Part IA

Vectors and Matrices

Year
2023
2022
2021
2020
2019
2018
2017
2016
2015
2014
2013
2012
2011
2010
2009
2008

Paper 1, Section I

1A Vectors and Matrices

The principal value of the logarithm of a complex variable is defined to have its argument in the range $(-\pi, \pi]$.

(a) Evaluate $\log(-i)$, stating both the principal value and the other possible values.

(b) Show that i^{-2i} represents an infinite set of real numbers, which should be specified.

(c) By writing $z = \tan w$ in terms of exponentials, show that

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$

Use this result to evaluate the principal value of

$$\tan^{-1}\left(\frac{2\sqrt{3}-3i}{7}\right).$$

Paper 1, Section I

2C Vectors and Matrices

For an $n \times n$ complex matrix A, define the *Hermitian conjugate* $A^\dagger$. State the conditions (i) for A to be unitary (ii) for A to be Hermitian.

Let A, B, C and D be $n \times n$ complex matrices and $\mathbf{x}$ a complex n-vector. A matrix N is defined to be normal if $N^\dagger N = NN^\dagger$.

(a) For A nonsingular, show that $B = A^{-1}A^\dagger$ is unitary if and only if A is normal.

(b) Let C be normal. Show that $|C\mathbf{x}| = 0$ if and only if $|C^\dagger \mathbf{x}| = 0$.

(c) Let D be normal. Deduce from part (b) that if $\mathbf{e}$ is an eigenvector of D with eigenvalue λ then $\mathbf{e}$ is also an eigenvector of $D^\dagger$ and find the corresponding eigenvalue.

Paper 1, Section II

5A Vectors and Matrices

(a) The position vector $\mathbf{r}$ of a general point on a surface in $\mathbb{R}^3$ is given by the equations below

(i) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$,

(ii) $\mathbf{r}=\mathbf{b}+\lambda(\mathbf{d}-\mathbf{b})+\mu(\mathbf{f}-\mathbf{b})$,

(iii) $|\mathbf{r}-\mathbf{c}|=\rho$.

Identify each surface and describe the meaning of the constant vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{f}$ and $\mathbf{n}$, together with the scalars λ,μ and $\rho>0$.

(b) Find the equation for the line of intersection of the planes $2x+3y-z=3$ and $x-3y+4z=3$ in the form $\mathbf{r}\times\mathbf{m}=\mathbf{u}\times\mathbf{m}$, where $\mathbf{m}$ is a unit vector and $\mathbf{u}\cdot\mathbf{m}=0$.

Find the minimum distance from this line to the line that is inclined at equal angles to the positive x-, y-, and z- axes and passes through the origin.

(c) The intersection of the surface $\mathbf{r}\cdot\mathbf{n}=p$, where $\mathbf{n}$ is a unit vector and p is a real number, and the sphere of radius A centred on the point with position vector $\mathbf{g}$ is a circle of radius R.

Find the position vector $\mathbf{h}$ of the centre of the circle and determine R as a function of $A,p,\mathbf{g}$ and $\mathbf{n}$. Discuss geometrically the condition on R to be real.

Paper 1, Section II

6C Vectors and Matrices

(a) Consider the matrix

$$M=\begin{pmatrix}2&1&0\\0&1&-1\\0&2&4\end{pmatrix}.$$

Determine whether or not M is diagonalisable.

(b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities. Is the converse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

(c) State the Cayley–Hamilton theorem for an $n\times n$ matrix A and prove it for the case that A is a 2×2 diagonalisable matrix.

Suppose B is an $n\times n$ matrix satisfying $B^k=0$ for some $k>n$ (where 0 denotes the zero matrix). Show that $B^n=0$.

Paper 1, Section II

7B Vectors and Matrices

(a) Let A be an $n \times n$ non-singular matrix and let $G = A^\dagger A$ and $H = AA^\dagger$, where dagger denotes the Hermitian conjugate.

By considering $|A\mathbf{x}|$ for a vector $\mathbf{x}$, or otherwise, show that the eigenvalues of G are positive real numbers.

Show that if $\mathbf{e}_i$ is an eigenvector of G with eigenvalue λ_i then $\mathbf{f}_i = A\mathbf{e}_i$ is an eigenvector of H. What is the value of $|\mathbf{f}_i|/|\mathbf{e}_i|$?

(b) Using part (a), explain how to construct unitary matrices U and V from the eigenvectors of G and H such that $V^\dagger AU = D$, where D is a diagonal matrix to be specified in terms of the eigenvalues of G. [You may assume that, for any $n \times n$ Hermitian matrix, it is possible to find n orthogonal eigenvectors.]

(c) Find U, V and D for the case

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix} .$$

Paper 1, Section II

8B Vectors and Matrices

Define what it means for an $n \times n$ real matrix to be *orthogonal*.

Show that the eigenvalues of an orthogonal matrix have unit modulus, and show that eigenvectors with distinct eigenvalues are orthogonal.

Let Q be a 3×3 orthogonal matrix with $\det Q = -1$. Show that -1 is an eigenvalue of Q.

Let $\mathbf{n}$ be a nonzero vector satisfying $Q\mathbf{n} = -\mathbf{n}$ and consider the plane Π through the origin that is perpendicular to $\mathbf{n}$. Show that Q maps Π to itself.

Show that Q acts on Π as a rotation through some angle θ, and show that $\cos\theta = \frac{1}{2}(\operatorname{tr} Q + 1)$.

Show also that $\det(Q - I) = 4(\cos\theta - 1)$.

[*You may quote the form of relationship between two matrix representations A and A' of a linear map α with respect to different bases, but should explain results derived from it.*]

END OF PAPER

Paper 1, Section I
1B Vectors and Matrices

(a) Consider the equation

$$|z-a|+|z-b|=c\,,$$

for $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$, $a \neq b$, and $c \in \mathbb{R}$, $c>0$.

For each of the following cases, either show the equation has no solutions for z or give a rough sketch of the set of solutions:

(i) $c<|a-b|$,

(ii) $c=|a-b|$,

(iii) $c>|a-b|$.

(b) Let ω be the solution to $\omega^3=1$ with $\mathrm{Im}(\omega)>0$. Calculate the following:

(i) $(1+\omega)^{10^6}$,

(ii) all values of $(1+\omega)^{1+\omega}$.

Paper 1, Section I
2B Vectors and Matrices

Consider the equation

$$M^T J M = J\,, \qquad (*)$$

where M is a 2×2 real matrix and

$$J=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

(a) (i) What are the possible values of $\det(M)$ if M satisfies $(*)$? Justify your answer.

(ii) Suppose that $(*)$ holds when $M=M_1$ and when $M=M_2$. Show that $(*)$ also holds when $M=M_1M_2$ and when $M=(M_1)^{-1}$.

(b) Show that if M satisfies $(*)$ and its first entry satisfies $M_{11}>0$ then M takes one of the forms

$$\begin{pmatrix}a(u) & b(u)\\ c(u) & d(u)\end{pmatrix}, \qquad \begin{pmatrix}a(u) & -b(u)\\ c(u) & -d(u)\end{pmatrix},$$

where $u \in \mathbb{R}$ and $a(u), b(u), c(u), d(u)$ are hyperbolic functions whose form you should determine.

Paper 1, Section II
5B Vectors and Matrices

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be vectors in $\mathbb{R}^3$.

(a) (i) Define the scalar product $\mathbf{A}\cdot\mathbf{B}$ and the vector product $\mathbf{A}\times\mathbf{B}$, expressing the products in terms of vector components.

(ii) Obtain expressions for $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ as linear combinations of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

(iii) Now suppose that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. By considering the expression $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})$, obtain an expression for $\mathbf{D}$ as a linear combination of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

(b) Again suppose that the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent, and that they are position vectors of points on a sphere S that passes through the origin $\mathbf{O}$. By writing the position vector of the centre of S in the form

$$\mathbf{P} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}\,,$$

obtain three linear equations for α, β, γ in terms of $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Hence find $\mathbf{P}$ when $\mathbf{A} = (1,0,0), \mathbf{B} = (1,1,0), \mathbf{C} = (0,1,2)$ in Cartesian coordinates.

Paper 1, Section II
6B Vectors and Matrices

(a) (i) Find, with brief justification, the 2×2 matrix R representing an anticlockwise rotation through angle θ in the xy-plane and the 2×2 matrix M representing reflection in the x-axis in the xy-plane.

(ii) Show that $MRM = R^a$, where a is an integer that you should determine.

(iii) Can $R^a = MR^b$ for some integers a, b? Justify your answer.

(b) Now let $n \geqslant 3$ be an integer and $\theta = \frac{2\pi}{n}$. Consider matrices of the form

$$M^{m_1}R^{n_1}M^{m_2}R^{n_2}\dots M^{m_k}R^{n_k}\,,$$

where $k \geqslant 1$ is an integer and $m_i \geqslant 0, n_i \geqslant 0$ are integers for $i = 1,\dots,k$.

Show that there are precisely $2n$ distinct matrices of this form, and give explicit expressions for them as 2×2 matrices.

Paper 1, Section II
7B Vectors and Matrices

An $n \times n$ complex matrix P is called an *orthogonal projection matrix* if $P^2 = P = P^\dagger$, where † denotes the Hermitian conjugate. An $n \times n$ complex matrix A is *positive semidefinite* if $(\mathbf{x}, A\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{C}^n$. [Recall that the standard inner product on $\mathbb{C}^n$ is defined by $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\dagger \mathbf{y}$.]

(a) Show that the eigenvalues of any $n \times n$ Hermitian matrix are real.

(b) Show that every $n \times n$ orthogonal projection matrix is positive semidefinite.

(c) If P is an $n \times n$ orthogonal projection matrix, show that every vector $\mathbf{v} \in \mathbb{C}^n$ can be written in the form $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, where $\mathbf{v}_0$ is in the kernel of P, $P\mathbf{v}_1 = \mathbf{v}_1$ and $(\mathbf{v}_0, \mathbf{v}_1) = 0$.

