Part IB

Complex Methods

Paper 1, Section I

3B Complex Analysis OR Complex Methods

(a) What is the Laurent series of $e^{1/z}$ about $z_0 = 0$?

(b) Let $\rho > 0$. Show that for all large enough $n \in \mathbb{N}$, all zeros of the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n}$$

lie in the open disc $\{z : |z| < \rho\}$.

Paper 1, Section II

12G Complex Analysis OR Complex Methods

(a) Let $f(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$ for $|z-1| < 1$. By differentiating $z\exp(-f(z))$, show that f is an analytic branch of logarithm on the disc $D(1,1)$ with $f(1) = 0$. Use scaling and the function f to show that for every point a in the domain $D = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$, there is an analytic branch of logarithm on a small neighbourhood of a whose imaginary part lies in $(0, 2\pi)$.

(b) For $z \in D$, let $\theta(z)$ be the unique value of the argument of z in the interval $(0, 2\pi)$. Define the function $L\colon D \to \mathbb{C}$ by $L(z) = \log|z| + i\theta(z)$. Briefly explain using part (a) why L is an analytic branch of logarithm on D. For $\alpha \in (-1, 1)$ write down an analytic branch of z^α on D.

(c) State the residue theorem. Evaluate the integral

$$I = \int_0^\infty \frac{x^\alpha}{(x+1)^2}\, dx$$

where $\alpha \in (-1, 1)$.

Paper 2, Section II

12B Complex Analysis OR Complex Methods

(a) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic, and is bounded in the half-plane $\{z : \mathrm{Re}(z) > 0\}$. Prove that, for any real number $c > 0$, there is a positive real constant M such that

$$|f(z_1) - f(z_2)| \leqslant M|z_1 - z_2|$$

whenever $z_1, z_2 \in \mathbb{C}$ satisfy $\mathrm{Re}(z_1) > c$, $\mathrm{Re}(z_2) > c$, and $|z_1 - z_2| < c$.

(b) Let the functions $g, h : \mathbb{C} \to \mathbb{C}$ both be analytic.

(i) State Liouville's Theorem.

(ii) Show that if g is not constant, then $g(\mathbb{C})$ is dense in $\mathbb{C}$.

(iii) Show that if $|h(z)| \leqslant |\mathrm{Re}(z)|^{-1/2}$ for all $z \in \mathbb{C}$, then h is constant.

Paper 3, Section I

3B Complex Methods

Let $f = u + iv$ be an analytic function in a connected open set $D \subset \mathbb{C}$, where $u(x, y)$ and $v(x, y)$ are real-valued functions on D, with $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, for $z \in D$.

(a) Show that $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$, and state the Cauchy–Riemann equations.

(b) Suppose there are real constants a, b and c such that $a^2 + b^2 \neq 0$ and

$$au(x, y) + bv(x, y) = c, \qquad z \in D.$$

Show that f is constant on D.

Paper 4, Section II

12B Complex Methods

Let $B : [0, \infty) \to \mathbb{R}^{n \times p}$ be a $n \times p$ matrix-valued function. The Laplace transform $\mathcal{L}\{B\}$ of B is defined componentwise on the matrix element functions of B.

(a) Show that if A is a constant $n \times n$ matrix and $B : [0, \infty) \to \mathbb{R}^{n \times p}$ is an $n \times p$ matrix-valued function, then $\mathcal{L}\{AB\} = A\mathcal{L}\{B\}$.

(b) Consider the ODE given by

$$y'(t) = Ay(t) + g(t), \quad y(0) = y_0 \in \mathbb{R}^n, \quad t \geqslant 0, \qquad (*)$$

where A is a constant $n \times n$ matrix, and $g : [0, \infty) \to \mathbb{R}^n$ is a vector-valued function whose Laplace transform $G(s) = \mathcal{L}\{g\}(s)$ exists for all but one $s \in \mathbb{C}$. Show that

$$Y(s) = (sI - A)^{-1}(y_0 + G(s)),$$

and that

$$\mathcal{L}\{e^{tA}\}(s) = (sI - A)^{-1},$$

for all s that are not eigenvalues of A, where $Y = \mathcal{L}\{y\}$ is the Laplace transform of the solution y of $(*)$. You may assume that y exists and is the unique solution to the ODE for all $t \geqslant 0$ with solution $y(t) = e^{tA} y_0$ when $g = 0$.

(c) Consider the ODE

$$y'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} y(t) + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t \geqslant 0.$$

Determine the integer values $n \in \mathbb{N}$ such that $\lim_{t \to \infty} e^{-nt} y(t)$ exists and is a finite and nonzero vector in $\mathbb{R}^2$.

Paper 1, Section I
3G Complex Analysis or Complex Methods

Show that $f(z) = \frac{z}{\sin z}$ has a removable singularity at $z = 0$. Find the radius of convergence of the power series of f at the origin.

Paper 1, Section II
12G Complex Analysis or Complex Methods

(a) Let $\Omega \subset \mathbb{C}$ be an open set such that there is $z_0 \in \Omega$ with the property that for any $z \in \Omega$, the line segment $[z_0, z]$ connecting z_0 to z is completely contained in Ω. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma f(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a triangle contained in Ω. Given $w \in \Omega$, let

$$g(w) = \int_{[z_0,w]} f(z)\, dz.$$

Explain briefly why g is a holomorphic function such that $g'(w) = f(w)$ for all $w \in \Omega$.

(b) Fix $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ and let $\mathcal{D} \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ such that the line segment connecting z to z_0 does not pass through the origin. Show that there exists a holomorphic function $h : \mathcal{D} \to \mathbb{C}$ such that $h(z)^2 = z$ for all $z \in \mathcal{D}$. [You may assume that the integral of $1/z$ over the boundary of any triangle contained in $\mathcal{D}$ is zero.]

(c) Show that there exists a holomorphic function f defined in a neighbourhood U of the origin such that $f(z)^2 = \cos z$ for all $z \in U$. Is it possible to find a holomorphic function f defined on the disk $|z| < 2$ such that $f(z)^2 = \cos z$ for all z in the disk? Justify your answer.

Paper 2, Section II
12A Complex Analysis or Complex Methods

(a) Let $R = P/Q$ be a rational function, where deg $Q \geqslant$ deg $P + 2$, and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} R(x)e^{ix}\, dx$$

in terms of residues. Briefly justify your answer.

[You may assume that the polynomials P and Q do not have any common factors.]

(b) Explicitly evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^4}\, dx\,.$$

Paper 3, Section I
3A Complex Methods

The function $f(x)$ has Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{-2ki}{p^2 + k^2},$$

where $p > 0$ is a real constant. Using contour integration, calculate $f(x)$ for $x > 0$.

[Jordan's lemma and the residue theorem may be used without proof.]

Paper 4, Section II
12A Complex Methods

The Laplace transform $F(s)$ of a function $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

(a) For $f(t) = t^n$ for n a non-negative integer, show that

$$\begin{aligned} L\{f(t)\} &= F(s) = \frac{n!}{s^{n+1}}, \\ L\{e^{at} f(t)\} &= F(s-a) = \frac{n!}{(s-a)^{n+1}}. \end{aligned}$$

(b) Use contour integration to find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+2)^2}.$$

(c) Verify the result in part (b) by using the results in part (a) and the convolution theorem.

(d) Use Laplace transforms to solve the differential equation

$$\frac{d^4}{dt^4}[f(t)] + 4\frac{d^3}{dt^3}[f(t)] + 4\frac{d^2}{dt^2}[f(t)] = 0,$$

subject to the initial conditions

$$f(0) = \frac{d}{dt} f(0) = \frac{d^2}{dt^2} f(0) = 0 \text{ and } \frac{d^3}{dt^3} f(0) = 1.$$

Paper 1, Section I

3B Complex Analysis or Complex Methods

Let $x > 0$, $x \neq 2$, and let C_x denote the positively oriented circle of radius x centred at the origin. Define

$$g(x) = \oint_{C_x} \frac{z^2 + e^z}{z^2(z-2)}\, dz.$$

Evaluate $g(x)$ for $x \in (0, \infty) \setminus \{2\}$.

Paper 1, Section II

12G Complex Analysis or Complex Methods

(a) State a theorem establishing Laurent series of analytic functions on suitable domains. Give a formula for the n^{th} Laurent coefficient.

Define the notion of *isolated singularity*. State the classification of an isolated singularity in terms of Laurent coefficients.

Compute the Laurent series of

$$f(z) = \frac{1}{z(z-1)}$$

on the annuli $A_1 = \{z : 0 < |z| < 1\}$ and $A_2 = \{z : 1 < |z|\}$. Using this example, comment on the statement that Laurent coefficients are unique. Classify the singularity of f at 0.

(b) Let U be an open subset of the complex plane, let $a \in U$ and let $U' = U \setminus \{a\}$. Assume that f is an analytic function on U' with $|f(z)| \to \infty$ as $z \to a$. By considering the Laurent series of $g(z) = \frac{1}{f(z)}$ at a, classify the singularity of f at a in terms of the Laurent coefficients. [You may assume that a continuous function on U that is analytic on U' is analytic on U.]

