Part IB

Complex Analysis

Year

2023
2022
2021
2020
2019
2018
2017
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2013
2012
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2008
2007
2006
2005

Paper 1, Section I

3B Complex Analysis OR Complex Methods

(a) What is the Laurent series of $e^{1/z}$ about $z_0 = 0$?

(b) Let $\rho > 0$. Show that for all large enough $n \in \mathbb{N}$, all zeros of the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n}$$

lie in the open disc $\{z : |z| < \rho\}$.

Paper 1, Section II

12G Complex Analysis OR Complex Methods

(a) Let $f(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$ for $|z-1| < 1$. By differentiating $z\exp(-f(z))$, show that f is an analytic branch of logarithm on the disc $D(1,1)$ with $f(1) = 0$. Use scaling and the function f to show that for every point a in the domain $D = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$, there is an analytic branch of logarithm on a small neighbourhood of a whose imaginary part lies in $(0, 2\pi)$.

(b) For $z \in D$, let $\theta(z)$ be the unique value of the argument of z in the interval $(0, 2\pi)$. Define the function $L\colon D \to \mathbb{C}$ by $L(z) = \log|z| + i\theta(z)$. Briefly explain using part (a) why L is an analytic branch of logarithm on D. For $\alpha \in (-1, 1)$ write down an analytic branch of z^α on D.

(c) State the residue theorem. Evaluate the integral

$$I = \int_0^\infty \frac{x^\alpha}{(x+1)^2}\, dx$$

where $\alpha \in (-1, 1)$.

Paper 2, Section II

12B Complex Analysis OR Complex Methods

(a) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic, and is bounded in the half-plane $\{z : \mathrm{Re}(z) > 0\}$. Prove that, for any real number $c > 0$, there is a positive real constant M such that

$$|f(z_1) - f(z_2)| \leqslant M|z_1 - z_2|$$

whenever $z_1, z_2 \in \mathbb{C}$ satisfy $\mathrm{Re}(z_1) > c$, $\mathrm{Re}(z_2) > c$, and $|z_1 - z_2| < c$.

(b) Let the functions $g, h : \mathbb{C} \to \mathbb{C}$ both be analytic.

(i) State Liouville's Theorem.

(ii) Show that if g is not constant, then $g(\mathbb{C})$ is dense in $\mathbb{C}$.

(iii) Show that if $|h(z)| \leqslant |\mathrm{Re}(z)|^{-1/2}$ for all $z \in \mathbb{C}$, then h is constant.

Paper 4, Section I

3G Complex Analysis

Define what it means for two domains in $\mathbb{C}$ to be *conformally equivalent*.

For each of the following pairs of domains, determine whether they are conformally equivalent. Justify your answers.

(i) $\mathbb{C} \setminus \{0\}$ and $\{z \in \mathbb{C} : 0 < |z| < 1\}$;

(ii) $\mathbb{C}$ and $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$;

(iii) $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0,\ |z| < 1\}$ and $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Paper 3, Section II

13G Complex Analysis

State Rouché's theorem. State the open mapping theorem and prove it using Rouché's theorem. Show that if f is a non-constant holomorphic function on a domain Ω, then $|f|$ has no local maximum on Ω.

Let Ω be a bounded domain in $\mathbb{C}$, and let $\overline{\Omega}$ denote the closure of Ω. Let $f\colon \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Show that if $|f(z)| \leqslant M$ for all $z \in \partial\Omega$, then $|f(z)| \leqslant M$ for all $z \in \Omega$, where $\partial\Omega = \overline{\Omega} \setminus \Omega$ is the boundary of Ω.

Consider the unbounded domain $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$. Let $f\colon \overline{\Omega} \to \mathbb{C}$ be a continuous function that is holomorphic on Ω. Assume that f is bounded both on Ω and on its boundary $\partial\Omega$. Show that if $|f(z)| \leqslant M$ for all $z \in \partial\Omega$, then $|f(z)| \leqslant M$ for all $z \in \Omega$. [*Hint: Consider for large $n \in \mathbb{N}$ and for a large disc $D(0,R)$ the function $z \mapsto \big(f(z)\big)^n/z$ on $D(0,R) \cap \Omega$.*] Is the boundedness assumption of f on Ω necessary? Justify your answer.

Paper 1, Section I
3G Complex Analysis or Complex Methods

Show that $f(z) = \frac{z}{\sin z}$ has a removable singularity at $z = 0$. Find the radius of convergence of the power series of f at the origin.

Paper 1, Section II
12G Complex Analysis or Complex Methods

(a) Let $\Omega \subset \mathbb{C}$ be an open set such that there is $z_0 \in \Omega$ with the property that for any $z \in \Omega$, the line segment $[z_0, z]$ connecting z_0 to z is completely contained in Ω. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma f(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a triangle contained in Ω. Given $w \in \Omega$, let

$$g(w) = \int_{[z_0,w]} f(z)\, dz.$$

Explain briefly why g is a holomorphic function such that $g'(w) = f(w)$ for all $w \in \Omega$.

(b) Fix $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ and let $\mathcal{D} \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ such that the line segment connecting z to z_0 does not pass through the origin. Show that there exists a holomorphic function $h : \mathcal{D} \to \mathbb{C}$ such that $h(z)^2 = z$ for all $z \in \mathcal{D}$. [You may assume that the integral of $1/z$ over the boundary of any triangle contained in $\mathcal{D}$ is zero.]

(c) Show that there exists a holomorphic function f defined in a neighbourhood U of the origin such that $f(z)^2 = \cos z$ for all $z \in U$. Is it possible to find a holomorphic function f defined on the disk $|z| < 2$ such that $f(z)^2 = \cos z$ for all z in the disk? Justify your answer.

Paper 2, Section II
12A Complex Analysis or Complex Methods

(a) Let $R = P/Q$ be a rational function, where deg $Q \geqslant$ deg $P + 2$, and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} R(x) e^{ix}\, dx$$

in terms of residues. Briefly justify your answer.

[You may assume that the polynomials P and Q do not have any common factors.]

(b) Explicitly evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^4}\, dx\,.$$

Paper 4, Section I
3G Complex Analysis

Show that there is no bijective holomorphic map $f : D(0,1) \setminus \{0\} \to A$, where $D(0,1)$ is the disc $\{z \in \mathbb{C} : |z| < 1\}$ and A is the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

[*Hint: Consider an extension of f to the whole disc.*]

Paper 3, Section II
13G Complex Analysis

Let $U \subset \mathbb{C}$ be a (non-empty) connected open set and let f_n be a sequence of holomorphic functions defined on U. Suppose that f_n converges uniformly to a function f on every compact subset of U. Show that f is holomorphic in U. Furthermore, show that f_n' converges uniformly to f' on every compact subset of U.

Suppose in addition that f is not identically zero and that for each n, there is a unique $c_n \in U$ such that $f_n(c_n) = 0$. Show that there is at most one $c \in U$ such that $f(c) = 0$. Find an example such that f has no zeros in U. Give a necessary and sufficient condition on the c_n for this to happen in general.

Paper 1, Section I

3B Complex Analysis or Complex Methods

Let $x > 0$, $x \neq 2$, and let C_x denote the positively oriented circle of radius x centred at the origin. Define

$$g(x) = \oint_{C_x} \frac{z^2 + e^z}{z^2(z-2)}\, dz.$$

Evaluate $g(x)$ for $x \in (0, \infty) \setminus \{2\}$.