(d) If A and B are distinct $n \times n$ Hermitian matrices, show that there is an orthogonal projection matrix P such that $\mathrm{Tr}(PA) \neq \mathrm{Tr}(PB)$.

(e) If P and Q are $n \times n$ orthogonal projection matrices, is PQ necessarily a positive semidefinite matrix? Justify your answer.

Paper 1, Section II
8B Vectors and Matrices

Let M be an $n \times n$ complex matrix with columns $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{C}^n$. Verify that $\mathbf{c}_1 = M\mathbf{e}_1, \ldots, \mathbf{c}_n = M\mathbf{e}_n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{C}^n$. If P is also an $n \times n$ complex matrix, show that PM has columns $P\mathbf{c}_1, \ldots, P\mathbf{c}_n$.

For an $n \times n$ complex matrix A with characteristic polynomial

$$\chi_A(x) = (-1)^n (x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)\,,$$

consider the matrix

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}.$$

Show that there exists an invertible matrix S such that

$$S^{-1}AS = C$$

if and only if there is a vector $\mathbf{v} \in \mathbb{C}^n$ such that $\mathbf{v}, A\mathbf{v}, \ldots, A^{n-1}\mathbf{v}$ are linearly independent. You may assume that $\chi_A(A) = 0$ (the Cayley-Hamilton theorem).

[*Hint: consider the columns of S.*]

END OF PAPER

Paper 1, Section I

1C Vectors and Matrices

(a) Find all complex solutions to the equation $z^i = 1$.

(b) Write down an equation for the numbers z which describe, in the complex plane, a circle with radius 5 centred at $c = 5i$. Find the points on the circle at which it intersects the line passing through c and $z_0 = \frac{15}{4}$.

Paper 1, Section I

2B Vectors and Matrices

The matrix

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Find the matrix A' that represents Φ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad C' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Paper 1, Section II

5C Vectors and Matrices

Using the standard formula relating products of the Levi-Civita symbol ϵ_{ijk} to products of the Kronecker δ_{ij}, prove

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,.$$

Define the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$ in terms of the dot and cross product. Show that

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2\,.$$

Given a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for $\mathbb{R}^3$ which is not necessarily orthonormal, let

$$\mathbf{e}_1' = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}\,, \quad \mathbf{e}_2' = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}\,, \quad \mathbf{e}_3' = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}\,.$$

Show that $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ is also a basis for $\mathbb{R}^3$. [You may assume that three linearly independent vectors in $\mathbb{R}^3$ form a basis.]

The vectors $\mathbf{e}_1''$, $\mathbf{e}_2''$, $\mathbf{e}_3''$ are constructed from $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ in the same way that $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ are constructed from $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Show that

$$\mathbf{e}_1'' = \mathbf{e}_1\,, \quad \mathbf{e}_2'' = \mathbf{e}_2\,, \quad \mathbf{e}_3'' = \mathbf{e}_3\,.$$

An infinite lattice consists of all points with position vectors given by

$$\mathbf{R} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \quad \text{with } n_1, n_2, n_3 \in \mathbb{Z}\,.$$

Find all points with position vectors $\mathbf{K}$ such that $\mathbf{K} \cdot \mathbf{R}$ is an integer for all integers n_1, n_2, n_3.

Paper 1, Section II

6A Vectors and Matrices

(a) For an $n \times n$ matrix A define the *characteristic polynomial* χ_A and the *characteristic equation.*

The Cayley–Hamilton theorem states that every $n \times n$ matrix satisfies its own characteristic equation. Verify this in the case $n = 2$.

(b) Define the adjugate matrix $\operatorname{adj}(A)$ of an $n \times n$ matrix A in terms of the minors of A. You may assume that

$$A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A) I \,,$$

where I is the $n \times n$ identity matrix. Show that if A and B are non-singular $n \times n$ matrices then

$$\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A) \,. \qquad (*)$$

(c) Let M be an arbitrary $n \times n$ matrix. Explain why

(i) there is an $\alpha > 0$ such that $M - tI$ is non-singular for $0 < t < \alpha$;

(ii) the entries of $\operatorname{adj}(M - tI)$ are polynomials in t.

Using parts (i) and (ii), or otherwise, show that $(*)$ holds for all matrices A, B.

(d) The characteristic polynomial of the arbitrary $n \times n$ matrix A is

$$\chi_A(z) = (-1)^n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \,.$$

By considering $\operatorname{adj}(A - tI)$, or otherwise, show that

$$\operatorname{adj}(A) = (-1)^{n-1} A^{n-1} - c_{n-1} A^{n-2} - \cdots - c_2 A - c_1 I \,.$$

[You may assume the Cayley–Hamilton theorem.]

Paper 1, Section II

7A Vectors and Matrices

Let A be a real, symmetric $n \times n$ matrix.

We say that A is *positive semi-definite* if $\mathbf{x}^T A \mathbf{x} \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that A is positive semi-definite if and only if all the eigenvalues of A are non-negative. [You may quote results from the course, provided that they are clearly stated.]

We say that A has a *principal square root* B if $A = B^2$ for some symmetric, positive semi-definite $n \times n$ matrix B. If such a B exists we write $B = \sqrt{A}$. Show that if A is positive semi-definite then $\sqrt{A}$ exists.

Let M be a real, non-singular $n \times n$ matrix. Show that $M^T M$ is symmetric and positive semi-definite. Deduce that $\sqrt{M^T M}$ exists and is non-singular. By considering the matrix

$$M\left(\sqrt{M^T M}\right)^{-1},$$

or otherwise, show $M = RP$ for some orthogonal $n \times n$ matrix R and a symmetric, positive semi-definite $n \times n$ matrix P.

Describe the transformation RP geometrically in the case $n = 3$.

Paper 1, Section II

8B Vectors and Matrices

(a) Consider the matrix

$$A = \begin{pmatrix} \mu & 1 & 1 \\ 2 & -\mu & 0 \\ -\mu & 2 & 1 \end{pmatrix}.$$

Find the kernel of A for each real value of the constant μ. Hence find how many solutions $\mathbf{x} \in \mathbb{R}^3$ there are to

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

depending on the value of μ. [There is no need to find expressions for the solution(s).]

(b) Consider the reflection map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\Phi : \mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}$$

where $\mathbf{n}$ is a unit vector normal to the plane of reflection.

(i) Find the matrix H which corresponds to the map Φ in terms of the components of $\mathbf{n}$.

(ii) Prove that a reflection in a plane with unit normal $\mathbf{n}$ followed by a reflection in a plane with unit normal vector $\mathbf{m}$ (both containing the origin) is equivalent to a rotation along the line of intersection of the planes with an angle twice that between the planes.

[*Hint: Choose your coordinate axes carefully.*]

(iii) Briefly explain why a rotation followed by a reflection or vice-versa can never be equivalent to another rotation.

Paper 1, Section I

1C Vectors and Matrices

Given a non-zero complex number $z = x+iy$, where x and y are real, find expressions for the real and imaginary parts of the following functions of z in terms of x and y:

(i) e^z ,

(ii) $\sin z$,

(iii) $\dfrac{1}{z} - \dfrac{1}{\bar{z}}$,

(iv) $z^3 - z^2\bar{z} - z\bar{z}^2 + \bar{z}^3$,

where $\bar{z}$ is the complex conjugate of z.

Now assume $x > 0$ and find expressions for the real and imaginary parts of all solutions to

(v) $w = \log z$.

Paper 1, Section II

5C Vectors and Matrices

(a) Let A, B, and C be three distinct points in the plane $\mathbb{R}^2$ which are not collinear, and let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be their position vectors.

Show that the set L_{AB} of points in $\mathbb{R}^2$ equidistant from A and B is given by an equation of the form

$$\mathbf{n}_{AB} \cdot \mathbf{x} = p_{AB} \, ,$$

where $\mathbf{n}_{AB}$ is a unit vector and p_{AB} is a scalar, to be determined. Show that L_{AB} is perpendicular to $\overrightarrow{AB}$.

Show that if $\mathbf{x}$ satisfies

$$\mathbf{n}_{AB} \cdot \mathbf{x} = p_{AB} \quad \text{and} \quad \mathbf{n}_{BC} \cdot \mathbf{x} = p_{BC}$$

then

$$\mathbf{n}_{CA} \cdot \mathbf{x} = p_{CA} \, .$$

How do you interpret this result geometrically?

(b) Let $\mathbf{a}$ and $\mathbf{u}$ be constant vectors in $\mathbb{R}^3$. Explain why the vectors $\mathbf{x}$ satisfying

$$\mathbf{x} \times \mathbf{u} = \mathbf{a} \times \mathbf{u}$$

describe a line in $\mathbb{R}^3$. Find an expression for the shortest distance between two lines $\mathbf{x} \times \mathbf{u}_k = \mathbf{a}_k \times \mathbf{u}_k$, where $k = 1, 2$.

Paper 1, Section II

6A Vectors and Matrices

What does it mean to say an $n \times n$ matrix is *Hermitian*?

What does it mean to say an $n \times n$ matrix is *unitary*?

Show that the eigenvalues of a Hermitian matrix are real and that eigenvectors corresponding to distinct eigenvalues are orthogonal.

Suppose that A is an $n \times n$ Hermitian matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding normalised eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let U denote the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Show directly that U is unitary and $UDU^\dagger = A$, where D is a diagonal matrix you should specify.

If U is unitary and D diagonal, must it be the case that $UDU^\dagger$ is Hermitian? Give a proof or counterexample.

Find a unitary matrix U and a diagonal matrix D such that

$$UDU^\dagger = \begin{pmatrix} 2 & 0 & 3i \\ 0 & 2 & 0 \\ -3i & 0 & 2 \end{pmatrix}.$$

Paper 1, Section I
1C Vectors and Matrices

(a) If

$$x + iy = \sum_{a=0}^{200} i^a + \prod_{b=1}^{50} i^b,$$

where $x, y \in \mathbb{R}$, what is the value of xy?

(b) Evaluate

$$\frac{(1+i)^{2019}}{(1-i)^{2017}}.$$

(c) Find a complex number z such that

$$i^{i^z} = 2.$$

(d) Interpret geometrically the curve defined by the set of points satisfying

$$\log z = i \log \overline{z}$$

in the complex z-plane.