Now let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with $|f(z)| \to \infty$ as $z \to \infty$. By considering Laurent series at 0 of $f(z)$ and of $h(z) = f\left(\frac{1}{z}\right)$, show that f is a polynomial.

(c) Classify, giving reasons, the singularity at the origin of each of the following functions and in each case compute the residue:

$$g(z) = \frac{\exp(z) - 1}{z \log(z+1)} \qquad \text{and} \qquad h(z) = \sin(z)\sin(1/z)\ .$$

Paper 2, Section II

12B Complex Analysis or Complex Methods

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $a > 0$, $b > 0$ be constants. Show that if

$$|f(z)| \leqslant a|z|^{n/2} + b$$

for all $z \in \mathbb{C}$, where n is a positive odd integer, then f must be a polynomial with degree not exceeding $\lfloor n/2 \rfloor$ (closest integer part rounding down).

Does there exist a function f, analytic in $\mathbb{C} \setminus \{0\}$, such that $|f(z)| \geqslant 1/\sqrt{|z|}$ for all nonzero z? Justify your answer.

(b) State Liouville's Theorem and use it to show the following.

(i) If u is a positive harmonic function on $\mathbb{R}^2$, then u is a constant function.

(ii) Let $L = \{z \,|\, z = ax + b, x \in \mathbb{R}\}$ be a line in $\mathbb{C}$ where $a, b \in \mathbb{C}$, $a \neq 0$. If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(\mathbb{C}) \cap L = \emptyset$, then f is a constant function.

Paper 3, Section I

3B Complex Methods

Find the value of A for which the function

$$\phi(x, y) = x \cosh y \sin x + Ay \sinh y \cos x$$

satisfies Laplace's equation. For this value of A, find a complex analytic function of which ϕ is the real part.

Paper 4, Section II

12B Complex Methods

Let $f(t)$ be defined for $t \geqslant 0$. Define the *Laplace transform* $\widehat{f}(s)$ of f. Find an expression for the Laplace transform of $\frac{df}{dt}$ in terms of $\widehat{f}$.

Three radioactive nuclei decay sequentially, so that the numbers $N_i(t)$ of the three types obey the equations

$$\begin{aligned}
\frac{dN_1}{dt} &= -\lambda_1 N_1 \,, \\
\frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \,, \\
\frac{dN_3}{dt} &= \lambda_2 N_2 - \lambda_3 N_3 \,,
\end{aligned}$$

where $\lambda_3 > \lambda_2 > \lambda_1 > 0$ are constants. Initially, at $t = 0$, $N_1 = N$, $N_2 = 0$ and $N_3 = n$. Using Laplace transforms, find $N_3(t)$.

By taking an appropriate limit, find $N_3(t)$ when $\lambda_2 = \lambda_1 = \lambda > 0$ and $\lambda_3 > \lambda$.

Paper 1, Section I

3G Complex Analysis or Complex Methods

Let D be the open disc with centre $e^{2\pi i/6}$ and radius 1, and let L be the open lower half plane. Starting with a suitable Möbius map, find a conformal equivalence (or conformal bijection) of $D \cap L$ onto the open unit disc.

Paper 1, Section II

12G Complex Analysis or Complex Methods

Let $\ell(z)$ be an analytic branch of $\log z$ on a domain $D \subset \mathbb{C} \setminus \{0\}$. Write down an analytic branch of $z^{1/2}$ on D. Show that if $\psi_1(z)$ and $\psi_2(z)$ are two analytic branches of $z^{1/2}$ on D, then either $\psi_1(z) = \psi_2(z)$ for all $z \in D$ or $\psi_1(z) = -\psi_2(z)$ for all $z \in D$.

Describe the principal value or branch $\sigma_1(z)$ of $z^{1/2}$ on $D_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leqslant 0\}$. Describe a branch $\sigma_2(z)$ of $z^{1/2}$ on $D_2 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$.

Construct an analytic branch $\varphi(z)$ of $\sqrt{1-z^2}$ on $\mathbb{C} \setminus \{x \in \mathbb{R} : -1 \leqslant x \leqslant 1\}$ with $\varphi(2i) = \sqrt{5}$. [If you choose to use σ_1 and σ_2 in your construction, then you may assume without proof that they are analytic.]

Show that for $0 < |z| < 1$ we have $\varphi(1/z) = -i\sigma_1(1-z^2)/z$. Hence find the first three terms of the Laurent series of $\varphi(1/z)$ about 0.

Set $f(z) = \varphi(z)/(1+z^2)$ for $|z| > 1$ and $g(z) = f(1/z)/z^2$ for $0 < |z| < 1$. Compute the residue of g at 0 and use it to compute the integral

$$\int_{|z|=2} f(z)\, dz\,.$$

Paper 2, Section II

12B Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{1}{z(z-2)}\,,$$

find the Laurent expansions

(i) about $z = 0$ in the annulus $0 < |z| < 2$,

(ii) about $z = 0$ in the annulus $2 < |z| < \infty$,

(iii) about $z = 1$ in the annulus $0 < |z-1| < 1$.

What is the nature of the singularity of f, if any, at $z = 0$, $z = \infty$ and $z = 1$?

Using an integral of f, or otherwise, evaluate

$$\int_0^{2\pi} \frac{2-\cos\theta}{5-4\cos\theta}\, d\theta\,.$$

Paper 1, Section I

2F Complex Analysis or Complex Methods

What is the *Laurent series* for a function f defined in an annulus A? Find the Laurent series for $f(z) = \frac{10}{(z+2)(z^2+1)}$ on the annuli

$$A_1 = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \quad \text{and}$$
$$A_2 = \{z \in \mathbb{C} \,|\, 1 < |z| < 2\}\,.$$

Paper 1, Section II

13F Complex Analysis or Complex Methods

State and prove Jordan's lemma.

What is the *residue* of a function f at an isolated singularity a? If $f(z) = \frac{g(z)}{(z-a)^k}$ with k a positive integer, g analytic, and $g(a) \neq 0$, derive a formula for the residue of f at a in terms of derivatives of g.

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2}\, dx\,.$$

Paper 2, Section II

13D Complex Analysis or Complex Methods

Let C_1 and C_2 be smooth curves in the complex plane, intersecting at some point p. Show that if the map $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, then it preserves the angle between C_1 and C_2 at p, provided $f'(p) \neq 0$. Give an example that illustrates why the condition $f'(p) \neq 0$ is important.

Show that $f(z) = z + 1/z$ is a one-to-one conformal map on each of the two regions $|z| > 1$ and $0 < |z| < 1$, and find the image of each region.

Hence construct a one-to-one conformal map from the unit disc to the complex plane with the intervals $(-\infty, -1/2]$ and $[1/2, \infty)$ removed.

Paper 3, Section I

4D Complex Methods

By considering the transformation $w = i(1-z)/(1+z)$, find a solution to Laplace's equation $\nabla^2 \phi = 0$ inside the unit disc $D \subset \mathbb{C}$, subject to the boundary conditions

$$\phi\Big|_{|z|=1} = \begin{cases} \phi_0 & \text{for } \arg(z) \in (0, \pi) \\ -\phi_0 & \text{for } \arg(z) \in (\pi, 2\pi)\,, \end{cases}$$

where ϕ_0 is constant. Give your answer in terms of $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$.

Paper 4, Section II

14D Complex Methods

(a) Using the Bromwich contour integral, find the inverse Laplace transform of $1/s^2$.

The temperature $u(r,t)$ of mercury in a spherical thermometer bulb $r \leqslant a$ obeys the radial heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$$

with unit diffusion constant. At $t = 0$ the mercury is at a uniform temperature u_0 equal to that of the surrounding air. For $t > 0$ the surrounding air temperature lowers such that at the edge of the thermometer bulb

$$\frac{1}{k}\frac{\partial u}{\partial r}\bigg|_{r=a} = u_0 - u(a,t) - t\,,$$

where k is a constant.

(b) Find an explicit expression for $U(r,s) = \int_0^\infty e^{-st}\, u(r,t)\, dt$.

(c) Show that the temperature of the mercury at the centre of the thermometer bulb at late times is

$$u(0,t) \approx u_0 - t + \frac{a}{3k} + \frac{a^2}{6}\,.$$

[*You may assume that the late time behaviour of* $u(r,t)$ *is determined by the singular part of* $U(r,s)$ *at* $s = 0$.]

Paper 1, Section I

2A Complex Analysis or Complex Methods

(a) Show that

$$w = \log(z)$$

is a conformal mapping from the right half z-plane, $\mathrm{Re}(z) > 0$, to the strip

$$S = \left\{ w \,:\, -\frac{\pi}{2} < \mathrm{Im}(w) < \frac{\pi}{2} \right\},$$

for a suitably chosen branch of $\log(z)$ that you should specify.