Paper 1, Section II

12G Complex Analysis or Complex Methods

(a) State a theorem establishing Laurent series of analytic functions on suitable domains. Give a formula for the n^{th} Laurent coefficient.

Define the notion of *isolated singularity*. State the classification of an isolated singularity in terms of Laurent coefficients.

Compute the Laurent series of

$$f(z) = \frac{1}{z(z-1)}$$

on the annuli $A_1 = \{z : 0 < |z| < 1\}$ and $A_2 = \{z : 1 < |z|\}$. Using this example, comment on the statement that Laurent coefficients are unique. Classify the singularity of f at 0.

(b) Let U be an open subset of the complex plane, let $a \in U$ and let $U' = U \setminus \{a\}$. Assume that f is an analytic function on U' with $|f(z)| \to \infty$ as $z \to a$. By considering the Laurent series of $g(z) = \frac{1}{f(z)}$ at a, classify the singularity of f at a in terms of the Laurent coefficients. [You may assume that a continuous function on U that is analytic on U' is analytic on U.]

Now let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function with $|f(z)| \to \infty$ as $z \to \infty$. By considering Laurent series at 0 of $f(z)$ and of $h(z) = f\left(\frac{1}{z}\right)$, show that f is a polynomial.

(c) Classify, giving reasons, the singularity at the origin of each of the following functions and in each case compute the residue:

$$g(z) = \frac{\exp(z) - 1}{z \log(z+1)} \qquad \text{and} \qquad h(z) = \sin(z)\sin(1/z)\ .$$

Paper 2, Section II
12B Complex Analysis or Complex Methods

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $a > 0$, $b > 0$ be constants. Show that if

$$|f(z)| \leqslant a|z|^{n/2} + b$$

for all $z \in \mathbb{C}$, where n is a positive odd integer, then f must be a polynomial with degree not exceeding $\lfloor n/2 \rfloor$ (closest integer part rounding down).

Does there exist a function f, analytic in $\mathbb{C} \setminus \{0\}$, such that $|f(z)| \geqslant 1/\sqrt{|z|}$ for all nonzero z? Justify your answer.

(b) State Liouville's Theorem and use it to show the following.

(i) If u is a positive harmonic function on $\mathbb{R}^2$, then u is a constant function.

(ii) Let $L = \{z \,|\, z = ax + b, x \in \mathbb{R}\}$ be a line in $\mathbb{C}$ where $a, b \in \mathbb{C}$, $a \neq 0$. If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(\mathbb{C}) \cap L = \emptyset$, then f is a constant function.

Paper 4, Section I
3G Complex Analysis

Let f be a holomorphic function on a neighbourhood of $a \in \mathbb{C}$. Assume that f has a zero of order k at a with $k \geqslant 1$. Show that there exist $\varepsilon > 0$ and $\delta > 0$ such that for any b with $0 < |b| < \varepsilon$ there are exactly k distinct values of $z \in D(a, \delta)$ with $f(z) = b$.

Paper 3, Section II
13G Complex Analysis

Let γ be a curve (not necessarily closed) in $\mathbb{C}$ and let $[\gamma]$ denote the image of γ. Let $\phi\colon [\gamma] \to \mathbb{C}$ be a continuous function and define

$$f(z) = \int_\gamma \frac{\phi(\lambda)}{\lambda - z}\, d\lambda$$

for $z \in \mathbb{C} \setminus [\gamma]$. Show that f has a power series expansion about every $a \notin [\gamma]$.

Using Cauchy's Integral Formula, show that a holomorphic function has complex derivatives of all orders. [Properties of power series may be assumed without proof.] Let f be a holomorphic function on an open set U that contains the closed disc $\overline{D}(a, r)$. Obtain an integral formula for the derivative of f on the open disc $D(a, r)$ in terms of the values of f on the boundary of the disc.

Show that if holomorphic functions f_n on an open set U converge locally uniformly to a holomorphic function f on U, then f_n' converges locally uniformly to f'.

Let D_1 and D_2 be two overlapping closed discs. Let f be a holomorphic function on some open neighbourhood of $D = D_1 \cap D_2$. Show that there exist open neighbourhoods U_j of D_j and holomorphic functions f_j on U_j, $j = 1, 2$, such that $f(z) = f_1(z) + f_2(z)$ on $U_1 \cap U_2$.

Paper 1, Section I

3G Complex Analysis or Complex Methods

Let D be the open disc with centre $e^{2\pi i/6}$ and radius 1, and let L be the open lower half plane. Starting with a suitable Möbius map, find a conformal equivalence (or conformal bijection) of $D \cap L$ onto the open unit disc.

Paper 1, Section II

12G Complex Analysis or Complex Methods

Let $\ell(z)$ be an analytic branch of $\log z$ on a domain $D \subset \mathbb{C} \setminus \{0\}$. Write down an analytic branch of $z^{1/2}$ on D. Show that if $\psi_1(z)$ and $\psi_2(z)$ are two analytic branches of $z^{1/2}$ on D, then either $\psi_1(z) = \psi_2(z)$ for all $z \in D$ or $\psi_1(z) = -\psi_2(z)$ for all $z \in D$.

Describe the principal value or branch $\sigma_1(z)$ of $z^{1/2}$ on $D_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leqslant 0\}$. Describe a branch $\sigma_2(z)$ of $z^{1/2}$ on $D_2 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$.

Construct an analytic branch $\varphi(z)$ of $\sqrt{1-z^2}$ on $\mathbb{C} \setminus \{x \in \mathbb{R} : -1 \leqslant x \leqslant 1\}$ with $\varphi(2i) = \sqrt{5}$. [If you choose to use σ_1 and σ_2 in your construction, then you may assume without proof that they are analytic.]

Show that for $0 < |z| < 1$ we have $\varphi(1/z) = -i\sigma_1(1-z^2)/z$. Hence find the first three terms of the Laurent series of $\varphi(1/z)$ about 0.

Set $f(z) = \varphi(z)/(1+z^2)$ for $|z| > 1$ and $g(z) = f(1/z)/z^2$ for $0 < |z| < 1$. Compute the residue of g at 0 and use it to compute the integral

$$\int_{|z|=2} f(z)\, dz\,.$$

Paper 2, Section II

12B Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{1}{z(z-2)}\,,$$

find the Laurent expansions

(i) about $z = 0$ in the annulus $0 < |z| < 2$,

(ii) about $z = 0$ in the annulus $2 < |z| < \infty$,

(iii) about $z = 1$ in the annulus $0 < |z-1| < 1$.

What is the nature of the singularity of f, if any, at $z = 0$, $z = \infty$ and $z = 1$?

Using an integral of f, or otherwise, evaluate

$$\int_0^{2\pi} \frac{2-\cos\theta}{5-4\cos\theta}\, d\theta\,.$$

Paper 1, Section I

2F Complex Analysis or Complex Methods

What is the *Laurent series* for a function f defined in an annulus A? Find the Laurent series for $f(z) = \frac{10}{(z+2)(z^2+1)}$ on the annuli

$$A_1 = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \quad \text{and}$$
$$A_2 = \{z \in \mathbb{C} \,|\, 1 < |z| < 2\}.$$

Paper 1, Section II

13F Complex Analysis or Complex Methods

State and prove Jordan's lemma.

