distance $\rho(z_1, z_2)$ between z_1 and z_2 is given by $\rho(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2, taken in an appropriate order.

Paper 4, Section II

15F Geometry

Suppose that D is the unit disc, with Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{1 - (x^2 + y^2)}\,.$$

[*Note that this is not a multiple of the Poincaré metric.*] Show that the diameters of D are, with appropriate parametrization, geodesics.

Show that distances between points in D are bounded, but areas of regions in D are unbounded.

Paper 1, Section I

2G Geometry

What is an *ideal hyperbolic triangle*? State a formula for its area.

Compute the area of a hyperbolic disk of hyperbolic radius ρ. Hence, or otherwise, show that no hyperbolic triangle completely contains a hyperbolic circle of hyperbolic radius 2.

Paper 3, Section I

2G Geometry

Write down the equations for geodesic curves on a surface. Use these to describe all the geodesics on a circular cylinder, and draw a picture illustrating your answer.

Paper 2, Section II

12G Geometry

What is meant by *stereographic projection* from the unit sphere in $\mathbb{R}^3$ to the complex plane? Briefly explain why a spherical triangle cannot map to a Euclidean triangle under stereographic projection.

Derive an explicit formula for stereographic projection. Hence, or otherwise, prove that if a Möbius map corresponds via stereographic projection to a rotation of the sphere, it has two fixed points p and q which satisfy $p\bar{q} = -1$. Give, with justification:

(i) a Möbius transformation which fixes a pair of points $p, q \in \mathbb{C}$ satisfying $p\bar{q} = -1$ but which does not arise from a rotation of the sphere;

(ii) an isometry of the sphere (for the spherical metric) which does not correspond to any Möbius transformation under stereographic projection.

Paper 3, Section II
12G Geometry

Consider a tessellation of the two-dimensional sphere, that is to say a decomposition of the sphere into polygons each of which has at least three sides. Let E, V and F denote the numbers of edges, vertices and faces in the tessellation, respectively. State Euler's formula. Prove that $2E \geqslant 3F$. Deduce that not all the vertices of the tessellation have valence $\geqslant 6$.

By considering the plane $\{z = 1\} \subset \mathbb{R}^3$, or otherwise, deduce the following: if Σ is a finite set of straight lines in the plane $\mathbb{R}^2$ with the property that every intersection point of two lines is an intersection point of at least three, then all the lines in Σ meet at a single point.

Paper 4, Section II
12G Geometry

Let $U \subset \mathbb{R}^2$ be an open set. Let $\Sigma \subset \mathbb{R}^3$ be a surface locally given as the graph of an infinitely-differentiable function $f : U \to \mathbb{R}$. Compute the Gaussian curvature of Σ in terms of f.

Deduce that if $\widehat{\Sigma} \subset \mathbb{R}^3$ is a compact surface without boundary, its Gaussian curvature is not everywhere negative.

Give, with brief justification, a compact surface $\widehat{\Sigma} \subset \mathbb{R}^3$ without boundary whose Gaussian curvature must change sign.

1/I/2G **Geometry**

Show that any element of $SO(3,\mathbb{R})$ is a rotation, and that it can be written as the product of two reflections.

2/II/12G **Geometry**

Show that the area of a spherical triangle with angles α, β, γ is $\alpha+\beta+\gamma-\pi$. Hence derive the formula for the area of a convex spherical n-gon.

Deduce Euler's formula $F-E+V=2$ for a decomposition of a sphere into F convex polygons with a total of E edges and V vertices.

A sphere is decomposed into convex polygons, comprising m quadrilaterals, n pentagons and p hexagons, in such a way that at each vertex precisely three edges meet. Show that there are at most 7 possibilities for the pair (m,n), and that at least 3 of these do occur.

3/I/2G **Geometry**

A smooth surface in $\mathbb{R}^3$ has parametrization

$$\sigma(u,v)=\left(u-\frac{u^3}{3}+uv^2,\, v-\frac{v^3}{3}+u^2v,\, u^2-v^2\right).$$

Show that a unit normal vector at the point $\sigma(u,v)$ is

$$\left(\frac{-2u}{1+u^2+v^2},\frac{2v}{1+u^2+v^2},\frac{1-u^2-v^2}{1+u^2+v^2}\right)$$

and that the curvature is $\dfrac{-4}{(1+u^2+v^2)^4}$.

