# Part IB

# Further Analysis

**2/I/4E Further Analysis**

Let $\tau$ be the topology on $\mathbb{N}$ consisting of the empty set and all sets $X \subset \mathbb{N}$ such that $\mathbb{N} \setminus X$ is finite. Let $\sigma$ be the usual topology on $\mathbb{R}$, and let $\rho$ be the topology on $\mathbb{R}$ consisting of the empty set and all sets of the form $(x, \infty)$ for some real $x$.

(i) Prove that all continuous functions $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \sigma)$ are constant.

(ii) Give an example with proof of a non-constant function $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \rho)$ that is continuous.

**2/II/15E Further Analysis**

(i) Let $X$ be the set of all infinite sequences $(\epsilon_1, \epsilon_2, \ldots)$ such that $\epsilon_i \in \{0, 1\}$ for all $i$. Let $\tau$ be the collection of all subsets $Y \subset X$ such that, for every $(\epsilon_1, \epsilon_2, \ldots) \in Y$ there exists $n$ such that $(\eta_1, \eta_2, \ldots) \in Y$ whenever $\eta_1 = \epsilon_1, \eta_2 = \epsilon_2, \ldots, \eta_n = \epsilon_n$. Prove that $\tau$ is a topology on $X$.

(ii) Let a distance $d$ be defined on $X$ by

$$d\Big((\epsilon_1, \epsilon_2, \ldots), (\eta_1, \eta_2, \ldots)\Big) = \sum_{n=1}^{\infty} 2^{-n} |\epsilon_n - \eta_n| \, .$$

Prove that $d$ is a metric and that the topology arising from $d$ is the same as $\tau$.

**3/I/5E Further Analysis**

Let $C$ be the contour that goes once round the boundary of the square

$$\{z : -1 \leqslant \operatorname{Re} z \leqslant 1, -1 \leqslant \operatorname{Im} z \leqslant 1\}$$

in an anticlockwise direction. What is $\displaystyle\int_C \frac{dz}{z}$? Briefly justify your answer.

Explain why the integrals along each of the four edges of the square are equal. Deduce that $\displaystyle\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}$.

3/II/17E **Further Analysis**

(i) Explain why the formula

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

defines a function that is analytic on the domain $\mathbb{C} \setminus \mathbb{Z}$. [You need not give full details, but should indicate what results are used.]

Show also that $f(z+1) = f(z)$ for every $z$ such that $f(z)$ is defined.

(ii) Write $\log z$ for $\log r + i\theta$ whenever $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$. Let $g$ be defined by the formula

$$g(z) = f\left(\frac{1}{2\pi i} \log z\right) .$$

Prove that $g$ is analytic on $\mathbb{C} \setminus \{0, 1\}$.

[*Hint: What would be the effect of redefining* $\log z$ *to be* $\log r + i\theta$ *when* $z = re^{i\theta}$, $r > 0$ *and* $0 \leqslant \theta < 2\pi$ ?]

(iii) Determine the nature of the singularity of $g$ at $z = 1$.

4/I/4E **Further Analysis**

(i) Let $D$ be the open unit disc of radius 1 about the point $3 + 3i$. Prove that there is an analytic function $f : D \to \mathbb{C}$ such that $f(z)^2 = z$ for every $z \in D$.

(ii) Let $D' = \mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Im}\ z = 0, \mathrm{Re}\ z \leqslant 0\}$. Explain briefly why there is at most one extension of $f$ to a function that is analytic on $D'$.

(iii) Deduce that $f$ cannot be extended to an analytic function on $\mathbb{C} \setminus \{0\}$.

4/II/14E **Further Analysis**

(i) State and prove Rouché's theorem.

[*You may assume the principle of the argument.*]

(ii) Let $0 < c < 1$. Prove that the polynomial $p(z) = z^3 + icz + 8$ has three roots with modulus less than 3. Prove that one root $\alpha$ satisfies $\mathrm{Re}\ \alpha > 0, \mathrm{Im}\ \alpha > 0$; another, $\beta$, satisfies $\mathrm{Re}\ \beta > 0, \mathrm{Im}\ \beta < 0$; and the third, $\gamma$, has $\mathrm{Re}\ \gamma < 0$.

