Part IB

Analysis II

Year

Paper 3, Section I

2E Analysis II

(a) Let $A \subset \mathbb{R}$. What does it mean for a function $f : A \to \mathbb{R}$ to be *uniformly continuous*?

(b) Which of the following functions are uniformly continuous? Briefly justify your answers.

(i) $f(x) = x^2$ on $\mathbb{R}$.

(ii) $f(x) = \sqrt{x}$ on $[0, \infty)$.

(iii) $f(x) = \cos(1/x)$ on $[1, \infty)$.

Paper 4, Section I

3E Analysis II

Let $A \subset \mathbb{R}$. What does it mean to say that a sequence of real-valued functions on A is *uniformly convergent*?

(i) If a sequence (f_n) of real-valued functions on A converges uniformly to f, and each f_n is continuous, must f also be continuous?

(ii) Let $f_n(x) = e^{-nx}$. Does the sequence (f_n) converge uniformly on $[0, 1]$?

(iii) If a sequence (f_n) of real-valued functions on $[-1, 1]$ converges uniformly to f, and each f_n is differentiable, must f also be differentiable?

Give a proof or counterexample in each case.

Paper 2, Section I

3E Analysis II

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (x^{1/3} + y^2, y^5)$$

where $x^{1/3}$ denotes the unique real cube root of $x \in \mathbb{R}$.

(a) At what points is f continuously differentiable? Calculate its derivative there.

(b) Show that f has a local differentiable inverse near any (x, y) with $xy \neq 0$.

You should justify your answers, stating accurately any results that you require.

Paper 1, Section II

11E Analysis II

Let $A \subset \mathbb{R}^n$ be an open subset. State what it means for a function $f : A \to \mathbb{R}^m$ to be *differentiable* at a point $p \in A$, and define its derivative $Df(p)$.

State and prove the chain rule for the derivative of $g \circ f$, where $g : \mathbb{R}^m \to \mathbb{R}^r$ is a differentiable function.

Let $M = M_n(\mathbb{R})$ be the vector space of $n \times n$ real-valued matrices, and $V \subset M$ the open subset consisting of all invertible ones. Let $f : V \to V$ be given by $f(A) = A^{-1}$.

(a) Show that f is differentiable at the identity matrix, and calculate its derivative.

(b) For $C \in V$, let $l_C, r_C : M \to M$ be given by $l_C(A) = CA$ and $r_C(A) = AC$. Show that $r_C \circ f \circ l_C = f$ on V. Hence or otherwise, show that f is differentiable at any point of V, and calculate $Df(C)(h)$ for $h \in M$.

Paper 4, Section II

12E Analysis II

(a) (i) Show that a compact metric space must be complete.

(ii) If a metric space is complete and bounded, must it be compact? Give a proof or counterexample.

(b) A metric space (X, d) is said to be totally bounded if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $\{x_1, \ldots, x_N\} \subset X$ such that $X = \bigcup_{i=1}^{N} B_\epsilon(x_i)$.

(i) Show that a compact metric space is totally bounded.

(ii) Show that a complete, totally bounded metric space is compact.

[*Hint: If (x_n) is Cauchy, then there is a subsequence (x_{n_j}) such that*

$$\sum_j d(x_{n_{j+1}}, x_{n_j}) < \infty .]$$

(iii) Consider the space $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{R}$, with the metric

$$d(f,g) = \min \left\{ \int_0^1 |f(t) - g(t)| dt,\ 1 \right\} .$$

Is this space compact? Justify your answer.

Paper 3, Section II

12E Analysis II

(a) Carefully state the Picard–Lindelöf theorem on solutions to ordinary differential equations.

(b) Let $X = C([1,b],\mathbb{R}^n)$ be the set of continuous functions from a closed interval $[1,b]$ to $\mathbb{R}^n$, and let $||\cdot||$ be a norm on $\mathbb{R}^n$.

(i) Let $f \in X$. Show that for any $c \in [0,\infty)$ the norm

$$||f||_c = \sup_{t\in[1,b]} ||f(t)t^{-c}||$$

is Lipschitz equivalent to the usual sup norm on X.

(ii) Assume that $F : [1,b]\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable, i.e. there exists $M > 0$ such that

$$\|F(t,x) - F(t,y)\| \leqslant M\|x-y\|$$

for all $t \in [1,b]$ and all $x, y \in \mathbb{R}^n$. Define $\varphi : X \to X$ by

$$\varphi(f)(t) = \int_1^t F(l, f(l))\, dl$$

for $t \in [1,b]$.

Show that there is a choice of c such that φ is a contraction on $(X, ||\cdot||_c)$. Deduce that for any $y_0 \in \mathbb{R}^n$, the differential equation

$$Df(t) = F(t, f(t))$$

has a unique solution on $[1,b]$ with $f(1) = y_0$.

Paper 2, Section II

12E Analysis II

(a) (i) Define what it means for two norms on a vector space to be *Lipschitz equivalent*.

(ii) Show that any two norms on a finite-dimensional vector space are Lipschitz equivalent.

(iii) Show that if two norms $||\cdot||$, $||\cdot||'$ on a vector space V are Lipschitz equivalent then the following holds: for any sequence (v_n) in V, (v_n) is Cauchy with respect to $||\cdot||$ if and only if it is Cauchy with respect to $||\cdot||'$.

(b) Let V be the vector space of real sequences $x = (x_i)$ such that $\sum |x_i| < \infty$. Let

$$||x||_\infty = \sup\{|x_i| : i \in \mathbb{N}\},$$

and for $1 \leqslant p < \infty$, let

$$||x||_p = \Big(\sum |x_i|^p\Big)^{1/p}.$$

You may assume that $||\cdot||_\infty$ and $||\cdot||_p$ are well-defined norms on V.

(i) Show that $||\cdot||_p$ is not Lipschitz equivalent to $||\cdot||_\infty$ for any $1 \leqslant p < \infty$.

(ii) Are there any p, q with $1 \leqslant p < q < \infty$ such that $||\cdot||_p$ and $||\cdot||_q$ are Lipschitz equivalent? Justify your answer.

Paper 3, Section I

2F Analysis II

For a continuous function $f = (f_1, f_2, \ldots, f_m) : [0,1] \to \mathbb{R}^m$, define

$$\int_0^1 f(t)\,dt = \left(\int_0^1 f_1(t)\,dt, \int_0^1 f_2(t)\,dt, \ldots, \int_0^1 f_m(t)\,dt\right).$$

Show that

$$\left\|\int_0^1 f(t)\,dt\right\|_2 \leqslant \int_0^1 \|f(t)\|_2\,dt$$

for every continuous function $f : [0,1] \to \mathbb{R}^m$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^m$.

Find all continuous functions $f : [0,1] \to \mathbb{R}^m$ with the property that

$$\left\|\int_0^1 f(t)\,dt\right\| = \int_0^1 \|f(t)\|\,dt$$

regardless of the norm $\|\cdot\|$ on $\mathbb{R}^m$.

[*Hint: start by analysing the case when* $\|\cdot\|$ *is the Euclidean norm* $\|\cdot\|_2$.]

Paper 2, Section I

3F Analysis II

Show that $\|f\|_1 = \int_0^1 |f(x)|\,dx$ defines a norm on the space $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

Let $\mathcal{S}$ be the set of continuous functions $g : [0,1] \to \mathbb{R}$ with $g(0) = g(1) = 0$. Show that for each continuous function $f : [0,1] \to \mathbb{R}$, there is a sequence $g_n \in \mathcal{S}$ with $\sup_{x\in[0,1]} |g_n(x)| \leqslant \sup_{x\in[0,1]} |f(x)|$ such that $\|f - g_n\|_1 \to 0$ as $n \to \infty$.

Show that if $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 f(x)g(x)\,dx = 0$ for every $g \in \mathcal{S}$ then $f = 0$.

Paper 4, Section I

3F Analysis II

State the Bolzano–Weierstrass theorem in $\mathbb{R}$. Use it to deduce the Bolzano–Weierstrass theorem in $\mathbb{R}^n$.

Let D be a closed, bounded subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a function. Let $\mathcal{S}$ be the set of points in D where f is discontinuous. For $\rho > 0$ and $z \in \mathbb{R}^n$, let $B_\rho(z)$ denote the ball $\{x \in \mathbb{R}^n : \|x - z\| < \rho\}$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x \in D$, $y \in D \setminus \cup_{z \in \mathcal{S}} B_\epsilon(z)$ and $\|x - y\| < \delta$.

(If you use the fact that a continuous function on a compact metric space is uniformly continuous, you must prove it.)

Paper 1, Section II

11F Analysis II

Let $U \subset \mathbb{R}^n$ be a non-empty open set and let $f : U \to \mathbb{R}^n$.

(a) What does it mean to say that f is *differentiable*? What does it mean to say that f is a C^1 function?

If f is differentiable, show that f is continuous.

State the inverse function theorem.

(b) Suppose that U is convex, f is C^1 and that its derivative $Df(a)$ at a satisfies $\|Df(a) - I\| < 1$ for all $a \in U$, where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map and $\|\cdot\|$ denotes the operator norm. Show that f is injective.

Explain why $f(U)$ is an open subset of $\mathbb{R}^n$.

Must it be true that $f(U) = \mathbb{R}^n$? What if $U = \mathbb{R}^n$? Give proofs or counter-examples as appropriate.

(c) Find the largest set $U \subset \mathbb{R}^2$ such that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x^2 - y^2, 2xy)$ satisfies $\|Df(a) - I\| < 1$ for every $a \in U$.

Paper 4, Section II
12F Analysis II

(a) Define what it means for a metric space (X, d) to be *complete*. Give a metric d on the interval $I = (0, 1]$ such that (I, d) is complete and such that a subset of I is open with respect to d if and only if it is open with respect to the Euclidean metric on I. Be sure to prove that d has the required properties.

(b) Let (X, d) be a complete metric space.

(i) If $Y \subset X$, show that Y taken with the subspace metric is complete if and only if Y is closed in X.

(ii) Let $f : X \to X$ and suppose that there is a number $\lambda \in (0, 1)$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for every $x, y \in X$. Show that there is a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Deduce that if (a_n) is a sequence of points in X converging to a point $a \neq x_0$, then there are integers ℓ and $m \geqslant \ell$ such that $f(a_m) \neq a_n$ for every $n \geqslant \ell$.

Paper 3, Section II
12F Analysis II

(a) Let $A \subset \mathbb{R}^m$ and let $f, f_n : A \to \mathbb{R}$ be functions for $n = 1, 2, 3, \ldots$. What does it mean to say that the sequence (f_n) *converges uniformly to f on A*? What does it mean to say that f is *uniformly continuous*?

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Determine whether each of the following statements is true or false. Give reasons for your answers.

(i) If $f_n(x) = f(x + 1/n)$ for each $n = 1, 2, 3, \ldots$ and each $x \in \mathbb{R}$, then $f_n \to f$ uniformly on $\mathbb{R}$.

(ii) If $g_n(x) = (f(x + 1/n))^2$ for each $n = 1, 2, 3, \ldots$ and each $x \in \mathbb{R}$, then $g_n \to (f)^2$ uniformly on $\mathbb{R}$.

(c) Let A be a closed, bounded subset of $\mathbb{R}^m$. For each $n = 1, 2, 3, \ldots$, let $g_n : A \to \mathbb{R}$ be a continuous function such that $(g_n(x))$ is a decreasing sequence for each $x \in A$. If $\delta \in \mathbb{R}$ is such that for each n there is $x_n \in A$ with $g_n(x_n) \geqslant \delta$, show that there is $x_0 \in A$ such that $\lim_{n\to\infty} g_n(x_0) \geqslant \delta$.

Deduce the following: If $f_n : A \to \mathbb{R}$ is a continuous function for each $n = 1, 2, 3, \ldots$ such that $(f_n(x))$ is a decreasing sequence for each $x \in A$, and if the pointwise limit of (f_n) is a continuous function $f : A \to \mathbb{R}$, then $f_n \to f$ uniformly on A.