What is the *residue* of a function f at an isolated singularity a? If $f(z) = \frac{g(z)}{(z-a)^k}$ with k a positive integer, g analytic, and $g(a) \neq 0$, derive a formula for the residue of f at a in terms of derivatives of g.

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2}\, dx\,.$$

Paper 2, Section II

13D Complex Analysis or Complex Methods

Let C_1 and C_2 be smooth curves in the complex plane, intersecting at some point p. Show that if the map $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, then it preserves the angle between C_1 and C_2 at p, provided $f'(p) \neq 0$. Give an example that illustrates why the condition $f'(p) \neq 0$ is important.

Show that $f(z) = z + 1/z$ is a one-to-one conformal map on each of the two regions $|z| > 1$ and $0 < |z| < 1$, and find the image of each region.

Hence construct a one-to-one conformal map from the unit disc to the complex plane with the intervals $(-\infty, -1/2]$ and $[1/2, \infty)$ removed.

Paper 4, Section I

4F Complex Analysis

State the Cauchy Integral Formula for a disc. If $f : D(z_0; r) \to \mathbb{C}$ is a holomorphic function such that $|f(z)| \leqslant |f(z_0)|$ for all $z \in D(z_0; r)$, show using the Cauchy Integral Formula that f is constant.

Paper 3, Section II

13F Complex Analysis

Define the *winding number* $n(\gamma, w)$ of a closed path $\gamma : [a, b] \to \mathbb{C}$ around a point $w \in \mathbb{C}$ which does not lie on the image of γ. [You do not need to justify its existence.]

If f is a meromorphic function, define the *order* of a zero z_0 of f and of a pole w_0 of f. State the Argument Principle, and explain how it can be deduced from the Residue Theorem.

How many roots of the polynomial

$$z^4 + 10z^3 + 4z^2 + 10z + 5$$

lie in the right-hand half plane?

Paper 1, Section I

2A Complex Analysis or Complex Methods

(a) Show that

$$w = \log(z)$$

is a conformal mapping from the right half z-plane, $\mathrm{Re}(z) > 0$, to the strip

$$S = \left\{ w \, : \, -\frac{\pi}{2} < \mathrm{Im}(w) < \frac{\pi}{2} \right\},$$

for a suitably chosen branch of $\log(z)$ that you should specify.

(b) Show that

$$w = \frac{z-1}{z+1}$$

is a conformal mapping from the right half z-plane, $\mathrm{Re}(z) > 0$, to the unit disc

$$D = \{ w \, : \, |w| < 1 \}.$$

(c) Deduce a conformal mapping from the strip S to the disc D.

Paper 1, Section II
13A Complex Analysis or Complex Methods

(a) Let C be a rectangular contour with vertices at $\pm R + \pi\, i$ and $\pm R - \pi\, i$ for some $R > 0$ taken in the anticlockwise direction. By considering

$$\lim_{R\to\infty} \oint_C \frac{e^{iz^2/4\pi}}{e^{z/2} - e^{-z/2}} dz,$$

show that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix^2/4\pi} dx = 2\pi e^{\pi i/4}.$$

(b) By using a semi-circular contour in the upper half plane, calculate

$$\int_0^\infty \frac{x \sin(\pi x)}{x^2 + a^2}\, dx$$

for $a > 0$.

[You may use Jordan's Lemma without proof.]

Paper 2, Section II

13A Complex Analysis or Complex Methods

(a) Let $f(z)$ be a complex function. Define the *Laurent series* of $f(z)$ about $z = z_0$, and give suitable formulae in terms of integrals for calculating the coefficients of the series.

(b) Calculate, by any means, the first 3 terms in the Laurent series about $z = 0$ for

$$f(z) = \frac{1}{e^{2z} - 1}.$$

Indicate the range of values of $|z|$ for which your series is valid.

(c) Let

$$g(z) = \frac{1}{2z} + \sum_{k=1}^{m} \frac{z}{z^2 + \pi^2 k^2}.$$

Classify the singularities of $F(z) = f(z) - g(z)$ for $|z| < (m+1)\pi$.

(d) By considering

$$\oint_{C_R} \frac{F(z)}{z^2} dz$$

where $C_R = \{|z| = R\}$ for some suitably chosen $R > 0$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Paper 4, Section I

4F Complex Analysis

(a) Let $\Omega \subset \mathbb{C}$ be open, $a \in \Omega$ and suppose that $D_\rho(a) = \{z \in \mathbb{C} : |z - a| \leqslant \rho\} \subset \Omega$. Let $f : \Omega \to \mathbb{C}$ be analytic.

State the Cauchy integral formula expressing $f(a)$ as a contour integral over $C = \partial D_\rho(a)$. Give, without proof, a similar expression for $f'(a)$.

If additionally $\Omega = \mathbb{C}$ and f is bounded, deduce that f must be constant.

(b) If $g = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic where u, v are real, and if $u^2(z) - u(z) \geqslant v^2(z)$ for all $z \in \mathbb{C}$, show that g is constant.

Paper 3, Section II

13F Complex Analysis

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : D \to \mathbb{C}$ be analytic.

(a) If there is a point $a \in D$ such that $|f(z)| \leqslant |f(a)|$ for all $z \in D$, prove that f is constant.

(b) If $f(0) = 0$ and $|f(z)| \leqslant 1$ for all $z \in D$, prove that $|f(z)| \leqslant |z|$ for all $z \in D$.

(c) Show that there is a constant C independent of f such that if $f(0) = 1$ and $f(z) \notin (-\infty, 0]$ for all $z \in D$ then $|f(z)| \leqslant C$ whenever $|z| \leqslant 1/2$.

[*Hint: you may find it useful to consider the principal branch of the map* $z \mapsto z^{1/2}$.]

(d) Does the conclusion in (c) hold if we replace the hypothesis $f(z) \notin (-\infty, 0]$ for $z \in D$ with the hypothesis $f(z) \neq 0$ for $z \in D$, and keep all other hypotheses? Justify your answer.

Paper 1, Section I

2A Complex Analysis or Complex Methods

Let $F(z) = u(x, y)+i\,v(x, y)$ where $z = x+i\,y$. Suppose $F(z)$ is an analytic function of z in a domain $\mathcal{D}$ of the complex plane.

Derive the Cauchy-Riemann equations satisfied by u and v.

For $u = \dfrac{x}{x^2 + y^2}$ find a suitable function v and domain $\mathcal{D}$ such that $F = u + i\,v$ is analytic in $\mathcal{D}$.

Paper 2, Section II

13A Complex Analysis or Complex Methods

State the residue theorem.

By considering

$$\oint_C \frac{z^{1/2} \log z}{1 + z^2} dz$$

with C a suitably chosen contour in the upper half plane or otherwise, evaluate the real integrals

$$\int_0^\infty \frac{x^{1/2} \log x}{1 + x^2} dx$$

and

$$\int_0^\infty \frac{x^{1/2}}{1 + x^2} dx$$

where $x^{1/2}$ is taken to be the positive square root.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(a) Let $f(z)$ be defined on the complex plane such that $zf(z) \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic on an open set containing $\mathrm{Im}(z) \geqslant -c$, where c is a positive real constant.

