**Part IB**

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# Analysis and Topology

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**Year**

**Paper 2, Section I**

**2G Analysis and Topology**

Show that a topological space $X$ is connected if and only if every continuous integer-valued function on $X$ is constant.

Let $\mathcal{A}$ be a family of connected subsets of a topological space $X$ such that $\bigcup_{A\in\mathcal{A}} A = X$. Assume that $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. Prove that $X$ is connected.

Deduce, or otherwise show, that if $X$ and $Y$ are connected topological spaces, then $X \times Y$ is also connected in the product topology.

**Paper 4, Section I**

**2G Analysis and Topology**

Let $(f_n)$ be a sequence of continuous real-valued functions on a topological space $X$. Assume that there is a function $f\colon X \to \mathbb{R}$ such that every $x \in X$ has a neighbourhood $U$ on which $(f_n)$ converges to $f$ uniformly. Show that $f$ is continuous at every $x \in X$. Further show that $(f_n)$ converges to $f$ uniformly on every compact subset of $X$.

**Paper 1, Section II**

**10G Analysis and Topology**

Define the terms *Cauchy sequence* and *complete metric space*. Prove that every Cauchy sequence in a metric space is bounded.

Show that a metric space $(M, d)$ is complete if and only if given any sequence $(F_n)$ of non-empty, closed subsets of $M$ satisfying

- $F_n \supset F_{n+1}$ for all $n \in \mathbb{N}$ and
- $\operatorname{diam} F_n = \sup\{d(x,y) : x, y \in F_n\} \to 0$ as $n \to \infty$,

the intersection $\bigcap_{n\in\mathbb{N}} F_n$ is non-empty.

State the contraction mapping theorem.

Let $(\Lambda, \rho)$ and $(M, d)$ be non-empty metric spaces, and assume that $(M, d)$ is complete. Let $T\colon \Lambda \times M \to M$ be a function with the following properties:

- there exists $0 \leqslant k < 1$ such that $d(T(\lambda, x), T(\lambda, y)) \leqslant k d(x, y)$ for all $\lambda \in \Lambda$ and all $x, y \in M$;
- for each $x \in M$, the function $\Lambda \to M$, given by $\lambda \mapsto T(\lambda, x)$, is continuous.

Show that there is a unique function $x^*\colon \Lambda \to M$ such that $T(\lambda, x^*(\lambda)) = x^*(\lambda)$ for all $\lambda \in \Lambda$. Show further that the function $x^*$ is continuous.

**Paper 2, Section II**

**10G Analysis and Topology**

Define the notion of *uniform convergence* for a sequence $(f_n)$ of real-valued functions on an arbitrary set $S$ and the notion of *uniform continuity* for a function $h\colon M \to N$ between metric spaces.

Let $C_0(\mathbb{R}^d)$ denote the set of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying $f(x) \to 0$ as $\|x\| \to \infty$, i.e. for all $\varepsilon > 0$ there exists $K > 0$ such that $|f(x)| < \varepsilon$ whenever $\|x\| > K$ (where $\|x\|$ denotes the usual Euclidean length of $x$). Briefly explain why every function in $C_0(\mathbb{R}^d)$ is bounded. Prove that $C_0(\mathbb{R}^d)$ is a complete metric space in the uniform metric. Is it true that every member of $C_0(\mathbb{R}^d)$ is uniformly continuous? Give a proof or counterexample.

Let $\varepsilon\colon \mathbb{R} \to [0,\infty)$ be a continuous function with $\varepsilon(0) = 0$. For $n \in \mathbb{N}$ define $f_n\colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \varepsilon(x/n)}$. Must $(f_n)$ converge pointwise? Must $(f_n)$ converge uniformly? Do your answers change if we further assume that for some $M \geqslant 0$ and for all $t \in \mathbb{R}$ we have $\varepsilon(t) \leqslant M|t|$? Justify your answers.