Paper 1, Section I
2A Vectors and Matrices

If A is an n by n matrix, define its *determinant* $\det A$.

Find the following in terms of $\det A$ and a scalar λ, clearly showing your argument:

(i) $\det B$, where B is obtained from A by multiplying one row by λ.

(ii) $\det(\lambda A)$.

(iii) $\det C$, where C is obtained from A by switching row k and row l $(k \neq l)$.

(iv) $\det D$, where D is obtained from A by adding λ times column l to column k $(k \neq l)$.

Paper 1, Section II

5C Vectors and Matrices

(a) Use index notation to prove $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.
Hence simplify

(i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$,

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

(b) Give the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the simultaneous equations

$$\mathbf{x} + \mathbf{y} = 2\mathbf{a}, \qquad \mathbf{x} \cdot \mathbf{y} = c \qquad (c < \mathbf{a} \cdot \mathbf{a}).$$

Show in particular that $\mathbf{x}$ and $\mathbf{y}$ must lie at opposite ends of a diameter of a sphere whose centre and radius should be found.

(c) If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair are also perpendicular to each other. Show also that the sum of the lengths squared of two opposite edges is the same for each pair.

Paper 1, Section II

6B Vectors and Matrices

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the standard basis vectors of $\mathbb{R}^3$. A second set of vectors $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ are defined with respect to the standard basis by

$$\mathbf{f}_j = \sum_{i=1}^{3} P_{ij}\mathbf{e}_i, \qquad j = 1, 2, 3.$$

The P_{ij} are the elements of the 3×3 matrix P. State the condition on P under which the set $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ forms a basis of $\mathbb{R}^3$.

Define the matrix A that, for a given linear transformation α, gives the relation between the components of any vector $\mathbf{v}$ and those of the corresponding $\alpha(\mathbf{v})$, with the components specified with respect to the standard basis.

Show that the relation between the matrix A and the matrix $\tilde{A}$ of the same transformation with respect to the second basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is

$$\tilde{A} = P^{-1}AP.$$

Consider the matrix

$$A = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -1 & -1 \\ 0 & 6 & 4 \end{pmatrix}.$$

Find a matrix P such that $B = P^{-1}AP$ is diagonal. Give the elements of B and demonstrate explicitly that the relation between A and B holds.

Give the elements of A^nP for any positive integer n.

Paper 1, Section II

7B Vectors and Matrices

(a) Let A be an $n \times n$ matrix. Define the characteristic polynomial $\chi_A(z)$ of A. [Choose a sign convention such that the coefficient of z^n in the polynomial is equal to $(-1)^n$.] State and justify the relation between the characteristic polynomial and the eigenvalues of A. Why does A have at least one eigenvalue?

(b) Assume that A has n distinct eigenvalues. Show that $\chi_A(A) = 0$. [Each term $c_r z^r$ in $\chi_A(z)$ corresponds to a term $c_r A^r$ in $\chi_A(A)$.]

(c) For a general $n \times n$ matrix B and integer $m \geqslant 1$, show that $\chi_{B^m}(z^m) = \prod_{l=1}^{m} \chi_B(\omega_l z)$, where $\omega_l = e^{2\pi i l/m}$, $(l = 1, \ldots, m)$. [*Hint: You may find it helpful to note the factorization of* $z^m - 1$.]

Prove that if B^m has an eigenvalue λ then B has an eigenvalue μ where $\mu^m = \lambda$.

Paper 1, Section II

8A Vectors and Matrices

The exponential of a square matrix M is defined as

$$\exp M = I + \sum_{n=1}^{\infty} \frac{M^n}{n!},$$

where I is the identity matrix. [You do not have to consider issues of convergence.]

(a) Calculate the elements of R and S, where

$$R = \exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}, \qquad S = \exp \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$$

and θ is a real number.

(b) Show that $RR^T = I$ and that

$$SJS = J, \qquad \text{where} \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(c) Consider the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Calculate:
(i) $\exp(xA)$,
(ii) $\exp(xB)$.

(d) Defining

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

find the elements of the following matrices, where N is a natural number:
(i)

$$\sum_{n=1}^{N} \left(\exp(xA) C [\exp(xA)]^T\right)^n,$$

(ii)

$$\sum_{n=1}^{N} \left(\exp(xB) C \exp(xB)\right)^n.$$

[Your answers to parts (a), (c) and (d) should be in closed form, i.e. not given as series.]

Paper 1, Section I

1C Vectors and Matrices

For $z, w \in \mathbb{C}$ define the *principal value* of z^w. State de Moivre's theorem.

Hence solve the equations

$$\text{(i) } z^6 = \sqrt{3} + i, \quad \text{(ii) } z^{1/6} = \sqrt{3} + i, \quad \text{(iii) } i^z = \sqrt{3} + i, \quad \text{(iv) } \left(e^{5i\pi/2}\right)^z = \sqrt{3} + i\,.$$

[In each expression, the principal value is to be taken.]

Paper 1, Section I

2A Vectors and Matrices

The map $\mathbf{\Phi}(\mathbf{x}) = \alpha(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{x})$ is defined for $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$ and α is a real constant.

(i) Find the values of α for which the inverse map $\mathbf{\Phi}^{-1}$ exists, as well as the inverse map itself in these cases.

(ii) When $\mathbf{\Phi}$ is not invertible, find its image and kernel. What is the value of the rank and the value of the nullity of $\mathbf{\Phi}$?

(iii) Let $\mathbf{y} = \mathbf{\Phi}(\mathbf{x})$. Find the components A_{ij} of the matrix A such that $y_i = A_{ij}x_j$. When $\mathbf{\Phi}$ is invertible, find the components of the matrix B such that $x_i = B_{ij}y_j$.

Paper 1, Section II

5C Vectors and Matrices

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be non-zero real vectors. Define the *inner product* $\mathbf{x} \cdot \mathbf{y}$ in terms of the components x_i and y_i, and define the *norm* $|\mathbf{x}|$. Prove that $\mathbf{x} \cdot \mathbf{y} \leqslant |\mathbf{x}|\,|\mathbf{y}|$. When does equality hold? Express the angle between $\mathbf{x}$ and $\mathbf{y}$ in terms of their inner product.

Use suffix notation to expand $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c})$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be given unit vectors in $\mathbb{R}^3$, and let $\mathbf{m} = (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$. Obtain expressions for the angle between $\mathbf{m}$ and each of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $|\mathbf{m}|$. Calculate $|\mathbf{m}|$ for the particular case when the angles between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all equal to θ, and check your result for an example with $\theta = 0$ and an example with $\theta = \pi/2$.

Consider three planes in $\mathbb{R}^3$ passing through the points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, with unit normals $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. State a condition that must be satisfied for the three planes to intersect at a single point, and find the intersection point.

Paper 1, Section II
6B Vectors and Matrices

(a) Consider the matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

representing a rotation about the z-axis through an angle θ.

Show that R has three eigenvalues in $\mathbb{C}$ each with modulus 1, of which one is real and two are complex (in general), and give the relation of the real eigenvector and the two complex eigenvalues to the properties of the rotation.

Now consider the rotation composed of a rotation by angle $\pi/2$ about the z-axis followed by a rotation by angle $\pi/2$ about the x-axis. Determine the rotation axis $\mathbf{n}$ and the magnitude of the angle of rotation ϕ.

(b) A surface in $\mathbb{R}^3$ is given by

$$7x^2 + 4xy + 3y^2 + 2xz + 3z^2 = 1.$$

By considering a suitable eigenvalue problem, show that the surface is an ellipsoid, find the lengths of its semi-axes and find the position of the two points on the surface that are closest to the origin.

Paper 1, Section II

7B Vectors and Matrices

Let A be a real symmetric $n \times n$ matrix.

(a) Prove the following:

(i) Each eigenvalue of A is real and there is a corresponding real eigenvector.

(ii) Eigenvectors corresponding to different eigenvalues are orthogonal.

(iii) If there are n distinct eigenvalues then the matrix is diagonalisable.

Assuming that A has n distinct eigenvalues, explain briefly how to choose (up to an arbitrary scalar factor) the vector v such that $\dfrac{v^T A v}{v^T v}$ is maximised.

(b) A scalar λ and a non-zero vector v such that

$$Av = \lambda B v$$

are called, for a specified $n \times n$ matrix B, respectively a *generalised eigenvalue* and a *generalised eigenvector* of A.

Assume the matrix B is real, symmetric and positive definite (i.e. $(u^*)^T B u > 0$ for all non-zero complex vectors u).

Prove the following:

(i) If λ is a generalised eigenvalue of A then it is a root of $\det(A - \lambda B) = 0$.

(ii) Each generalised eigenvalue of A is real and there is a corresponding real generalised eigenvector.

(iii) Two generalised eigenvectors u, v, corresponding to different generalised eigenvalues, are orthogonal in the sense that $u^T B v = 0$.

(c) Find, up to an arbitrary scalar factor, the vector v such that the value of $F(v) = \dfrac{v^T A v}{v^T B v}$ is maximised, and the corresponding value of $F(v)$, where

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Paper 1, Section II

8A Vectors and Matrices

What is the definition of an *orthogonal matrix* M?

Write down a 2×2 matrix R representing the rotation of a 2-dimensional vector (x, y) by an angle θ around the origin. Show that R is indeed orthogonal.

Take a matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where a, b, c are real. Suppose that the 2×2 matrix $B = RAR^T$ is diagonal. Determine all possible values of θ.

Show that the diagonal entries of B are the eigenvalues of A and express them in terms of the determinant and trace of A.

Using the above results, or otherwise, find the elements of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{2N}$$

as a function of N, where N is a natural number.

Paper 1, Section I

1A Vectors and Matrices

Consider $z \in \mathbb{C}$ with $|z| = 1$ and $\arg z = \theta$, where $\theta \in [0, \pi)$.