3/II/12G **Geometry**

Let D be the unit disc model of the hyperbolic plane, with metric

$$\frac{4\,|d\zeta|^2}{(1-|\zeta|^2)^2}.$$

(i) Show that the group of Möbius transformations mapping D to itself is the group of transformations

$$\zeta \mapsto \omega \frac{\zeta - \lambda}{\bar{\lambda}\zeta - 1},$$

where $|\lambda| < 1$ and $|\omega| = 1$.

(ii) Assuming that the transformations in (i) are isometries of D, show that any hyperbolic circle in D is a Euclidean circle.

(iii) Let P and Q be points on the unit circle with $\angle POQ = 2\alpha$. Show that the hyperbolic distance from O to the hyperbolic line PQ is given by

$$2\tanh^{-1}\left(\frac{1-\sin\alpha}{\cos\alpha}\right).$$

(iv) Deduce that if $a > 2\tanh^{-1}(2-\sqrt{3})$ then no hyperbolic open disc of radius a is contained in a hyperbolic triangle.

4/II/12G **Geometry**

Let $\gamma\colon [a,b] \to S$ be a curve on a smoothly embedded surface $S \subset \mathbf{R}^3$. Define the energy of γ. Show that if γ is a stationary point for the energy for proper variations of γ, then γ satisfies the geodesic equations

$$\frac{d}{dt}(E\dot{\gamma}_1 + F\dot{\gamma}_2) = \frac{1}{2}(E_u\dot{\gamma}_1^2 + 2F_u\dot{\gamma}_1\dot{\gamma}_2 + G_u\dot{\gamma}_2^2)$$

$$\frac{d}{dt}(F\dot{\gamma}_1 + G\dot{\gamma}_2) = \frac{1}{2}(E_v\dot{\gamma}_1^2 + 2F_v\dot{\gamma}_1\dot{\gamma}_2 + G_v\dot{\gamma}_2^2)$$

where $\gamma = (\gamma_1, \gamma_2)$ in terms of a smooth parametrization (u,v) for S, with first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$.

Now suppose that for every c, d the curves $u = c$, $v = d$ are geodesics.

(i) Show that $(F/\sqrt{G})_v = (\sqrt{G})_u$ and $(F/\sqrt{E})_u = (\sqrt{E})_v$.

(ii) Suppose moreover that the angle between the curves $u = c$, $v = d$ is independent of c and d. Show that $E_v = 0 = G_u$.

1/I/2A Geometry

State the Gauss–Bonnet theorem for spherical triangles, and deduce from it that for each convex polyhedron with F faces, E edges, and V vertices, $F - E + V = 2$.

2/II/12A Geometry

(i) The spherical circle with centre $P \in S^2$ and radius r, $0 < r < \pi$, is the set of all points on the unit sphere S^2 at spherical distance r from P. Find the circumference of a spherical circle with spherical radius r. Compare, for small r, with the formula for a Euclidean circle and comment on the result.

(ii) The cross ratio of four distinct points z_i in $\mathbf{C}$ is

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}\,.$$

Show that the cross-ratio is a real number if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

[*You may assume that Möbius transformations preserve the cross-ratio.*]

3/I/2A Geometry

Let l be a line in the Euclidean plane $\mathbf{R}^2$ and P a point on l. Denote by ρ the reflection in l and by τ the rotation through an angle α about P. Describe, in terms of l, P, and α, a line fixed by the composition $\tau\rho$ and show that $\tau\rho$ is a reflection.

3/II/12A Geometry

For a parameterized smooth embedded surface $\sigma : V \to U \subset \mathbf{R}^3$, where V is an open domain in $\mathbf{R}^2$, define the *first fundamental form*, the *second fundamental form*, and the *Gaussian curvature* K. State the Gauss–Bonnet formula for a compact embedded surface $S \subset \mathbf{R}^3$ having Euler number $e(S)$.

Let S denote a surface defined by rotating a curve

$$\eta(u) = (r + a\sin u, 0, b\cos u) \qquad 0 \le u \le 2\pi\,,$$

about the z-axis. Here a, b, r are positive constants, such that $a^2 + b^2 = 1$ and $a < r$. By considering a smooth parameterization, find the first fundamental form and the second fundamental form of S.