**Paper 3, Section II**

**11G Analysis and Topology**

Let $f\colon U \to \mathbb{R}^n$ be a function where $U$ is an open subset of $\mathbb{R}^m$, and let $a \in U$. Define what it means that $f$ *is differentiable at* $a$ and define *the derivative of* $f$ *at* $a$. Define what it means that $f$ *is continuously differentiable at* $a$. Show that a linear map $\mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable at every point of $\mathbb{R}^m$.

State and prove the mean value inequality. Let $U$ be an open, connected subset of $\mathbb{R}^m$. Let $f\colon U \to \mathbb{R}^n$ be a differentiable function such that $Df|_a$ is the zero map for all $a \in U$. Show that $f$ is a constant function.

State the inverse function theorem. Consider the curve $C$ in $\mathbb{R}^2$ defined by the equation

$$x^2 + y + \cos(xy) = 1.$$

Show that there exist an open neighbourhood $U$ of $(0,0)$ in $\mathbb{R}^2$, an open interval $I$ in $\mathbb{R}$ containing 0 and a continuous function $g\colon I \to \mathbb{R}$ such that $U \cap C$ is the graph of $g$, *i.e.*,

$$\{(x,y) \in \mathbb{R}^2 : x \in I,\ y = g(x)\} = U \cap C.$$

**Paper 4, Section II**

**10G Analysis and Topology**

Define the notions of *compact space, Hausdorff space* and *homeomorphism.*

Let $X$ be a topological space and $R$ be an equivalence relation on $X$. Define the quotient space $X/R$ and show that the quotient map $q\colon X \to X/R$ is continuous. Let $Y$ be another topological space and $f\colon X \to Y$ be a continuous function such that $f(x) = f(y)$ whenever $xRy$ in $X$. Show that the unique function $F\colon X/R \to Y$ with $F \circ q = f$ is continuous.

Show that the quotient of a compact space is compact. Give an example to show that the quotient of a Hausdorff space need not be Hausdorff.

Let $f\colon X \to Y$ be a continuous bijection from the compact space $X$ to the Hausdorff space $Y$. Carefully quoting any necessary results, show that $f$ is a homeomorphism.

Let $X = [0,1]^2$ be the closed unit square in $\mathbb{R}^2$. Define an equivalence relation $R$ on $X$ by $(x_1, y_1)R(x_2, y_2)$ if and only if one of the following holds:

(i) $x_1 = x_2$ and $y_1 = y_2$, or

(ii) $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = y_2$, or

(iii) $y_1 = y_2 \in \{0, 1\}$.

Show that the quotient space $X/R$ is homeomorphic to the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

**Paper 2, Section I**

**2G Analysis and Topology**

Let $f : (M, d) \to (N, e)$ be a homeomorphism between metric spaces. Show that $d'(x, y) = e(f(x), f(y))$ defines a metric on $M$ that is equivalent to $d$. Construct a metric on $\mathbb{R}$ which is equivalent to the standard metric but in which $\mathbb{R}$ is not complete.

**Paper 4, Section I**

**2G Analysis and Topology**

Define the *closure* of a subspace $Z$ of a topological space $X$, and what it means for $Z$ to be *dense*. What does it mean for a topological space $Y$ to be *Hausdorff*?

Assume that $Y$ is Hausdorff, and that $Z$ is a dense subspace of $X$. Show that if two continuous maps $f, g : X \to Y$ agree on $Z$, they must agree on the whole of $X$. Does this remain true if you drop the assumption that $Y$ is Hausdorff?

**Paper 1, Section II**

**10G Analysis and Topology**

Let $X$ and $Y$ be metric spaces. Determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) Let $f_n$ and $f$ be real-valued functions on $X$ and let $A, B$ be two subsets of $X$ such that $X = A \cup B$. If $f_n$ converges uniformly to $f$ on both $A$ and $B$, then $f_n$ converges uniformly to $f$ on $X$.

(b) If the sequences of real-valued functions $f_n$ and $g_n$ converge uniformly on $X$ to $f$ and $g$ respectively, then $f_n g_n$ converges uniformly to $fg$ on $X$.