(a) Prove algebraically that the modulus of $1 + z$ is $2 \cos \frac{1}{2}\theta$ and that the argument is $\frac{1}{2}\theta$. Obtain these results geometrically using the Argand diagram.

(b) Obtain corresponding results algebraically and geometrically for $1 - z$.

Paper 1, Section I

2C Vectors and Matrices

Let A and B be real $n \times n$ matrices.

Show that $(AB)^T = B^T A^T$.

For any square matrix, the *matrix exponential* is defined by the series

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} .$$

Show that $(e^A)^T = e^{A^T}$. [You are not required to consider issues of convergence.]

Calculate, in terms of A and A^T, the matrices Q_0, Q_1 and Q_2 in the series for the matrix product

$$e^{tA}\, e^{tA^T} = \sum_{k=0}^{\infty} Q_k t^k , \quad \text{where} \quad t \in \mathbb{R}.$$

Hence obtain a relation between A and A^T which necessarily holds if e^{tA} is an orthogonal matrix.

Paper 1, Section II

5A Vectors and Matrices

(a) Define the *vector product* $\mathbf{x} \times \mathbf{y}$ of the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. Use suffix notation to prove that

$$\mathbf{x} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{x}\,(\mathbf{x} \cdot \mathbf{y}) - \mathbf{y}\,(\mathbf{x} \cdot \mathbf{x}).$$

(b) The vectors $\mathbf{x}_{n+1}$ $(n = 0, 1, 2, \dots)$ are defined by $\mathbf{x}_{n+1} = \lambda\,\mathbf{a} \times \mathbf{x}_n$, where $\mathbf{a}$ and $\mathbf{x}_0$ are fixed vectors with $|\mathbf{a}| = 1$ and $\mathbf{a} \times \mathbf{x}_0 \neq \mathbf{0}$, and λ is a positive constant.

(i) Write $\mathbf{x}_2$ as a linear combination of $\mathbf{a}$ and $\mathbf{x}_0$. Further, for $n \geqslant 1$, express $\mathbf{x}_{n+2}$ in terms of λ and $\mathbf{x}_n$. Show, for $n \geqslant 1$, that $|\mathbf{x}_n| = \lambda^n\,|\mathbf{a} \times \mathbf{x}_0|$.

(ii) Let X_n be the point with position vector $\mathbf{x}_n$ $(n = 0, 1, 2, \dots)$. Show that $X_1, X_2, \dots$ lie on a pair of straight lines.

(iii) Show that the line segment X_nX_{n+1} $(n \geqslant 1)$ is perpendicular to $X_{n+1}X_{n+2}$. Deduce that X_nX_{n+1} is parallel to $X_{n+2}X_{n+3}$.

Show that $\mathbf{x}_n \to \mathbf{0}$ as $n \to \infty$ if $\lambda < 1$, and give a sketch to illustrate the case $\lambda = 1$.

(iv) The straight line through the points X_{n+1} and X_{n+2} makes an angle θ with the straight line through the points X_n and X_{n+3}. Find $\cos\theta$ in terms of λ.

Paper 1, Section II
6B Vectors and Matrices

(a) Show that the eigenvalues of any real $n \times n$ square matrix A are the same as the eigenvalues of A^T.

The eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the eigenvalues of A^TA are $\mu_1, \mu_2, \ldots, \mu_n$. Determine, by means of a proof or a counterexample, whether the following are necessary valid:

(i) $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i^2$;

(ii) $\prod_{i=1}^{n} \mu_i = \prod_{i=1}^{n} \lambda_i^2$.

(b) The 3×3 matrix B is given by

$$B = I + \mathbf{m}\mathbf{n}^T ,$$

where $\mathbf{m}$ and $\mathbf{n}$ are orthogonal real unit vectors and I is the 3×3 identity matrix.

(i) Show that $\mathbf{m} \times \mathbf{n}$ is an eigenvector of B, and write down a linearly independent eigenvector. Find the eigenvalues of B and determine whether B is diagonalisable.

(ii) Find the eigenvectors and eigenvalues of B^TB.

Paper 1, Section II
7B Vectors and Matrices

(a) Show that a square matrix A is anti-symmetric if and only if $\mathbf{x}^T A\mathbf{x} = 0$ for every vector $\mathbf{x}$.

(b) Let A be a real anti-symmetric $n \times n$ matrix. Show that the eigenvalues of A are imaginary or zero, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal (in the sense that $\mathbf{x}^\dagger \mathbf{y} = 0$, where the dagger denotes the hermitian conjugate).

(c) Let A be a non-zero real 3×3 anti-symmetric matrix. Show that there is a real non-zero vector $\mathbf{a}$ such that $A\mathbf{a} = \mathbf{0}$.

Now let $\mathbf{b}$ be a real vector orthogonal to $\mathbf{a}$. Show that $A^2\mathbf{b} = -\theta^2\mathbf{b}$ for some real number θ.

The matrix e^A is defined by the exponential series $I + A + \frac{1}{2!}A^2 + \cdots$. Express $e^A\mathbf{a}$ and $e^A\mathbf{b}$ in terms of $\mathbf{a}, \mathbf{b}$, $A\mathbf{b}$ and θ.

[*You are not required to consider issues of convergence.*]

Paper 1, Section II
8C Vectors and Matrices

(a) Given $\mathbf{y} \in \mathbb{R}^3$ consider the linear transformation T which maps

$$\mathbf{x} \mapsto T\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + \mathbf{x} \times \mathbf{y}\,.$$

Express T as a matrix with respect to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and determine the rank and the dimension of the kernel of T for the cases (i) $\mathbf{y} = c_1\mathbf{e}_1$, where c_1 is a fixed number, and (ii) $\mathbf{y} \cdot \mathbf{e}_1 = 0$.

(b) Given that the equation

$$A\,B\,\mathbf{x} = \mathbf{d}\,,$$

where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 1 \\ -3 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix},$$

has a solution, show that $4k = 1$.

Paper 1, Section I

1A Vectors and Matrices

Let $z \in \mathbb{C}$ be a solution of

$$z^2 + bz + 1 = 0,$$

where $b \in \mathbb{R}$ and $|b| \leqslant 2$. For which values of b do the following hold?

(i) $|e^z| < 1$.

(ii) $|e^{iz}| = 1$.

(iii) $\mathrm{Im}(\cosh z) = 0$.

Paper 1, Section I

2C Vectors and Matrices

Write down the general form of a 2×2 rotation matrix. Let R be a real 2×2 matrix with positive determinant such that $|R\mathbf{x}| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^2$. Show that R is a rotation matrix.

Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that any real 2×2 matrix A which satisfies $AJ = JA$ can be written as $A = \lambda R$, where λ is a real number and R is a rotation matrix.

Paper 1, Section II

5A Vectors and Matrices

(a) Use suffix notation to prove that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

(b) Show that the equation of the plane through three non-colinear points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}),$$

where $\mathbf{r}$ is the position vector of a point in this plane.

Find a unit vector normal to the plane in the case $\mathbf{a}=(2,0,1)$, $\mathbf{b}=(0,4,0)$ and $\mathbf{c}=(1,-1,2)$.

(c) Let $\mathbf{r}$ be the position vector of a point in a given plane. The plane is a distance d from the origin and has unit normal vector $\mathbf{n}$, where $\mathbf{n}\cdot\mathbf{r}\geqslant 0$. Write down the equation of this plane.

This plane intersects the sphere with centre at $\mathbf{p}$ and radius q in a circle with centre at $\mathbf{m}$ and radius ρ. Show that

$$\mathbf{m}-\mathbf{p}=\gamma\,\mathbf{n}.$$

Find γ in terms of q and ρ. Hence find ρ in terms of $\mathbf{n}$, d, $\mathbf{p}$ and q.

Paper 1, Section II

6B Vectors and Matrices

The $n \times n$ real symmetric matrix M has eigenvectors of unit length $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Prove that the eigenvalues are real and that $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$.

Let $\mathbf{x}$ be any (real) unit vector. Show that

$$\mathbf{x}^{\mathrm{T}} M \mathbf{x} \leqslant \lambda_1 .$$

What can be said about $\mathbf{x}$ if $\mathbf{x}^{\mathrm{T}} M \mathbf{x} = \lambda_1$?

Let S be the set of all (real) unit vectors of the form

$$\mathbf{x} = (0, x_2, \ldots, x_n) .$$

Show that $\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in S$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Deduce that

$$\operatorname*{Max}_{\mathbf{x} \in S} \mathbf{x}^{\mathrm{T}} M \mathbf{x} \geqslant \lambda_2 .$$

The $(n-1) \times (n-1)$ matrix A is obtained by removing the first row and the first column of M. Let μ be the greatest eigenvalue of A. Show that

$$\lambda_1 \geqslant \mu \geqslant \lambda_2 .$$

Paper 1, Section II

7B Vectors and Matrices

What does it mean to say that a matrix can be diagonalised? Given that the $n \times n$ real matrix M has n eigenvectors satisfying $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$, explain how to obtain the diagonal form Λ of M. Prove that Λ is indeed diagonal. Obtain, with proof, an expression for the trace of M in terms of its eigenvalues.

The elements of M are given by

$$M_{ij} = \begin{cases} 0 & \text{for } i = j , \\ 1 & \text{for } i \neq j . \end{cases}$$

Determine the elements of M^2 and hence show that, if λ is an eigenvalue of M, then

$$\lambda^2 = (n-1) + (n-2)\lambda .$$

Assuming that M can be diagonalised, give its diagonal form.

Paper 1, Section II
8C Vectors and Matrices

(a) Show that the equations

$$\begin{aligned} 1 + s + t &= a \\ 1 - s + t &= b \\ 1 - 2t &= c \end{aligned}$$

determine s and t uniquely if and only if $a + b + c = 3$.

Write the following system of equations

$$\begin{aligned} 5x + 2y - z &= 1 + s + t \\ 2x + 5y - z &= 1 - s + t \\ -x - y + 8z &= 1 - 2t \end{aligned}$$

in matrix form $A\mathbf{x} = \mathbf{b}$. Use Gaussian elimination to solve the system for x, y, and z. State a relationship between the rank and the kernel of a matrix. What is the rank and what is the kernel of A?