(iii) For sufficiently small $c$, prove that $\mathrm{Im}\ \gamma > 0$.

[*You may use results from the course if you state them precisely.*]

2/I/4E **Further Analysis**

Let $\tau_1$ be the collection of all subsets $A \subset \mathbb{N}$ such that $A = \emptyset$ or $\mathbb{N} \setminus A$ is finite. Let $\tau_2$ be the collection of all subsets of $\mathbb{N}$ of the form $I_n = \{n, n+1, n+2, \ldots\}$, together with the empty set. Prove that $\tau_1$ and $\tau_2$ are both topologies on $\mathbb{N}$.

Show that a function $f$ from the topological space $(\mathbb{N}, \tau_1)$ to the topological space $(\mathbb{N}, \tau_2)$ is continuous if and only if one of the following alternatives holds:

(i) $f(n) \to \infty$ as $n \to \infty$;

(ii) there exists $N \in \mathbb{N}$ such that $f(n) = N$ for all but finitely many $n$ and $f(n) \leqslant N$ for all $n$.

2/II/13E **Further Analysis**

(a) Let $f\colon [1, \infty) \to \mathbb{C}$ be defined by $f(t) = t^{-1}e^{2\pi i t}$ and let $X$ be the image of $f$. Prove that $X \cup \{0\}$ is compact and path-connected. [*Hint: you may find it helpful to set* $s = t^{-1}$.]

(b) Let $g\colon [1, \infty) \to \mathbb{C}$ be defined by $g(t) = (1 + t^{-1})e^{2\pi i t}$, let $Y$ be the image of $g$ and let $\overline{D}$ be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Prove that $Y \cup \overline{D}$ is connected. Explain briefly why it is not path-connected.

3/I/3E **Further Analysis**

(a) Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leqslant 1 + |z|^{1/2}$ for every $z$. Prove that $f$ is constant.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\mathrm{Re}\,(f(z)) \geqslant 0$ for every $z$. Prove that $f$ is constant.

3/II/13E **Further Analysis**

(a) State Taylor's Theorem.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ be defined whenever $|z - z_0| < r$. Suppose that $z_k \to z_0$ as $k \to \infty$, that no $z_k$ equals $z_0$ and that $f(z_k) = g(z_k)$ for every $k$. Prove that $a_n = b_n$ for every $n \geqslant 0$.

(c) Let $D$ be a domain, let $z_0 \in D$ and let $(z_k)$ be a sequence of points in $D$ that converges to $z_0$, but such that no $z_k$ equals $z_0$. Let $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be analytic functions such that $f(z_k) = g(z_k)$ for every $k$. Prove that $f(z) = g(z)$ for every $z \in D$.

(d) Let $D$ be the domain $\mathbb{C} \setminus \{0\}$. Give an example of an analytic function $f\colon D \to \mathbb{C}$ such that $f(n^{-1}) = 0$ for every positive integer $n$ but $f$ is not identically 0.

(e) Show that any function with the property described in (d) must have an essential singularity at the origin.

4/I/4E **Further Analysis**

(a) State and prove Morera's Theorem.

(b) Let $D$ be a domain and for each $n \in \mathbb{N}$ let $f_n : D \to \mathbb{C}$ be an analytic function. Suppose that $f : D \to \mathbb{C}$ is another function and that $f_n \to f$ uniformly on $D$. Prove that $f$ is analytic.

4/II/13E **Further Analysis**

(a) State the residue theorem and use it to deduce the principle of the argument, in a form that involves winding numbers.

(b) Let $p(z) = z^5 + z$. Find all $z$ such that $|z| = 1$ and $\operatorname{Im}(p(z)) = 0$. Calculate $\operatorname{Re}(p(z))$ for each such $z$. [*It will be helpful to set* $z = e^{i\theta}$. *You may use the addition formulae* $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ *and* $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$.]