(c) Let $X$ be the rectangle $[1, 2] \times [1, 2] \subset \mathbb{R}^2$ and let $f_n : X \to \mathbb{R}$ be given by

$$f_n(x, y) = \frac{1 + nx}{1 + ny}.$$

Then $f_n$ converges uniformly on $X$.

(d) Let $A$ be a subset of $X$ and $x_0$ a point such that any neighbourhood of $x_0$ contains a point of $A$ different from $x_0$. Suppose the functions $f_n : A \to Y$ converge uniformly on $A$ and, for each $n$, $\lim_{x \to x_0} f_n(x) = y_n$. If $Y$ is complete, then the sequence $y_n$ converges.

(e) Let $f_n$ converge uniformly on $X$ to a bounded function $f$ and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then the composition $g \circ f_n$ converges uniformly to $g \circ f$ on $X$.

**Paper 2, Section II**
**10G Analysis and Topology**

State the inverse function theorem for a function $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose $F$ is a differentiable bijection with $F^{-1}$ also differentiable. Show that the derivative of $F$ at any point in $\mathbb{R}^n$ is a linear isomorphism.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map such that its derivative is invertible at any point in $\mathbb{R}^n$. Is $F(\mathbb{R}^n)$ open? Is $F(\mathbb{R}^n)$ closed? Justify your answers.

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x, y, z) = (x + y + z, zy + zx + xy, xyz).$$

Determine the set $C$ of points $p \in \mathbb{R}^3$ for which $F$ fails to admit a differentiable local inverse around $p$. Is the set $\mathbb{R}^3 \setminus C$ connected? Justify your answer.

**Paper 3, Section II**
**11G Analysis and Topology**

Define a *contraction mapping* between two metric spaces. State and prove the contraction mapping theorem. Use this to show that the equation $x = \cos x$ has a unique real solution.

State the mean value inequality. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by

$$f(x, y) = \left( \frac{\cos x + \cos y - 1}{2}, \cos x - \cos y \right).$$

Prove that $f$ has a fixed point. [*Hint: Find a suitable subset of* $\mathbb{R}^2$ *on which* $f$ *is a contraction mapping.*]

**Paper 4, Section II**
**10G Analysis and Topology**

Define what it means for a topological space to be *connected*. Describe without proof the connected subspaces of $\mathbb{R}$ with the standard topology. Define what it means for a topological space to be *path connected*, and show that path connectedness implies connectedness.

Given metric spaces $A$ and $B$, let $C(A, B)$ be the space of continuous bounded functions from $A$ to $B$ with the topology induced by the uniform metric.

(a) For $n \in \mathbb{N}$, let $I_n \subset \mathbb{R}$ be

$$I_n = [1, 2] \cup [3, 4] \cup \ldots \cup [2n - 1, 2n]$$

with the subspace topology. For fixed $m, n \in \mathbb{N}$, how many connected components does $C(I_n, I_m)$ have?

(b) (i) Give an example of a closed bounded subspace of $\mathbb{R}^2$ which is connected but not path connected, justifying your answer. Call your example $S$.

(ii) Show that $C([0, 1], S)$ is not path connected.

(iii) Is $C([0, 1], S)$ connected? Briefly justify your answer.

**Paper 2, Section I**
**2F Analysis and Topology**

Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and let $C([0,1])$ denote the set of continuous real-valued functions on $[0,1]$. Given $f \in C([0,1])$, define the function $Tf$ by the expression

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

(a) Prove that $T$ is a continuous map $C([0,1]) \to C([0,1])$ with the uniform metric on $C([0,1])$.

(b) Let $d_1$ be the metric on $C([0,1])$ given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Is $T$ continuous with respect to $d_1$?

**Paper 4, Section I**
**2F Analysis and Topology**

Let $X$ be a topological space with an equivalence relation, $\tilde{X}$ the set of equivalence classes, $\pi : X \to \tilde{X}$, the quotient map taking a point in $X$ to its equivalence class.

(a) Define the *quotient topology* on $\tilde{X}$ and check it is a topology.

(b) Prove that if $Y$ is a topological space, a map $f : \tilde{X} \to Y$ is continuous if and only if $f \circ \pi$ is continuous.