Let C_1 be the horizontal contour running from $-\infty - ic$ to $+\infty - ic$ and let

$$F(\lambda) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-\lambda} dz.$$

By evaluating the integral, show that $F(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(b) Let $g(z)$ be defined on the complex plane such that $z\, g(z) \to 0$ as $|z| \to \infty$ with $\mathrm{Im}(z) \geqslant -c$. Suppose $g(z)$ is analytic at all points except $z = \alpha_+$ and $z = \alpha_-$ which are simple poles with $\mathrm{Im}(\alpha_+) > c$ and $\mathrm{Im}(\alpha_-) < -c$.

Let C_2 be the horizontal contour running from $-\infty + ic$ to $+\infty + ic$, and let

$$H(\lambda) = \frac{1}{2\pi i} \int_{C_1} \frac{g(z)}{z-\lambda} dz,$$
$$J(\lambda) = -\frac{1}{2\pi i} \int_{C_2} \frac{g(z)}{z-\lambda} dz.$$

(i) Show that $H(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(ii) Show that $J(\lambda)$ is analytic for $\mathrm{Im}(\lambda) < c$.

(iii) Show that if $-c < \mathrm{Im}(\lambda) < c$ then $H(\lambda) + J(\lambda) = g(\lambda)$.

[You should be careful to make sure you consider all points in the required regions.]

Paper 4, Section I

4F Complex Analysis

Let D be a star-domain, and let f be a continuous complex-valued function on D. Suppose that for every triangle T contained in D we have

$$\int_{\partial T} f(z)\, dz \;=\; 0\ .$$

Show that f has an antiderivative on D.

If we assume instead that D is a domain (not necessarily a star-domain), does this conclusion still hold? Briefly justify your answer.

Paper 3, Section II

13F Complex Analysis

Let f be an entire function. Prove Taylor's theorem, that there exist complex numbers $c_0, c_1, \ldots$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all z. [You may assume Cauchy's Integral Formula.]

For a positive real r, let $M_r = \sup\{|f(z)| : |z| = r\}$. Explain why we have

$$|c_n| \leqslant \frac{M_r}{r^n}$$

for all n.

Now let n and r be fixed. For which entire functions f do we have $|c_n| = \dfrac{M_r}{r^n}$?

Paper 1, Section I

2A Complex Analysis or Complex Methods

Classify the singularities of the following functions at both $z = 0$ and at the point at infinity on the extended complex plane:

$$\begin{aligned} f_1(z) &= \frac{e^z}{z \sin^2 z}, \\ f_2(z) &= \frac{1}{z^2(1-\cos z)}, \\ f_3(z) &= z^2 \sin(1/z). \end{aligned}$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

Let $a = N + 1/2$ for a positive integer N. Let C_N be the anticlockwise contour defined by the square with its four vertices at $a \pm ia$ and $-a \pm ia$. Let

$$I_N = \oint_{C_N} \frac{dz}{z^2 \sin(\pi z)}.$$

Show that $1/\sin(\pi z)$ is uniformly bounded on the contours C_N as $N \to \infty$, and hence that $I_N \to 0$ as $N \to \infty$.

Using this result, establish that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Paper 1, Section II

13A Complex Analysis or Complex Methods

Let $w = u + iv$ and let $z = x + iy$, for u, v, x, y real.

(a) Let A be the map defined by $w = \sqrt{z}$, using the principal branch. Show that A maps the region to the left of the parabola $y^2 = 4(1 - x)$ on the $z-$plane, with the negative real axis $x \in (-\infty, 0]$ removed, into the vertical strip of the $w-$plane between the lines $u = 0$ and $u = 1$.

(b) Let B be the map defined by $w = \tan^2(z/2)$. Show that B maps the vertical strip of the $z-$plane between the lines $x = 0$ and $x = \pi/2$ into the region inside the unit circle on the $w-$plane, with the part $u \in (-1, 0]$ of the negative real axis removed.

(c) Using the results of parts (a) and (b), show that the map C, defined by $w = \tan^2(\pi\sqrt{z}/4)$, maps the region to the left of the parabola $y^2 = 4(1 - x)$ on the $z-$plane, *including* the negative real axis, onto the unit disc on the $w-$plane.

Paper 4, Section I

4G Complex Analysis

State carefully Rouché's theorem. Use it to show that the function $z^4 + 3 + e^{iz}$ has exactly one zero $z = z_0$ in the quadrant

$$\{z \in \mathbb{C} \mid 0 < \arg(z) < \pi/2\},$$

and that $|z_0| \leqslant \sqrt{2}$.

Paper 3, Section II

13G Complex Analysis

(a) Prove Cauchy's theorem for a triangle.

(b) Write down an expression for the winding number $I(\gamma, a)$ of a closed, piecewise continuously differentiable curve γ about a point $a \in \mathbb{C}$ which does not lie on γ.

(c) Let $U \subset \mathbb{C}$ be a domain, and $f \colon U \to \mathbb{C}$ a holomorphic function with no zeroes in U. Suppose that for infinitely many positive integers k the function f has a holomorphic k-th root. Show that there exists a holomorphic function $F \colon U \to \mathbb{C}$ such that $f = \exp F$.

Paper 1, Section I

2B Complex Analysis or Complex Methods

Consider the analytic (holomorphic) functions f and g on a nonempty domain Ω where g is nowhere zero. Prove that if $|f(z)| = |g(z)|$ for all z in Ω then there exists a real constant α such that $f(z) = e^{i\alpha} g(z)$ for all z in Ω.

Paper 2, Section II

13B Complex Analysis or Complex Methods

(i) A function $f(z)$ has a pole of order m at $z = z_0$. Derive a general expression for the residue of $f(z)$ at $z = z_0$ involving f and its derivatives.

(ii) Using contour integration along a contour in the upper half-plane, determine the value of the integral

$$I = \int_0^\infty \frac{(\ln x)^2}{(1+x^2)^2} \mathrm{d}x.$$

Paper 1, Section II

13B Complex Analysis or Complex Methods

(i) Show that transformations of the complex plane of the form

$$\zeta = \frac{az+b}{cz+d},$$

always map circles and lines to circles and lines, where a, b, c and d are complex numbers such that $ad - bc \neq 0$.

(ii) Show that the transformation

$$\zeta = \frac{z-\alpha}{\bar{\alpha} z - 1}, \quad |\alpha| < 1,$$

maps the unit disk centered at $z = 0$ onto itself.

(iii) Deduce a conformal transformation that maps the non-concentric annular domain $\Omega = \{|z| < 1,\ |z - c| > c\}$, $0 < c < 1/2$, to a concentric annular domain.

Paper 4, Section I

4G Complex Analysis

Let f be a continuous function defined on a connected open set $D \subset \mathbb{C}$. Prove carefully that the following statements are equivalent.

(i) There exists a holomorphic function F on D such that $F'(z) = f(z)$.

(ii) $\int_\gamma f(z)dz = 0$ holds for every closed curve γ in D.

Paper 3, Section II

13G Complex Analysis

State the argument principle.

Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic injective function. Show that $f'(z) \neq 0$ for each z in U and that $f(U)$ is open.