4/II/12A **Geometry**

Write down the Riemannian metric for the upper half-plane model **H** of the hyperbolic plane. Describe, without proof, the group of isometries of **H** and the hyperbolic lines (i.e. the geodesics) on **H**.

Show that for any two hyperbolic lines ℓ_1, ℓ_2, there is an isometry of **H** which maps ℓ_1 onto ℓ_2.

Suppose that g is a composition of two reflections in hyperbolic lines which are ultraparallel (i.e. do not meet either in the hyperbolic plane or at its boundary). Show that g cannot be an element of finite order in the group of isometries of **H**.

[*Existence of a common perpendicular to two ultraparallel hyperbolic lines may be assumed. You might like to choose carefully which hyperbolic line to consider as a common perpendicular.*]

1/I/2H **Geometry**

Define the hyperbolic metric in the upper half-plane model H of the hyperbolic plane. How does one define the hyperbolic area of a region in H? State the Gauss–Bonnet theorem for hyperbolic triangles.

Let R be the region in H defined by

$$0 < x < \frac{1}{2}, \quad \sqrt{1-x^2} < y < 1.$$

Calculate the hyperbolic area of R.

2/II/12H **Geometry**

Let $\sigma : V \to U \subset \mathbf{R}^3$ denote a parametrized smooth embedded surface, where V is an open ball in $\mathbf{R}^2$ with coordinates (u, v). Explain briefly the geometric meaning of the *second fundamental form*

$$L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where $L = \sigma_{uu} \cdot \mathbf{N},\ M = \sigma_{uv} \cdot \mathbf{N},\ N = \sigma_{vv} \cdot \mathbf{N}$, with $\mathbf{N}$ denoting the unit normal vector to the surface U.

Prove that if the second fundamental form is identically zero, then $\mathbf{N}_u = \mathbf{0} = \mathbf{N}_v$ as vector-valued functions on V, and hence that $\mathbf{N}$ is a constant vector. Deduce that U is then contained in a plane given by $\mathbf{x} \cdot \mathbf{N} =$ constant.

3/I/2H **Geometry**

Show that the Gaussian curvature K at an arbitrary point (x, y, z) of the hyperboloid $z = xy$, as an embedded surface in $\mathbf{R}^3$, is given by the formula

$$K = -1/(1 + x^2 + y^2)^2.$$

3/II/12H **Geometry**

Describe the stereographic projection map from the sphere S^2 to the extended complex plane $\mathbf{C}_\infty$, positioned equatorially. Prove that $w, z \in \mathbf{C}_\infty$ correspond to antipodal points on S^2 if and only if $w = -1/\bar{z}$. State, without proof, a result which relates the rotations of S^2 to a certain group of Möbius transformations on $\mathbf{C}_\infty$.

Show that any circle in the complex plane corresponds, under stereographic projection, to a circle on S^2. Let C denote any circle in the complex plane other than the unit circle; show that C corresponds to a great circle on S^2 if and only if its intersection with the unit circle consists of two points, one of which is the negative of the other.

[*You may assume the result that a Möbius transformation on the complex plane sends circles and straight lines to circles and straight lines.*]

4/II/12H **Geometry**

Describe the hyperbolic lines in both the disc and upper half-plane models of the hyperbolic plane. Given a hyperbolic line l and a point $P \notin l$, we define

$$d(P, l) := \inf_{Q \in l} \rho(P, Q),$$

where ρ denotes the hyperbolic distance. Show that $d(P, l) = \rho(P, Q')$, where Q' is the unique point of l for which the hyperbolic line segment PQ' is perpendicular to l.

Suppose now that L_1 is the positive imaginary axis in the upper half-plane model of the hyperbolic plane, and L_2 is the semicircle with centre $a > 0$ on the real line, and radius r, where $0 < r < a$. For any $P \in L_2$, show that the hyperbolic line through P which is perpendicular to L_1 is a semicircle centred on the origin of radius $\leqslant a + r$, and prove that

$$d(P, L_1) \geqslant \frac{a - r}{a + r}.$$

For arbitrary hyperbolic lines L_1, L_2 in the hyperbolic plane, we define

$$d(L_1, L_2) := \inf_{P \in L_1, Q \in L_2} \rho(P, Q).$$

If L_1 and L_2 are *ultraparallel* (i.e. hyperbolic lines which do not meet, either inside the hyperbolic plane or at its boundary), prove that $d(L_1, L_2)$ is strictly positive.