(c) If $X$ is Hausdorff, is it true that $\tilde{X}$ is also Hausdorff? Justify your answer.

**Paper 1, Section II**
**10F Analysis and Topology**

Let $f : X \to Y$ be a map between metric spaces. Prove that the following two statements are equivalent:

(i) $f^{-1}(A) \subset X$ is open whenever $A \subset Y$ is open.

(ii) $f(x_n) \to f(a)$ for any sequence $x_n \to a$.

For $f : X \to Y$ as above, determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) If $X$ is compact and $f$ is continuous, then $f$ is uniformly continuous.

(b) If $X$ is compact and $f$ is continuous, then $Y$ is compact.

(c) If $X$ is connected, $f$ is continuous and $f(X)$ is dense in $Y$, then $Y$ is connected.

(d) If the set $\{(x,y) \in X \times Y :\ y = f(x)\}$ is closed in $X \times Y$ and $Y$ is compact, then $f$ is continuous.

**Paper 2, Section II**

**10F Analysis and Topology**

Let $k_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions satisfying the following properties:

1. $k_n(x) \geqslant 0$ for all $n$ and $x \in \mathbb{R}$ and there is $R > 0$ such that $k_n$ vanishes outside $[-R, R]$ for all $n$;
2. each $k_n$ is continuous and
$$\int_{-\infty}^{\infty} k_n(t)\,dt = 1;$$
3. given $\varepsilon > 0$ and $\delta > 0$, there exists a positive integer $N$ such that if $n \geqslant N$, then
$$\int_{-\infty}^{-\delta} k_n(t)\,dt + \int_{\delta}^{\infty} k_n(t)\,dt < \varepsilon.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and set

$$f_n(x) := \int_{-\infty}^{\infty} k_n(t) f(x-t)\,dt.$$

Show that $f_n$ converges uniformly to $f$ on any compact subset of $\mathbb{R}$.

Let $g : [0,1] \to \mathbb{R}$ be a continuous function with $g(0) = g(1) = 0$. Show that there is a sequence of polynomials $p_n$ such that $p_n$ converges uniformly to $g$ on $[0,1]$. [*Hint: consider the functions*

$$k_n(t) = \begin{cases} (1-t^2)^n / c_n & t \in [-1,1] \\ 0 & \text{otherwise,} \end{cases}$$

*where $c_n$ is a suitably chosen constant.*]

**Paper 3, Section II**

**11F Analysis and Topology**

Define the terms *connected* and *path-connected* for a topological space. Prove that the interval $[0, 1]$ is connected and that if a topological space is path-connected, then it is connected.

Let $X$ be an open subset of Euclidean space $\mathbb{R}^n$. Show that $X$ is connected if and only if $X$ is path-connected.

Let $X$ be a topological space with the property that every point has a neighbourhood homeomorphic to an open set in $\mathbb{R}^n$. Assume $X$ is connected; must $X$ be also path-connected? Briefly justify your answer.

Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x, 0) : \ x \in (0, 1]\}, \quad B = \{(0, y) : \ y \in [1/2, 1]\}, \text{ and}$$

$$C_n = \{(1/n, y) : \ y \in [0, 1]\} \text{ for } n \geqslant 1.$$

Let

$$X = A \cup B \cup \bigcup_{n \geqslant 1} C_n$$

with the subspace topology. Is $X$ path-connected? Is $X$ connected? Justify your answers.

**Paper 4, Section II**

**10F Analysis and Topology**

(a) Let $g : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for each $t \in [0,1]$, the partial derivatives $D_i g(t,x)$ $(i = 1, \ldots, n)$ of $x \mapsto g(t,x)$ exist and are continuous on $[0,1] \times \mathbb{R}^n$. Define $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x) = \int_0^1 g(t,x)\, dt.$$

Show that $G$ has continuous partial derivatives $D_i G$ given by

$$D_i G(x) = \int_0^1 D_i g(t,x)\, dt$$

for $i = 1, \ldots, n$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function, that is, partial derivatives $D_{i_1} D_{i_2} \cdots D_{i_k} f$ exist and are continuous for all $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1,2\}$ . Show that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$f(x_1, x_2) = f(x_1, 0) + x_2\, D_2 f(x_1, 0) + x_2^2\, h(x_1, x_2),$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is an infinitely differentiable function.