Paper 2, Section II

12F Analysis II

(a) Let (X, d) be a metric space, A a non-empty subset of X and $f : A \to \mathbb{R}$. Define what it means for f to be *Lipschitz*. If f is Lipschitz with Lipschitz constant L and if

$$F(x) = \inf_{y \in A} \left(f(y) + L d(x, y) \right)$$

for each $x \in X$, show that $F(x) = f(x)$ for each $x \in A$ and that $F : X \to \mathbb{R}$ is Lipschitz with Lipschitz constant L. (Be sure to justify that $F(x) \in \mathbb{R}$, i.e. that the infimum is finite for every $x \in X$.)

(b) What does it mean to say that two norms on a vector space are *Lipschitz equivalent*?

Let V be an n-dimensional real vector space equipped with a norm $\| \cdot \|$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V. Show that the map $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(x_1, x_2, \ldots, x_n) = \|x_1 e_1 + x_2 e_2 + \ldots + x_n e_n\|$ is continuous. Deduce that any two norms on V are Lipschitz equivalent.

(c) Prove that for each positive integer n and each $a \in (0, 1]$, there is a constant $C > 0$ with the following property: for every polynomial p of degree $\leqslant n$, there is a point $y \in [0, a]$ such that

$$\sup_{x \in [0,1]} |p'(x)| \leqslant C |p(y)|,$$

where p' is the derivative of p.

Paper 3, Section I

2G Analysis II

What does it mean to say that a metric space is *complete*? Which of the following metric spaces are complete? Briefly justify your answers.

(i) $[0,1]$ with the Euclidean metric.

(ii) $\mathbb{Q}$ with the Euclidean metric.

(iii) The subset
$$\{\,(0,0)\,\} \cup \{\,(x, \sin(1/x)) \mid x > 0\,\} \subset \mathbb{R}^2$$
with the metric induced from the Euclidean metric on $\mathbb{R}^2$.

Write down a metric on $\mathbb{R}$ with respect to which $\mathbb{R}$ is not complete, justifying your answer.

[*You may assume throughout that* $\mathbb{R}$ *is complete with respect to the Euclidean metric.*]

Paper 2, Section I

3G Analysis II

Let $X \subset \mathbb{R}$. What does it mean to say that a sequence of real-valued functions on X is *uniformly convergent*?

Let f, f_n $(n \geqslant 1)\colon \mathbb{R} \to \mathbb{R}$ be functions.

(a) Show that if each f_n is continuous, and (f_n) converges uniformly on $\mathbb{R}$ to f, then f is also continuous.

(b) Suppose that, for every $M > 0$, (f_n) converges uniformly on $[-M, M]$. Need (f_n) converge uniformly on $\mathbb{R}$? Justify your answer.

Paper 4, Section I

3G Analysis II

State the chain rule for the composition of two differentiable functions $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable. For $c \in \mathbb{R}$, let $g(x) = f(x, c-x)$. Compute the derivative of g. Show that if $\partial f/\partial x = \partial f/\partial y$ throughout $\mathbb{R}^2$, then $f(x,y) = h(x+y)$ for some function $h\colon \mathbb{R} \to \mathbb{R}$.

Paper 1, Section II

11G Analysis II

What does it mean to say that a real-valued function on a metric space is *uniformly continuous*? Show that a continuous function on a closed interval in $\mathbb{R}$ is uniformly continuous.

What does it mean to say that a real-valued function on a metric space is *Lipschitz*? Show that if a function is Lipschitz then it is uniformly continuous.

Which of the following statements concerning continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ are true and which are false? Justify your answers.

(i) If f is bounded then f is uniformly continuous.

(ii) If f is differentiable and f' is bounded, then f is uniformly continuous.

(iii) There exists a sequence of uniformly continuous functions converging pointwise to f.

Paper 2, Section II

12G Analysis II

Let V be a real vector space. What is a *norm* on V? Show that if $\|-\|$ is a norm on V, then the maps $T_v \colon x \mapsto x + v$ (for $v \in V$) and $m_a \colon x \mapsto ax$ (for $a \in \mathbb{R}$) are continuous with respect to the norm.

Let $B \subset V$ be a subset containing 0. Show that there exists at most one norm on V for which B is the open unit ball.

Suppose that B satisfies the following two properties:

- if $v \in V$ is a nonzero vector, then the line $\mathbb{R}v \subset V$ meets B in a set of the form $\{tv \ : \ -\lambda < t < \lambda\}$ for some $\lambda > 0$;
- if $x,\, y \in B$ and $s,\, t > 0$ then $(s+t)^{-1}(sx + ty) \in B$.

Show that there exists a norm $\|-\|_B$ for which B is the open unit ball.

Identify $\|-\|_B$ in the following two cases:

(i) $V = \mathbb{R}^n$, $B = \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ -1 < x_i < 1 \ \text{ for all } i \,\}$.

(ii) $V = \mathbb{R}^2$, B the interior of the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$.

Let $C \subset \mathbb{R}^2$ be the set

$$C = \{\, (x_1, x_2) \in \mathbb{R}^2 \ : \ |x_1| < 1,\ |x_2| < 1,\ \text{and } (|x_1| - 1)^2 + (|x_2| - 1)^2 > 1 \,\}\,.$$

Is there a norm on $\mathbb{R}^2$ for which C is the open unit ball? Justify your answer.

Paper 4, Section II

12G Analysis II

Let $U \subset \mathbb{R}^m$ be a nonempty open set. What does it mean to say that a function $f \colon U \to \mathbb{R}^n$ is *differentiable*?

Let $f \colon U \to \mathbb{R}$ be a function, where $U \subset \mathbb{R}^2$ is open. Show that if the first partial derivatives of f exist and are continuous on U, then f is differentiable on U.

Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \dfrac{x^3 + 2y^4}{x^2 + y^2} & (x, y) \neq (0, 0). \end{cases}$$

Determine, with proof, where f is differentiable.

Paper 3, Section II

12G Analysis II

What is a *contraction map* on a metric space X? State and prove the contraction mapping theorem.

Let (X, d) be a complete non-empty metric space. Show that if $f \colon X \to X$ is a map for which some iterate f^k $(k \geqslant 1)$ is a contraction map, then f has a unique fixed point. Show that f itself need not be a contraction map.

Let $f \colon [0, \infty) \to [0, \infty)$ be the function

$$f(x) = \frac{1}{3}\left(x + \sin x + \frac{1}{x+1}\right) .$$

Show that f has a unique fixed point.

Paper 3, Section I
2G Analysis II

(a) Let X be a subset of $\mathbb{R}$. What does it mean to say that a sequence of functions $f_n\colon X \to \mathbb{R}$ $(n \in \mathbb{N})$ is *uniformly convergent*?

(b) Which of the following sequences of functions are uniformly convergent? Justify your answers.

(i) $f_n\colon (0,1) \to \mathbb{R}$, $\quad f_n(x) = \dfrac{1-x^n}{1-x}$.

(ii) $f_n\colon (0,1) \to \mathbb{R}$, $\quad f_n(x) = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}\, x^k$.

(iii) $f_n\colon \mathbb{R} \to \mathbb{R}$, $\quad f_n(x) = x/n$.

(iv) $f_n\colon [0,\infty) \to \mathbb{R}$, $\quad f_n(x) = xe^{-nx}$.

Paper 4, Section I
3G Analysis II

(a) What does it mean to say that a mapping $f\colon X \to X$ from a metric space to itself is a *contraction*?

(b) State carefully the contraction mapping theorem.

(c) Let $(a_1, a_2, a_3) \in \mathbb{R}^3$. By considering the metric space $(\mathbb{R}^3, d)$ with

$$d(x,y) = \sum_{i=1}^{3} |x_i - y_i|\,,$$

or otherwise, show that there exists a unique solution $(x_1, x_2, x_3) \in \mathbb{R}^3$ of the system of equations

$$\begin{aligned}
x_1 &= a_1 + \frac{1}{6}\left(\sin x_2 + \sin x_3\right),\\
x_2 &= a_2 + \frac{1}{6}\left(\sin x_1 + \sin x_3\right),\\
x_3 &= a_3 + \frac{1}{6}\left(\sin x_1 + \sin x_2\right).
\end{aligned}$$

Paper 2, Section I

3G Analysis II

(a) What does it mean to say that the function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at the point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$? Show from your definition that if f is differentiable at x, then f is continuous at x.

(b) Suppose that there are functions $g_j\colon \mathbb{R} \to \mathbb{R}^m$ $(1 \leqslant j \leqslant n)$ such that for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$f(x) = \sum_{j=1}^{n} g_j(x_j).$$

Show that f is differentiable at x if and only if each g_j is differentiable at x_j.

(c) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = |x|^{3/2} + |y|^{1/2}.$$

Determine at which points $(x, y) \in \mathbb{R}^2$ the function f is differentiable.

Paper 1, Section II

11G Analysis II

Let (X, d) be a metric space.

(a) What does it mean to say that $(x_n)_n$ is a *Cauchy sequence* in X? Show that if $(x_n)_n$ is a Cauchy sequence, then it converges if it contains a convergent subsequence.

(b) Let $(x_n)_n$ be a Cauchy sequence in X.

(i) Show that for every $m \geqslant 1$, the sequence $(d(x_m, x_n))_n$ converges to some $d_m \in \mathbb{R}$.

(ii) Show that $d_m \to 0$ as $m \to \infty$.

(iii) Let $(y_n)_n$ be a subsequence of $(x_n)_n$. If ℓ, m are such that $y_\ell = x_m$, show that $d(y_\ell, y_n) \to d_m$ as $n \to \infty$.

(iv) Show also that for every m and n,

$$d_m - d_n \leqslant d(x_m, x_n) \leqslant d_m + d_n.$$

(v) Deduce that $(x_n)_n$ has a subsequence $(y_n)_n$ such that for every m and n,

$$d(y_{m+1}, y_m) \leqslant \frac{1}{3} d(y_m, y_{m-1})$$

and

$$d(y_{m+1}, y_{n+1}) \leqslant \frac{1}{2} d(y_m, y_n).$$

(c) Suppose that every closed subset Y of X has the property that every contraction mapping $Y \to Y$ has a fixed point. Prove that X is complete.

Paper 4, Section II
12G Analysis II

(a) Let V be a real vector space. What does it mean to say that two norms on V are *Lipschitz equivalent*? Prove that every norm on $\mathbb{R}^n$ is Lipschitz equivalent to the Euclidean norm. Hence or otherwise, show that any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous.

(b) Let $f\colon U \to V$ be a linear map between normed real vector spaces. We say that f is *bounded* if there exists a constant C such that for all $u \in U$, $\|f(u)\| \leqslant C\,\|u\|$. Show that f is bounded if and only if f is continuous.

(c) Let ℓ^2 denote the space of sequences $(x_n)_{n\geqslant 1}$ of real numbers such that $\sum_{n\geqslant 1} x_n^2$ is convergent, with the norm $\|(x_n)_n\| = \left(\sum_{n\geqslant 1} x_n^2\right)^{1/2}$. Let $e_m \in \ell^2$ be the sequence $e_m = (x_n)_n$ with $x_m = 1$ and $x_n = 0$ if $n \neq m$. Let w be the sequence $(2^{-n})_n$. Show that the subset $\{w\} \cup \{e_m \mid m \geqslant 1\}$ is linearly independent. Let $V \subset \ell^2$ be the subspace it spans, and consider the linear map $f\colon V \to \mathbb{R}$ defined by

$$f(w) = 1, \qquad f(e_m) = 0 \quad \text{for all } m \geqslant 1.$$

Is f continuous? Justify your answer.

Paper 3, Section II
12G Analysis II

Let X be a metric space.

(a) What does it mean to say that a function $f\colon X \to \mathbb{R}$ is *uniformly continuous*? What does it mean to say that f is *Lipschitz*? Show that if f is Lipschitz then it is uniformly continuous. Show also that if $(x_n)_n$ is a Cauchy sequence in X, and f is uniformly continuous, then the sequence $(f(x_n))_n$ is convergent.

(b) Let $f\colon X \to \mathbb{R}$ be continuous, and X be sequentially compact. Show that f is uniformly continuous. Is f necessarily Lipschitz? Justify your answer.

(c) Let Y be a dense subset of X, and let $g\colon Y \to \mathbb{R}$ be a continuous function. Show that there exists at most one continuous function $f\colon X \to \mathbb{R}$ such that for all $y \in Y$, $f(y) = g(y)$. Prove that if g is uniformly continuous, then such a function f exists, and is uniformly continuous.