(b) Show that

$$w = \frac{z-1}{z+1}$$

is a conformal mapping from the right half z-plane, $\mathrm{Re}(z) > 0$, to the unit disc

$$D = \{ w \,:\, |w| < 1 \}.$$

(c) Deduce a conformal mapping from the strip S to the disc D.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(a) Let C be a rectangular contour with vertices at $\pm R + \pi\, i$ and $\pm R - \pi\, i$ for some $R > 0$ taken in the anticlockwise direction. By considering

$$\lim_{R\to\infty} \oint_C \frac{e^{iz^2/4\pi}}{e^{z/2} - e^{-z/2}} dz,$$

show that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix^2/4\pi} dx = 2\pi e^{\pi i/4}.$$

(b) By using a semi-circular contour in the upper half plane, calculate

$$\int_0^\infty \frac{x \sin(\pi x)}{x^2 + a^2}\, dx$$

for $a > 0$.

[You may use Jordan's Lemma without proof.]

Paper 2, Section II

13A Complex Analysis or Complex Methods

(a) Let $f(z)$ be a complex function. Define the *Laurent series* of $f(z)$ about $z = z_0$, and give suitable formulae in terms of integrals for calculating the coefficients of the series.

(b) Calculate, by any means, the first 3 terms in the Laurent series about $z = 0$ for

$$f(z) = \frac{1}{e^{2z} - 1}.$$

Indicate the range of values of $|z|$ for which your series is valid.

(c) Let

$$g(z) = \frac{1}{2z} + \sum_{k=1}^{m} \frac{z}{z^2 + \pi^2 k^2}.$$

Classify the singularities of $F(z) = f(z) - g(z)$ for $|z| < (m+1)\pi$.

(d) By considering

$$\oint_{C_R} \frac{F(z)}{z^2} dz$$

where $C_R = \{|z| = R\}$ for some suitably chosen $R > 0$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Paper 3, Section I

4A Complex Methods

(a) Let $f(z) = (z^2 - 1)^{1/2}$. Define the branch cut of $f(z)$ as $[-1, 1]$ such that

$$f(x) = +\sqrt{x^2 - 1} \qquad x > 1.$$

Show that $f(z)$ is an odd function.

(b) Let $g(z) = \left[(z-2)(z^2-1)\right]^{1/2}$.

(i) Show that $z = \infty$ is a branch point of $g(z)$.

(ii) Define the branch cuts of $g(z)$ as $[-1, 1] \cup [2, \infty)$ such that

$$g(x) = e^{\pi i/2} \sqrt{|x-2||x^2-1|} \qquad x \in (1, 2).$$

Find $g(0_{\pm})$, where 0_{+} denotes $z = 0$ just above the branch cut, and 0_{-} denotes $z = 0$ just below the branch cut.

Paper 4, Section II

14A Complex Methods

(a) Find the Laplace transform of

$$y(t) = \frac{e^{-a^2/4t}}{\sqrt{\pi t}},$$

for $a \in \mathbb{R}$, $a \neq 0$.

[*You may use without proof that*

$$\int_0^\infty \exp\left(-c^2x^2 - \frac{c^2}{x^2}\right)dx = \frac{\sqrt{\pi}}{2|c|}e^{-2c^2}.]$$

(b) By using the Laplace transform, show that the solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \\ u(x,t) &\quad \text{bounded}, \end{aligned}$$

can be written as

$$u(x,t) = \int_{-\infty}^{\infty} K(|x-\xi|,t)f(\xi)d\xi$$

for some $K(|x-\xi|,t)$ to be determined.

[*You may use without proof that a particular solution to*

$$y''(x) - sy(x) + f(x) = 0$$

is given by

$$y(x) = \frac{e^{-\sqrt{s}\,x}}{2\sqrt{s}}\int_0^x e^{\sqrt{s}\,\xi}f(\xi)\,d\xi - \frac{e^{\sqrt{s}\,x}}{2\sqrt{s}}\int_0^x e^{-\sqrt{s}\,\xi}f(\xi)\,d\xi.]$$

Paper 1, Section I

2A Complex Analysis or Complex Methods

Let $F(z) = u(x,y) + i\,v(x,y)$ where $z = x + i\,y$. Suppose $F(z)$ is an analytic function of z in a domain $\mathcal{D}$ of the complex plane.

Derive the Cauchy-Riemann equations satisfied by u and v.

For $u = \dfrac{x}{x^2+y^2}$ find a suitable function v and domain $\mathcal{D}$ such that $F = u + i\,v$ is analytic in $\mathcal{D}$.

Paper 2, Section II

13A Complex Analysis or Complex Methods

State the residue theorem.

By considering

$$\oint_C \frac{z^{1/2}\log z}{1+z^2}dz$$

with C a suitably chosen contour in the upper half plane or otherwise, evaluate the real integrals

$$\int_0^\infty \frac{x^{1/2}\log x}{1+x^2}dx$$

and

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

where $x^{1/2}$ is taken to be the positive square root.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(a) Let $f(z)$ be defined on the complex plane such that $zf(z) \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic on an open set containing $\mathrm{Im}(z) \geqslant -c$, where c is a positive real constant.

Let C_1 be the horizontal contour running from $-\infty - ic$ to $+\infty - ic$ and let

$$F(\lambda) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\lambda}dz.$$

By evaluating the integral, show that $F(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(b) Let $g(z)$ be defined on the complex plane such that $z\,g(z) \to 0$ as $|z| \to \infty$ with $\mathrm{Im}(z) \geqslant -c$. Suppose $g(z)$ is analytic at all points except $z = \alpha_+$ and $z = \alpha_-$ which are simple poles with $\mathrm{Im}(\alpha_+) > c$ and $\mathrm{Im}(\alpha_-) < -c$.

Let C_2 be the horizontal contour running from $-\infty + ic$ to $+\infty + ic$, and let

$$H(\lambda) = \frac{1}{2\pi i}\int_{C_1} \frac{g(z)}{z-\lambda}dz,$$
$$J(\lambda) = -\frac{1}{2\pi i}\int_{C_2} \frac{g(z)}{z-\lambda}dz.$$

(i) Show that $H(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(ii) Show that $J(\lambda)$ is analytic for $\mathrm{Im}(\lambda) < c$.

(iii) Show that if $-c < \mathrm{Im}(\lambda) < c$ then $H(\lambda) + J(\lambda) = g(\lambda)$.

[You should be careful to make sure you consider all points in the required regions.]

Paper 3, Section I

4A Complex Methods

By using the Laplace transform, show that the solution to

$$y'' - 4y' + 3y = t\,e^{-3t},$$

subject to the conditions $y(0) = 0$ and $y'(0) = 1$, is given by

$$y(t) = \frac{37}{72}e^{3t} - \frac{17}{32}e^{t} + \left(\frac{5}{288} + \frac{1}{24}t\right)e^{-3t}$$

when $t \geqslant 0$.

Paper 4, Section II

14A Complex Methods

By using Fourier transforms and a conformal mapping

$$w = \sin\left(\frac{\pi z}{a}\right)$$

with $z = x + iy$ and $w = \xi + i\eta$, and a suitable real constant a, show that the solution to

$$\begin{aligned} \nabla^2\phi &= 0 \qquad -2\pi \leqslant x \leqslant 2\pi,\ y \geqslant 0,\\ \phi(x,0) &= f(x) \qquad -2\pi \leqslant x \leqslant 2\pi,\\ \phi(\pm 2\pi, y) &= 0 \qquad y > 0,\\ \phi(x,y) &\to 0 \qquad y \to \infty,\ -2\pi \leqslant x \leqslant 2\pi, \end{aligned}$$

is given by

$$\phi(\xi,\eta) = \frac{\eta}{\pi}\int_{-1}^{1} \frac{F(\xi')}{\eta^2 + (\xi - \xi')^2}\,d\xi',$$

where $F(\xi')$ is to be determined.

In the case of $f(x) = \sin\left(\frac{x}{4}\right)$, give $F(\xi')$ explicitly as a function of ξ'. [You need not evaluate the integral.]

Paper 1, Section I

2A Complex Analysis or Complex Methods

Classify the singularities of the following functions at both $z = 0$ and at the point at infinity on the extended complex plane:

$$\begin{aligned} f_1(z) &= \frac{e^z}{z \sin^2 z}, \\ f_2(z) &= \frac{1}{z^2(1-\cos z)}, \\ f_3(z) &= z^2 \sin(1/z). \end{aligned}$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

Let $a = N + 1/2$ for a positive integer N. Let C_N be the anticlockwise contour defined by the square with its four vertices at $a \pm ia$ and $-a \pm ia$. Let

$$I_N = \oint_{C_N} \frac{dz}{z^2 \sin(\pi z)}.$$

Show that $1/\sin(\pi z)$ is uniformly bounded on the contours C_N as $N \to \infty$, and hence that $I_N \to 0$ as $N \to \infty$.

Using this result, establish that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Paper 1, Section II

13A Complex Analysis or Complex Methods

Let $w = u + iv$ and let $z = x + iy$, for u, v, x, y real.

(a) Let A be the map defined by $w = \sqrt{z}$, using the principal branch. Show that A maps the region to the left of the parabola $y^2 = 4(1 - x)$ on the $z-$plane, with the negative real axis $x \in (-\infty, 0]$ removed, into the vertical strip of the $w-$plane between the lines $u = 0$ and $u = 1$.