[*Hint: You may use the fact that if* $u : \mathbb{R} \to \mathbb{R}$ *is infinitely differentiable, then*

$$u(1) = u(0) + u'(0) + \int_0^1 (1-t) u''(t)\, dt.]$$

**Paper 2, Section I**

**2E Analysis and Topology**

Let $\tau$ be the collection of subsets of $\mathbb{C}$ of the form $\mathbb{C}\backslash f^{-1}(0)$, where $f$ is an arbitrary complex polynomial. Show that $\tau$ is a topology on $\mathbb{C}$.

Given topological spaces $X$ and $Y$, define the *product topology* on $X \times Y$. Equip $\mathbb{C}^2$ with the topology given by the product of $(\mathbb{C}, \tau)$ with itself. Let $g$ be an arbitrary two-variable complex polynomial. Is the subset $\mathbb{C}^2\backslash g^{-1}(0)$ always open in this topology? Justify your answer.

**Paper 1, Section II**

**10E Analysis and Topology**

State what it means for a function $f : \mathbb{R}^m \to \mathbb{R}^r$ to be *differentiable* at a point $x \in \mathbb{R}^m$, and define its derivative $f'(x)$.

Let $\mathcal{M}_n$ be the vector space of $n \times n$ real-valued matrices, and let $p : \mathcal{M}_n \to \mathcal{M}_n$ be given by $p(A) = A^3 - 3A - I$. Show that $p$ is differentiable at any $A \in \mathcal{M}_n$, and calculate its derivative.

State the inverse function theorem for a function $f$. In the case when $f(0) = 0$ and $f'(0) = I$, prove the existence of a continuous local inverse function in a neighbourhood of 0. [The rest of the proof of the inverse function theorem is not expected.]

Show that there exists a positive $\epsilon$ such that there is a continuously differentiable function $q : D_\epsilon(I) \to \mathcal{M}_n$ such that $p \circ q = \mathrm{id}|_{D_\epsilon(I)}$. Is it possible to find a continuously differentiable inverse to $p$ on the whole of $\mathcal{M}_n$? Justify your answer.

**Paper 2, Section II**

**10E Analysis and Topology**

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$, and let $d_1, d_\infty$ be the metrics on it given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx \qquad \text{and} \qquad d_\infty(f,g) = \max_{x\in[0,1]}|f(x) - g(x)|.$$

Show that $\mathrm{id} : (C[0,1], d_\infty) \to (C[0,1], d_1)$ is a continuous map. Do $d_1$ and $d_\infty$ induce the same topology on $C[0,1]$? Justify your answer.

Let $d$ denote for any $m \in \mathbb{N}$ the uniform metric on $\mathbb{R}^m$: $d((x_i),(y_i)) = \max_i |x_i - y_i|$. Let $\mathcal{P}_n \subset C[0,1]$ be the subspace of real polynomials of degree at most $n$. Define a *Lipschitz map* between two metric spaces, and show that evaluation at a point gives a Lipschitz map $(C[0,1], d_\infty) \to (\mathbb{R}, d)$. Hence or otherwise find a bijection from $(\mathcal{P}_n, d_\infty)$ to $(\mathbb{R}^{n+1}, d)$ which is Lipschitz and has a Lipschitz inverse.

Let $\tilde{\mathcal{P}}_n \subset \mathcal{P}_n$ be the subset of polynomials with values in the range $[-1,1]$.

(i) Show that $(\tilde{\mathcal{P}}_n, d_\infty)$ is compact.

(ii) Show that $d_1$ and $d_\infty$ induce the same topology on $\tilde{\mathcal{P}}_n$.

Any theorems that you use should be clearly stated.

[*You may use the fact that for distinct constants $a_i$, the following matrix is invertible:*

$$\begin{pmatrix} 1 & a_0 & a_0^2 & \dots & a_0^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{pmatrix}.]$$