[A subset $Y \subset X$ is *dense* if for any nonempty open subset $U \subset X$, the intersection $U \cap Y$ is nonempty.]

Paper 2, Section II

12G Analysis II

(a) What is a *norm* on a real vector space?

(b) Let $L(\mathbb{R}^m, \mathbb{R}^n)$ be the space of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Show that

$$\|A\| = \sup_{0 \neq x \in \mathbb{R}^m} \frac{\|Ax\|}{\|x\|}, \qquad A \in L(\mathbb{R}^m, \mathbb{R}^n),$$

defines a norm on $L(\mathbb{R}^m, \mathbb{R}^n)$, and that if $B \in L(\mathbb{R}^\ell, \mathbb{R}^m)$ then $\|AB\| \leqslant \|A\| \, \|B\|$.

(c) Let M_n be the space of $n \times n$ real matrices, identified with $L(\mathbb{R}^n, \mathbb{R}^n)$ in the usual way. Let $U \subset M_n$ be the subset

$$U = \{X \in M_n \mid I - X \text{ is invertible}\}\,.$$

Show that U is an open subset of M_n which contains the set $V = \{X \in M_n \mid \|X\| < 1\}$.

(d) Let $f\colon U \to M_n$ be the map $f(X) = (I - X)^{-1}$. Show carefully that the series $\sum_{k=0}^{\infty} X^k$ converges on V to $f(X)$. Hence or otherwise, show that f is twice differentiable at 0, and compute its first and second derivatives there.

Paper 3, Section I

2G Analysis II

Define what is meant by a *uniformly continuous* function f on a subset E of a metric space. Show that every continuous function on a closed, bounded interval is uniformly continuous. [You may assume the Bolzano–Weierstrass theorem.]

Suppose that a function $g : [0, \infty) \to \mathbb{R}$ is continuous and tends to a finite limit at ∞. Is g necessarily uniformly continuous on $[0, \infty)$? Give a proof or a counterexample as appropriate.

Paper 4, Section I

3G Analysis II

Define what is meant for two norms on a vector space to be *Lipschitz equivalent*.

Let $C_c^1([-1, 1])$ denote the vector space of continuous functions $f : [-1, 1] \to \mathbb{R}$ with continuous first derivatives and such that $f(x) = 0$ for x in some neighbourhood of the end-points -1 and 1. Which of the following four functions $C_c^1([-1, 1]) \to \mathbb{R}$ define norms on $C_c^1([-1, 1])$ (give a brief explanation)?

$$p(f) = \sup |f|, \qquad q(f) = \sup(|f| + |f'|),$$

$$r(f) = \sup |f'|, \qquad s(f) = \left| \int_{-1}^{1} f(x)dx \right|.$$

Among those that define norms, which pairs are Lipschitz equivalent? Justify your answer.

Paper 2, Section I

3G Analysis II

Show that the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x, y, z) = (x - y - z,\ x^2 + y^2 + z^2,\ xyz)$$

is differentiable everywhere and find its derivative.

Stating accurately any theorem that you require, show that f has a differentiable local inverse at a point (x, y, z) if and only if

$$(x + y)(x + z)(y - z) \neq 0.$$

Paper 1, Section II

11G Analysis II

Define what it means for a sequence of functions $f_n : [0,1] \to \mathbb{R}$ to converge *uniformly on* $[0,1]$ to a function f.

Let $f_n(x) = n^p x e^{-n^q x}$, where p, q are positive constants. Determine all the values of (p,q) for which $f_n(x)$ converges pointwise on $[0,1]$. Determine all the values of (p,q) for which $f_n(x)$ converges uniformly on $[0,1]$.

Let now $f_n(x) = e^{-nx^2}$. Determine whether or not f_n converges uniformly on $[0,1]$.

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that the sequence $x^n f(x)$ is uniformly convergent on $[0,1]$ if and only if $f(1) = 0$.

[If you use any theorems about uniform convergence, you should prove these.]

Paper 4, Section II

12G Analysis II

Consider the space ℓ^∞ of bounded real sequences $x = (x_i)_{i=1}^\infty$ with the norm $\|x\|_\infty = \sup_i |x_i|$. Show that for every bounded sequence $x^{(n)}$ in ℓ^∞ there is a subsequence $x^{(n_j)}$ which converges in every coordinate, i.e. the sequence $(x_i^{(n_j)})_{j=1}^\infty$ of real numbers converges for each i. Does every bounded sequence in ℓ^∞ have a convergent subsequence? Justify your answer.

Let $\ell^1 \subset \ell^\infty$ be the subspace of real sequences $x = (x_i)_{i=1}^\infty$ such that $\sum_{i=1}^\infty |x_i|$ converges. Is ℓ^1 complete in the norm $\|\cdot\|_\infty$ (restricted from ℓ^∞ to ℓ^1)? Justify your answer.

Suppose that (x_i) is a real sequence such that, for every $(y_i) \in \ell^\infty$, the series $\sum_{i=1}^\infty x_i y_i$ converges. Show that $(x_i) \in \ell^1$.

Suppose now that (x_i) is a real sequence such that, for every $(y_i) \in \ell^1$, the series $\sum_{i=1}^\infty x_i y_i$ converges. Show that $(x_i) \in \ell^\infty$.

Paper 3, Section II

12G Analysis II

Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be *differentiable at* $x \in \mathbb{R}^n$ with derivative $Df(x)$.

State and prove the *chain rule* for the derivative of $g \circ f$, where $g : \mathbb{R}^m \to \mathbb{R}^p$ is a differentiable function.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c - x)$ where c is a constant. Show that g is differentiable and find its derivative in terms of the partial derivatives of f. Show that if $D_1 f(x, y) = D_2 f(x, y)$ holds everywhere in $\mathbb{R}^2$, then $f(x, y) = h(x + y)$ for some differentiable function h.

Paper 2, Section II

12G Analysis II

Let E, F be normed spaces with norms $\|\cdot\|_E$, $\|\cdot\|_F$. Show that for a map $f : E \to F$ and $a \in E$, the following two statements are equivalent:

(i) For every given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(x) - f(a)\|_F < \varepsilon$ whenever $\|x - a\|_E < \delta$.

(ii) $f(x_n) \to f(a)$ for each sequence $x_n \to a$.

We say that f is continuous at a if (i), or equivalently (ii), holds.

Let now $(E, \|\cdot\|_E)$ be a normed space. Let $A \subset E$ be a non-empty closed subset and define $d(x, A) = \inf\{\|x - a\|_E : a \in A\}$. Show that

$$|d(x, A) - d(y, A)| \leqslant \|x - y\|_E \text{ for all } x, y \in E.$$

In the case when $E = \mathbb{R}^n$ with the standard Euclidean norm, show that there exists $a \in A$ such that $d(x, A) = \|x - a\|$.

Let A, B be two disjoint closed sets in $\mathbb{R}^n$. Must there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$? Must there exist $a \in A$ and $b \in B$ such that $d(a, b) \leqslant d(x, y)$ for all $x \in A$ and $y \in B$? For each answer, give a proof or counterexample as appropriate.

Paper 3, Section I

2F Analysis II

Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a differentiable function on U such that $\| Df|_x \| \leqslant M$ for some constant M and all $x \in U$, where $\| Df|_x \|$ denotes the operator norm of the linear map $Df|_x$. Let $[a,b] = \{ta + (1-t)b : 0 \leqslant t \leqslant 1\}$ $(a, b, \in \mathbb{R}^n)$ be a straight-line segment contained in U. Prove that $|f(b) - f(a)| \leqslant M\|b-a\|$, where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Prove that if U is an open ball and $Df|_x = 0$ for each $x \in U$, then f is constant on U.

Paper 4, Section I

3F Analysis II

Define a *contraction mapping* and state the contraction mapping theorem.

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$ endowed with the uniform norm. Show that the map $A : C[0,1] \to C[0,1]$ defined by

$$Af(x) = \int_0^x f(t)dt$$

is not a contraction mapping, but that $A \circ A$ is.

Paper 2, Section I

3F Analysis II

Define what is meant by a *uniformly continuous* function on a set $E \subset \mathbb{R}$.

If f and g are uniformly continuous functions on $\mathbb{R}$, is the (pointwise) product fg necessarily uniformly continuous on $\mathbb{R}$?

Is a uniformly continuous function on $(0,1)$ necessarily bounded?

Is $\cos(1/x)$ uniformly continuous on $(0,1)$?

Justify your answers.

Paper 1, Section II

11F Analysis II

Define what it means for two norms on a real vector space V to be Lipschitz equivalent. Show that if two norms on V are Lipschitz equivalent and $F \subset V$, then F is closed in one norm if and only if F is closed in the other norm.

Show that if V is finite-dimensional, then any two norms on V are Lipschitz equivalent.

Show that $\|f\|_1 = \int_0^1 |f(x)|dx$ is a norm on the space $C[0,1]$ of continuous real-valued functions on $[0,1]$. Is the set $S = \{f \in C[0,1] : f(1/2) = 0\}$ closed in the norm $\|\cdot\|_1$?

Determine whether or not the norm $\|\cdot\|_1$ is Lipschitz equivalent to the uniform norm $\|\cdot\|_\infty$ on $C[0,1]$.

[You may assume the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}^n$.]

Paper 4, Section II

12F Analysis II

Let $U \subset \mathbb{R}^2$ be an open set. Define what it means for a function $f : U \to \mathbb{R}$ to be *differentiable* at a point $(x_0, y_0) \in U$.

Prove that if the partial derivatives $D_1 f$ and $D_2 f$ exist on U and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

If f is differentiable on U must $D_1 f$, $D_2 f$ be continuous at (x_0, y_0)? Give a proof or counterexample as appropriate.

The function $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$h(x,y) = xy\sin(1/x) \quad \text{for } x \neq 0, \qquad h(0,y) = 0.$$

Determine all the points (x, y) at which h is differentiable.

Paper 3, Section II

12F Analysis II

Let f_n, $n = 1, 2, \ldots$, be continuous functions on an open interval (a, b). Prove that if the sequence (f_n) converges to f uniformly on (a, b) then the function f is continuous on (a, b).

If instead (f_n) is only known to converge pointwise to f and f is continuous, must (f_n) be uniformly convergent? Justify your answer.

Suppose that a function f has a continuous derivative on (a, b) and let

$$g_n(x) = n\left(f(x + \frac{1}{n}) - f(x)\right).$$

Stating clearly any standard results that you require, show that the functions g_n converge uniformly to f' on each interval $[\alpha, \beta] \subset (a, b)$.

Paper 2, Section II

12F Analysis II

Let X, Y be subsets of $\mathbb{R}^n$ and define $X + Y = \{x + y : x \in X,\ y \in Y\}$. For each of the following statements give a proof or a counterexample (with justification) as appropriate.

(i) If each of X, Y is bounded and closed, then $X + Y$ is bounded and closed.

(ii) If X is bounded and closed and Y is closed, then $X + Y$ is closed.

(iii) If X, Y are both closed, then $X + Y$ is closed.

(iv) If X is open and Y is closed, then $X + Y$ is open.

[The Bolzano–Weierstrass theorem in $\mathbb{R}^n$ may be assumed without proof.]

Paper 3, Section I

2F Analysis II

For each of the following sequences of functions on $[0,1]$, indexed by $n = 1, 2, \ldots$, determine whether or not the sequence has a pointwise limit, and if so, determine whether or not the convergence to the pointwise limit is uniform.

1. $f_n(x) = 1/(1 + n^2x^2)$
2. $g_n(x) = nx(1-x)^n$
3. $h_n(x) = \sqrt{n}x(1-x)^n$

Paper 4, Section I

3F Analysis II

State and prove the chain rule for differentiable mappings $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^k$.

Suppose now $F : \mathbb{R}^2 \to \mathbb{R}^2$ has image lying on the unit circle in $\mathbb{R}^2$. Prove that the determinant $\det(DF|_x)$ vanishes for every $x \in \mathbb{R}^2$.

Paper 2, Section I

3F Analysis II

Let $\mathcal{C}[a,b]$ denote the vector space of continuous real-valued functions on the interval $[a,b]$, and let $\mathcal{C}'[a,b]$ denote the subspace of continuously differentiable functions.