For which values of x, y, and z is it possible to solve the above system for s and t?

(b) Define a *unitary* $n \times n$ matrix. Let A be a real symmetric $n \times n$ matrix, and let I be the $n \times n$ identity matrix. Show that $|(A + iI)\mathbf{x}|^2 = |A\mathbf{x}|^2 + |\mathbf{x}|^2$ for arbitrary $\mathbf{x} \in \mathbb{C}^n$, where $|\mathbf{x}|^2 = \sum_{j=1}^{n} |x_j|^2$. Find a similar expression for $|(A - iI)\mathbf{x}|^2$. Prove that $(A - iI)(A + iI)^{-1}$ is well-defined and is a unitary matrix.

Paper 1, Section I

1B Vectors and Matrices

(a) Describe geometrically the curve

$$|\alpha z + \beta \bar{z}| = \sqrt{\alpha\beta}\,(z + \bar{z}) + (\alpha - \beta)^2,$$

where $z \in \mathbb{C}$ and α, β are positive, distinct, real constants.

(b) Let θ be a real number not equal to an integer multiple of 2π. Show that

$$\sum_{m=1}^{N} \sin(m\theta) = \frac{\sin\theta + \sin(N\theta) - \sin(N\theta + \theta)}{2(1 - \cos\theta)},$$

and derive a similar expression for $\sum_{m=1}^{N} \cos(m\theta)$.

Paper 1, Section I

2C Vectors and Matrices

Precisely one of the four matrices specified below is not orthogonal. Which is it? Give a brief justification.

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -\sqrt{3} & \sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \\ -2 & 0 & \sqrt{2} \end{pmatrix} \qquad \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix} \qquad \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -2 & 1 \\ -\sqrt{6} & 0 & \sqrt{6} \\ 1 & 1 & 1 \end{pmatrix} \qquad \frac{1}{9}\begin{pmatrix} 7 & -4 & -4 \\ -4 & 1 & -8 \\ -4 & -8 & 1 \end{pmatrix}$$

Given that the four matrices represent transformations of $\mathbb{R}^3$ corresponding (in no particular order) to a rotation, a reflection, a combination of a rotation and a reflection, and none of these, identify each matrix. Explain your reasoning.

[*Hint: For* ***two*** *of the matrices, A and B say, you may find it helpful to calculate* $\det(A - I)$ *and* $\det(B - I)$*, where I is the identity matrix.*]

Paper 1, Section II

5B Vectors and Matrices

(i) State and prove the Cauchy–Schwarz inequality for vectors in $\mathbb{R}^n$. Deduce the inequalities

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}|+|\mathbf{b}| \quad \text{and} \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}| \leqslant |\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|$$

for $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^n$.

(ii) Show that every point on the intersection of the planes

$$\mathbf{x}\cdot\mathbf{a} = A, \quad \mathbf{x}\cdot\mathbf{b} = B,$$

where $\mathbf{a} \neq \mathbf{b}$, satisfies

$$|\mathbf{x}|^2 \geqslant \frac{(A-B)^2}{|\mathbf{a}-\mathbf{b}|^2}.$$

What happens if $\mathbf{a} = \mathbf{b}$?

(iii) Using your results from part (i), or otherwise, show that for any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$,

$$|\mathbf{x}_1 - \mathbf{y}_1| - |\mathbf{x}_1 - \mathbf{y}_2| \leqslant |\mathbf{x}_2 - \mathbf{y}_1| + |\mathbf{x}_2 - \mathbf{y}_2|.$$

Paper 1, Section II

6C Vectors and Matrices

(i) Consider the map from $\mathbb{R}^4$ to $\mathbb{R}^3$ represented by the matrix

$$\begin{pmatrix} \alpha & 1 & 1 & -1 \\ 2 & -\alpha & 0 & -2 \\ -\alpha & 2 & 1 & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. Find the image and kernel of the map for each value of α.

(ii) Show that any linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ may be written in the form $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ for some fixed vector $\mathbf{a} \in \mathbb{R}^n$. Show further that $\mathbf{a}$ is uniquely determined by f.

It is given that $n = 4$ and that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

lie in the kernel of f. Determine the set of possible values of $\mathbf{a}$.

Paper 1, Section II
7A Vectors and Matrices

(i) Find the eigenvalues and eigenvectors of the following matrices and show that both are diagonalisable:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 4 & -3 \\ -4 & 10 & -4 \\ -3 & 4 & 1 \end{pmatrix}.$$

(ii) Show that, if two real $n \times n$ matrices can both be diagonalised using the same basis transformation, then they commute.

(iii) Suppose now that two real $n \times n$ matrices C and D commute and that D has n distinct eigenvalues. Show that for any eigenvector $\mathbf{x}$ of D the vector $C\mathbf{x}$ is a scalar multiple of $\mathbf{x}$. Deduce that there exists a common basis transformation that diagonalises both matrices.

(iv) Show that A and B satisfy the conditions in (iii) and find a matrix S such that both of the matrices $S^{-1}AS$ and $S^{-1}BS$ are diagonal.

Paper 1, Section II
8A Vectors and Matrices

(a) A matrix is called *normal* if $A^\dagger A = AA^\dagger$. Let A be a normal $n \times n$ complex matrix.

(i) Show that for any vector $\mathbf{x} \in \mathbb{C}^n$,

$$|A\mathbf{x}| = |A^\dagger \mathbf{x}|.$$

(ii) Show that $A - \lambda I$ is also normal for any $\lambda \in \mathbb{C}$, where I denotes the identity matrix.

(iii) Show that if $\mathbf{x}$ is an eigenvector of A with respect to the eigenvalue $\lambda \in \mathbb{C}$, then $\mathbf{x}$ is also an eigenvector of $A^\dagger$, and determine the corresponding eigenvalue.

(iv) Show that if $\mathbf{x}_\lambda$ and $\mathbf{x}_\mu$ are eigenvectors of A with respect to distinct eigenvalues λ and μ respectively, then $\mathbf{x}_\lambda$ and $\mathbf{x}_\mu$ are orthogonal.

(v) Show that if A has a basis of eigenvectors, then A can be diagonalised using an orthonormal basis. Justify your answer.

[You may use standard results provided that they are clearly stated.]

(b) Show that any matrix A satisfying $A^\dagger = A$ is normal, and deduce using results from (a) that its eigenvalues are real.

(c) Show that any matrix A satisfying $A^\dagger = -A$ is normal, and deduce using results from (a) that its eigenvalues are purely imaginary.

(d) Show that any matrix A satisfying $A^\dagger = A^{-1}$ is normal, and deduce using results from (a) that its eigenvalues have unit modulus.

Paper 1, Section I

1B Vectors and Matrices

(a) Let

$$z = 2 + 2i.$$

(i) Compute z^4.

(ii) Find all complex numbers w such that $w^4 = z$.

(b) Find all the solutions of the equation

$$e^{2z^2} - 1 = 0.$$

(c) Let $z = x + iy$, $\bar{z} = x - iy$, $x, y \in \mathbb{R}$. Show that the equation of any line, and of any circle, may be written respectively as

$$Bz + \bar{B}\bar{z} + C = 0 \qquad \text{and} \qquad z\bar{z} + \bar{B}z + B\bar{z} + C = 0,$$

for some complex B and real C.

Paper 1, Section I

2A Vectors and Matrices

(a) What is meant by an eigenvector and the corresponding eigenvalue of a matrix A?

(b) Let A be the matrix

$$A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 3 & -3 & -1 \end{pmatrix}.$$

Find the eigenvalues and the corresponding eigenspaces of A and determine whether or not A is diagonalisable.

Paper 1, Section II

5B Vectors and Matrices

(i) For vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Show that the plane $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ and the line $(\mathbf{r} - \mathbf{b}) \times \mathbf{m} = \mathbf{0}$, where $\mathbf{m} \cdot \mathbf{n} \neq 0$, intersect at the point

$$\mathbf{r} = \frac{(\mathbf{a} \cdot \mathbf{n})\mathbf{m} + \mathbf{n} \times (\mathbf{b} \times \mathbf{m})}{\mathbf{m} \cdot \mathbf{n}},$$

and only at that point. What happens if $\mathbf{m} \cdot \mathbf{n} = 0$?

(ii) Explain why the distance between the planes $(\mathbf{r} - \mathbf{a}_1) \cdot \hat{\mathbf{n}} = 0$ and $(\mathbf{r} - \mathbf{a}_2) \cdot \hat{\mathbf{n}} = 0$ is $\left|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \hat{\mathbf{n}}\right|$, where $\hat{\mathbf{n}}$ is a unit vector.

(iii) Find the shortest distance between the lines $(3 + s, 3s, 4 - s)$ and $(-2, 3 + t, 3 - t)$ where $s, t \in \mathbb{R}$. [*You may wish to consider two appropriately chosen planes and use the result of part (ii).*]

Paper 1, Section II

6A Vectors and Matrices

Let A be a real $n \times n$ symmetric matrix.

(i) Show that all eigenvalues of A are real, and that the eigenvectors of A with respect to different eigenvalues are orthogonal. Assuming that any real symmetric matrix can be diagonalised, show that there exists an orthonormal basis $\{\mathbf{y}_i\}$ of eigenvectors of A.

(ii) Consider the linear system

$$A\mathbf{x} = \mathbf{b}.$$

Show that this system has a solution if and only if $\mathbf{b} \cdot \mathbf{h} = 0$ for every vector $\mathbf{h}$ in the kernel of A. Let $\mathbf{x}$ be such a solution. Given an eigenvector of A with non-zero eigenvalue, determine the component of $\mathbf{x}$ in the direction of this eigenvector. Use this result to find the general solution of the linear system, in the form

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{y}_i.$$

Paper 1, Section II

7C Vectors and Matrices

Let $\mathcal{A}\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map

$$\mathcal{A}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{i\theta} + w \\ w\mathrm{e}^{-i\phi} + z \end{pmatrix},$$

where θ and ϕ are real constants. Write down the matrix A of $\mathcal{A}$ with respect to the standard basis of $\mathbb{C}^2$ and show that $\det A = 2i \sin \frac{1}{2}(\theta - \phi) \exp\big(\frac{1}{2} i(\theta - \phi)\big)$.