[*The equivalence of the disc and upper half-plane models of the hyperbolic plane, and standard facts about the metric and isometries of these models, may be quoted without proof.*]

1/I/2A **Geometry**

Let $\sigma : \mathbf{R}^2 \to \mathbf{R}^3$ be the map defined by

$$\sigma(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u),$$

where $0 < b < a$. Describe briefly the image $T = \sigma(\mathbf{R}^2) \subset \mathbf{R}^3$. Let V denote the open subset of $\mathbf{R}^2$ given by $0 < u < 2\pi$, $0 < v < 2\pi$; prove that the restriction $\sigma|_V$ defines a smooth parametrization of a certain open subset (which you should specify) of T. Hence, or otherwise, prove that T is a smooth embedded surface in $\mathbf{R}^3$.

[*You may assume that the image under σ of any open set $B \subset \mathbf{R}^2$ is open in T.*]

2/II/12A **Geometry**

Let U be an open subset of $\mathbf{R}^2$ equipped with a Riemannian metric. For $\gamma : [0, 1] \to U$ a smooth curve, define what is meant by its *length* and *energy.* Prove that *length*$(\gamma)^2 \leq$ *energy*(γ), with equality if and only if $\dot{\gamma}$ has constant norm with respect to the metric.

Suppose now U is the upper half plane model of the hyperbolic plane, and P, Q are points on the positive imaginary axis. Show that a smooth curve γ joining P and Q represents an absolute minimum of the length of such curves if and only if $\gamma(t) = i\,v(t)$, with v a smooth monotonic real function.

Suppose that a smooth curve γ joining the above points P and Q represents a stationary point for the energy under proper variations; deduce from an appropriate form of the Euler–Lagrange equations that γ must be of the above form, with $\dot{v}/v$ constant.

3/I/2A **Geometry**

Write down the Riemannian metric on the disc model Δ of the hyperbolic plane. Given that the length minimizing curves passing through the origin correspond to diameters, show that the hyperbolic circle of radius ρ centred on the origin is just the Euclidean circle centred on the origin with Euclidean radius $\tanh(\rho/2)$. Prove that the hyperbolic area is $2\pi(\cosh\rho - 1)$.

State the Gauss–Bonnet theorem for the area of a hyperbolic triangle. Given a hyperbolic triangle and an interior point P, show that the distance from P to the nearest side is at most $\cosh^{-1}(3/2)$.

3/II/12A **Geometry**

Describe geometrically the stereographic projection map π from the unit sphere S^2 to the extended complex plane $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, positioned equatorially, and find a formula for π.

Show that any Möbius transformation $T \neq 1$ on $\mathbf{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of S^2 through a non-zero angle has exactly two fixed points z_1 and z_2, where $z_2 = -1/\bar{z}_1$. If now T is a Möbius transformation with two fixed points z_1 and z_2 satisfying $z_2 = -1/\bar{z}_1$, prove that **either** T corresponds to a rotation of S^2, **or** one of the fixed points, say z_1, is an *attractive* fixed point, i.e. for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

[*You may assume the fact that any rotation of S^2 corresponds to some Möbius transformation of $\mathbf{C}_\infty$ under the stereographic projection map.*]

4/II/12A **Geometry**

Given a parametrized smooth embedded surface $\sigma : V \to U \subset \mathbf{R}^3$, where V is an open subset of $\mathbf{R}^2$ with coordinates (u, v), and a point $P \in U$, define what is meant by the *tangent space* at P, the *unit normal* $\mathbf{N}$ at P, and the *first fundamental form*

$$Edu^2 + 2F du\, dv + G dv^2.$$

[*You need not show that your definitions are independent of the parametrization.*]

The *second fundamental form* is defined to be

$$Ldu^2 + 2M du\, dv + N dv^2,$$

where $L = \sigma_{uu} \cdot \mathbf{N}$, $M = \sigma_{uv} \cdot \mathbf{N}$ and $N = \sigma_{vv} \cdot \mathbf{N}$. Prove that the partial derivatives of $\mathbf{N}$ (considered as a vector-valued function of u, v) are of the form $\mathbf{N}_u = a\sigma_u + b\sigma_v$, $\mathbf{N}_v = c\sigma_u + d\sigma_v$, where

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Explain briefly the significance of the determinant $ad - bc$.