(c) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed path $\theta \mapsto e^{i\theta}$. Use your answer to (b) to give a rough sketch of the path $p \circ \gamma$, paying particular attention to where it crosses the real axis.

(d) Hence, or otherwise, determine for every real $t$ the number of $z$ (counted with multiplicity) such that $|z| < 1$ and $p(z) = t$. (You need not give rigorous justifications for your calculations.)

2/I/4G **Further Analysis**

Let the function $f = u + iv$ be analytic in the complex plane $\mathbb{C}$ with $u$,$v$ real-valued. Prove that, if $uv$ is bounded above everywhere on $\mathbb{C}$, then $f$ is constant.

2/II/13G **Further Analysis**

(a) Given a topology $\mathcal{T}$ on $X$, a collection $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every non-empty set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$. Prove that a collection $\mathcal{B}$ is a basis for some topology if it satisfies:

(i) the union of all sets in $\mathcal{B}$ is $X$;

(ii) if $x \in B_1 \cap B_2$ for two sets $B_1$ and $B_2$ in $\mathcal{B}$, then there is a set $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

(b) On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consider the dictionary order given by

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. Given points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z} \in \mathbb{R}^2 : \ \mathbf{x} < \mathbf{z} < \mathbf{y}\}.$$

Show that the sets $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ form a basis of a topology.

(c) Show that this topology on $\mathbb{R}^2$ does not have a countable basis.

3/I/3G **Further Analysis**

Let $f : X \to Y$ be a continuous map between topological spaces. Let

$$G_f = \{(x, f(x)) : \ x \in X\}.$$

(a) Show that if $Y$ is Hausdorff, then $G_f$ is closed in $X \times Y$.

(b) Show that if $X$ is compact, then $G_f$ is also compact.

3/II/13G **Further Analysis**

(a) Let $f$ and $g$ be two analytic functions on a domain $D$ and let $\gamma \subset D$ be a simple closed curve homotopic in $D$ to a point. If $|g(z)| < |f(z)|$ for every $z$ in $\gamma$, prove that $\gamma$ encloses the same number of zeros of $f$ as of $f+g$.

(b) Let $g$ be an analytic function on the disk $|z| < 1+\epsilon$, for some $\epsilon > 0$. Suppose that $g$ maps the closed unit disk into the open unit disk (both centred at 0). Prove that $g$ has exactly one fixed point in the open unit disk.

(c) Prove that, if $|a| < 1$, then

$$z^m \left( \frac{z-a}{1-\bar{a}z} \right)^n - a$$

has $m+n$ zeros in $|z| < 1$.

4/I/4G **Further Analysis**

(a) Let $X$ be a topological space and suppose $X = C \cup D$, where $C$ and $D$ are disjoint nonempty open subsets of $X$. Show that, if $Y$ is a connected subset of $X$, then $Y$ is entirely contained in either $C$ or $D$.

(b) Let $X$ be a topological space and let $\{A_n\}$ be a sequence of connected subsets of $X$ such that $A_n \cap A_{n+1} \neq \emptyset$, for $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n \geqslant 1} A_n$ is connected.

4/II/13G **Further Analysis**

A function $f$ is said to be analytic at $\infty$ if there exists a real number $r > 0$ such that $f$ is analytic for $|z| > r$ and $\lim_{z \to 0} f(1/z)$ is finite (i.e. $f(1/z)$ has a removable singularity at $z = 0$). $f$ is said to have a pole at $\infty$ if $f(1/z)$ has a pole at $z = 0$. Suppose that $f$ is a meromorphic function on the extended plane $\mathbb{C}_\infty$, that is, $f$ is analytic at each point of $\mathbb{C}_\infty$ except for poles.

(a) Show that if $f$ has a pole at $z = \infty$, then there exists $r > 0$ such that $f(z)$ has no poles for $r < |z| < \infty$.

(b) Show that the number of poles of $f$ is finite.

(c) By considering the Laurent expansions around the poles show that $f$ is in fact a rational function, i.e. of the form $p/q$, where $p$ and $q$ are polynomials.

(d) Deduce that the only bijective meromorphic maps of $\mathbb{C}_\infty$ onto itself are the Möbius maps.