Let $\mathcal{R}\colon \mathbb{C}^2 \to \mathbb{R}^4$ be the invertible map

$$\mathcal{R}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} z \\ \operatorname{Im} z \\ \operatorname{Re} w \\ \operatorname{Im} w \end{pmatrix}$$

and define a linear map $\mathcal{B}\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $\mathcal{B} = \mathcal{R}\mathcal{A}\mathcal{R}^{-1}$. Find the image of each of the standard basis vectors of $\mathbb{R}^4$ under both $\mathcal{R}^{-1}$ and $\mathcal{B}$. Hence, or otherwise, find the matrix B of $\mathcal{B}$ with respect to the standard basis of $\mathbb{R}^4$ and verify that $\det B = |\det A|^2$.

Paper 1, Section II

8C Vectors and Matrices

Let A and B be complex $n \times n$ matrices.

(i) The *commutator* of A and B is defined to be

$$[A, B] \equiv AB - BA.$$

Show that $[A, A] = 0$; $[A, B] = -[B, A]$; and $[A, \lambda B] = \lambda [A, B]$ for $\lambda \in \mathbb{C}$. Show further that the trace of $[A, B]$ vanishes.

(ii) A *skew-Hermitian* matrix S is one which satisfies $S^\dagger = -S$, where $\dagger$ denotes the Hermitian conjugate. Show that if A and B are skew-Hermitian then so is $[A, B]$.

(iii) Let $\mathcal{M}$ be the linear map from $\mathbb{R}^3$ to the set of 2×2 complex matrices given by

$$\mathcal{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xM_1 + yM_2 + zM_3,$$

where

$$M_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Prove that for any $\mathbf{a} \in \mathbb{R}^3$, $\mathcal{M}(\mathbf{a})$ is traceless and skew-Hermitian. By considering pairs such as $[M_1, M_2]$, or otherwise, show that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$\mathcal{M}(\mathbf{a} \times \mathbf{b}) = \big[\mathcal{M}(\mathbf{a}), \mathcal{M}(\mathbf{b})\big].$$

(iv) Using the result of part (iii), or otherwise, prove that if C is a traceless skew-Hermitian 2×2 matrix then there exist matrices A, B such that $C = [A, B]$. [You may use geometrical properties of vectors in $\mathbb{R}^3$ without proof.]

Paper 1, Section I
1C Vectors and Matrices

(a) State de Moivre's theorem and use it to derive a formula for the roots of order n of a complex number $z = a+ib$. Using this formula compute the cube roots of $z = -8$.

(b) Consider the equation $|z + 3i| = 3|z|$ for $z \in \mathbb{C}$. Give a geometric description of the set S of solutions and sketch S as a subset of the complex plane.

Paper 1, Section I
2A Vectors and Matrices
Let A be a real 3×3 matrix.

(i) For $B = R_1 A$ with

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

find an angle θ_1 so that the element $b_{31} = 0$, where b_{ij} denotes the ij^{th} entry of the matrix B.

(ii) For $C = R_2 B$ with $b_{31} = 0$ and

$$R_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

show that $c_{31} = 0$ and find an angle θ_2 so that $c_{21} = 0$.

(iii) For $D = R_3 C$ with $c_{31} = c_{21} = 0$ and

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix}$$

show that $d_{31} = d_{21} = 0$ and find an angle θ_3 so that $d_{32} = 0$.

(iv) Deduce that any real 3×3 matrix can be written as a product of an orthogonal matrix and an upper triangular matrix.

Paper 1, Section II

5C Vectors and Matrices

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy-Schwarz inequality, and deduce the triangle inequality. Under what condition does equality hold in the Cauchy-Schwarz inequality?

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{z} + \mathbf{z} \cdot \mathbf{x}\,.$$

Show that for any fixed, linearly independent vectors $\mathbf{x}$ and $\mathbf{y}$, the minimum of S over $\mathbf{z}$ is attained when $\mathbf{z} = \lambda(\mathbf{x} + \mathbf{y})$ for some $\lambda \in \mathbb{R}$, and that for this value of λ we have

(i) $\lambda \leqslant -\frac{1}{2}$ (for any choice of $\mathbf{x}$ and $\mathbf{y}$);

(ii) $\lambda = -1$ and $S = -\frac{3}{2}$ in the case where $\mathbf{x} \cdot \mathbf{y} = \cos\frac{2\pi}{3}$.

Paper 1, Section II

6A Vectors and Matrices

Define the kernel and the image of a linear map α from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ be a basis of $\mathbb{R}^m$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ a basis of $\mathbb{R}^n$. Explain how to represent α by a matrix A relative to the given bases.

A second set of bases $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_m\}$ and $\{\mathbf{f}'_1, \mathbf{f}'_2, \ldots, \mathbf{f}'_n\}$ is now used to represent α by a matrix A'. Relate the elements of A' to the elements of A.

Let β be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by

$$\beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.$$

Either find one or more $\mathbf{x}$ in $\mathbb{R}^2$ such that

$$\beta\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

or explain why one cannot be found.

Let γ be a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$\gamma \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \gamma \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the kernel of γ.

Paper 1, Section II

7B Vectors and Matrices

(a) Let $\lambda_1, \ldots, \lambda_d$ be distinct eigenvalues of an $n \times n$ matrix A, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. Prove that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is linearly independent.

(b) Consider the quadric surface Q in $\mathbb{R}^3$ defined by

$$2x^2 - 4xy + 5y^2 - z^2 + 6\sqrt{5}y = 0.$$

Find the position of the origin $\tilde{O}$ and orthonormal coordinate basis vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ and $\tilde{\mathbf{e}}_3$, for a coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ in which Q takes the form

$$\alpha\tilde{x}^2 + \beta\tilde{y}^2 + \gamma\tilde{z}^2 = 1.$$

Also determine the values of α, β and γ, and describe the surface geometrically.

Paper 1, Section II

8B Vectors and Matrices

(a) Let A and A' be the matrices of a linear map L on $\mathbb{C}^2$ relative to bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. In this question you may assume without proof that A and A' are similar.

(i) State how the matrix A of L relative to the basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is constructed from L and $\mathcal{B}$. Also state how A may be used to compute $L\mathbf{v}$ for any $\mathbf{v} \in \mathbb{C}^2$.

(ii) Show that A and A' have the same characteristic equation.

(iii) Show that for any $k \neq 0$ the matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & c/k \\ bk & d \end{pmatrix}$$

are similar. [*Hint: if* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *is a basis then so is* $\{k\mathbf{e}_1, \mathbf{e}_2\}$.]

(b) Using the results of (a), or otherwise, prove that any 2×2 complex matrix M with equal eigenvalues is similar to one of

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad \text{with } a \in \mathbb{C}.$$

(c) Consider the matrix

$$B(r) = \frac{1}{2} \begin{pmatrix} 1+r & 1-r & 1 \\ 1-r & 1+r & -1 \\ -1 & 1 & 2r \end{pmatrix}.$$

Show that there is a real value $r_0 > 0$ such that $B(r_0)$ is an orthogonal matrix. Show that $B(r_0)$ is a rotation and find the axis and angle of the rotation.

Paper 1, Section I
1C Vectors and Matrices

(a) Let R be the set of all $z \in \mathbb{C}$ with real part 1. Draw a picture of R and the image of R under the map $z \mapsto e^z$ in the complex plane.

(b) For each of the following equations, find all complex numbers z which satisfy it:

(i) $e^z = e$,

(ii) $(\log z)^2 = -\dfrac{\pi^2}{4}$.

(c) Prove that there is no complex number z satisfying $|z| - z = i$.

Paper 1, Section I
2A Vectors and Matrices

Define what is meant by the terms *rotation*, *reflection*, *dilation* and *shear*. Give examples of real 2×2 matrices representing each of these.

Consider the three 2×2 matrices

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad C = AB\,.$$

Identify the three matrices in terms of your definitions above.

Paper 1, Section II

5C Vectors and Matrices

The equation of a plane Π in $\mathbb{R}^3$ is

$$\mathbf{x} \cdot \mathbf{n} = d\,,$$

where d is a constant scalar and $\mathbf{n}$ is a unit vector normal to Π. What is the distance of the plane from the origin O?

A sphere S with centre $\mathbf{p}$ and radius r satisfies the equation

$$|\mathbf{x} - \mathbf{p}|^2 = r^2\,.$$

Show that the intersection of Π and S contains exactly one point if $|\mathbf{p} \cdot \mathbf{n} - d| = r$.

The tetrahedron $OABC$ is defined by the vectors $\mathbf{a} = \vec{OA}$, $\mathbf{b} = \vec{OB}$, and $\mathbf{c} = \vec{OC}$ with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$. What does the condition $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$ imply about the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$? A sphere T_r with radius $r > 0$ lies inside the tetrahedron and intersects each of the three faces OAB, OBC, and OCA in exactly one point. Show that the centre P of T_r satisfies

$$\vec{OP} = r\frac{|\mathbf{b} \times \mathbf{c}|\mathbf{a} + |\mathbf{c} \times \mathbf{a}|\mathbf{b} + |\mathbf{a} \times \mathbf{b}|\mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}\,.$$

Given that the vector $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is orthogonal to the plane Ψ of the face ABC, obtain an equation for Ψ. What is the distance of Ψ from the origin?

Paper 1, Section II

6A Vectors and Matrices

Explain why the number of solutions $\mathbf{x}$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is 0, 1 or infinity, where A is a real 3×3 matrix and $\mathbf{x}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. State necessary and sufficient conditions on A and $\mathbf{b}$ for each of these possibilities to hold.

Let A and B be real 3×3 matrices. Give necessary and sufficient conditions on A for there to exist a unique real 3×3 matrix X satisfying $AX = B$.

Find X when

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$$

Paper 1, Section II
7B Vectors and Matrices

(a) Consider the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Determine whether or not M is diagonalisable.