1/I/3G **Geometry**

Using the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

define the length of a curve and the area of a region in the upper half-plane $H = \{x + iy : y > 0\}$.

Find the hyperbolic area of the region $\{(x, y) \in H : 0 < x < 1, y > 1\}$.

1/II/14G **Geometry**

Show that for every hyperbolic line L in the hyperbolic plane H there is an isometry of H which is the identity on L but not on all of H. Call it the *reflection* R_L.

Show that every isometry of H is a composition of reflections.

3/I/3G **Geometry**

State Euler's formula for a convex polyhedron with F faces, E edges, and V vertices.

Show that any regular polyhedron whose faces are pentagons has the same number of vertices, edges and faces as the dodecahedron.

3/II/15G **Geometry**

Let a, b, c be the lengths of a right-angled triangle in spherical geometry, where c is the hypotenuse. Prove the Pythagorean theorem for spherical geometry in the form

$$\cos c = \cos a \cos b\,.$$

Now consider such a spherical triangle with the sides a, b replaced by $\lambda a, \lambda b$ for a positive number λ. Show that the above formula approaches the usual Pythagorean theorem as λ approaches zero.

1/I/4F **Geometry**

Describe the geodesics (that is, hyperbolic straight lines) in **either** the disc model **or** the half-plane model of the hyperbolic plane. Explain what is meant by the statements that two hyperbolic lines are parallel, and that they are ultraparallel.

Show that two hyperbolic lines l and l' have a unique common perpendicular if and only if they are ultraparallel.

[*You may assume standard results about the group of isometries of whichever model of the hyperbolic plane you use.*]

1/II/13F **Geometry**

Write down the Riemannian metric in the half-plane model of the hyperbolic plane. Show that Möbius transformations mapping the upper half-plane to itself are isometries of this model.

Calculate the hyperbolic distance from ib to ic, where b and c are positive real numbers. Assuming that the hyperbolic circle with centre ib and radius r is a Euclidean circle, find its Euclidean centre and radius.

Suppose that a and b are positive real numbers for which the points ib and $a + ib$ of the upper half-plane are such that the hyperbolic distance between them coincides with the Euclidean distance. Obtain an expression for b as a function of a. Hence show that, for any b with $0 < b < 1$, there is a unique positive value of a such that the hyperbolic distance between ib and $a + ib$ coincides with the Euclidean distance.

3/I/4F **Geometry**

Show that any isometry of Euclidean 3-space which fixes the origin can be written as a composite of at most three reflections in planes through the origin, and give (with justification) an example of an isometry for which three reflections are necessary.

3/II/14F **Geometry**

State and prove the Gauss–Bonnet formula for the area of a spherical triangle. Deduce a formula for the area of a spherical n-gon with angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. For what range of values of α does there exist a (convex) regular spherical n-gon with angle α?

Let Δ be a spherical triangle with angles π/p, π/q and π/r where p, q, r are integers, and let G be the group of isometries of the sphere generated by reflections in the three sides of Δ. List the possible values of (p, q, r), and in each case calculate the order of the corresponding group G. If $(p, q, r) = (2, 3, 5)$, show how to construct a regular dodecahedron whose group of symmetries is G.

[*You may assume that the images of* Δ *under the elements of* G *form a tessellation of the sphere.*]

1/I/4E **Geometry**

Show that any finite group of orientation-preserving isometries of the Euclidean plane is cyclic.

Show that any finite group of orientation-preserving isometries of the hyperbolic plane is cyclic.

[*You may assume that given any non-empty finite set E in the hyperbolic plane, or the Euclidean plane, there is a unique smallest closed disc that contains E. You may also use any general fact about the hyperbolic plane without proof providing that it is stated carefully.*]

1/II/13E **Geometry**

Let $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$, and let $\mathbb{H}$ have the hyperbolic metric ρ derived from the line element $|dz|/y$. Let Γ be the group of Möbius maps of the form $z \mapsto (az+b)/(cz+d)$, where a, b, c and d are real and $ad - bc = 1$. Show that every g in Γ is an isometry of the metric space $(\mathbb{H}, \rho)$. For z and w in $\mathbb{H}$, let

$$h(z,w) = \frac{|z-w|^2}{\operatorname{Im}(z)\operatorname{Im}(w)}.$$

Show that for every g in Γ, $h\big(g(z), g(w)\big) = h(z,w)$. By considering $z = iy$, where $y > 1$, and $w = i$, or otherwise, show that for all z and w in $\mathbb{H}$,

$$\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\operatorname{Im}(z)\operatorname{Im}(w)}.$$

By considering points i, iy, where $y > 1$ and $s + it$, where $s^2 + t^2 = 1$, or otherwise, derive Pythagoras' Theorem for hyperbolic geometry in the form $\cosh a \cosh b = \cosh c$, where a, b and c are the lengths of sides of a right-angled triangle whose hypotenuse has length c.