2/I/4B **Further Analysis**

Define the terms *connected* and *path connected* for a topological space. If a topological space $X$ is path connected, prove that it is connected.

Consider the following subsets of $\mathbb{R}^2$:

$$I = \{(x,0) : \ 0 \le x \le 1\}, \quad A = \{(0,y) : \ \tfrac{1}{2} \le y \le 1\}, \text{ and}$$

$$J_n = \{(n^{-1}, \ y) : \ 0 \le y \le 1\} \quad \text{for } n \ge 1.$$

Let

$$X = A \cup I \cup \bigcup_{n \ge 1} J_n$$

with the subspace (metric) topology. Prove that $X$ is connected.

[*You may assume that any interval in* $\mathbb{R}$ *(with the usual topology) is connected.*]

2/II/13A **Further Analysis**

State Liouville's Theorem. Prove it by considering

$$\int_{|z|=R} \frac{f(z)\, dz}{(z-a)(z-b)}$$

and letting $R \to \infty$.

Prove that, if $g(z)$ is a function analytic on all of $\mathbb{C}$ with real and imaginary parts $u(z)$ and $v(z)$, then either of the conditions:

$$\text{(i) } u + v \geqslant 0 \text{ for all } z; \quad \text{or} \quad \text{(ii) } uv \geqslant 0 \text{ for all } z,$$

implies that $g(z)$ is constant.

3/I/3B **Further Analysis**

State a version of Rouché's Theorem. Find the number of solutions (counted with multiplicity) of the equation

$$z^4 = a(z-1)(z^2-1) + \tfrac{1}{2}$$

inside the open disc $\{z : |z| < \sqrt{2}\}$, for the cases $a = \frac{1}{3}, 12$ and $5$.

[*Hint: For the case* $a = 5$, *you may find it helpful to consider the function* $(z^2-1)(z-2)(z-3)$.]

**3/II/13B Further Analysis**

If $X$ and $Y$ are topological spaces, describe the open sets in the *product topology* on $X \times Y$. If the topologies on $X$ and $Y$ are induced from metrics, prove that the same is true for the product.

What does it mean to say that a topological space is *compact*? If the topologies on $X$ and $Y$ are compact, prove that the same is true for the product.

**4/I/4A Further Analysis**

Let $f(z)$ be analytic in the disc $|z| < R$. Assume the formula

$$f'(z_0) = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{f(z)\,dz}{(z-z_0)^2}, \quad 0 \leqslant |z_0| < r < R.$$

By combining this formula with a complex conjugate version of Cauchy's Theorem, namely

$$0 = \int\limits_{|z|=r} \overline{f(z)}\,d\bar{z},$$

prove that

$$f'(0) = \frac{1}{\pi r} \int_0^{2\pi} u(\theta) e^{-i\theta} d\theta,$$

where $u(\theta)$ is the real part of $f(re^{i\theta})$.

**4/II/13B Further Analysis**

Let $\Delta^* = \{z : 0 < |z| < r\}$ be a punctured disc, and $f$ an analytic function on $\Delta^*$. What does it mean to say that $f$ has the origin as (i) a removable singularity, (ii) a pole, and (iii) an essential singularity? State criteria for (i), (ii), (iii) to occur, in terms of the Laurent series for $f$ at 0.

Suppose now that the origin is an essential singularity for $f$. Given any $w \in \mathbb{C}$, show that there exists a sequence $(z_n)$ of points in $\Delta^*$ such that $z_n \to 0$ and $f(z_n) \to w$. [*You may assume the fact that an isolated singularity is removable if the function is bounded in some open neighbourhood of the singularity.*]

State the Open Mapping Theorem. Prove that if $f$ is analytic and injective on $\Delta^*$, then the origin cannot be an essential singularity. By applying this to the function $g(1/z)$, or otherwise, deduce that if $g$ is an injective analytic function on $\mathbb{C}$, then $g$ is linear of the form $az + b$, for some non-zero complex number $a$. [*Here, you may assume that $g$ injective implies that its derivative $g'$ is nowhere vanishing.*]