Stating clearly any theorems that you require, show that for each $a \in U$ and a sufficiently small $r > 0$,

$$g(w) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{zf'(z)}{f(z) - w} dz$$

defines a holomorphic function on some open disc D about $f(a)$.

Show that g is the inverse for the restriction of f to $g(D)$.

Paper 1, Section I

2B Complex Analysis or Complex Methods

Let $f(z)$ be an analytic/holomorphic function defined on an open set D, and let $z_0 \in D$ be a point such that $f'(z_0) \neq 0$. Show that the transformation $w = f(z)$ preserves the angle between smooth curves intersecting at z_0. Find such a transformation $w = f(z)$ that maps the second quadrant of the unit disc (i.e. $|z| < 1$, $\pi/2 < \arg(z) < \pi$) to the region in the first quadrant of the complex plane where $|w| > 1$ (i.e. the region in the first quadrant *outside* the unit circle).

Paper 1, Section II

13B Complex Analysis or Complex Methods

By choice of a suitable contour show that for $a > b > 0$

$$\int_0^{2\pi} \frac{\sin^2 \theta d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right].$$

Hence evaluate

$$\int_0^1 \frac{(1-x^2)^{1/2} x^2 dx}{1+x^2}$$

using the substitution $x = \cos(\theta/2)$.

Paper 2, Section II

13B Complex Analysis or Complex Methods

By considering a rectangular contour, show that for $0 < a < 1$ we have

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = \frac{\pi}{\sin \pi a}.$$

Hence evaluate

$$\int_0^{\infty} \frac{dt}{t^{5/6}(1+t)}.$$

Paper 4, Section I

4G Complex Analysis

Let f be an entire function. State Cauchy's Integral Formula, relating the nth derivative of f at a point z with the values of f on a circle around z.

State Liouville's Theorem, and deduce it from Cauchy's Integral Formula.

Let f be an entire function, and suppose that for some k we have that $|f(z)| \leqslant |z|^k$ for all z. Prove that f is a polynomial.

Paper 3, Section II

13G Complex Analysis

State the Residue Theorem precisely.

Let D be a star-domain, and let γ be a closed path in D. Suppose that f is a holomorphic function on D, having no zeros on γ. Let N be the number of zeros of f inside γ, counted with multiplicity (i.e. order of zero and winding number). Show that

$$N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz \ .$$

[The Residue Theorem may be used without proof.]

Now suppose that g is another holomorphic function on D, also having no zeros on γ and with $|g(z)| < |f(z)|$ on γ. Explain why, for any $0 \leqslant t \leqslant 1$, the expression

$$I(t) = \int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\, dz$$

is well-defined. By considering the behaviour of the function $I(t)$ as t varies, deduce Rouché's Theorem.

For each n, let p_n be the polynomial $\sum_{k=0}^{n} \frac{z^k}{k!}$. Show that, as n tends to infinity, the smallest modulus of the roots of p_n also tends to infinity.

[You may assume any results on convergence of power series, provided that they are stated clearly.]

Paper 1, Section I

2D Complex Analysis or Complex Methods

Classify the singularities (in the finite complex plane) of the following functions:

(i) $\dfrac{1}{(\cosh z)^2}$;

(ii) $\dfrac{1}{\cos(1/z)}$;

(iii) $\dfrac{1}{\log z} \quad (-\pi < \arg z < \pi)$;

(iv) $\dfrac{z^{\frac{1}{2}} - 1}{\sin \pi z} \quad (-\pi < \arg z < \pi)$.

Paper 1, Section II

13E Complex Analysis or Complex Methods

Suppose $p(z)$ is a polynomial of even degree, all of whose roots satisfy $|z| < R$. Explain why there is a holomorphic (*i.e.* analytic) function $h(z)$ defined on the region $R < |z| < \infty$ which satisfies $h(z)^2 = p(z)$. We write $h(z) = \sqrt{p(z)}$.

By expanding in a Laurent series or otherwise, evaluate

$$\int_C \sqrt{z^4 - z}\, dz$$

where C is the circle of radius 2 with the anticlockwise orientation. (Your answer will be well-defined up to a factor of ± 1, depending on which square root you pick.)

Paper 2, Section II

13D Complex Analysis or Complex Methods

Let

$$I = \oint_C \frac{e^{iz^2/\pi}}{1+e^{-2z}} dz\,,$$

where C is the rectangle with vertices at $\pm R$ and $\pm\ R + i\pi$, traversed anti-clockwise.

(i) Show that $I = \dfrac{\pi(1+i)}{\sqrt{2}}$.

(ii) Assuming that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$, show that

$$\int_{-\infty}^{\infty} e^{ix^2/\pi} dx = \frac{\pi(1+i)}{\sqrt{2}}\,.$$

(iii) Justify briefly the assumption that the contribution to I from the vertical sides of the rectangle is negligible in the limit $R \to \infty$.

Paper 4, Section I

4E Complex Analysis

State Rouché's theorem. How many roots of the polynomial $z^8 + 3z^7 + 6z^2 + 1$ are contained in the annulus $1 < |z| < 2$?

Paper 3, Section II

13E Complex Analysis

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk, and let C be its boundary (the unit circle), with the anticlockwise orientation. Suppose $\phi : C \to \mathbb{C}$ is continuous. Stating clearly any theorems you use, show that

$$g_\phi(w) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{z - w} dz$$

is an analytic function of w for $w \in D$.

Now suppose ϕ is the restriction of a holomorphic function F defined on some annulus $1 - \epsilon < |z| < 1 + \epsilon$. Show that $g_\phi(w)$ is the restriction of a holomorphic function defined on the open disc $|w| < 1 + \epsilon$.

Let $f_\phi : [0, 2\pi] \to \mathbb{C}$ be defined by $f_\phi(\theta) = \phi(e^{i\theta})$. Express the coefficients in the power series expansion of g_ϕ centered at 0 in terms of f_ϕ.

Let $n \in \mathbb{Z}$. What is g_ϕ in the following cases?

1. $\phi(z) = z^n$.
2. $\phi(z) = \overline{z}^n$.
3. $\phi(z) = (\text{Re } z)^2$.

Paper 1, Section I

2A Complex Analysis or Complex Methods

Find a conformal transformation $\zeta = \zeta(z)$ that maps the domain D, $0 < \arg z < \frac{3\pi}{2}$, on to the strip $0 < \mathrm{Im}(\zeta) < 1$.

Hence find a bounded harmonic function ϕ on D subject to the boundary conditions $\phi = 0, A$ on $\arg z = 0, \frac{3\pi}{2}$, respectively, where A is a real constant.

Paper 2, Section II

13A Complex Analysis or Complex Methods

By a suitable choice of contour show that, for $-1 < \alpha < 1$,

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,\mathrm{d}x \quad = \quad \frac{\pi}{2\cos(\alpha\pi/2)}\,.$$

Paper 1, Section II

13A Complex Analysis or Complex Methods

Using Cauchy's integral theorem, write down the value of a holomorphic function $f(z)$ where $|z| < 1$ in terms of a contour integral around the unit circle, $\zeta = e^{i\theta}$.

By considering the point $1/\overline{z}$, or otherwise, show that

$$f(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1-|z|^2}{|\zeta - z|^2}\,\mathrm{d}\theta\,.$$

By setting $z = re^{i\alpha}$, show that for any harmonic function $u(r,\alpha)$,

$$u(r,\alpha) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\frac{1-r^2}{1-2r\cos(\alpha-\theta)+r^2}\,\mathrm{d}\theta$$

if $r < 1$.