(b) Let B be the map defined by $w = \tan^2(z/2)$. Show that B maps the vertical strip of the $z-$plane between the lines $x = 0$ and $x = \pi/2$ into the region inside the unit circle on the $w-$plane, with the part $u \in (-1, 0]$ of the negative real axis removed.

(c) Using the results of parts (a) and (b), show that the map C, defined by $w = \tan^2(\pi\sqrt{z}/4)$, maps the region to the left of the parabola $y^2 = 4(1 - x)$ on the $z-$plane, *including* the negative real axis, onto the unit disc on the $w-$plane.

Paper 3, Section I

4A Complex Methods

The function $f(x)$ has Fourier transform

$$\tilde{f}(k) \quad = \quad \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{-2ki}{p^2 + k^2},$$

where $p > 0$ is a real constant. Using contour integration, calculate $f(x)$ for $x < 0$. [Jordan's lemma and the residue theorem may be used without proof.]

Paper 4, Section II

14A Complex Methods

(a) Show that the Laplace transform of the Heaviside step function $H(t-a)$ is

$$\int_0^\infty H(t-a)e^{-pt}dt = \frac{e^{-ap}}{p},$$

for $a > 0$.

(b) Derive an expression for the Laplace transform of the second derivative of a function $f(t)$ in terms of the Laplace transform of $f(t)$ and the properties of $f(t)$ at $t = 0$.

(c) A bar of length L has its end at $x = L$ fixed. The bar is initially at rest and straight. The end at $x = 0$ is given a small fixed transverse displacement of magnitude a at $t = 0^+$. You may assume that the transverse displacement $y(x,t)$ of the bar satisfies the wave equation with some wave speed c, and so the tranverse displacement $y(x,t)$ is the solution to the problem:

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} && \text{for } 0 < x < L \text{ and } t > 0, \\
y(x,0) &= \frac{\partial y}{\partial t}(x,0) = 0 && \text{for } 0 < x < L, \\
y(0,t) = a;\ y(L,t) &= 0 && \text{for } t > 0.
\end{aligned}$$

(i) Show that the Laplace transform $Y(x,p)$ of $y(x,t)$, defined as

$$Y(x,p) = \int_0^\infty y(x,t)e^{-pt}dt,$$

is given by

$$Y(x,p) = \frac{a \sinh\left[\frac{p}{c}(L-x)\right]}{p \sinh\left[\frac{pL}{c}\right]}.$$

(ii) By use of the binomial theorem or otherwise, express $y(x,t)$ as an infinite series.

(iii) Plot the transverse displacement of the midpoint of the bar $y(L/2,t)$ against time.

Paper 1, Section I

2B Complex Analysis or Complex Methods

Consider the analytic (holomorphic) functions f and g on a nonempty domain Ω where g is nowhere zero. Prove that if $|f(z)| = |g(z)|$ for all z in Ω then there exists a real constant α such that $f(z) = e^{i\alpha} g(z)$ for all z in Ω.

Paper 2, Section II

13B Complex Analysis or Complex Methods

(i) A function $f(z)$ has a pole of order m at $z = z_0$. Derive a general expression for the residue of $f(z)$ at $z = z_0$ involving f and its derivatives.

(ii) Using contour integration along a contour in the upper half-plane, determine the value of the integral

$$I = \int_0^\infty \frac{(\ln x)^2}{(1+x^2)^2} \mathrm{d}x.$$

Paper 1, Section II

13B Complex Analysis or Complex Methods

(i) Show that transformations of the complex plane of the form

$$\zeta = \frac{az+b}{cz+d},$$

always map circles and lines to circles and lines, where a, b, c and d are complex numbers such that $ad - bc \neq 0$.

(ii) Show that the transformation

$$\zeta = \frac{z-\alpha}{\bar{\alpha}z - 1}, \quad |\alpha| < 1,$$

maps the unit disk centered at $z = 0$ onto itself.

(iii) Deduce a conformal transformation that maps the non-concentric annular domain $\Omega = \{|z| < 1,\ |z-c| > c\}$, $0 < c < 1/2$, to a concentric annular domain.

Paper 3, Section I

4B Complex Methods

Find the Fourier transform of the function

$$f(x)=\frac{1}{1+x^2}, \quad x\in\mathbb{R},$$

using an appropriate contour integration. Hence find the Fourier transform of its derivative, $f'(x)$, and evaluate the integral

$$I=\int_{-\infty}^{\infty}\frac{4x^2}{(1+x^2)^4}dx\cdot$$

Paper 4, Section II

14B Complex Methods

(i) State and prove the convolution theorem for Laplace transforms of two real-valued functions.

(ii) Let the function $f(t)$, $t\geqslant 0$, be equal to 1 for $0\leqslant t\leqslant a$ and zero otherwise, where a is a positive parameter. Calculate the Laplace transform of f. Hence deduce the Laplace transform of the convolution $g=f*f$. Invert this Laplace transform to obtain an explicit expression for $g(t)$.

[*Hint: You may use the notation* $(t-a)_+=H(t-a)\cdot(t-a)$.]

Paper 1, Section I

2B Complex Analysis or Complex Methods

Let $f(z)$ be an analytic/holomorphic function defined on an open set D, and let $z_0 \in D$ be a point such that $f'(z_0) \neq 0$. Show that the transformation $w = f(z)$ preserves the angle between smooth curves intersecting at z_0. Find such a transformation $w = f(z)$ that maps the second quadrant of the unit disc (i.e. $|z| < 1$, $\pi/2 < \arg(z) < \pi$) to the region in the first quadrant of the complex plane where $|w| > 1$ (i.e. the region in the first quadrant *outside* the unit circle).

Paper 1, Section II

13B Complex Analysis or Complex Methods

By choice of a suitable contour show that for $a > b > 0$

$$\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right].$$

Hence evaluate

$$\int_0^1 \frac{(1-x^2)^{1/2}x^2 dx}{1+x^2}$$

using the substitution $x = \cos(\theta/2)$.

Paper 2, Section II

13B Complex Analysis or Complex Methods

By considering a rectangular contour, show that for $0 < a < 1$ we have

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = \frac{\pi}{\sin \pi a}.$$

Hence evaluate

$$\int_0^{\infty} \frac{dt}{t^{5/6}(1+t)}.$$

Paper 3, Section I

4B Complex Methods

Find the most general cubic form

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

which satisfies Laplace's equation, where a, b, c and d are all real. Hence find an analytic function $f(z) = f(x+iy)$ which has such a u as its real part.

Paper 4, Section II

14B Complex Methods

Find the Laplace transforms of t^n for n a positive integer and $H(t-a)$ where $a > 0$ and $H(t)$ is the Heaviside step function.

Consider a semi-infinite string which is initially at rest and is fixed at one end. The string can support wave-like motions, and for $t > 0$ it is allowed to fall under gravity. Therefore the deflection $y(x,t)$ from its initial location satisfies

$$\frac{\partial^2}{\partial t^2} y = c^2 \frac{\partial^2}{\partial x^2} y + g \quad \text{for} \quad x > 0,\ t > 0$$

with

$$y(0,t) = y(x,0) = \frac{\partial}{\partial t} y(x,0) = 0 \quad \text{and} \quad y(x,t) \to \frac{gt^2}{2} \text{ as } x \to \infty,$$

where g is a constant. Use Laplace transforms to find $y(x,t)$.

[The convolution theorem for Laplace transforms may be quoted without proof.]

Paper 1, Section I

2D Complex Analysis or Complex Methods

Classify the singularities (in the finite complex plane) of the following functions:

(i) $\dfrac{1}{(\cosh z)^2}$;

(ii) $\dfrac{1}{\cos(1/z)}$;

(iii) $\dfrac{1}{\log z} \quad (-\pi < \arg z < \pi)$;

(iv) $\dfrac{z^{\frac{1}{2}} - 1}{\sin \pi z} \quad (-\pi < \arg z < \pi)$.

Paper 1, Section II

13E Complex Analysis or Complex Methods

Suppose $p(z)$ is a polynomial of even degree, all of whose roots satisfy $|z| < R$. Explain why there is a holomorphic (*i.e.* analytic) function $h(z)$ defined on the region $R < |z| < \infty$ which satisfies $h(z)^2 = p(z)$. We write $h(z) = \sqrt{p(z)}$.

By expanding in a Laurent series or otherwise, evaluate

$$\int_C \sqrt{z^4 - z}\, dz$$

where C is the circle of radius 2 with the anticlockwise orientation. (Your answer will be well-defined up to a factor of ± 1, depending on which square root you pick.)

Paper 2, Section II

13D Complex Analysis or Complex Methods

Let

$$I = \oint_C \frac{e^{iz^2/\pi}}{1+e^{-2z}} dz\,,$$

where C is the rectangle with vertices at $\pm R$ and $\pm\ R + i\pi$, traversed anti-clockwise.

(i) Show that $I = \dfrac{\pi(1+i)}{\sqrt{2}}$.