Show that $\|f\|_1 = \max|f| + \max|f'|$ defines a norm on $\mathcal{C}'[a,b]$. Show furthermore that the map $\Phi : f \mapsto f'((a+b)/2)$ takes the closed unit ball $\{\|f\|_1 \leqslant 1\} \subset \mathcal{C}'[a,b]$ to a bounded subset of $\mathbb{R}$.

If instead we had used the norm $\|f\|_0 = \max|f|$ restricted from $\mathcal{C}[a,b]$ to $\mathcal{C}'[a,b]$, would Φ take the closed unit ball $\{\|f\|_0 \leqslant 1\} \subset \mathcal{C}'[a,b]$ to a bounded subset of $\mathbb{R}$? Justify your answer.

Paper 1, Section II
11F Analysis II

Define what it means for a sequence of functions $k_n : A \to \mathbb{R}$, $n = 1, 2, \ldots$, to *converge uniformly* on an interval $A \subset \mathbb{R}$.

By considering the functions $k_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, or otherwise, show that uniform convergence of a sequence of differentiable functions does not imply uniform convergence of their derivatives.

Now suppose $k_n(x)$ is continuously differentiable on A for each n, that $k_n(x_0)$ converges as $n \to \infty$ for some $x_0 \in A$, and moreover that the derivatives $k_n'(x)$ converge uniformly on A. Prove that $k_n(x)$ converges to a continuously differentiable function $k(x)$ on A, and that

$$k'(x) = \lim_{n\to\infty} k_n'(x).$$

Hence, or otherwise, prove that the function

$$\sum_{n=1}^{\infty} \frac{x^n \sin(nx)}{n^3+1}$$

is continuously differentiable on $(-1, 1)$.

Paper 4, Section II
12F Analysis II

State the contraction mapping theorem.

A metric space (X, d) is bounded if $\{d(x,y) \mid x, y \in X\}$ is a bounded subset of $\mathbb{R}$. Suppose (X, d) is complete and bounded. Let $\mathrm{Maps}(X, X)$ denote the set of continuous maps from X to itself. For $f, g \in \mathrm{Maps}(X, X)$, let

$$\delta(f, g) \;=\; \sup_{x\in X} d(f(x), g(x)).$$

Prove that $(\mathrm{Maps}(X, X), \delta)$ is a complete metric space. Is the subspace $\mathcal{C} \subset \mathrm{Maps}(X, X)$ of contraction mappings a complete subspace?

Let $\tau : \mathcal{C} \to X$ be the map which associates to any contraction its fixed point. Prove that τ is continuous.

Paper 3, Section II

12F Analysis II

For each of the following statements, provide a proof or justify a counterexample.

1. The norms $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$ on $\mathbb{R}^n$ are Lipschitz equivalent.

2. The norms $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ on the vector space of sequences $(x_i)_{i \geqslant 1}$ with $\sum |x_i| < \infty$ are Lipschitz equivalent.

3. Given a linear function $\phi : V \to W$ between normed real vector spaces, there is some N for which $\|\phi(x)\| \leqslant N$ for every $x \in V$ with $\|x\| \leqslant 1$.

4. Given a linear function $\phi : V \to W$ between normed real vector spaces for which there is some N for which $\|\phi(x)\| \leqslant N$ for every $x \in V$ with $\|x\| \leqslant 1$, then ϕ is continuous.

5. The uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is complete on the vector space of continuous real-valued functions f on $\mathbb{R}$ for which $f(x) = 0$ for $|x|$ sufficiently large.

6. The uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ is complete on the vector space of continuous real-valued functions f on $\mathbb{R}$ which are bounded.

Paper 2, Section II

12F Analysis II

Let $f : U \to \mathbb{R}$ be continuous on an open set $U \subset \mathbb{R}^2$. Suppose that on U the partial derivatives $D_1 f$, $D_2 f$, $D_1 D_2 f$ and $D_2 D_1 f$ exist and are continuous. Prove that $D_1 D_2 f \; = \; D_2 D_1 f$ on U.

If f is infinitely differentiable, and $m \in \mathbb{N}$, what is the maximum number of distinct m-th order partial derivatives that f may have on U ?

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \begin{cases} \dfrac{xy(x^4 - y^4)}{x^4 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

For each of f and g, determine whether they are (i) differentiable, (ii) infinitely differentiable at the origin. Briefly justify your answers.

Paper 3, Section I
2E Analysis II

Let $C[0,1]$ be the set of continuous real-valued functions on $[0,1]$ with the uniform norm. Suppose $T : C[0,1] \to C[0,1]$ is defined by

$$T(f)(x) = \int_0^x f(t^3)\,dt\,,$$

for all $x \in [0,1]$ and $f \in C[0,1]$. Is T a contraction mapping? Does T have a unique fixed point? Justify your answers.

Paper 4, Section I
3E Analysis II

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a bilinear function. Show that f is differentiable at any point in $\mathbb{R}^n \times \mathbb{R}^m$ and find its derivative.

Paper 2, Section I
3E Analysis II

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. What does it mean to say that f is differentiable at a point $(x,y) \in \mathbb{R}^2$? Prove directly from this definition, that if f is differentiable at (x,y), then f is continuous at (x,y).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function:

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \text{ and } y \text{ are rational} \\ 0 & \text{otherwise.} \end{cases}$$

For which points $(x,y) \in \mathbb{R}^2$ is f differentiable? Justify your answer.

Paper 1, Section II

11E Analysis II

State the inverse function theorem for a function $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose F is a differentiable bijection with F^{-1} also differentiable. Show that the derivative of F at any point in $\mathbb{R}^n$ is a linear isomorphism.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous. Assume there is a point $(a, b) \in \mathbb{R}^2$ for which $f(a, b) = 0$ and $\frac{\partial f}{\partial x}(a, b) \neq 0$. Prove that there exist open sets $U \subset \mathbb{R}^2$ and $W \subset \mathbb{R}$ containing (a, b) and b, respectively, such that for every $y \in W$ there exists a unique x such that $(x, y) \in U$ and $f(x, y) = 0$. Moreover, if we define $g : W \to \mathbb{R}$ by $g(y) = x$, prove that g is differentiable with continuous derivative. Find the derivative of g at b in terms of $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$.

Paper 4, Section II

12E Analysis II

State and prove the Bolzano-Weierstrass theorem in $\mathbb{R}^n$. [You may assume the Bolzano-Weierstrass theorem in $\mathbb{R}$.]

Let $X \subset \mathbb{R}^n$ be a subset and let $f : X \to X$ be a mapping such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$, where d is the Euclidean distance in $\mathbb{R}^n$. Prove that if X is closed and bounded, then f is a bijection. Is this result still true if we drop the boundedness assumption on X? Justify your answer.

Paper 3, Section II

12E Analysis II

Let f_n be a sequence of continuous functions on the interval $[0, 1]$ such that $f_n(x) \to f(x)$ for each x. For the three statements:

(a) $f_n \to f$ uniformly on $[0, 1]$;

(b) f is a continuous function;

(c) $\int_0^1 f_n(x)\, dx \to \int_0^1 f(x)\, dx$ as $n \to \infty$;

say which of the six possible implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(b) \Rightarrow (a)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (a)$, $(c) \Rightarrow (b)$ are true and which false, giving in each case a proof or counter-example.

Paper 2, Section II

12E Analysis II

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Fix $a \in \mathbb{R}^n$ and prove that the following two statements are equivalent:

(i) Given $\varepsilon > 0$ there is $\delta > 0$ such that $\|f(x) - f(a)\| < \varepsilon$ whenever $\|x - a\| < \delta$ (we use the standard norm in Euclidean space).

(ii) $f(x_n) \to f(a)$ for any sequence $x_n \to a$.

We say that f is continuous if (i) (or equivalently (ii)) holds for every $a \in \mathbb{R}^n$.

Let E and F be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For $f : \mathbb{R}^n \to \mathbb{R}^m$ as above, determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) If $f^{-1}(F)$ is closed whenever F is closed, then f is continuous.

(b) If f is continuous, then $f^{-1}(F)$ is closed whenever F is closed.

(c) If f is continuous, then $f(E)$ is open whenever E is open.

(d) If f is continuous, then $f(E)$ is bounded whenever E is bounded.

(e) If f is continuous and $f^{-1}(F)$ is bounded whenever F is bounded, then $f(E)$ is closed whenever E is closed.

Paper 3, Section I

2E Analysis II

Suppose f is a uniformly continuous mapping from a metric space X to a metric space Y. Prove that $f(x_n)$ is a Cauchy sequence in Y for every Cauchy sequence x_n in X.

Let f be a continuous mapping between metric spaces and suppose that f has the property that $f(x_n)$ is a Cauchy sequence whenever x_n is a Cauchy sequence. Is it true that f must be uniformly continuous? Justify your answer.

Paper 4, Section I

3E Analysis II

Let $B[0,1]$ denote the set of bounded real-valued functions on $[0,1]$. A distance d on $B[0,1]$ is defined by

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Given that $(B[0,1], d)$ is a metric space, show that it is complete. Show that the subset $C[0,1] \subset B[0,1]$ of continuous functions is a closed set.

Paper 2, Section I

3E Analysis II

Define differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a > 0$ be a constant. For which points $(x,y) \in \mathbb{R}^2$ is

$$f(x,y) = |x|^a + |x-y|$$

differentiable? Justify your answer.

Paper 1, Section II

11E Analysis II

What is meant by saying that a sequence of functions f_n converges uniformly to a function f?

Let f_n be a sequence of differentiable functions on $[a,b]$ with f_n' continuous and such that $f_n(x_0)$ converges for some point $x_0 \in [a,b]$. Assume in addition that f_n' converges uniformly on $[a,b]$. Prove that f_n converges uniformly to a differentiable function f on $[a,b]$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$. [You may assume that the uniform limit of continuous functions is continuous.]

Show that the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for $s > 1$ and is uniformly convergent on $[1+\varepsilon, \infty)$ for any $\varepsilon > 0$. Show that $\zeta(s)$ is differentiable on $(1,\infty)$ and

$$\zeta'(s) = -\sum_{n=2}^{\infty} \frac{\log n}{n^s}.$$

[You may use the Weierstrass M-test provided it is clearly stated.]

Paper 4, Section II

12E Analysis II

Define a contraction mapping and state the contraction mapping theorem.

Let (X,d) be a non-empty complete metric space and let $\phi : X \to X$ be a map. Set $\phi^1 = \phi$ and $\phi^{n+1} = \phi \circ \phi^n$. Assume that for some integer $r \geqslant 1$, ϕ^r is a contraction mapping. Show that ϕ has a unique fixed point y and that any $x \in X$ has the property that $\phi^n(x) \to y$ as $n \to \infty$.

Let $C[0,1]$ be the set of continuous real-valued functions on $[0,1]$ with the uniform norm. Suppose $T : C[0,1] \to C[0,1]$ is defined by

$$T(f)(x) = \int_0^x f(t)\,dt$$

for all $x \in [0,1]$ and $f \in C[0,1]$. Show that T is not a contraction mapping but that T^2 is.

Paper 3, Section II

12E Analysis II

Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

Assume f is differentiable at x and let $D_x f$ denote the derivative of f at x. Show that

$$D_x f(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

for any $v \in \mathbb{R}^n$.

Assume now that f is such that for some fixed x and for every $v \in \mathbb{R}^n$ the limit

$$\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists. Is it true that f is differentiable at x? Justify your answer.

Let M_k denote the set of all $k \times k$ real matrices which is identified with $\mathbb{R}^{k^2}$. Consider the function $f : M_k \to M_k$ given by $f(A) = A^3$. Explain why f is differentiable. Show that the derivative of f at the matrix A is given by

$$D_A f(H) = HA^2 + AHA + A^2 H$$

for any matrix $H \in M_k$. State carefully the inverse function theorem and use it to prove that there exist open sets U and V containing the identity matrix such that given $B \in V$ there exists a unique $A \in U$ such that $A^3 = B$.

Paper 2, Section II

12E Analysis II

What is meant by saying that two norms on a real vector space are Lipschitz equivalent?

Show that any two norms on $\mathbb{R}^n$ are Lipschitz equivalent. [You may assume that a continuous function on a closed bounded set in $\mathbb{R}^n$ has closed bounded image.]