Assuming that the function $v(r,\alpha)$, which is the conjugate harmonic function to $u(r,\alpha)$, can be written as

$$v(r,\alpha) \quad = \quad v(0) + \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\frac{r\sin(\alpha-\theta)}{1-2r\cos(\alpha-\theta)+r^2}\,\mathrm{d}\theta\,,$$

deduce that

$$f(z) \quad = \quad iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\frac{\zeta+z}{\zeta-z}\,\mathrm{d}\theta\,.$$

[You may use the fact that on the unit circle, $\zeta = 1/\overline{\zeta}$, and hence

$$\frac{\zeta}{\zeta - 1/\overline{z}} = -\frac{\overline{z}}{\overline{\zeta}-\overline{z}}\,. \quad]$$

Paper 4, Section I

4E Complex Analysis

Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $h(i) \neq h(-i)$. Does there exist a holomorphic function f defined in $|z| < 1$ for which $f'(z) = \dfrac{h(z)}{1+z^2}$? Does there exist a holomorphic function f defined in $|z| > 1$ for which $f'(z) = \dfrac{h(z)}{1+z^2}$? Justify your answers.

Paper 3, Section II

13E Complex Analysis

Let $D(a, R)$ denote the disc $|z - a| < R$ and let $f : D(a, R) \to \mathbb{C}$ be a holomorphic function. Using Cauchy's integral formula show that for every $r \in (0, R)$

$$f(a) = \int_0^1 f(a + re^{2\pi i t})\, dt\,.$$

Deduce that if for every $z \in D(a, R)$, $|f(z)| \leqslant |f(a)|$, then f is constant.

Let $f : D(0, 1) \to D(0, 1)$ be holomorphic with $f(0) = 0$. Show that $|f(z)| \leqslant |z|$ for all $z \in D(0, 1)$. Moreover, show that if $|f(w)| = |w|$ for some $w \neq 0$, then there exists λ with $|\lambda| = 1$ such that $f(z) = \lambda z$ for all $z \in D(0, 1)$.

Paper 1, Section I

2A Complex Analysis or Complex Methods

Derive the Cauchy-Riemann equations satisfied by the real and imaginary parts of a complex analytic function $f(z)$.

If $|f(z)|$ is constant on $|z| < 1$, prove that $f(z)$ is constant on $|z| < 1$.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(i) Let $-1 < \alpha < 0$ and let

$$\begin{aligned} f(z) &= \frac{\log(z-\alpha)}{z} \text{ where } -\pi \leqslant \arg(z-\alpha) < \pi\,, \\ g(z) &= \frac{\log z}{z} \quad\quad \text{ where } -\pi \leqslant \arg(z) < \pi\,. \end{aligned}$$

Here the logarithms take their principal values. Give a sketch to indicate the positions of the branch cuts implied by the definitions of $f(z)$ and $g(z)$.

(ii) Let $h(z) = f(z) - g(z)$. Explain why $h(z)$ is analytic in the annulus $1 \leqslant |z| \leqslant R$ for any $R > 1$. Obtain the first three terms of the Laurent expansion for $h(z)$ around $z = 0$ in this annulus and hence evaluate

$$\oint_{|z|=2} h(z) dz\,.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(i) Let C be an anticlockwise contour defined by a square with vertices at $z = x+iy$ where

$$|x| = |y| = \left(2N + \frac{1}{2}\right)\pi,$$

for large integer N. Let

$$I = \oint_C \frac{\pi \cot z}{(z+\pi a)^4} dz.$$

Assuming that $I \to 0$ as $N \to \infty$, prove that, if a is not an integer, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^4} = \frac{\pi^4}{3\sin^2(\pi a)}\left(\frac{3}{\sin^2(\pi a)} - 2\right).$$

(ii) Deduce the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^4}.$$

(iii) Briefly justify the assumption that $I \to 0$ as $N \to \infty$.

[*Hint: For part (iii) it is sufficient to consider, at most, one vertical side of the square and one horizontal side and to use a symmetry argument for the remaining sides.*]

Paper 4, Section I

4E Complex Analysis

Let $f(z)$ be an analytic function in an open subset U of the complex plane. Prove that f has derivatives of all orders at any point z in U. [You may assume Cauchy's integral formula provided it is clearly stated.]

Paper 3, Section II

13E Complex Analysis

Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma g(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a rectangle in $\mathbb{C}$ with sides parallel to the real and imaginary axes. Prove that g is analytic.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous. Suppose in addition that f is analytic at every point $z \in \mathbb{C}$ with non-zero imaginary part. Show that f is analytic at every point in $\mathbb{C}$.

Let $\mathbb{H}$ be the upper half-plane of complex numbers z with positive imaginary part $\Im(z) > 0$. Consider a continuous function $F : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ such that F is analytic on $\mathbb{H}$ and $F(\mathbb{R}) \subset \mathbb{R}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} F(z) & \text{if } \Im(z) \geqslant 0 \\ \overline{F(\overline{z})} & \text{if } \Im(z) \leqslant 0. \end{cases}$$

Show that f is analytic.

Paper 1, Section I

2A Complex Analysis or Complex Methods

(a) Write down the definition of the complex derivative of the function $f(z)$ of a single complex variable.

(b) Derive the Cauchy-Riemann equations for the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $f(z)$, where $z = x + iy$ and

$$f(z) = u(x, y) + iv(x, y).$$

(c) State necessary and sufficient conditions on $u(x, y)$ and $v(x, y)$ for the function $f(z)$ to be complex differentiable.

Paper 1, Section II

13A Complex Analysis or Complex Methods

Calculate the following real integrals by using contour integration. Justify your steps carefully.

(a)

$$I_1 = \int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx, \qquad a > 0,$$

(b)

$$I_2 = \int_0^\infty \frac{x^{1/2} \log x}{1 + x^2}\, dx.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(a) Prove that a complex differentiable map, $f(z)$, is conformal, i.e. preserves angles, provided a certain condition holds on the first complex derivative of $f(z)$.

(b) Let D be the region

$$D := \{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z-2| < 2\}.$$

Draw the region D. It might help to consider the two sets

$$C(1) := \{z \in \mathbb{C} : |z-1| = 1\},$$
$$C(2) := \{z \in \mathbb{C} : |z-2| = 2\}.$$

(c) For the transformations below identify the images of D.

Step 1: The first map is $f_1(z) = \dfrac{z-1}{z}$,

Step 2: The second map is the composite $f_2 f_1$ where $f_2(z) = (z - \frac{1}{2})i$,

Step 3: The third map is the composite $f_3 f_2 f_1$ where $f_3(z) = e^{2\pi z}$.

(d) Write down the inverse map to the composite $f_3 f_2 f_1$, explaining any choices of branch.

[*The composite* $f_2 f_1$ *means* $f_2(f_1(z))$.]

Paper 4, Section I

4G Complex Analysis

State the principle of the argument for meromorphic functions and show how it follows from the Residue theorem.