(b) Prove that if A and B are similar matrices then A and B have the same eigenvalues with the same corresponding algebraic multiplicities.

Is the converse true? Give either a proof (if true) or a counterexample with a brief reason (if false).

(c) State the Cayley-Hamilton theorem for a complex matrix A and prove it in the case when A is a 2×2 diagonalisable matrix.

Suppose that an $n \times n$ matrix B has $B^k = \mathbf{0}$ for some $k > n$ (where $\mathbf{0}$ denotes the zero matrix). Show that $B^n = \mathbf{0}$.

Paper 1, Section II
8B Vectors and Matrices

(a) (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(ii) Show that the quadric $\mathcal{Q}$ in $\mathbb{R}^3$ defined by

$$3x^2 + 2xy + 2y^2 + 2xz + 2z^2 = 1$$

is an ellipsoid. Find the matrix B of a linear transformation of $\mathbb{R}^3$ that will map $\mathcal{Q}$ onto the unit sphere $x^2 + y^2 + z^2 = 1$.

(b) Let P be a real orthogonal matrix. Prove that:

(i) as a mapping of vectors, P preserves inner products;

(ii) if λ is an eigenvalue of P then $|\lambda| = 1$ and λ^* is also an eigenvalue of P.

Now let Q be a real orthogonal 3×3 matrix having $\lambda = 1$ as an eigenvalue of algebraic multiplicity 2. Give a geometrical description of the action of Q on $\mathbb{R}^3$, giving a reason for your answer. [You may assume that orthogonal matrices are always diagonalisable.]

Paper 1, Section I

1C Vectors and Matrices

For $z, a \in \mathbb{C}$ define the *principal value* of $\log z$ and hence of z^a. Hence find all solutions to

(i) $z^{\mathrm{i}} = 1$

(ii) $z^{\mathrm{i}} + \overline{z}^{\,\mathrm{i}} = 2\mathrm{i}\,,$

and sketch the curve $|z^{\mathrm{i}+1}| = 1$.

Paper 1, Section I

2A Vectors and Matrices

The matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad , \qquad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad .$$

Find the matrix A' that represents Φ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \quad , \qquad C' = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \quad .$$

Paper 1, Section II
5C Vectors and Matrices

Explain why each of the equations

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{b} \tag{1}$$

$$\mathbf{x} \times \mathbf{c} = \mathbf{d} \tag{2}$$

describes a straight line, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constant vectors in $\mathbb{R}^3$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero, $\mathbf{c} \cdot \mathbf{d} = 0$ and λ is a real parameter. Describe the geometrical relationship of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ to the relevant line, assuming that $\mathbf{d} \neq \mathbf{0}$.

Show that the solutions of (2) satisfy an equation of the form (1), defining $\mathbf{a}$, $\mathbf{b}$ and $\lambda(\mathbf{x})$ in terms of $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{a} \cdot \mathbf{b} = 0$ and $|\mathbf{b}| = |\mathbf{c}|$. Deduce that the conditions on $\mathbf{c}$ and $\mathbf{d}$ are sufficient for (2) to have solutions.

For each of the lines described by (1) and (2), find the point $\mathbf{x}$ that is closest to a given fixed point $\mathbf{y}$.

Find the line of intersection of the two planes $\mathbf{x} \cdot \mathbf{m} = \mu$ and $\mathbf{x} \cdot \mathbf{n} = \nu$, where $\mathbf{m}$ and $\mathbf{n}$ are constant unit vectors, $\mathbf{m} \times \mathbf{n} \neq \mathbf{0}$ and μ and ν are constants. Express your answer in each of the forms (1) and (2), giving both $\mathbf{a}$ and $\mathbf{d}$ as linear combinations of $\mathbf{m}$ and $\mathbf{n}$.

Paper 1, Section II
6A Vectors and Matrices

The map $\Phi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n}) + \alpha(\mathbf{n} \cdot \mathbf{x})\,\mathbf{n}$ is defined for $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$ and α is a constant.

(a) Find the inverse map Φ^{-1}, when it exists, and determine the values of α for which it does.

(b) When Φ is not invertible, find its image and kernel, and explain geometrically why these subspaces are perpendicular.

(c) Let $\mathbf{y} = \Phi(\mathbf{x})$. Find the components A_{ij} of the matrix A such that $y_i = A_{ij} x_j$. When Φ is invertible, find the components of the matrix B such that $x_i = B_{ij} y_j$.

(d) Now let A be as defined in (c) for the case $\mathbf{n} = \frac{1}{\sqrt{3}}(1,1,1)$, and let

$$C = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}.$$

By analysing a suitable determinant, for all values of α find all vectors $\mathbf{x}$ such that $A\mathbf{x} = C\mathbf{x}$. Explain your results by interpreting A and C geometrically.

Paper 1, Section II

7B Vectors and Matrices

(a) Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -2 & 3 \end{pmatrix} .$$

(b) Under what conditions on the $3{\times}3$ matrix A and the vector $\mathbf{b}$ in $\mathbb{R}^3$ does the equation

$$A\mathbf{x} = \mathbf{b} \qquad (*)$$

have 0, 1, or infinitely many solutions for the vector $\mathbf{x}$ in $\mathbb{R}^3$? Give clear, concise arguments to support your answer, explaining why just these three possibilities are allowed.

(c) Using the results of (a), or otherwise, find all solutions to $(*)$ when

$$A = M - \lambda I \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

in each of the cases $\lambda = 0,\ 1,\ 2$.

Paper 1, Section II

8B Vectors and Matrices

(a) Let M be a real symmetric $n{\times}n$ matrix. Prove the following.

(i) Each eigenvalue of M is real.

(ii) Each eigenvector can be chosen to be real.

(iii) Eigenvectors with different eigenvalues are orthogonal.

(b) Let A be a real antisymmetric $n{\times}n$ matrix. Prove that each eigenvalue of A^2 is real and is less than or equal to zero.

If $-\lambda^2$ and $-\mu^2$ are distinct, non-zero eigenvalues of A^2, show that there exist orthonormal vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{w}$, $\mathbf{w}'$ with

$$A\mathbf{u} = \lambda\mathbf{u}' , \qquad A\mathbf{w} = \mu\mathbf{w}' ,$$

$$A\mathbf{u}' = -\lambda\mathbf{u} , \qquad A\mathbf{w}' = -\mu\mathbf{w} .$$

Paper 1, Section I

1A Vectors and Matrices

Let A be the matrix representing a linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ of $\mathbb{R}^m$, so that $\Phi(\mathbf{b}_i) = A_{ji}\mathbf{c}_j$. Let $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ be another basis of $\mathbb{R}^n$ and let $\{\mathbf{c}'_1, \ldots, \mathbf{c}'_m\}$ be another basis of $\mathbb{R}^m$. Show that the matrix A' representing Φ with respect to these new bases satisfies $A' = C^{-1}A\,B$ with matrices B and C which should be defined.

Paper 1, Section I

2C Vectors and Matrices

(a) The complex numbers z_1 and z_2 satisfy the equations

$$z_1^3 = 1\,, \qquad z_2^9 = 512\,.$$

What are the possible values of $|z_1 - z_2|$? Justify your answer.

(b) Show that $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for all complex numbers z_1 and z_2. Does the inequality $|z_1 + z_2| + |z_1 - z_2| \leqslant 2\max\left(|z_1|, |z_2|\right)$ hold for all complex numbers z_1 and z_2? Justify your answer with a proof or a counterexample.

Paper 1, Section II

5A Vectors and Matrices

Let A and B be real $n \times n$ matrices.

(i) Define the trace of A, $\operatorname{tr}(A)$, and show that $\operatorname{tr}\left(A^T B\right) = \operatorname{tr}\left(B^T A\right)$.

(ii) Show that $\operatorname{tr}\left(A^T A\right) \geqslant 0$, with $\operatorname{tr}\left(A^T A\right) = 0$ if and only if A is the zero matrix. Hence show that

$$\left(\operatorname{tr}(A^T B)\right)^2 \leqslant \operatorname{tr}\left(A^T A\right)\operatorname{tr}\left(B^T B\right)\ .$$

Under what condition on A and B is equality achieved?

(iii) Find a basis for the subspace of 2×2 matrices X such that

$$\operatorname{tr}\left(A^T X\right) = \operatorname{tr}\left(B^T X\right) = \operatorname{tr}\left(C^T X\right) = 0\,,$$

$$\text{where} \qquad A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Paper 1, Section II

6C Vectors and Matrices

Let $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ be vectors in $\mathbb{R}^3$. Give a definition of the dot product, $\mathbf{a}_1 \cdot \mathbf{a}_2$, the cross product, $\mathbf{a}_1 \times \mathbf{a}_2$, and the triple product, $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$. Explain what it means to say that the three vectors are *linearly independent*.

Let $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ be vectors in $\mathbb{R}^3$. Let S be a 3×3 matrix with entries $S_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$. Show that

$$(\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3)(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3) \;=\; \det(S)\,.$$

Hence show that S is of maximal rank if and only if the sets of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are both linearly independent.

Now let $\{\mathbf{c}_1,\, \mathbf{c}_2,\, \ldots,\, \mathbf{c}_n\}$ and $\{\mathbf{d}_1,\, \mathbf{d}_2,\, \ldots,\, \mathbf{d}_n\}$ be sets of vectors in $\mathbb{R}^n$, and let T be an $n \times n$ matrix with entries $T_{ij} = \mathbf{c}_i \cdot \mathbf{d}_j$. Is it the case that T is of maximal rank if and only if the sets of vectors $\{\mathbf{c}_1,\, \mathbf{c}_2,\, \ldots,\, \mathbf{c}_n\}$ and $\{\mathbf{d}_1,\, \mathbf{d}_2,\, \ldots,\, \mathbf{d}_n\}$ are both linearly independent? Justify your answer with a proof or a counterexample.

Given an integer $n > 2$, is it always possible to find a set of vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ in $\mathbb{R}^n$ with the property that every pair is linearly independent and that every triple is linearly dependent? Justify your answer.