Show that $\|f\|_1 = \int_{-1}^{1} |f(x)|\, dx$ defines a norm on the space $C[-1,1]$ of continuous real-valued functions on $[-1,1]$. Is it Lipschitz equivalent to the uniform norm? Justify your answer. Prove that the normed space $(C[-1,1], \|\cdot\|_1)$ is not complete.

Paper 3, Section I

2G Analysis II

Consider the map $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$f(x,\, y,\, z) \;=\; (x+y+z,\; xy+yz+zx,\; xyz)\,.$$

Show that f is differentiable everywhere and find its derivative.

Stating carefully any theorem that you quote, show that f is locally invertible near a point (x,y,z) unless $(x-y)(y-z)(z-x)=0$.

Paper 2, Section I

3G Analysis II

Let $c>1$ be a real number, and let F_c be the space of sequences $\mathbf{a}=(a_1,a_2,\ldots)$ of real numbers a_i with $\sum_{r=1}^{\infty} c^{-r}|a_r|$ convergent. Show that $\|\mathbf{a}\|_c = \sum_{r=1}^{\infty} c^{-r}|a_r|$ defines a norm on F_c.

Let F denote the space of sequences $\mathbf{a}$ with $|a_i|$ bounded; show that $F \subset F_c$. If $c' > c$, show that the norms on F given by restricting to F the norms $\|\,.\,\|_c$ on F_c and $\|\,.\,\|_{c'}$ on $F_{c'}$ are not Lipschitz equivalent.

By considering sequences of the form $\mathbf{a}^{(n)} = (a,\, a^2,\, \ldots\,,\, a^n,\, 0,\, 0,\, \ldots)$ in F, for a an appropriate real number, or otherwise, show that F (equipped with the norm $\|\,.\,\|_c$) is not complete.

Paper 4, Section I

3G Analysis II

Let S denote the set of continuous real-valued functions on the interval $[0,1]$. For $f,g \in S$, set

$$d_1(f,g) = \sup\{|f(x)-g(x)| \,:\, x\in[0,1]\} \quad \text{and} \quad d_2(f,g) = \int_0^1 |f(x)-g(x)|\,dx\,.$$

Show that both d_1 and d_2 define metrics on S. Does the identity map on S define a continuous map of metric spaces $(S,d_1)\to(S,d_2)$? Does the identity map define a continuous map of metric spaces $(S,d_2)\to(S,d_1)$?

Paper 1, Section II

11G Analysis II

State and prove the contraction mapping theorem. Demonstrate its use by showing that the differential equation $f'(x) = f(x^2)$, with boundary condition $f(0) = 1$, has a unique solution on $[0, 1)$, with one-sided derivative $f'(0) = 1$ at zero.

Paper 2, Section II

12G Analysis II

Suppose the functions f_n $(n = 1, 2, \ldots)$ are defined on the open interval $(0, 1)$ and that f_n tends uniformly on $(0, 1)$ to a function f. If the f_n are continuous, show that f is continuous. If the f_n are differentiable, show by example that f need not be differentiable.

Assume now that each f_n is differentiable and the derivatives f_n' converge uniformly on $(0, 1)$. For any given $c \in (0, 1)$, we define functions $g_{c,n}$ by

$$g_{c,n}(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} & \text{for } x \neq c\,, \\ f_n'(c) & \text{for } x = c\,. \end{cases}$$

Show that each $g_{c,n}$ is continuous. Using the general principle of uniform convergence (the Cauchy criterion) and the Mean Value Theorem, or otherwise, prove that the functions $g_{c,n}$ converge uniformly to a continuous function g_c on $(0, 1)$, where

$$g_c(x) = \frac{f(x) - f(c)}{x - c} \qquad \text{for } x \neq c\ .$$

Deduce that f is differentiable on $(0, 1)$.

Paper 3, Section II
12G Analysis II

Let $f : U \to \mathbf{R}^n$ be a map on an open subset $U \subset \mathbf{R}^m$. Explain what it means for f to be differentiable on U. If $g : V \to \mathbf{R}^m$ is a differentiable map on an open subset $V \subset \mathbf{R}^p$ with $g(V) \subset U$, state and prove the Chain Rule for the derivative of the composite fg.

Suppose now $F : \mathbf{R}^n \to \mathbf{R}$ is a differentiable function for which the partial derivatives $D_1F(\mathbf{x}) = D_2F(\mathbf{x}) = \ldots = D_nF(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$. By considering the function $G : \mathbf{R}^n \to \mathbf{R}$ given by

$$G(y_1, \ldots, y_n) = F\left(y_1, \ldots, y_{n-1}, y_n - \sum_{i=1}^{n-1} y_i\right),$$

or otherwise, show that there exists a differentiable function $h : \mathbf{R} \to \mathbf{R}$ with $F(x_1, \ldots, x_n) = h(x_1 + \cdots + x_n)$ at all points of $\mathbf{R}^n$.

Paper 4, Section II
12G Analysis II

What does it mean to say that a function f on an interval in $\mathbf{R}$ is *uniformly continuous*? Assuming the Bolzano–Weierstrass theorem, show that any continuous function on a finite closed interval is uniformly continuous.

Suppose that f is a continuous function on the real line, and that $f(x)$ tends to finite limits as $x \to \pm\infty$; show that f is uniformly continuous.

If f is a uniformly continuous function on $\mathbf{R}$, show that $f(x)/x$ is bounded as $x \to \pm\infty$. If g is a continuous function on $\mathbf{R}$ for which $g(x)/x \to 0$ as $x \to \pm\infty$, determine whether g is necessarily uniformly continuous, giving proof or counterexample as appropriate.

Paper 2, Section I

3E Analysis II

State and prove the contraction mapping theorem. Let $f(x) = e^{-x}$. By considering $f(f(x))$ and using the contraction mapping theorem, show that there is a unique real number x such that $x = e^{-x}$.

Paper 4, Section I

3E Analysis II

Let $(s_n)_{n=1}^{\infty}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let $s : \mathbb{R} \to \mathbb{R}$ be another continuous function. What does it mean to say that $s_n \to s$ uniformly? Give examples (without proof) of a sequence (s_n) of nonzero functions which converges to 0 uniformly, and of a sequence which converges to 0 pointwise but *not* uniformly. Show that if $s_n \to s$ uniformly then

$$\int_{-1}^{1} s_n(x)\,dx \to \int_{-1}^{1} s(x)\,dx.$$

Give an example of a continuous function $s : \mathbb{R} \to \mathbb{R}$ with $s(x) \geqslant 0$ for all x, $s(x) \to 0$ as $|x| \to \infty$ but for which $\int_{-\infty}^{\infty} s(x)\,dx$ does not converge. For each positive integer n define $s_n(x)$ to be equal to $s(x)$ if $|x| \leqslant n$, and to be $s(n)\min(1, ||x|-n|^{-2})$ for $|x| > n$. Show that the functions s_n are continuous, tend uniformly to s, and furthermore that $\int_{-\infty}^{\infty} s_n(x)\,dx$ exists and is finite for all n.

Paper 3, Section I

3E Analysis II

What is meant by a *norm* on $\mathbb{R}^n$? For $\mathbf{x} \in \mathbb{R}^n$ write

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|,$$

$$\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}.$$

Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms. [*You may assume the Cauchy-Schwarz inequality.*]

Find the smallest constant C_n such that $\|x\|_1 \leqslant C_n\|x\|_2$ for all $x \in \mathbb{R}^n$, and also the smallest constant C_n' such that $\|x\|_2 \leqslant C_n'\|x\|_1$ for all $x \in \mathbb{R}^n$.

Paper 1, Section II

11E Analysis II

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \|2^n x\|,$$

where $\|t\|$ is the distance from t to the nearest integer. Prove that f is continuous. [*Results about uniform convergence may not be used unless they are clearly stated and proved.*]

Suppose now that $g : \mathbb{R} \to \mathbb{R}$ is a function which is differentiable at some point x, and let $(u_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty}$ be two sequences of real numbers with $u_n \leqslant x \leqslant v_n$ for all n, $u_n \neq v_n$ and $u_n, v_n \to x$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \frac{g(v_n) - g(u_n)}{v_n - u_n}$$

exists.

By considering appropriate sequences of rationals with denominator 2^{-n}, or otherwise, show that f is nowhere differentiable.

Paper 3, Section II

13E Analysis II

What does it mean for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ of several variables to be differentiable at a point $\mathbf{x}$? State and prove the chain rule for functions of several variables. For each of the following two functions from $\mathbb{R}^2$ to $\mathbb{R}$, give with proof the set of points at which it is differentiable:

$$g_1(x,y) = \begin{cases} (x^2 - y^2)\sin\frac{1}{x^2-y^2} & \text{if } x \neq \pm y \\ 0 & \text{otherwise;} \end{cases}$$

$$g_2(x,y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{x^2+y^2} & \text{if at least one of } x, y \text{ is not } 0 \\ 0 & \text{if } x = y = 0. \end{cases}$$

Paper 2, Section II

13E Analysis II

Let $U \subseteq \mathbb{R}^n$ be a set. What does it mean to say that U is *open*? Show that if U is open and if $f : U \to \{0, 1\}$ is a continuous function then f is also differentiable, and that its derivative is zero.

Suppose that $g : U \to \mathbb{R}$ is differentiable and that $\|(Dg)|_x\| \leqslant M$ for all x, where $(Dg)|_x$ denotes the derivative of g at x and $\| \cdot \|$ is the operator norm. Suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and that the line segment $[\mathbf{a}, \mathbf{b}] = \{\lambda \mathbf{a} + (1 - \lambda)\mathbf{b} : \lambda \in [0, 1]\}$ lies wholly in U. Prove that $|g(\mathbf{a}) - g(\mathbf{b})| \leqslant M\|\mathbf{a} - \mathbf{b}\|$.

Let $\ell_1, \ldots, \ell_k$ be (infinite) lines in $\mathbb{R}^3$, and write $V = \mathbb{R}^3 \setminus (\ell_1 \cup \cdots \cup \ell_k)$. If $\mathbf{a}, \mathbf{b} \in V$, show that there is some $\mathbf{c} \in V$ such that the line segments $[\mathbf{a}, \mathbf{c}]$ and $[\mathbf{c}, \mathbf{b}]$ both lie inside V. [*You may assume without proof that* $\mathbb{R}^3$ *may not be written as the union of finitely many planes.*]

Show that if $V \to \{0, 1\}$ is a continuous function then f is constant on V.

Paper 4, Section II

13E Analysis II

Let (X, d) be a metric space with at least two points. If $f : X \to \mathbb{R}$ is a function, write

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} + \sup_z |f(z)|,$$

provided that this supremum is finite. Let $\mathrm{Lip}(X) = \{f : \mathrm{Lip}(f) \text{ is defined}\}$. Show that $\mathrm{Lip}(X)$ is a vector space over $\mathbb{R}$, and that Lip is a norm on it.

Now let $X = \mathbb{R}$. Suppose that $(f_i)_{i=1}^{\infty}$ is a sequence of functions with $\mathrm{Lip}(f_i) \leqslant 1$ and with the property that the sequence $f_i(q)$ converges as $i \to \infty$ for every rational number q. Show that the f_i converge pointwise to a function f satisfying $\mathrm{Lip}(f) \leqslant 1$.

Suppose now that $(f_i)_{i=1}^{\infty}$ are any functions with $\mathrm{Lip}(f_i) \leqslant 1$. Show that there is a subsequence $f_{i_1}, f_{i_2}, \ldots$ which converges pointwise to a function f with $\mathrm{Lip}(f) \leqslant 1$.

1/II/11F **Analysis II**

State and prove the Contraction Mapping Theorem.

Let (X, d) be a nonempty complete metric space and $f: X \to X$ a mapping such that, for some $k > 0$, the kth iterate f^k of f (that is, f composed with itself k times) is a contraction mapping. Show that f has a unique fixed point.

Now let X be the space of all continuous real-valued functions on $[0,1]$, equipped with the uniform norm $\|h\|_\infty = \sup\{|h(t)| : t \in [0,1]\}$, and let $\phi: \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition

$$|\phi(x,t) - \phi(y,t)| \leqslant M|x-y|$$

for all $t \in [0,1]$ and all $x, y \in \mathbb{R}$, where M is a constant. Let $F: X \to X$ be defined by

$$F(h)(t) = g(t) + \int_0^t \phi(h(s), s)\, ds\ ,$$

where g is a fixed continuous function on $[0,1]$. Show by induction on n that

$$|F^n(h)(t) - F^n(k)(t)| \leqslant \frac{M^n t^n}{n!} \|h-k\|_\infty$$

for all $h, k \in X$ and all $t \in [0,1]$. Deduce that the integral equation

$$f(t) = g(t) + \int_0^t \phi(f(s), s)\, ds$$

has a unique continuous solution f on $[0,1]$.