Paper 3, Section II

13G Complex Analysis

State Morera's theorem. Suppose f_n $(n = 1, 2, \dots)$ are analytic functions on a domain $U \subset \mathbf{C}$ and that f_n tends locally uniformly to f on U. Show that f is analytic on U. Explain briefly why the derivatives f_n' tend locally uniformly to f'.

Suppose now that the f_n are nowhere vanishing and f is not identically zero. Let a be any point of U; show that there exists a closed disc $\overline{\Delta} \subset U$ with centre a, on which the convergence of f_n and f_n' are both uniform, and where f is nowhere zero on $\overline{\Delta} \setminus \{a\}$. By considering

$$\frac{1}{2\pi i} \int_C \frac{f_n'(w)}{f_n(w)} \, dw$$

(where C denotes the boundary of $\overline{\Delta}$), or otherwise, deduce that $f(a) \neq 0$.

Paper 1, Section I
3D Complex Analysis or Complex Methods

Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, be an analytic function of z in a domain D of the complex plane. Derive the Cauchy–Riemann equations relating the partial derivatives of u and v.

For $u = e^{-x}\cos y$, find v and hence $f(z)$.

Paper 1, Section II
13D Complex Analysis or Complex Methods

Consider the real function $f(t)$ of a real variable t defined by the following contour integral in the complex s-plane:

$$f(t) = \frac{1}{2\pi i}\int_\Gamma \frac{e^{st}}{(s^2+1)s^{1/2}}\,ds,$$

where the contour Γ is the line $s = \gamma + iy, -\infty < y < \infty$, for constant $\gamma > 0$. By closing the contour appropriately, show that

$$f(t) = \sin(t - \pi/4) + \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}dr}{(r^2+1)r^{1/2}}$$

when $t > 0$ and is zero when $t < 0$. You should justify your evaluation of the inversion integral over all parts of the contour.

By expanding $(r^2+1)^{-1}\ r^{-1/2}$ as a power series in r, and assuming that you may integrate the series term by term, show that the two leading terms, as $t \to \infty$, are

$$f(t) \sim \sin(t - \pi/4) + \frac{1}{(\pi t)^{1/2}} + \cdots .$$

[*You may assume that* $\int_0^\infty x^{-1/2}e^{-x}dx = \pi^{1/2}$.]

Paper 2, Section II

14D Complex Analysis or Complex Methods

Show that both the following transformations from the z-plane to the ζ-plane are conformal, except at certain critical points which should be identified in both planes, and in each case find a domain in the z-plane that is mapped onto the upper half ζ-plane:

$$\begin{aligned}\text{(i) } \zeta &= z + \frac{b^2}{z}\,; \\ \text{(ii) } \zeta &= \cosh\frac{\pi z}{b}\,,\end{aligned}$$

where b is real and positive.

Paper 4, Section I

4E Complex Analysis

State Rouché's Theorem. How many complex numbers z are there with $|z| \leqslant 1$ and $2z = \sin z$?

Paper 3, Section II

14E Complex Analysis

For each positive real number R write $B_R = \{z \in \mathbb{C} : |z| \leqslant R\}$. If F is holomorphic on some open set containing B_R, we define

$$J(F,R) = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\, d\theta.$$

1. If F_1, F_2 are both holomorphic on some open set containing B_R, show that $J(F_1F_2, R) = J(F_1, R) + J(F_2, R)$.

2. Suppose that $F(0) = 1$ and that F does not vanish on some open set containing B_R. By showing that there is a holomorphic branch of logarithm of F and then considering $z^{-1}\log F(z)$, prove that $J(F,R) = 0$.

3. Suppose that $|w| < R$. Prove that the function $\psi_{W,R}(z) = R(z-w)/(R^2 - \overline{w}z)$ has modulus 1 on $|z| = R$ and hence that it satisfies $J(\psi_{W,R}, R) = 0$.

Suppose now that $F : \mathbb{C} \to \mathbb{C}$ is holomorphic and not identically zero, and let R be such that no zeros of F satisfy $|z| = R$. Briefly explain why there are only finitely many zeros of F in B_R and, assuming these are listed with the correct multiplicity, derive a formula for $J(F,R)$ in terms of the zeros, R, and $F(0)$.

Suppose that F has a zero at every lattice point (point with integer coordinates) except for $(0,0)$. Show that there is a constant $c > 0$ such that $|F(z_i)| > e^{c|z_i|^2}$ for a sequence $z_1, z_2, \ldots$ of complex numbers tending to infinity.

1/I/3C Complex Analysis or Complex Methods

Given that $f(z)$ is an analytic function, show that the mapping $w = f(z)$

(a) preserves angles between smooth curves intersecting at z if $f'(z) \neq 0$;

(b) has Jacobian given by $|f'(z)|^2$.

1/II/13C Complex Analysis or Complex Methods

By a suitable choice of contour show the following:

(a)

$$\int_0^\infty \frac{x^{1/n}}{1+x^2}\, dx \;=\; \frac{\pi}{2\cos(\pi/2n)}\,,$$

where $n > 1$,

(b)

$$\int_0^\infty \frac{x^{1/2}\log x}{1+x^2}\, dx \;=\; \frac{\pi^2}{2\sqrt{2}}\,.$$

2/II/14C Complex Analysis or Complex Methods

Let $f(z) = 1/(e^z - 1)$. Find the first three terms in the Laurent expansion for $f(z)$ valid for $0 < |z| < 2\pi$.

Now let n be a positive integer, and define

$$f_1(z) = \frac{1}{z} + \sum_{r=1}^{n} \frac{2z}{z^2 + 4\pi^2 r^2}\,,$$
$$f_2(z) = f(z) - f_1(z)\,.$$

Show that the singularities of f_2 in $\{z : |z| < 2(n+1)\pi\}$ are all removable. By expanding f_1 as a Laurent series valid for $|z| > 2n\pi$, and f_2 as a Taylor series valid for $|z| < 2(n+1)\pi$, find the coefficients of z^j for $-1 \leq j \leq 1$ in the Laurent series for f valid for $2n\pi < |z| < 2(n+1)\pi$.

By estimating an appropriate integral around the contour $|z| = (2n+1)\pi$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}\,.$$

3/II/14E Complex Analysis

State and prove Rouché's theorem, and use it to count the number of zeros of $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ inside the annulus $\{z : 1 < |z| < 2\}$.

Let $(p_n)_{n=1}^{\infty}$ be a sequence of polynomials of degree at most d with the property that $p_n(z)$ converges uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Prove that there is a polynomial p of degree at most d such that $p_n \to p$ uniformly on compact subsets of $\mathbb{C}$. [*If you use any results about uniform convergence of analytic functions, you should prove them.*]

Suppose that p has d distinct roots $z_1, \ldots, z_d$. Using Rouché's theorem, or otherwise, show that for each i there is a sequence $(z_{i,n})_{n=1}^{\infty}$ such that $p_n(z_{i,n}) = 0$ and $z_{i,n} \to z_i$ as $n \to \infty$.

4/I/4E Complex Analysis

Suppose that f and g are two functions which are analytic on the whole complex plane $\mathbb{C}$. Suppose that there is a sequence of distinct points $z_1, z_2, \ldots$ with $|z_i| \leqslant 1$ such that $f(z_i) = g(z_i)$. Show that $f(z) = g(z)$ for all $z \in \mathbb{C}$. [*You may assume any results on Taylor expansions you need, provided they are clearly stated.*]

What happens if the assumption that $|z_i| \leqslant 1$ is dropped?