3/I/4E **Geometry**

State Euler's formula for a graph $\mathcal{G}$ with F faces, E edges and V vertices on the surface of a sphere.

Suppose that every face in $\mathcal{G}$ has at least three edges, and that at least three edges meet at every vertex of $\mathcal{G}$. Let F_n be the number of faces of $\mathcal{G}$ that have exactly n edges ($n \geqslant 3$), and let V_m be the number of vertices at which exactly m edges meet ($m \geqslant 3$). By expressing $6F - \sum_n nF_n$ in terms of the V_j, or otherwise, show that every convex polyhedron has at least four faces each of which is a triangle, a quadrilateral or a pentagon.

3/II/14E Geometry

Show that every isometry of Euclidean space $\mathbb{R}^3$ is a composition of reflections in planes.

What is the smallest integer N such that every isometry f of $\mathbb{R}^3$ with $f(0) = 0$ can be expressed as the composition of at most N reflections? Give an example of an isometry that needs this number of reflections and justify your answer.

Describe (geometrically) all twelve orientation-reversing isometries of a regular tetrahedron.

1/I/4B **Geometry**

Write down the Riemannian metric on the disc model Δ of the hyperbolic plane. What are the geodesics passing through the origin? Show that the hyperbolic circle of radius ρ centred on the origin is just the Euclidean circle centred on the origin with Euclidean radius $\tanh(\rho/2)$.

Write down an isometry between the upper half-plane model H of the hyperbolic plane and the disc model Δ, under which $i \in H$ corresponds to $0 \in \Delta$. Show that the hyperbolic circle of radius ρ centred on i in H is a Euclidean circle with centre $i \cosh \rho$ and of radius $\sinh \rho$.

1/II/13B **Geometry**

Describe geometrically the stereographic projection map ϕ from the unit sphere S^2 to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, and find a formula for ϕ. Show that any rotation of S^2 about the z-axis corresponds to a Möbius transformation of $\mathbb{C}_\infty$. You are given that the rotation of S^2 defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

corresponds under ϕ to a Möbius transformation of $\mathbb{C}_\infty$; deduce that any rotation of S^2 about the x-axis also corresponds to a Möbius transformation.

Suppose now that $u, v \in \mathbb{C}$ correspond under ϕ to distinct points $P, Q \in S^2$, and let d denote the angular distance from P to Q on S^2. Show that $-\tan^2(d/2)$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in some order (which you should specify). [*You may assume that the cross-ratio is invariant under Möbius transformations.*]

3/I/4B **Geometry**

State and prove the Gauss–Bonnet theorem for the area of a spherical triangle.

Suppose $\mathbf{D}$ is a regular dodecahedron, with centre the origin. Explain how each face of $\mathbf{D}$ gives rise to a spherical pentagon on the 2-sphere S^2. For each such spherical pentagon, calculate its angles and area.

3/II/14B **Geometry**

Describe the hyperbolic lines in the upper half-plane model H of the hyperbolic plane. The group $G = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$ acts on H via Möbius transformations, which you may assume are isometries of H. Show that G acts transitively on the hyperbolic lines. Find explicit formulae for the reflection in the hyperbolic line L in the cases (i) L is a vertical line $x = a$, and (ii) L is the unit semi-circle with centre the origin. Verify that the composite T of a reflection of type (ii) followed afterwards by one of type (i) is given by $T(z) = -z^{-1} + 2a$.

Suppose now that L_1 and L_2 are distinct hyperbolic lines in the hyperbolic plane, with R_1, R_2 denoting the corresponding reflections. By considering different models of the hyperbolic plane, or otherwise, show that

(a) $R_1 R_2$ has infinite order if L_1 and L_2 are parallel or ultraparallel, and

(b) $R_1 R_2$ has finite order if and only if L_1 and L_2 meet at an angle which is a rational multiple of π.