2/I/3F **Analysis II**

Explain what is meant by the statement that a sequence (f_n) of functions defined on an interval $[a,b]$ converges uniformly to a function f. If (f_n) converges uniformly to f, and each f_n is continuous on $[a,b]$, prove that f is continuous on $[a,b]$.

Now suppose additionally that (x_n) is a sequence of points of $[a,b]$ converging to a limit x. Prove that $f_n(x_n) \to f(x)$.

2/II/13F Analysis II

Let $(u_n(x) : n = 0, 1, 2, \ldots)$ be a sequence of real-valued functions defined on a subset E of $\mathbb{R}$. Suppose that for all n and all $x \in E$ we have $|u_n(x)| \leqslant M_n$, where $\sum_{n=0}^{\infty} M_n$ converges. Prove that $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly on E.

Now let $E = \mathbb{R} \setminus \mathbb{Z}$, and consider the series $\sum_{n=0}^{\infty} u_n(x)$, where $u_0(x) = 1/x^2$ and

$$u_n(x) = 1/(x-n)^2 + 1/(x+n)^2$$

for $n > 0$. Show that the series converges uniformly on $E_R = \{x \in E : |x| < R\}$ for any real number R. Deduce that $f(x) = \sum_{n=0}^{\infty} u_n(x)$ is a continuous function on E. Does the series converge uniformly on E? Justify your answer.

3/I/3F Analysis II

Explain what it means for a function $f(x, y)$ of two variables to be differentiable at a point (x_0, y_0). If f is differentiable at (x_0, y_0), show that for any α the function g_α defined by

$$g_\alpha(t) = f(x_0 + t \cos \alpha, y_0 + t \sin \alpha)$$

is differentiable at $t = 0$, and find its derivative in terms of the partial derivatives of f at (x_0, y_0).

Consider the function f defined by

$$\begin{aligned} f(x,y) &= (x^2y + xy^2)/(x^2+y^2) \qquad &((x,y) \neq (0,0)) \\ &= 0 \qquad &((x,y) = (0,0)). \end{aligned}$$

Is f differentiable at $(0, 0)$? Justify your answer.

3/II/13F Analysis II

Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a function, and (x_0, y_0) a point of $\mathbb{R}^2$. Prove that if the partial derivatives of f exist in some open disc around (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Now let X denote the vector space of all $(n \times n)$ real matrices, and let $f \colon X \to \mathbb{R}$ be the function assigning to each matrix its determinant. Show that f is differentiable at the identity matrix I, and that $Df|_I$ is the linear map $H \mapsto \operatorname{tr} H$. Deduce that f is differentiable at any invertible matrix A, and that $Df|_A$ is the linear map $H \mapsto \det A \operatorname{tr}(A^{-1}H)$.

Show also that if K is a matrix with $\|K\| < 1$, then $(I + K)$ is invertible. Deduce that f is twice differentiable at I, and find $D^2f|_I$ as a bilinear map $X \times X \to \mathbb{R}$.

[*You may assume that the norm* $\| - \|$ *on* X *is complete, and that it satisfies the inequality* $\|AB\| \leqslant \|A\|.\|B\|$ *for any two matrices* A *and* B.]

4/I/3F **Analysis II**

Let X be the vector space of all continuous real-valued functions on the unit interval $[0,1]$. Show that the functions

$$\|f\|_1 = \int_0^1 |f(t)|\, dt \qquad \text{and} \qquad \|f\|_\infty = \sup\{|f(t)| : 0 \leqslant t \leqslant 1\}$$

both define norms on X.

Consider the sequence (f_n) defined by $f_n(t) = nt^n(1-t)$. Does (f_n) converge in the norm $\|-\|_1$? Does it converge in the norm $\|-\|_\infty$? Justify your answers.

4/II/13F **Analysis II**

Explain what it means for two norms on a real vector space to be Lipschitz equivalent. Show that if two norms are Lipschitz equivalent, then one is complete if and only if the other is.

Let $\|-\|$ be an arbitrary norm on the finite-dimensional space $\mathbb{R}^n$, and let $\|-\|_2$ denote the standard (Euclidean) norm. Show that for every $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_2 = 1$, we have

$$\|\mathbf{x}\| \leqslant \|\mathbf{e}_1\| + \|\mathbf{e}_2\| + \cdots + \|\mathbf{e}_n\|$$

where $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is the standard basis for $\mathbb{R}^n$, and deduce that the function $\|-\|$ is continuous with respect to $\|-\|_2$. Hence show that there exists a constant $m > 0$ such that $\|\mathbf{x}\| \geqslant m$ for all $\mathbf{x}$ with $\|\mathbf{x}\|_2 = 1$, and deduce that $\|-\|$ and $\|-\|_2$ are Lipschitz equivalent.

[*You may assume the Bolzano–Weierstrass Theorem.*]

1/II/11H Analysis II

Define what it means for a function $f : \mathbb{R}^a \to \mathbb{R}^b$ to be differentiable at a point $p \in \mathbb{R}^a$ with derivative a linear map $Df|_p$.

State the Chain Rule for differentiable maps $f : \mathbb{R}^a \to \mathbb{R}^b$ and $g : \mathbb{R}^b \to \mathbb{R}^c$. Prove the Chain Rule.

Let $\|x\|$ denote the standard Euclidean norm of $x \in \mathbb{R}^a$. Find the partial derivatives $\frac{\partial f}{\partial x_i}$ of the function $f(x) = \|x\|$ where they exist.

2/I/3H Analysis II

For integers a and b, define $d(a,b)$ to be 0 if $a = b$, or $\frac{1}{2^n}$ if $a \neq b$ and n is the largest non-negative integer such that $a - b$ is a multiple of 2^n. Show that d is a metric on the integers $\mathbb{Z}$.

Does the sequence $x_n = 2^n - 1$ converge in this metric?

2/II/13H Analysis II

Show that the limit of a uniformly convergent sequence of real valued continuous functions on $[0,1]$ is continuous on $[0,1]$.

Let f_n be a sequence of continuous functions on $[0,1]$ which converge point-wise to a continuous function. Suppose also that the integrals $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$. Must the functions f_n converge uniformly to f? Prove or give a counterexample.

Let f_n be a sequence of continuous functions on $[0,1]$ which converge point-wise to a function f. Suppose that f is integrable and that the integrals $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$. Is the limit f necessarily continuous? Prove or give a counterexample.

3/I/3H Analysis II

Define uniform continuity for a real-valued function on an interval in the real line. Is a uniformly continuous function on the real line necessarily bounded?

Which of the following functions are uniformly continuous on the real line?

(i) $f(x) = x \sin x$,

(ii) $f(x) = e^{-x^4}$.

Justify your answers.

3/II/13H **Analysis II**

Let V be the real vector space of continuous functions $f:[0,1]\to\mathbb{R}$. Show that defining

$$||f|| = \int_0^1 |f(x)|dx$$

makes V a normed vector space.

Define $f_n(x) = \sin nx$ for positive integers n. Is the sequence (f_n) convergent to some element of V? Is (f_n) a Cauchy sequence in V? Justify your answers.

4/I/3H **Analysis II**

Define uniform convergence for a sequence $f_1, f_2, \ldots$ of real-valued functions on the interval $(0,1)$.

For each of the following sequences of functions on $(0,1)$, find the pointwise limit function. Which of these sequences converge uniformly on $(0,1)$?

(i) $f_n(x) = \log\,(x + \frac{1}{n})$,

(ii) $f_n(x) = \cos\,(\frac{x}{n})$.

Justify your answers.

4/II/13H **Analysis II**

State and prove the Contraction Mapping Theorem.

Find numbers a and b, with $a < 0 < b$, such that the mapping $T : C[a,b] \to C[a,b]$ defined by

$$T(f)(x) = 1 + \int_0^x 3t\, f(t)\, dt$$

is a contraction, in the sup norm on $C[a,b]$. Deduce that the differential equation

$$\frac{dy}{dx} = 3xy, \qquad \text{with } y = 1 \text{ when } x = 0,$$

has a unique solution in some interval containing 0.

1/II/11F **Analysis II**

Let a_n and b_n be sequences of real numbers for $n \geqslant 1$ such that $|a_n| \leqslant c/n^{1+\epsilon}$ and $|b_n| \leqslant c/n^{1+\epsilon}$ for all $n \geqslant 1$, for some constants $c > 0$ and $\epsilon > 0$. Show that the series

$$f(x) = \sum_{n \geqslant 1} a_n \cos nx + \sum_{n \geqslant 1} b_n \sin nx$$

converges uniformly to a continuous function on the real line. Show that f is periodic in the sense that $f(x + 2\pi) = f(x)$.

Now suppose that $|a_n| \leqslant c/n^{2+\epsilon}$ and $|b_n| \leqslant c/n^{2+\epsilon}$ for all $n \geqslant 1$, for some constants $c > 0$ and $\epsilon > 0$. Show that f is differentiable on the real line, with derivative

$$f'(x) = -\sum_{n \geqslant 1} n a_n \sin nx + \sum_{n \geqslant 1} n b_n \cos nx.$$

[*You may assume the convergence of standard series.*]

2/I/3F **Analysis II**

Define uniform convergence for a sequence $f_1, f_2, \ldots$ of real-valued functions on an interval in $\mathbf{R}$. If (f_n) is a sequence of continuous functions converging uniformly to a (necessarily continuous) function f on a closed interval $[a, b]$, show that

$$\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx$$

as $n \to \infty$.

Which of the following sequences of functions $f_1, f_2, \ldots$ converges uniformly on the open interval $(0, 1)$? Justify your answers.

(i) $f_n(x) = 1/(nx)$;

(ii) $f_n(x) = e^{-x/n}$.

2/II/13F Analysis II

For a smooth mapping $F : \mathbf{R}^2 \to \mathbf{R}^2$, the Jacobian $J(F)$ at a point (x, y) is defined as the determinant of the derivative DF, viewed as a linear map $\mathbf{R}^2 \to \mathbf{R}^2$. Suppose that F maps into a curve in the plane, in the sense that F is a composition of two smooth mappings $\mathbf{R}^2 \to \mathbf{R} \to \mathbf{R}^2$. Show that the Jacobian of F is identically zero.

Conversely, let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a smooth mapping whose Jacobian is identically zero. Write $F(x, y) = (f(x, y), g(x, y))$. Suppose that $\partial f/\partial y|_{(0,0)} \neq 0$. Show that $\partial f/\partial y \neq 0$ on some open neighbourhood U of $(0, 0)$ and that on U

$$(\partial g/\partial x,\ \partial g/\partial y) = e(x, y)\,(\partial f/\partial x,\ \partial f/\partial y)$$

for some smooth function e defined on U. Now suppose that $c : \mathbf{R} \to U$ is a smooth curve of the form $t \mapsto (t, \alpha(t))$ such that $F \circ c$ is constant. Write down a differential equation satisfied by α. Apply an existence theorem for differential equations to show that there is a neighbourhood V of $(0, 0)$ such that every point in V lies on a curve $t \mapsto (t, \alpha(t))$ on which F is constant.

[*A function is said to be smooth when it is infinitely differentiable. Detailed justification of the smoothness of the functions in question is not expected.*]

3/I/3F Analysis II

Define what it means for a function $f : \mathbf{R}^2 \to \mathbf{R}$ to be *differentiable* at a point (a, b). If the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are defined and continuous on a neighbourhood of (a, b), show that f is differentiable at (a, b).

3/II/13F Analysis II

State precisely the inverse function theorem for a smooth map F from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^2$.

Define $F : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$F(x, y) = (x^3 - x - y^2, y).$$

Determine the open subset of $\mathbf{R}^2$ on which F is locally invertible.