1/I/3F Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{2z}{z^2+1}\,,$$

determine the Taylor series of f around the point $z_0 = 1$, and give the largest r for which this series converges in the disc $|z-1| < r$.

1/II/13F Complex Analysis or Complex Methods

By integrating round the contour C_R, which is the boundary of the domain

$$D_R = \{z = re^{i\theta} : 0 < r < R, \quad 0 < \theta < \frac{\pi}{4}\}\,,$$

evaluate each of the integrals

$$\int_0^\infty \sin x^2 dx, \quad \int_0^\infty \cos x^2 dx\,.$$

[*You may use the relations* $\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$ *and* $\sin t \geq \frac{2}{\pi} t$ *for* $0 \leq t \leq \frac{\pi}{2}$ *.*]

2/II/14F Complex Analysis or Complex Methods

Let Ω be the half-strip in the complex plane,

$$\Omega = \{z = x + iy \in \mathbb{C} \;:\; -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0\}\,.$$

Find a conformal mapping that maps Ω onto the unit disc.

3/II/14H Complex Analysis

Say that a function on the complex plane $\mathbb{C}$ is *periodic* if $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z. If f is a periodic analytic function, show that f is constant.

If f is a meromorphic periodic function, show that the number of zeros of f in the square $[0,1) \times [0,1)$ is equal to the number of poles, both counted with multiplicities.

Define

$$f(z) = \frac{1}{z^2} + \sum_w \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right],$$

where the sum runs over all $w = a + bi$ with a and b integers, not both 0. Show that this series converges to a meromorphic periodic function on the complex plane.

4/I/4H Complex Analysis

State the argument principle.

Show that if f is an analytic function on an open set $U \subset \mathbb{C}$ which is one-to-one, then $f'(z) \neq 0$ for all $z \in U$.

1/I/3D Complex Analysis or Complex Methods

Let L be the Laplace operator, i.e., $L(g) = g_{xx} + g_{yy}$. Prove that if $f : \Omega \to \mathbf{C}$ is analytic in a domain Ω, then

$$L\big(|f(z)|^2\big) = 4|f'(z)|^2\,, \quad z \in \Omega.$$

1/II/13D Complex Analysis or Complex Methods

By integrating round the contour involving the real axis and the line $\mathrm{Im}(z) = 2\pi$, or otherwise, evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx\,, \quad 0 < a < 1.$$

Explain why the given restriction on the value a is necessary.

2/II/14D Complex Analysis or Complex Methods

Let Ω be the region enclosed between the two circles C_1 and C_2, where

$$C_1 = \{z \in \mathbf{C} : |z - i| = 1\}\,, \qquad C_2 = \{z \in \mathbf{C} : |z - 2i| = 2\}.$$

Find a conformal mapping that maps Ω onto the unit disc.

[*Hint: you may find it helpful first to map Ω to a strip in the complex plane.*]

3/II/14H **Complex Analysis**

Assuming the principle of the argument, prove that any polynomial of degree n has precisely n zeros in $\mathbf{C}$, counted with multiplicity.

Consider a polynomial $p(z) = z^4 + az^3 + bz^2 + cz + d$, and let R be a positive real number such that $|a|R^3 + |b|R^2 + |c|R + |d| < R^4$. Define a curve $\Gamma : [0,1] \to \mathbf{C}$ by

$$\Gamma(t) = \begin{cases} p(Re^{\pi i t}) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \ , \\ (2-2t)p(iR) + (2t-1)p(R) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \ . \end{cases}$$

Show that the winding number $n(\Gamma, 0) = 1$.

Suppose now that $p(z)$ has real coefficients, that $z^4 - bz^2 + d$ has no real zeros, and that the real zeros of $p(z)$ are all strictly negative. Show that precisely one of the zeros of $p(z)$ lies in the quadrant $\{x + iy : x > 0, y > 0\}$.

[*Standard results about winding numbers may be quoted without proof; in particular, you may wish to use the fact that if* $\gamma_i : [0,1] \to \mathbf{C}$*,* $i = 1, 2$*, are two closed curves with* $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t)|$ *for all* t*, then* $n(\gamma_1, 0) = n(\gamma_2, 0)$*.*]

4/I/4H **Complex Analysis**

State the principle of isolated zeros for an analytic function on a domain in $\mathbf{C}$.

Suppose f is an analytic function on $\mathbf{C} \setminus \{0\}$, which is real-valued at the points $1/n$, for $n = 1, 2, \ldots$, and does not have an essential singularity at the origin. Prove that $f(z) = \overline{f(\bar{z})}$ for all $z \in \mathbf{C} \setminus \{0\}$.

1/I/3F Complex Analysis or Complex Methods

State the Cauchy integral formula.

Using the Cauchy integral formula, evaluate

$$\int_{|z|=2} \frac{z^3}{z^2+1} dz.$$

1/II/13F Complex Analysis or Complex Methods

Determine a conformal mapping from $\Omega_0 = \mathbf{C} \setminus [-1,1]$ to the complex unit disc $\Omega_1 = \{z \in \mathbf{C} : |z| < 1\}$.

[*Hint: A standard method is first to map* Ω_0 *to* $\mathbf{C} \setminus (-\infty, 0]$, *then to the complex right half-plane* $\{z \in \mathbf{C} : \operatorname{Re} z > 0\}$ *and, finally, to* Ω_1.]

2/II/14F Complex Analysis or Complex Methods

Let $F = P/Q$ be a rational function, where $\deg Q \geqslant \deg P + 2$ and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} F(x) e^{ix} dx$$

in terms of residues and briefly sketch its proof.

Evaluate explicitly the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{4 + x^4} dx\,.$$

3/II/14A Complex Analysis

State the Cauchy integral formula, and use it to deduce Liouville's theorem.

Let f be a meromorphic function on the complex plane such that $|f(z)/z^n|$ is bounded outside some disc (for some fixed integer n). By considering Laurent expansions, or otherwise, show that f is a rational function in z.

4/I/4A Complex Analysis

Let $\gamma : [0,1] \to \mathbf{C}$ be a closed path, where all paths are assumed to be piecewise continuously differentiable, and let a be a complex number not in the image of γ. Write down an expression for the winding number $n(\gamma, a)$ in terms of a contour integral. From this characterization of the winding number, prove the following properties:

(a) If γ_1 and γ_2 are closed paths not passing through zero, and if $\gamma : [0,1] \to \mathbf{C}$ is defined by $\gamma(t) = \gamma_1(t)\gamma_2(t)$ for all t, then

$$n(\gamma, 0) = n(\gamma_1, 0) + n(\gamma_2, 0).$$

(b) If $\eta : [0,1] \to \mathbf{C}$ is a closed path whose image is contained in $\{\mathrm{Re}(z) > 0\}$, then $n(\eta, 0) = 0$.

(c) If γ_1 and γ_2 are closed paths and a is a complex number, not in the image of either path, such that

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t) - a|$$

for all t, then $n(\gamma_1, a) = n(\gamma_2, a)$.

[*You may wish here to consider the path defined by* $\eta(t) = 1 - (\gamma_1(t) - \gamma_2(t))/(\gamma_1(t) - a)$.]