(ii) Assuming that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$, show that

$$\int_{-\infty}^{\infty} e^{ix^2/\pi} dx = \frac{\pi(1+i)}{\sqrt{2}}\,.$$

(iii) Justify briefly the assumption that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$.

Paper 3, Section I

4D Complex Methods

Let $y(t) = 0$ for $t < 0$, and let $\lim_{t \to 0^+} y(t) = y_0$.

(i) Find the Laplace transforms of $H(t)$ and $tH(t)$, where $H(t)$ is the Heaviside step function.

(ii) Given that the Laplace transform of $y(t)$ is $\widehat{y}(s)$, find expressions for the Laplace transforms of $\dot{y}(t)$ and $y(t-1)$.

(iii) Use Laplace transforms to solve the equation

$$\dot{y}(t) - y(t-1) = H(t) - (t-1)H(t-1)$$

in the case $y_0 = 0$.

Paper 4, Section II

14D Complex Methods

Let C_1 and C_2 be the circles $x^2 + y^2 = 1$ and $5x^2 - 4x + 5y^2 = 0$, respectively, and let D be the (finite) region between the circles. Use the conformal mapping

$$w = \frac{z-2}{2z-1}$$

to solve the following problem:

$$\nabla^2 \phi = 0 \ \text{ in } D \text{ with } \phi = 1 \text{ on } C_1 \text{ and } \phi = 2 \text{ on } C_2.$$

Paper 1, Section I

2A Complex Analysis or Complex Methods

Find a conformal transformation $\zeta = \zeta(z)$ that maps the domain D, $0 < \arg z < \frac{3\pi}{2}$, on to the strip $0 < \operatorname{Im}(\zeta) < 1$.

Hence find a bounded harmonic function ϕ on D subject to the boundary conditions $\phi = 0, A$ on $\arg z = 0, \frac{3\pi}{2}$, respectively, where A is a real constant.

Paper 2, Section II

13A Complex Analysis or Complex Methods

By a suitable choice of contour show that, for $-1 < \alpha < 1$,

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2\cos(\alpha\pi/2)}\,.$$

Paper 1, Section II

13A Complex Analysis or Complex Methods

Using Cauchy's integral theorem, write down the value of a holomorphic function $f(z)$ where $|z| < 1$ in terms of a contour integral around the unit circle, $\zeta = e^{i\theta}$.

By considering the point $1/\overline{z}$, or otherwise, show that

$$f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1-|z|^2}{|\zeta - z|^2}\,\mathrm{d}\theta\,.$$

By setting $z = re^{i\alpha}$, show that for any harmonic function $u(r,\alpha)$,

$$u(r,\alpha) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\frac{1-r^2}{1-2r\cos(\alpha-\theta)+r^2}\,\mathrm{d}\theta$$

if $r < 1$.

Assuming that the function $v(r,\alpha)$, which is the conjugate harmonic function to $u(r,\alpha)$, can be written as

$$v(r,\alpha) \;=\; v(0) + \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\frac{r\sin(\alpha-\theta)}{1-2r\cos(\alpha-\theta)+r^2}\,\mathrm{d}\theta\,,$$

deduce that

$$f(z) \;=\; iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\frac{\zeta+z}{\zeta-z}\,\mathrm{d}\theta\,.$$

[You may use the fact that on the unit circle, $\zeta = 1/\overline{\zeta}$, and hence

$$\frac{\zeta}{\zeta - 1/\overline{z}} = -\frac{\overline{z}}{\overline{\zeta}-\overline{z}}\,.\quad]$$

Paper 3, Section I

4A Complex Methods

State the formula for the Laplace transform of a function $f(t)$, defined for $t \geqslant 0$.

Let $f(t)$ be periodic with period T (i.e. $f(t+T) = f(t)$). If $g(t)$ is defined to be equal to $f(t)$ in $[0,T]$ and zero elsewhere and its Laplace transform is $G(s)$, show that the Laplace transform of $f(t)$ is given by

$$F(s) = \frac{G(s)}{1-e^{-sT}}\,.$$

Hence, or otherwise, find the inverse Laplace transform of

$$F(s) = \frac{1}{s}\frac{1-e^{-sT/2}}{1-e^{-sT}}\,.$$

Paper 4, Section II

14A Complex Methods

State the convolution theorem for Fourier transforms.

The function $\phi(x,y)$ satisfies

$$\nabla^2\phi = 0$$

on the half-plane $y \geqslant 0$, subject to the boundary conditions

$$\phi \to 0 \;\text{ as }\; y \to \infty \;\text{ for all }\; x,$$

$$\phi(x,0) = \begin{cases} 1\,, & |x| \leqslant 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

Using Fourier transforms, show that

$$\phi(x,y) = \frac{y}{\pi}\int_{-1}^{1} \frac{1}{y^2+(x-t)^2}\,\mathrm{d}t\,,$$

and hence that

$$\phi(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{1+x}{y}\right)\right]\,.$$

Paper 1, Section I

2A Complex Analysis or Complex Methods

Derive the Cauchy-Riemann equations satisfied by the real and imaginary parts of a complex analytic function $f(z)$.

If $|f(z)|$ is constant on $|z| < 1$, prove that $f(z)$ is constant on $|z| < 1$.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(i) Let $-1 < \alpha < 0$ and let

$$\begin{aligned} f(z) &= \frac{\log(z-\alpha)}{z} \text{ where } -\pi \leqslant \arg(z-\alpha) < \pi\,, \\ g(z) &= \frac{\log z}{z} \qquad \text{where } -\pi \leqslant \arg(z) < \pi\,. \end{aligned}$$

Here the logarithms take their principal values. Give a sketch to indicate the positions of the branch cuts implied by the definitions of $f(z)$ and $g(z)$.

(ii) Let $h(z) = f(z) - g(z)$. Explain why $h(z)$ is analytic in the annulus $1 \leqslant |z| \leqslant R$ for any $R > 1$. Obtain the first three terms of the Laurent expansion for $h(z)$ around $z = 0$ in this annulus and hence evaluate

$$\oint_{|z|=2} h(z)dz\,.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(i) Let C be an anticlockwise contour defined by a square with vertices at $z = x + iy$ where

$$|x| = |y| = \left(2N + \frac{1}{2}\right)\pi,$$

for large integer N. Let

$$I = \oint_C \frac{\pi \cot z}{(z + \pi a)^4} dz.$$

Assuming that $I \to 0$ as $N \to \infty$, prove that, if a is not an integer, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^4} = \frac{\pi^4}{3\sin^2(\pi a)}\left(\frac{3}{\sin^2(\pi a)} - 2\right).$$

(ii) Deduce the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^4}.$$

(iii) Briefly justify the assumption that $I \to 0$ as $N \to \infty$.

[*Hint: For part (iii) it is sufficient to consider, at most, one vertical side of the square and one horizontal side and to use a symmetry argument for the remaining sides.*]

Paper 3, Section I

4D Complex Methods

Write down the function $\psi(u,v)$ that satisfies

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}=0\,, \qquad \psi(-\tfrac{1}{2},v)=-1, \quad \psi(\tfrac{1}{2},v)=1\,.$$

The circular arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ in the complex z-plane are defined by

$$|z+1|=1,\ z\neq 0 \ \text{ and } \ |z-1|=1,\ z\neq 0\,,$$

respectively. You may assume without proof that the mapping from the complex z-plane to the complex ζ-plane defined by

$$\zeta=\frac{1}{z}$$

takes $\mathcal{C}_1$ to the line $u=-\frac{1}{2}$ and $\mathcal{C}_2$ to the line $u=\frac{1}{2}$, where $\zeta=u+iv$, and that the region $\mathcal{D}$ in the z-plane exterior to both the circles $|z+1|=1$ and $|z-1|=1$ maps to the region in the ζ-plane given by $-\frac{1}{2}<u<\frac{1}{2}$.

Use the above mapping to solve the problem

$$\nabla^2\phi=0 \quad \text{in } \mathcal{D}, \quad \phi=-1 \text{ on } \mathcal{C}_1 \text{ and } \phi=1 \text{ on } \mathcal{C}_2.$$

Paper 4, Section II

14D Complex Methods

State and prove the convolution theorem for Laplace transforms.

Use Laplace transforms to solve

$$2f'(t)-\int_0^t (t-\tau)^2 f(\tau)\,d\tau = 4tH(t)$$

with $f(0)=0$, where $H(t)$ is the Heaviside function. You may assume that the Laplace transform, $\widehat{f}(s)$, of $f(t)$ exists for $\operatorname{Re} s$ sufficiently large.

Paper 1, Section I

2A Complex Analysis or Complex Methods

(a) Write down the definition of the complex derivative of the function $f(z)$ of a single complex variable.

(b) Derive the Cauchy-Riemann equations for the real and imaginary parts $u(x,y)$ and $v(x,y)$ of $f(z)$, where $z = x + iy$ and

$$f(z) = u(x,y) + iv(x,y).$$

(c) State necessary and sufficient conditions on $u(x,y)$ and $v(x,y)$ for the function $f(z)$ to be complex differentiable.