Let C be the curve $\{(x, y) \in \mathbf{R}^2 : x^3 - x - y^2 = 0\}$. Show that C is the union of the two subsets $C_1 = \{(x, y) \in C : x \in [-1, 0]\}$ and $C_2 = \{(x, y) \in C : x \geqslant 1\}$. Show that for each $y \in \mathbf{R}$ there is a unique $x = p(y)$ such that $(x, y) \in C_2$. Show that F is locally invertible at all points of C_2, and deduce that $p(y)$ is a smooth function of y.

[*A function is said to be smooth when it is infinitely differentiable.*]

4/I/3F Analysis II

Let V be the vector space of all sequences $(x_1, x_2, \ldots)$ of real numbers such that x_i converges to zero. Show that the function

$$|(x_1, x_2, \ldots)| = \max_{i \geqslant 1} |x_i|$$

defines a norm on V.

Is the sequence

$$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), \ldots$$

convergent in V? Justify your answer.

4/II/13F Analysis II

State precisely the contraction mapping theorem.

An ancient way to approximate the square root of a positive number a is to start with a guess $x > 0$ and then hope that the average of x and a/x gives a better guess. We can then repeat the procedure using the new guess. Justify this procedure as follows. First, show that all the guesses after the first one are greater than or equal to $\sqrt{a}$. Then apply the properties of contraction mappings to the interval $[\sqrt{a}, \infty)$ to show that the procedure always converges to $\sqrt{a}$.

Once the above procedure is close enough to $\sqrt{a}$, estimate how many more steps of the procedure are needed to get one more decimal digit of accuracy in computing $\sqrt{a}$.

1/II/11B **Analysis II**

Let $(f_n)_{n \geqslant 1}$ be a sequence of continuous real-valued functions defined on a set $E \subset \mathbf{R}$. Suppose that the functions f_n converge uniformly to a function f. Prove that f is continuous on E.

Show that the series $\sum_{n=1}^{\infty} 1/n^{1+x}$ defines a continuous function on the half-open interval $(0,1]$.

[*Hint: You may assume the convergence of standard series.*]

2/I/3B **Analysis II**

Define uniform continuity for a real-valued function defined on an interval in $\mathbf{R}$.

Is a uniformly continuous function on the interval $(0,1)$ necessarily bounded?

Is $1/x$ uniformly continuous on $(0,1)$?

Is $\sin(1/x)$ uniformly continuous on $(0,1)$?

Justify your answers.

2/II/13B **Analysis II**

Use the standard metric on $\mathbf{R}^n$ in this question.

(i) Let A be a nonempty closed subset of $\mathbf{R}^n$ and y a point in $\mathbf{R}^n$. Show that there is a point $x \in A$ which minimizes the distance to y, in the sense that $d(x,y) \leqslant d(a,y)$ for all $a \in A$.

(ii) Suppose that the set A in part (i) is convex, meaning that A contains the line segment between any two of its points. Show that point $x \in A$ described in part (i) is unique.

3/I/3B **Analysis II**

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function. What does it mean to say that f is *differentiable* at a point (a, b) in $\mathbf{R}^2$? Show that if f is differentiable at (a, b), then f is continuous at (a, b).

For each of the following functions, determine whether or not it is differentiable at $(0, 0)$. Justify your answers.

(i)

$$f(x,y) = \begin{cases} x^2y^2(x^2+y^2)^{-1} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(ii)

$$f(x,y) = \begin{cases} x^2(x^2+y^2)^{-1} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

3/II/13B **Analysis II**

Let f be a real-valued differentiable function on an open subset U of $\mathbf{R}^n$. Assume that $0 \notin U$ and that for all $x \in U$ and $\lambda > 0$, λx is also in U. Suppose that f is *homogeneous* of degree $c \in \mathbf{R}$, in the sense that $f(\lambda x) = \lambda^c f(x)$ for all $x \in U$ and $\lambda > 0$. By means of the Chain Rule or otherwise, show that

$$Df|_x(x) = cf(x)$$

for all $x \in U$. (Here $Df|_x$ denotes the derivative of f at x, viewed as a linear map $\mathbf{R}^n \to \mathbf{R}$.)

Conversely, show that any differentiable function f on U with $Df|_x(x) = cf(x)$ for all $x \in U$ must be homogeneous of degree c.

4/I/3B **Analysis II**

Let V be the vector space of continuous real-valued functions on $[0, 1]$. Show that the function

$$||f|| = \int_0^1 |f(x)|\, dx$$

defines a norm on V.

For $n = 1, 2, \ldots$, let $f_n(x) = e^{-nx}$. Is f_n a convergent sequence in the space V with this norm? Justify your answer.

4/II/13B **Analysis II**

Let $F : [-a,a] \times [x_0 - r, x_0 + r] \to \mathbf{R}$ be a continuous function. Let C be the maximum value of $|F(t,x)|$. Suppose there is a constant K such that

$$|F(t,x) - F(t,y)| \leqslant K|x-y|$$

for all $t \in [-a,a]$ and $x, y \in [x_0 - r, x_0 + r]$. Let $b < \min(a, r/C, 1/K)$. Show that there is a unique C^1 function $x : [-b,b] \to [x_0 - r, x_0 + r]$ such that

$$x(0) = x_0$$

and

$$\frac{dx}{dt} = F(t, x(t)).$$

[*Hint: First show that the differential equation with its initial condition is equivalent to the integral equation*

$$x(t) = x_0 + \int_0^t F(s, x(s))\, ds. \quad \Big]$$

1/I/4G **Analysis II**

Define what it means for a sequence of functions $F_n : (0,1) \to \mathbb{R}$, where $n = 1, 2, \ldots$, to converge uniformly to a function F.

For each of the following sequences of functions on $(0,1)$, find the pointwise limit function. Which of these sequences converge uniformly? Justify your answers.

(i) $F_n(x) = \frac{1}{n} e^x$

(ii) $F_n(x) = e^{-nx^2}$

(iii) $F_n(x) = \sum_{i=0}^{n} x^i$

1/II/15G **Analysis II**

State the axioms for a norm on a vector space. Show that the usual Euclidean norm on $\mathbb{R}^n$,

$$||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2},$$

satisfies these axioms.

Let U be any bounded convex open subset of $\mathbb{R}^n$ that contains 0 and such that if $x \in U$ then $-x \in U$. Show that there is a norm on $\mathbb{R}^n$, satisfying the axioms, for which U is the set of points in $\mathbb{R}^n$ of norm less than 1.

2/I/3G **Analysis II**

Consider a sequence of continuous functions $F_n : [-1,1] \to \mathbb{R}$. Suppose that the functions F_n converge uniformly to some continuous function F. Show that the integrals $\int_{-1}^{1} F_n(x)dx$ converge to $\int_{-1}^{1} F(x)dx$.

Give an example to show that, even if the functions $F_n(x)$ and $F(x)$ are differentiable, the derivatives $F_n'(0)$ need not converge to $F'(0)$.

2/II/14G Analysis II

Let X be a non-empty complete metric space. Give an example to show that the intersection of a descending sequence of non-empty closed subsets of X, $A_1 \supset A_2 \supset \cdots$, can be empty. Show that if we also assume that

$$\lim_{n\to\infty} \operatorname{diam}\,(A_n) = 0$$

then the intersection is not empty. Here the diameter $\operatorname{diam}(A)$ is defined as the supremum of the distances between any two points of a set A.

We say that a subset A of X is *dense* if it has nonempty intersection with every nonempty open subset of X. Let $U_1, U_2, \ldots$ be any sequence of dense open subsets of X. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not empty.

[*Hint: Look for a descending sequence of subsets* $A_1 \supset A_2 \supset \cdots$, *with* $A_i \subset U_i$, *such that the previous part of this problem applies.*]

3/I/4F Analysis II

Let X and X' be metric spaces with metrics d and d'. If $u = (x, x')$ and $v = (y, y')$ are any two points of $X \times X'$, prove that the formula

$$D(u, v) = \max\big\{d(x, y), d'(x', y')\big\}$$

defines a metric on $X \times X'$. If $X = X'$, prove that the diagonal Δ of $X \times X$ is closed in $X \times X$.

3/II/16F Analysis II

State and prove the contraction mapping theorem.

Let a be a positive real number, and take $X = [\sqrt{\frac{a}{2}},\ \infty)$. Prove that the function

$$f(x) = \frac{1}{2}\Big(x + \frac{a}{x}\Big)$$

is a contraction from X to X. Find the unique fixed point of f.

4/I/3F **Analysis II**

Let U, V be open sets in $\mathbb{R}^n, \mathbb{R}^m$, respectively, and let $f : U \to V$ be a map. What does it mean for f to be differentiable at a point u of U?

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the map given by

$$g(x, y) = |x| + |y| \,.$$

Prove that g is differentiable at all points (a, b) with $ab \neq 0$.

4/II/13F **Analysis II**

State the inverse function theorem for maps $f : U \to \mathbb{R}^2$, where U is a non-empty open subset of $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (x, x^3 + y^3 - 3xy) \,.$$

Find a non-empty open subset U of $\mathbb{R}^2$ such that f is locally invertible on U, and compute the derivative of the local inverse.

Let C be the set of all points (x, y) in $\mathbb{R}^2$ satisfying

$$x^3 + y^3 - 3xy = 0 \,.$$

Prove that f is locally invertible at all points of C except $(0, 0)$ and $(2^{2/3}, 2^{1/3})$. Deduce that, for each point (a, b) in C except $(0, 0)$ and $(2^{2/3}, 2^{1/3})$, there exist open intervals I, J containing a, b, respectively, such that for each x in I, there is a unique point y in J with (x, y) in C.

1/I/1F **Analysis II**

Let E be a subset of $\mathbb{R}^n$. Prove that the following conditions on E are equivalent:

(i) E is closed and bounded.

(ii) E has the Bolzano–Weierstrass property (i.e., every sequence in E has a subsequence convergent to a point of E).

(iii) Every continuous real-valued function on E is bounded.

[*The Bolzano–Weierstrass property for bounded closed intervals in* $\mathbb{R}^1$ *may be assumed.*]

1/II/10F **Analysis II**

Explain briefly what is meant by a *metric space*, and by a *Cauchy sequence* in a metric space.

A function $d : X \times X \to \mathbb{R}$ is called a pseudometric on X if it satisfies all the conditions for a metric except the requirement that $d(x, y) = 0$ implies $x = y$. If d is a pseudometric on X, show that the binary relation R on X defined by $x\,R\,y \Leftrightarrow d(x, y) = 0$ is an equivalence relation, and that the function d induces a metric on the set X/R of equivalence classes.

Now let (X, d) be a metric space. If (x_n) and (y_n) are Cauchy sequences in X, show that the sequence whose nth term is $d(x_n, y_n)$ is a Cauchy sequence of real numbers. Deduce that the function $\overline{d}$ defined by

$$\overline{d}((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$$

is a pseudometric on the set C of all Cauchy sequences in X. Show also that there is an isometric embedding (that is, a distance-preserving mapping) $X \to C/R$, where R is the equivalence relation on C induced by the pseudometric $\overline{d}$ as in the previous paragraph. Under what conditions on X is $X \to C/R$ bijective? Justify your answer.

2/I/1F **Analysis II**

Explain what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ to be *differentiable* at a point (a, b). Show that if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in a neighbourhood of (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Let

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad ((x, y) \neq (0, 0))$$

and $f(0, 0) = 0$. Do the partial derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Justify your answers.

2/II/10F **Analysis II**

Let V be the space of $n \times n$ real matrices. Show that the function

$$N(A) = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\}$$

(where $\| - \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$) defines a norm on V. Show also that this norm satisfies $N(AB) \leqslant N(A)N(B)$ for all A and B, and that if $N(A) < \epsilon$ then all entries of A have absolute value less than ϵ. Deduce that any function $f \colon V \to \mathbb{R}$ such that $f(A)$ is a polynomial in the entries of A is continuously differentiable.

Now let $d : V \to \mathbb{R}$ be the mapping sending a matrix to its determinant. By considering $d(I+H)$ as a polynomial in the entries of H, show that the derivative $d'(I)$ is the function $H \mapsto \operatorname{tr} H$. Deduce that, for any A, $d'(A)$ is the mapping $H \mapsto \operatorname{tr}((\operatorname{adj} A)H)$, where $\operatorname{adj} A$ is the adjugate of A, i.e. the matrix of its cofactors.

[*Hint: consider first the case when A is invertible. You may assume the results that the set U of invertible matrices is open in V and that its closure is the whole of V, and the identity* $(\operatorname{adj} A)A = \det A.I$.]