Paper 1, Section II

13A Complex Analysis or Complex Methods

Calculate the following real integrals by using contour integration. Justify your steps carefully.

(a)

$$I_1 = \int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx, \qquad a > 0,$$

(b)

$$I_2 = \int_0^\infty \frac{x^{1/2} \log x}{1 + x^2}\, dx.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(a) Prove that a complex differentiable map, $f(z)$, is conformal, i.e. preserves angles, provided a certain condition holds on the first complex derivative of $f(z)$.

(b) Let D be the region

$$D := \{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z-2| < 2\}.$$

Draw the region D. It might help to consider the two sets

$$C(1) := \{z \in \mathbb{C} : |z-1| = 1\},$$
$$C(2) := \{z \in \mathbb{C} : |z-2| = 2\}.$$

(c) For the transformations below identify the images of D.

Step 1: The first map is $f_1(z) = \dfrac{z-1}{z}$,

Step 2: The second map is the composite $f_2 f_1$ where $f_2(z) = (z - \frac{1}{2})i$,

Step 3: The third map is the composite $f_3 f_2 f_1$ where $f_3(z) = e^{2\pi z}$.

(d) Write down the inverse map to the composite $f_3 f_2 f_1$, explaining any choices of branch.

[*The composite $f_2 f_1$ means $f_2(f_1(z))$.*]

Paper 3, Section I

4A Complex Methods

(a) Prove that the real and imaginary parts of a complex differentiable function are harmonic.

(b) Find the most general harmonic polynomial of the form

$$u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

where a, b, c, d, x and y are real.

(c) Write down a complex analytic function of $z = x + iy$ of which $u(x, y)$ is the real part.

Paper 4, Section II

14A Complex Methods

A linear system is described by the differential equation

$$y'''(t) - y''(t) - 2\,y'(t) + 2\,y(t) \;=\; f(t)\,,$$

with initial conditions

$$y\,(0) \;=\; 0\,, \qquad y'(0) \;=\; 1\,, \qquad y''(0) \;=\; 1\,.$$

The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

You may assume the following Laplace transforms,

$$\begin{aligned}
\mathcal{L}[y(t)] &= \tilde{y}(s),\\
\mathcal{L}[y'(t)] &= s\tilde{y}(s) - y(0),\\
\mathcal{L}[y''(t)] &= s^2\tilde{y}(s) - sy(0) - y'(0),\\
\mathcal{L}[y'''(t)] &= s^3\tilde{y}(s) - s^2y(0) - sy'(0) - y''(0).
\end{aligned}$$

(a) Use Laplace transforms to determine the response, $y_1(t)$, of the system to the signal

$$f(t) = -2.$$

(b) Determine the response, $y_2(t)$, given that its Laplace transform is

$$\tilde{y}_2(s) = \frac{1}{s^2(s-1)^2}.$$

(c) Given that

$$y'''(t) - y''(t) - 2y'(t) + 2y(t) \;=\; g(t)$$

leads to the response with Laplace transform

$$\tilde{y}(s) \;=\; \frac{1}{s^2(s-1)^2}\,,$$

determine $g(t)$.

Paper 1, Section I
3D Complex Analysis or Complex Methods

Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, be an analytic function of z in a domain D of the complex plane. Derive the Cauchy–Riemann equations relating the partial derivatives of u and v.

For $u = e^{-x} \cos y$, find v and hence $f(z)$.

Paper 1, Section II
13D Complex Analysis or Complex Methods

Consider the real function $f(t)$ of a real variable t defined by the following contour integral in the complex s-plane:

$$f(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{st}}{(s^2 + 1)s^{1/2}} \, ds,$$

where the contour Γ is the line $s = \gamma + iy, -\infty < y < \infty$, for constant $\gamma > 0$. By closing the contour appropriately, show that

$$f(t) = \sin(t - \pi/4) + \frac{1}{\pi} \int_0^{\infty} \frac{e^{-rt} dr}{(r^2 + 1)r^{1/2}}$$

when $t > 0$ and is zero when $t < 0$. You should justify your evaluation of the inversion integral over all parts of the contour.

By expanding $(r^2 + 1)^{-1} \ r^{-1/2}$ as a power series in r, and assuming that you may integrate the series term by term, show that the two leading terms, as $t \to \infty$, are

$$f(t) \sim \sin(t - \pi/4) + \frac{1}{(\pi t)^{1/2}} + \cdots .$$

[*You may assume that* $\int_0^{\infty} x^{-1/2} e^{-x} dx = \pi^{1/2}$.]

Paper 2, Section II

14D Complex Analysis or Complex Methods

Show that both the following transformations from the z-plane to the ζ-plane are conformal, except at certain critical points which should be identified in both planes, and in each case find a domain in the z-plane that is mapped onto the upper half ζ-plane:

$$\begin{aligned} \text{(i) } \zeta &= z + \frac{b^2}{z}\,; \\ \text{(ii) } \zeta &= \cosh \frac{\pi z}{b}\,, \end{aligned}$$

where b is real and positive.

Paper 3, Section I
5D Complex Methods

Use the residue calculus to evaluate

$$\text{(i)} \quad \oint_C z e^{1/z} dz \quad \text{and} \quad \text{(ii)} \quad \oint_C \frac{z dz}{1-4z^2} ,$$

where C is the circle $|z| = 1$.

Paper 4, Section II
15D Complex Methods

The function $u(x, y)$ satisfies Laplace's equation in the half-space $y \geqslant 0$, together with boundary conditions

$$u(x, y) \to 0 \text{ as } y \to \infty \text{ for all } x \ ,$$
$$u(x, 0) = u_0(x), \text{ where } x\, u_0(x) \to 0 \text{ as } |x| \to \infty.$$

Using Fourier transforms, show that

$$u(x, y) = \int_{-\infty}^{\infty} u_0(t) v(x - t, y) dt \ ,$$

where

$$v(x, y) = \frac{y}{\pi(x^2 + y^2)} \ .$$

Suppose that $u_0(x) = (x^2 + a^2)^{-1}$. Using contour integration and the convolution theorem, or otherwise, show that

$$u(x, y) = \frac{y + a}{a\,[x^2 + (y + a)^2]} \ .$$

[*You may assume the convolution theorem of Fourier transforms, i.e. that if $\tilde{f}(k), \tilde{g}(k)$ are the Fourier transforms of two functions $f(x), g(x)$, then $\tilde{f}(k)\tilde{g}(k)$ is the Fourier transform of $\int_{-\infty}^{\infty} f(t) g(x-t) dt$.*]

1/I/3C **Complex Analysis or Complex Methods**

Given that $f(z)$ is an analytic function, show that the mapping $w = f(z)$

(a) preserves angles between smooth curves intersecting at z if $f'(z) \neq 0$;

(b) has Jacobian given by $|f'(z)|^2$.

1/II/13C **Complex Analysis or Complex Methods**

By a suitable choice of contour show the following:

(a)

$$\int_0^\infty \frac{x^{1/n}}{1+x^2}\,dx \;=\; \frac{\pi}{2\cos(\pi/2n)}\,,$$

where $n > 1$,

(b)

$$\int_0^\infty \frac{x^{1/2}\log x}{1+x^2}\,dx \;=\; \frac{\pi^2}{2\sqrt{2}}\,.$$

2/II/14C **Complex Analysis or Complex Methods**

Let $f(z) = 1/(e^z - 1)$. Find the first three terms in the Laurent expansion for $f(z)$ valid for $0 < |z| < 2\pi$.

Now let n be a positive integer, and define

$$\begin{aligned} f_1(z) &= \frac{1}{z} + \sum_{r=1}^{n} \frac{2z}{z^2 + 4\pi^2 r^2}\,, \\ f_2(z) &= f(z) - f_1(z)\,. \end{aligned}$$

Show that the singularities of f_2 in $\{z : |z| < 2(n+1)\pi\}$ are all removable. By expanding f_1 as a Laurent series valid for $|z| > 2n\pi$, and f_2 as a Taylor series valid for $|z| < 2(n+1)\pi$, find the coefficients of z^j for $-1 \leq j \leq 1$ in the Laurent series for f valid for $2n\pi < |z| < 2(n+1)\pi$.

By estimating an appropriate integral around the contour $|z| = (2n+1)\pi$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}\,.$$

3/I/5C **Complex Methods**

Using the contour integration formula for the inversion of Laplace transforms find the inverse Laplace transforms of the following functions:

$$\text{(a)} \qquad \frac{s}{s^2+a^2} \quad (a \text{ real and non–zero}), \qquad \text{(b)} \qquad \frac{1}{\sqrt{s}}\ .$$

[*You may use the fact that* $\int_{-\infty}^{\infty} e^{-bx^2}dx \;=\; \sqrt{\pi/b}$.]

4/II/15C **Complex Methods**

Let H be the domain $\mathbb{C} - \{x+iy : x \leq 0, y = 0\}$ (i.e., $\mathbb{C}$ cut along the negative x-axis). Show, by a suitable choice of branch, that the mapping

$$z \mapsto w \;=\; -i \log z$$

maps H onto the strip $S = \{z = x+iy, -\pi < x < \pi\}$.