3/I/1F **Analysis II**

Let V be the vector space of continuous real-valued functions on $[-1, 1]$. Show that the function

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx$$

defines a norm on V.

Let $f_n(x) = x^n$. Show that (f_n) is a Cauchy sequence in V. Is (f_n) convergent? Justify your answer.

3/II/11F **Analysis II**

State and prove the Contraction Mapping Theorem.

Let (X, d) be a bounded metric space, and let F denote the set of all continuous maps $X \to X$. Let $\rho \colon F \times F \to \mathbb{R}$ be the function

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}\ .$$

Show that ρ is a metric on F, and that (F, ρ) is complete if (X, d) is complete. [*You may assume that a uniform limit of continuous functions is continuous.*]

Now suppose that (X, d) is complete. Let $C \subseteq F$ be the set of contraction mappings, and let $\theta \colon C \to X$ be the function which sends a contraction mapping to its unique fixed point. Show that θ is continuous. [*Hint: fix $f \in C$ and consider $d(\theta(g), f(\theta(g)))$, where $g \in C$ is close to f.*]

4/I/1F Analysis II

Explain what it means for a sequence of functions (f_n) to converge uniformly to a function f on an interval. If (f_n) is a sequence of continuous functions converging uniformly to f on a finite interval $[a,b]$, show that

$$\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx \qquad \text{as } n \to \infty\ .$$

Let $f_n(x) = x\exp(-x/n)/n^2$, $x \geqslant 0$. Does $f_n \to 0$ uniformly on $[0,\infty)$? Does $\int_0^\infty f_n(x)\,dx \to 0$? Justify your answers.

4/II/10F Analysis II

Let $(f_n)_{n\geqslant 1}$ be a sequence of continuous complex-valued functions defined on a set $E \subseteq \mathbb{C}$, and converging uniformly on E to a function f. Prove that f is continuous on E.

State the Weierstrass M-test for uniform convergence of a series $\sum_{n=1}^\infty u_n(z)$ of complex-valued functions on a set E.

Now let $f(z) = \sum_{n=1}^\infty u_n(z)$, where

$$u_n(z) = n^{-2}\sec(\pi z/2n)\ .$$

Prove carefully that f is continuous on $\mathbb{C}\setminus\mathbb{Z}$.

[*You may assume the inequality* $|\cos z| \geqslant |\cos(\operatorname{Re} z)|$.]

1/I/1E **Analysis II**

Suppose that for each $n = 1, 2, \ldots$, the function $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) If $f_n \to f$ pointwise on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

(b) If $f_n \to f$ uniformly on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

In each case, give a proof or a counter-example (with justification).

1/II/10E **Analysis II**

Suppose that (X, d) is a metric space that has the Bolzano-Weierstrass property (that is, any sequence has a convergent subsequence). Let (Y, d') be any metric space, and suppose that f is a continuous map of X onto Y. Show that (Y, d') also has the Bolzano-Weierstrass property.

Show also that if f is a bijection of X onto Y, then $f^{-1} : Y \to X$ is continuous.

By considering the map $x \mapsto e^{ix}$ defined on the real interval $[-\pi/2, \pi/2]$, or otherwise, show that there exists a continuous choice of $\arg z$ for the complex number z lying in the right half-plane $\{x + iy : x > 0\}$.

2/I/1E **Analysis II**

Define what is meant by (i) a complete metric space, and (ii) a totally bounded metric space.

Give an example of a metric space that is complete but not totally bounded. Give an example of a metric space that is totally bounded but not complete.

Give an example of a continuous function that maps a complete metric space onto a metric space that is not complete. Give an example of a continuous function that maps a totally bounded metric space onto a metric space that is not totally bounded.

[*You need not justify your examples.*]

2/II/10E **Analysis II**

(a) Let f be a map of a complete metric space (X, d) into itself, and suppose that there exists some k in $(0,1)$, and some positive integer N, such that $d\big(f^N(x), f^N(y)\big) \leqslant k\, d(x,y)$ for all distinct x and y in X, where f^m is the mth iterate of f. Show that f has a unique fixed point in X.

(b) Let f be a map of a compact metric space (X, d) into itself such that $d\big(f(x), f(y)\big) < d(x,y)$ for all distinct x and y in X. By considering the function $d\big(f(x), x\big)$, or otherwise, show that f has a unique fixed point in X.

(c) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| < |x - y|$ for every distinct x and y in $\mathbb{R}^n$. Suppose that for some x, the orbit $O(x) = \{x, f(x), f^2(x), \ldots\}$ is bounded. Show that f maps the closure of $O(x)$ into itself, and deduce that f has a unique fixed point in $\mathbb{R}^n$.

[*The Contraction Mapping Theorem may be used without proof providing that it is correctly stated.*]

3/I/1E **Analysis II**

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f = (u, v)$, where u and v are defined by $u(0) = v(0) = 0$ and, for $t \neq 0$, $u(t) = t^2 \sin(1/t)$ and $v(t) = t^2 \cos(1/t)$. Show that f is differentiable on $\mathbb{R}$.

Show that for any real non-zero a, $||f'(a) - f'(0)|| > 1$, where we regard $f'(a)$ as the vector $\big(u'(a), v'(a)\big)$ in $\mathbb{R}^2$.

3/II/11E **Analysis II**

Show that if a, b and c are non-negative numbers, and $a \leqslant b+c$, then

$$\frac{a}{1+a} \leqslant \frac{b}{1+b} + \frac{c}{1+c}.$$

Deduce that if (X,d) is a metric space, then $d(x,y)/[1+d(x,y)]$ is a metric on X.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K_n = \{z \in D : |z| \leqslant (n-1)/n\}$. Let $\mathcal{F}$ be the class of continuous complex-valued functions on D and, for f and g in $\mathcal{F}$, define

$$\sigma(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{||f-g||_n}{1+||f-g||_n},$$

where $||f-g||_n = \sup\{|f(z)-g(z)| : z \in K_n\}$. Show that the series for $\sigma(f,g)$ converges, and that σ is a metric on $\mathcal{F}$.

For $|z| < 1$, let $s_k(z) = 1 + z + z^2 + \cdots + z^k$ and $s(z) = 1 + z + z^2 + \cdots$. Show that for $n \geqslant 2$, $||s_k - s||_n = n(1-\frac{1}{n})^{k+1}$. By considering the sums for $2 \leqslant n \leqslant N$ and $n > N$ separately, show that for each N,

$$\sigma(s_k, s) \leqslant \sum_{n=2}^{N} ||s_k - s||_n + 2^{-N},$$

and deduce that $\sigma(s_k, s) \to 0$ as $k \to \infty$.

4/I/1E **Analysis II**

(a) Let (X,d) be a metric space containing the point x_0, and let

$$U = \{x \in X : d(x,x_0) < 1\}, \quad K = \{x \in X : d(x,x_0) \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

(b) Let X be a normed space with norm $||\cdot||$, and let

$$U = \{x \in X : ||x|| < 1\}, \quad K = \{x \in X : ||x|| \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

4/II/10E **Analysis II**

(a) Let V be a finite-dimensional real vector space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on V. Show that a function $f : V \to \mathbb{R}$ is differentiable at a point a in V with respect to $||\cdot||_1$ if and only if it is differentiable at a with respect to $||\cdot||_2$, and that if this is so then the derivative $f'(a)$ of f is independent of the norm used. [*You may assume that all norms on a finite-dimensional vector space are equivalent.*]

(b) Let V_1, V_2 and V_3 be finite-dimensional normed real vector spaces with V_j having norm $||\cdot||_j$, $j = 1, 2, 3$, and let $f : V_1 \times V_2 \to V_3$ be a continuous bilinear mapping. Show that f is differentiable at any point (a, b) in $V_1 \times V_2$, and that $f'(a,b)(h,k) = f(h,b) + f(a,k)$. [*You may assume that* $\left(||u||_1^2 + ||v||_2^2\right)^{1/2}$ *is a norm on* $V_1 \times V_2$, *and that* $\{(x,y) \in V_1 \times V_2 : ||x||_1 = 1, ||y||_2 = 1\}$ *is compact.*]

1/I/1A **Analysis II**

Define uniform continuity for functions defined on a (bounded or unbounded) interval in $\mathbb{R}$.

Is it true that a real function defined and uniformly continuous on $[0,1]$ is necessarily bounded?

Is it true that a real function defined and with a bounded derivative on $[1,\infty)$ is necessarily uniformly continuous there?

Which of the following functions are uniformly continuous on $[1,\infty)$:

(i) x^2;

(ii) $\sin(x^2)$;

(iii) $\dfrac{\sin x}{x}$?

Justify your answers.

1/II/10A **Analysis II**

Show that each of the functions below is a metric on the set of functions $x(t) \in C[a,b]$:

$$d_1(x,y) = \sup_{t\in[a,b]} |x(t)-y(t)|,$$

$$d_2(x,y) = \left\{ \int_a^b |x(t)-y(t)|^2 dt \right\}^{1/2}.$$

Is the space complete in the d_1 metric? Justify your answer.

Show that the set of functions

$$x_n(t) = \begin{cases} 0, & -1 \leqslant t < 0 \\ nt, & 0 \leqslant t < 1/n \\ 1, & 1/n \leqslant t \leqslant 1 \end{cases}$$

is a Cauchy sequence with respect to the d_2 metric on $C[-1,1]$, yet does not tend to a limit in the d_2 metric in this space. Hence, deduce that this space is not complete in the d_2 metric.

2/I/1A Analysis II

State and prove the contraction mapping theorem.

Let $A = \{x, y, z\}$, let d be the discrete metric on A, and let d' be the metric given by: d' is symmetric and

$$d'(x,y) = 2,\ d'(x,z) = 2,\ d'(y,z) = 1,$$

$$d'(x,x) = d'(y,y) = d'(z,z) = 0.$$

Verify that d' is a metric, and that it is Lipschitz equivalent to d.

Define an appropriate function $f : A \to A$ such that f is a contraction in the d' metric, but not in the d metric.

2/II/10A Analysis II

Define total boundedness for metric spaces.

Prove that a metric space has the Bolzano–Weierstrass property if and only if it is complete and totally bounded.

3/I/1A Analysis II

Define what is meant by a norm on a real vector space.

(a) Prove that two norms on a vector space (not necessarily finite-dimensional) give rise to equivalent metrics if and only if they are Lipschitz equivalent.

(b) Prove that if the vector space V has an inner product, then for all $x, y \in V$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

in the induced norm.

Hence show that the norm on $\mathbb{R}^2$ defined by $\|x\| = \max\left(|x_1|, |x_2|\right)$, where $x = (x_1, x_2) \in \mathbb{R}^2$, cannot be induced by an inner product.

3/II/11A **Analysis II**

Prove that if all the partial derivatives of $f : \mathbb{R}^p \to \mathbb{R}$ (with $p \geqslant 2$) exist in an open set containing $(0, 0, \ldots, 0)$ and are continuous at this point, then f is differentiable at $(0, 0, \ldots, 0)$.

Let

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$f(x, y) = g(x) + g(y).$$

At which points of the plane is the partial derivative f_x continuous?

At which points is the function $f(x, y)$ differentiable? Justify your answers.

4/I/1A **Analysis II**

Let f be a mapping of a metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ for all distinct x, y in X.

Show that $f(x)$ and $d(x, f(x))$ are continuous functions of x.

Now suppose that (X, d) is compact and let

$$h = \inf_{x \in X} d(x, f(x)).$$

Show that we cannot have $h > 0$.

[*You may assume that a continuous function on a compact metric space is bounded and attains its bounds.*]

Deduce that f possesses a fixed point, and that it is unique.

4/II/10A **Analysis II**

Let $\{f_n\}$ be a pointwise convergent sequence of real-valued functions on a closed interval $[a, b]$. Prove that, if for every $x \in [a, b]$, the sequence $\{f_n(x)\}$ is monotonic in n, and if all the functions f_n, $n = 1, 2, \ldots$, and $f = \lim f_n$ are continuous, then $f_n \to f$ uniformly on $[a, b]$.

By considering a suitable sequence of functions $\{f_n\}$ on $[0, 1)$, show that if the interval is not closed but all other conditions hold, the conclusion of the theorem may fail.