How would a different choice of branch change the result?

Let G be the domain $\{z \in \mathbb{C} : |z| < 1, |z+i| > \sqrt{2}\}$. Find an analytic transformation that maps G to S, where S is the strip defined above.

1/I/3F Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{2z}{z^2+1},$$

determine the Taylor series of f around the point $z_0 = 1$, and give the largest r for which this series converges in the disc $|z-1| < r$.

1/II/13F Complex Analysis or Complex Methods

By integrating round the contour C_R, which is the boundary of the domain

$$D_R = \{z = re^{i\theta} : 0 < r < R, \quad 0 < \theta < \frac{\pi}{4}\},$$

evaluate each of the integrals

$$\int_0^\infty \sin x^2 dx, \quad \int_0^\infty \cos x^2 dx\,.$$

[*You may use the relations* $\displaystyle\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$ *and* $\sin t \geq \frac{2}{\pi} t$ *for* $0 \leq t \leq \frac{\pi}{2}$.]

2/II/14F Complex Analysis or Complex Methods

Let Ω be the half-strip in the complex plane,

$$\Omega = \{z = x + iy \in \mathbb{C} \;:\; -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0\}.$$

Find a conformal mapping that maps Ω onto the unit disc.

3/I/5F Complex Methods

Show that the function $\phi(x,y) = \tan^{-1}\dfrac{y}{x}$ is harmonic. Find its harmonic conjugate $\psi(x,y)$ and the analytic function $f(z)$ whose real part is $\phi(x,y)$. Sketch the curves $\phi(x,y) = C$ and $\psi(x,y) = K$.

4/II/15F Complex Methods

(i) Use the definition of the Laplace transform of $f(t)$:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt\,,$$

to show that, for $f(t) = t^n$,

$$L\{f(t)\} = F(s) = \frac{n!}{s^{n+1}}\,, \qquad L\{e^{at}f(t)\} = F(s-a) = \frac{n!}{(s-a)^{n+1}}\,.$$

(ii) Use contour integration to find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+1)^2}\,.$$

(iii) Verify the result in (ii) by using the results in (i) and the convolution theorem.

(iv) Use Laplace transforms to solve the differential equation

$$f^{(iv)}(t) + 2f'''(t) + f''(t) = 0,$$

subject to the initial conditions

$$f(0) = f'(0) = f''(0) = 0, \qquad f'''(0) = 1\,.$$

1/I/3D Complex Analysis or Complex Methods

Let L be the Laplace operator, i.e., $L(g) = g_{xx} + g_{yy}$. Prove that if $f : \Omega \to \mathbf{C}$ is analytic in a domain Ω, then

$$L\big(|f(z)|^2\big) = 4|f'(z)|^2\,, \quad z \in \Omega.$$

1/II/13D Complex Analysis or Complex Methods

By integrating round the contour involving the real axis and the line $\mathrm{Im}(z) = 2\pi$, or otherwise, evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx\,, \quad 0 < a < 1.$$

Explain why the given restriction on the value a is necessary.

2/II/14D Complex Analysis or Complex Methods

Let Ω be the region enclosed between the two circles C_1 and C_2, where

$$C_1 = \{z \in \mathbf{C} : |z - i| = 1\}\,, \qquad C_2 = \{z \in \mathbf{C} : |z - 2i| = 2\}.$$

Find a conformal mapping that maps Ω onto the unit disc.

[*Hint: you may find it helpful first to map Ω to a strip in the complex plane.*]

3/I/5D **Complex Methods**

The transformation

$$w = i\left(\frac{1-z}{1+z}\right)$$

maps conformally the interior of the unit disc D onto the upper half-plane H_+, and maps the upper and lower unit semicircles C_+ and C_- onto the positive and negative real axis $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Consider the Dirichlet problem in the upper half-plane:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0 \quad \text{in} \quad H_+; \qquad f(u,v) = \begin{cases} 1 & \text{on } \mathbb{R}_+, \\ 0 & \text{on } \mathbb{R}_-. \end{cases}$$

Its solution is given by the formula

$$f(u,v) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{u}{v}\right).$$

Using this result, determine the solution to the Dirichlet problem in the unit disc:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \quad \text{in} \quad D; \qquad F(x,y) = \begin{cases} 1 & \text{on } C_+, \\ 0 & \text{on } C_-. \end{cases}$$

Briefly explain your answer.

4/II/15D **Complex Methods**

Denote by $f * g$ the convolution of two functions, and by $\widehat{f}$ the Fourier transform, i.e.,

$$[f * g](x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt, \qquad \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,dx\,.$$

(a) Show that, for suitable functions f and g, the Fourier transform $\widehat{F}$ of the convolution $F = f * g$ is given by $\widehat{F} = \widehat{f} \cdot \widehat{g}$.

(b) Let

$$f_1(x) = \begin{cases} 1 & |x| \leqslant 1/2\,, \\ 0 & \text{otherwise.} \end{cases}$$

and let $f_2 = f_1 * f_1$ be the convolution of f_1 with itself. Find the Fourier transforms of f_1 and f_2, and, by applying Parseval's theorem, determine the value of the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin y}{y}\right)^4 dy\,.$$

1/I/3F **Complex Analysis or Complex Methods**

State the Cauchy integral formula.

Using the Cauchy integral formula, evaluate

$$\int_{|z|=2} \frac{z^3}{z^2+1} dz.$$

1/II/13F **Complex Analysis or Complex Methods**

Determine a conformal mapping from $\Omega_0 = \mathbf{C} \setminus [-1, 1]$ to the complex unit disc $\Omega_1 = \{z \in \mathbf{C} : |z| < 1\}$.

[*Hint: A standard method is first to map* Ω_0 *to* $\mathbf{C} \setminus (-\infty, 0]$, *then to the complex right half-plane* $\{z \in \mathbf{C} : \operatorname{Re} z > 0\}$ *and, finally, to* Ω_1.]

2/II/14F **Complex Analysis or Complex Methods**

Let $F = P/Q$ be a rational function, where $\deg Q \geqslant \deg P + 2$ and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} F(x) e^{ix} dx$$

in terms of residues and briefly sketch its proof.

Evaluate explicitly the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{4+x^4} dx\,.$$

3/I/5F **Complex Methods**

Define a harmonic function and state when the harmonic functions f and g are conjugate.

Let $\{u, v\}$ and $\{p, q\}$ be two pairs of harmonic conjugate functions. Prove that $\{p(u, v), q(u, v)\}$ are also harmonic conjugate.

4/II/15F **Complex Methods**

Determine the Fourier expansion of the function $f(x) = \sin \lambda x$, where $-\pi \leqslant x \leqslant \pi$, in the two cases where λ is an integer and λ is a real non-integer.

Using the Parseval identity in the case $\lambda = \frac{1}{2}$, find an explicit expression for the sum

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2 - 1)^2}.$$

1/I/5A **Complex Methods**

Determine the poles of the following functions and calculate their residues there.

$$\text{(i)} \quad \frac{1}{z^2+z^4}, \qquad \text{(ii)} \quad \frac{e^{1/z^2}}{z-1}, \qquad \text{(iii)} \quad \frac{1}{\sin(e^z)}.$$

1/II/16A **Complex Methods**

Let p and q be two polynomials such that

$$q(z)=\prod_{l=1}^{m}(z-\alpha_l),$$

where $\alpha_1,\ldots,\alpha_m$ are distinct non-real complex numbers and $\deg p \leqslant m-1$. Using contour integration, determine

$$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}e^{ix}dx\,,$$

carefully justifying all steps.

2/I/5A **Complex Methods**

Let the functions f and g be analytic in an open, nonempty domain Ω and assume that $g \neq 0$ there. Prove that if $|f(z)| \equiv |g(z)|$ in Ω then there exists $\alpha \in \mathbb{R}$ such that $f(z) \equiv e^{i\alpha}g(z)$.

2/II/16A **Complex Methods**

Prove by using the Cauchy theorem that if f is analytic in the open disc $\Omega=\{z\in\mathbb{C} : |z|<1\}$ then there exists a function g, analytic in Ω, such that $g'(z)=f(z)$, $z\in\Omega$.

4/I/5A **Complex Methods**

State and prove the Parseval formula.

[*You may use without proof properties of convolution, as long as they are precisely stated.*]

4/II/15A **Complex Methods**

(i) Show that the inverse Fourier transform of the function

$$\hat{g}(s) = \begin{cases} e^s - e^{-s}, & |s| \leqslant 1, \\ 0, & |s| \geqslant 1. \end{cases}$$

is

$$g(x) = \frac{2i}{\pi} \frac{1}{1+x^2} (x \sinh 1 \cos x - \cosh 1 \sin x)$$

(ii) Determine, by using Fourier transforms, the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

given in the strip $-\infty < x < \infty$, $0 < y < 1$, together with the boundary conditions

$$u(x,0) = g(x), \quad u(x,1) \equiv 0, \qquad -\infty < x < \infty,$$

where g has been given above.

[*You may use without proof properties of Fourier transforms.*]

1/I/7B **Complex Methods**

Let $u(x,y)$ and $v(x,y)$ be a pair of conjugate harmonic functions in a domain D. Prove that

$$U(x,y) = e^{-2uv}\cos(u^2 - v^2) \qquad \text{and} \qquad V(x,y) = e^{-2uv}\sin(u^2 - v^2)$$

also form a pair of conjugate harmonic functions in D.

1/II/16B **Complex Methods**

Sketch the region A which is the intersection of the discs

$$D_0 = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad D_1 = \{z \in \mathbb{C} : |z - (1+i)| < 1\}\,.$$

Find a conformal mapping that maps A onto the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also find a conformal mapping that maps A onto D_0.

[*Hint: You may find it useful to consider maps of the form* $w(z) = \frac{az+b}{cz+d}$.]

2/I/7B **Complex Methods**

(a) Using the residue theorem, evaluate

$$\int_{|z|=1} \left(z - \frac{1}{z}\right)^{2n} \frac{dz}{z}\,.$$

(b) Deduce that

$$\int_0^{2\pi} \sin^{2n} t\, dt = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}\,.$$

2/II/16B **Complex Methods**

(a) Show that if f satisfies the equation

$$f''(x) - x^2 f(x) = \mu f(x), \quad x \in \mathbb{R}, \tag{$*$}$$

where μ is a constant, then its Fourier transform $\widehat{f}$ satisfies the same equation, i.e.

$$\widehat{f}''(\lambda) - \lambda^2 \widehat{f}(\lambda) = \mu \widehat{f}(\lambda)\,.$$

(b) Prove that, for each $n \geq 0$, there is a polynomial $p_n(x)$ of degree n, unique up to multiplication by a constant, such that

$$f_n(x) = p_n(x) e^{-x^2/2}$$

is a solution of $(*)$ for some $\mu = \mu_n$.

(c) Using the fact that $g(x) = e^{-x^2/2}$ satisfies $\widehat{g} = cg$ for some constant c, show that the Fourier transform of f_n has the form

$$\widehat{f_n}(\lambda) = q_n(\lambda) e^{-\lambda^2/2}$$

where q_n is also a polynomial of degree n.

(d) Deduce that the f_n are eigenfunctions of the Fourier transform operator, i.e. $\widehat{f_n}(x) = c_n f_n(x)$ for some constants c_n.

4/I/8B **Complex Methods**

Find the Laurent series centred on 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in each of the domains

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$.

4/II/17B **Complex Methods**

Let

$$f(z) = \frac{z^m}{1+z^n}, \quad n > m+1, \quad m, n \in \mathbb{N},$$

and let C_R be the boundary of the domain

$$D_R = \{z = re^{i\theta} : \ 0 < r < R, \quad 0 < \theta < \tfrac{2\pi}{n}\}, \quad R > 1.$$

(a) Using the residue theorem, determine

$$\int_{C_R} f(z)\, dz.$$

(b) Show that the integral of $f(z)$ along the circular part γ_R of C_R tends to 0 as $R \to \infty$.

(c) Deduce that

$$\int_0^\infty \frac{x^m}{1+x^n}\, dx = \frac{\pi}{n \sin \frac{\pi(m+1)}{n}}\,.$$

1/I/7B **Complex Methods**

Using contour integration around a rectangle with vertices

$$-x,\ x,\ x+iy,\ -x+iy,$$

prove that, for all real y,

$$\int_{-\infty}^{+\infty} e^{-(x+iy)^2}\,dx = \int_{-\infty}^{+\infty} e^{-x^2}\,dx\,.$$

Hence derive that the function $f(x) = e^{-x^2/2}$ is an eigenfunction of the Fourier transform

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x)e^{-ixy}dx,$$

i.e. $\widehat{f}$ is a constant multiple of f.

1/II/16B **Complex Methods**

(a) Show that if f is an analytic function at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0, i.e. it preserves angles between paths passing through z_0.

(b) Let D be the disc given by $|z+i| < \sqrt{2}$, and let H be the half-plane given by $y > 0$, where $z = x+iy$. Construct a map of the domain $D \cap H$ onto H, and hence find a conformal mapping of $D \cap H$ onto the disc $\{z : |z| < 1\}$. [*Hint: You may find it helpful to consider a mapping of the form* $(az+b)/(cz+d)$, *where* $ad - bc \neq 0$.]

2/I/7B **Complex Methods**

Suppose that f is analytic, and that $|f(z)|^2$ is constant in an open disk D. Use the Cauchy–Riemann equations to show that $f(z)$ is constant in D.

2/II/16B Complex Methods

A function $f(z)$ has an isolated singularity at a, with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

(a) Define $\operatorname{res}(f, a)$, the residue of f at the point a.

(b) Prove that if a is a pole of order $k+1$, then

$$\operatorname{res}(f,a) = \lim_{z\to a} \frac{h^{(k)}(z)}{k!}, \quad \text{where} \quad h(z) = (z-a)^{k+1} f(z).$$

(c) Using the residue theorem and the formula above show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{k+1}} = \pi \frac{(2k)!}{(k!)^2} 4^{-k}, \quad k \geq 1.$$

4/I/8B Complex Methods

Let f be a function such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. Prove that

$$\int_{-\infty}^{+\infty} f(x+k)\overline{f(x+l)}\, dx = 0 \quad \text{for all integers } k \text{ and } l \text{ with } k \neq l,$$

if and only if

$$\int_{-\infty}^{+\infty} |\widehat{f}(t)|^2 e^{-imt} dt = 0 \quad \text{for all integers } m \neq 0,$$

where $\widehat{f}$ is the Fourier transform of f.

4/II/17B Complex Methods

(a) Using the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \frac{\pi}{2}$, show that, if f is continuous for large $|z|$, and if $f(z) \to 0$ as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z) e^{i\lambda z} dz = 0 \quad \text{for} \quad \lambda > 0,$$

where $\Gamma_R = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

(b) By integrating an appropriate function $f(z)$ along the contour formed by the semicircles Γ_R and Γ_r in the upper half-plane with the segments of the real axis $[-R, -r]$ and $[r, R]$, show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

1/I/7E **Complex Methods**

State the Cauchy integral formula.

Assuming that the function $f(z)$ is analytic in the disc $|z| < 1$, prove that, for every $0 < r < 1$, it is true that

$$\frac{d^n f(0)}{dz^n} = \frac{n!}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi, \qquad n = 0, 1, \ldots.$$

[*Taylor's theorem may be used if clearly stated.*]

1/II/16E **Complex Methods**

Let the function F be integrable for all real arguments x, such that

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \ ,$$

and assume that the series

$$f(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly for all $0 \leqslant \tau \leqslant 2\pi$.

Prove the Poisson summation formula

$$f(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{in\tau},$$

where $\hat{F}$ is the Fourier transform of F. [*Hint: You may show that*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-imx} f(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imx} F(x) dx$$

or, alternatively, prove that f is periodic and express its Fourier expansion coefficients explicitly in terms of $\hat{F}$.]

Letting $F(x) = e^{-|x|}$, use the Poisson summation formula to evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \ .$$

2/I/7E **Complex Methods**

A complex function is defined for every $z \in V$, where V is a non-empty open subset of $\mathbb{C}$, and it possesses a derivative at every $z \in V$. Commencing from a formal definition of derivative, deduce the Cauchy–Riemann equations.

2/II/16E **Complex Methods**

Let R be a rational function such that $\lim_{z\to\infty}\{zR(z)\} = 0$. Assuming that R has no real poles, use the residue calculus to evaluate

$$\int_{-\infty}^{\infty} R(x)dx.$$

Given that $n \geqslant 1$ is an integer, evaluate

$$\int_0^{\infty} \frac{dx}{1+x^{2n}}.$$

4/I/8F **Complex Methods**

Consider a conformal mapping of the form

$$f(z) = \frac{a+bz}{c+dz}, \quad z \in \mathbb{C}\ ,$$

where $a, b, c, d \in \mathbb{C}$, and $ad \neq bc$. You may assume $b \neq 0$. Show that any such $f(z)$ which maps the unit circle onto itself is necessarily of the form

$$f(z) = e^{i\psi}\,\frac{a+z}{1+\bar{a}z}.$$

[*Hint: Show that it is always possible to choose* $b = 1$.]

4/II/17F **Complex Methods**

State Jordan's Lemma.

Consider the integral

$$I = \oint_C dz \frac{z\ \sin(xz)}{(a^2+z^2)\sin \pi z},$$

for real x and a. The rectangular contour C runs from $+\infty + i\epsilon$ to $-\infty + i\epsilon$, to $-\infty - i\epsilon$, to $+\infty - i\epsilon$ and back to $+\infty + i\epsilon$, where ϵ is infinitesimal and positive. Perform the integral in two ways to show that

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{n \sin nx}{a^2+n^2} = -\pi \frac{\sinh ax}{\sinh a\pi},$$

for $|x| < \pi$.