Paper 1, Section II

7B Vectors and Matrices

Let A be a complex $n \times n$ matrix with an eigenvalue λ. Show directly from the definitions that:

(i) A^r has an eigenvalue λ^r for any integer $r \geqslant 1$; and

(ii) if A is invertible then $\lambda \neq 0$ and A^{-1} has an eigenvalue λ^{-1}.

For any complex $n \times n$ matrix A, let $\chi_A(t) = \det(A - tI)$. Using standard properties of determinants, show that:

(iii) $\chi_{A^2}(t^2) \;=\; \chi_A(t)\,\chi_A(-t)$; and

(iv) if A is invertible,

$$\chi_{A^{-1}}(t) \;=\; (\det A)^{-1}(-1)^n t^n \chi_A(t^{-1})\,.$$

Explain, including justifications, the relationship between the eigenvalues of A and the polynomial $\chi_A(t)$.

If A^4 has an eigenvalue μ, does it follow that A has an eigenvalue λ with $\lambda^4 = \mu$? Give a proof or counterexample.

Paper 1, Section II

8B Vectors and Matrices

Let R be a real orthogonal 3×3 matrix with a real eigenvalue λ corresponding to some real eigenvector. Show algebraically that $\lambda = \pm1$ and interpret this result geometrically.

Each of the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

has an eigenvalue $\lambda = 1$. Confirm this by finding as many independent eigenvectors as possible with this eigenvalue, for each matrix in turn.

Show that one of the matrices above represents a rotation, and find the axis and angle of rotation. Which of the other matrices represents a reflection, and why?

State, with brief explanations, whether the matrices M, N, P are diagonalisable (i) over the real numbers; (ii) over the complex numbers.

Paper 1, Section I

1C Vectors and Matrices

Describe geometrically the three sets of points defined by the following equations in the complex z plane:

(a) $z\overline{\alpha} + \overline{z}\alpha = 0$, where α is non-zero;

(b) $2|z - a| = z + \overline{z} + 2a$, where a is real and non-zero;

(c) $\log z = \mathrm{i} \log \overline{z}$.

Paper 1, Section I

2B Vectors and Matrices

Define the Hermitian conjugate $A^\dagger$ of an $n \times n$ complex matrix A. State the conditions (i) for A to be Hermitian (ii) for A to be unitary.

In the following, A, B, C and D are $n \times n$ complex matrices and $\mathbf{x}$ is a complex n-vector. A matrix N is defined to be *normal* if $N^\dagger N = NN^\dagger$.

(a) Let A be nonsingular. Show that $B = A^{-1}A^\dagger$ is unitary if and only if A is normal.

(b) Let C be normal. Show that $|C\mathbf{x}| = 0$ if and only if $|C^\dagger \mathbf{x}| = 0$.

(c) Let D be normal. Deduce from (b) that if $\mathbf{e}$ is an eigenvector of D with eigenvalue λ then $\mathbf{e}$ is also an eigenvector of $D^\dagger$ and find the corresponding eigenvalue.

Paper 1, Section II

5C Vectors and Matrices

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors. By using suffix notation, prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \tag{1}$$

and

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\mathbf{a}\ . \tag{2}$$

The three distinct points A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ lie on the surface of the unit sphere centred on the origin O. The *spherical distance* between the points A and B, denoted $\delta(A, B)$, is the length of the (shorter) arc of the circle with centre O passing through A and B. Show that

$$\cos \delta(A, B) = \mathbf{a} \cdot \mathbf{b}\ .$$

A *spherical triangle* with vertices A, B, C is a region on the sphere bounded by the three circular arcs AB, BC, CA. The interior angles of a spherical triangle at the vertices A, B, C are denoted α, β, γ, respectively.

By considering the normals to the planes OAB and OAC, or otherwise, show that

$$\cos \alpha = \frac{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})}{|\mathbf{a} \times \mathbf{b}||\mathbf{a} \times \mathbf{c}|}.$$

Using identities (1) and (2), prove that

$$\cos \delta(B, C) = \cos \delta(A, B) \cos \delta(A, C) + \sin \delta(A, B) \sin \delta(A, C) \cos \alpha$$

and

$$\frac{\sin \alpha}{\sin \delta(B, C)} = \frac{\sin \beta}{\sin \delta(A, C)} = \frac{\sin \gamma}{\sin \delta(A, B)}.$$

For an equilateral spherical triangle show that $\alpha > \pi/3$.

Paper 1, Section II

6B Vectors and Matrices

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the matrix equation $A\mathbf{x} = \mathbf{c}$ is 0, 1 or infinity, where A is a real 3×3 matrix and $\mathbf{c} \in \mathbb{R}^3$. State conditions on A and $\mathbf{c}$ that distinguish between these possibilities, and state the relationship that holds between any two solutions when there are infinitely many.

Consider the case

$$A = \begin{pmatrix} a & a & b \\ b & a & a \\ a & b & a \end{pmatrix} \quad \text{and } \mathbf{c} = \begin{pmatrix} 1 \\ c \\ 1 \end{pmatrix} .$$

Use row and column operations to find and factorize the determinant of A.

Find the kernel and image of the linear map represented by A for all values of a and b. Find the general solution to $A\mathbf{x} = \mathbf{c}$ for all values of a, b and c for which a solution exists.

Paper 1, Section II

7A Vectors and Matrices

Let A be an $n \times n$ Hermitian matrix. Show that all the eigenvalues of A are real.

Suppose now that A has n distinct eigenvalues.

(a) Show that the eigenvectors of A are orthogonal.

(b) Define the *characteristic polynomial* $P_A(t)$ of A. Let

$$P_A(t) = \sum_{r=0}^{n} a_r t^r \, .$$

Prove the matrix identity

$$\sum_{r=0}^{n} a_r A^r = 0 \, .$$

(c) What is the range of possible values of

$$\frac{\mathbf{x}^{\dagger} A \mathbf{x}}{\mathbf{x}^{\dagger}\mathbf{x}}$$

for non-zero vectors $\mathbf{x} \in \mathbb{C}^n$? Justify your answer.

(d) For any (not necessarily symmetric) real 2×2 matrix B with real eigenvalues, let $\lambda_{\max}(B)$ denote its maximum eigenvalue. Is it possible to find a constant C such that

$$\frac{\mathbf{x}^{\dagger} B \mathbf{x}}{\mathbf{x}^{\dagger}\mathbf{x}} \leqslant C \, \lambda_{\max}(B)$$

for all non-zero vectors $\mathbf{x} \in \mathbb{R}^2$ and all such matrices B? Justify your answer.

Paper 1, Section II

8A Vectors and Matrices

(a) Explain what is meant by saying that a 2×2 real transformation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves the scalar product with respect to the Euclidean metric $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{R}^2$.

Derive a description of all such matrices that uses a single real parameter together with choices of sign (± 1). Show that these matrices form a group.

(b) Explain what is meant by saying that a 2×2 real transformation matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves the scalar product with respect to the Minkowski metric $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{R}^2$.

Consider now the set of such matrices with $a > 0$. Derive a description of all matrices in this set that uses a single real parameter together with choices of sign (± 1). Show that these matrices form a group.

(c) What is the intersection of these two groups?

1/I/1B **Vectors and Matrices**

State de Moivre's Theorem. By evaluating

$$\sum_{r=1}^{n} e^{\,ir\theta}\ ,$$

or otherwise, show that

$$\sum_{r=1}^{n} \cos(r\theta) \;=\; \frac{\cos(n\theta) - \cos((n+1)\theta)}{2\,(1-\cos\theta)} \;-\; \frac{1}{2}\ .$$

Hence show that

$$\sum_{r=1}^{n} \cos\left(\frac{2\,p\,\pi\,r}{n+1}\right) = -1\ ,$$

where p is an integer in the range $1 \leqslant p \leqslant n$.

1/I/2A **Vectors and Matrices**

Let U be an $n \times n$ unitary matrix ($U^\dagger U = UU^\dagger = I$). Suppose that A and B are $n \times n$ Hermitian matrices such that $U = A + iB$.

Show that

(i) A and B commute,

(ii) $A^2 + B^2 \;=\; I$.

Find A and B in terms of U and $U^\dagger$, and hence show that A and B are uniquely determined for a given U.

1/II/5B **Vectors and Matrices**

(a) Use suffix notation to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}\, .$$

Hence, or otherwise, expand

(i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$,

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

(b) Write down the equation of the line that passes through the point $\mathbf{a}$ and is parallel to the unit vector $\hat{\mathbf{t}}$.

The lines L_1 and L_2 in three dimensions pass through $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively and are parallel to the unit vectors $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ respectively. Show that a necessary condition for L_1 and L_2 to intersect is

$$(\mathbf{a}_1 - \mathbf{a}_2) \cdot \left(\hat{\mathbf{t}}_1 \times \hat{\mathbf{t}}_2\right) = 0\, .$$

Why is this condition not sufficient?

In the case in which L_1 and L_2 are non-parallel and non-intersecting, find an expression for the shortest distance between them.

1/II/6A **Vectors and Matrices**

A real 3×3 matrix A with elements A_{ij} is said to be *upper triangular* if $A_{ij} = 0$ whenever $i > j$. Prove that if A and B are upper triangular 3×3 real matrices then so is the matrix product AB.

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that $A^3 + A^2 - A = I$. Write A^{-1} as a linear combination of A^2, A and I and hence compute A^{-1} explicitly.

For all integers n (including negative integers), prove that there exist coefficients α_n, β_n and γ_n such that

$$A^n = \alpha_n A^2 + \beta_n A + \gamma_n I.$$

For all integers n (including negative integers), show that

$$(A^n)_{11} = 1, \qquad (A^n)_{22} = (-1)^n, \qquad \text{and} \quad (A^n)_{23} = n(-1)^{n-1}.$$

Hence derive a set of 3 simultaneous equations for $\{\alpha_n, \beta_n, \gamma_n\}$ and find their solution.

1/II/7C **Vectors and Matrices**

Prove that any n orthonormal vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Let A be a real symmetric $n \times n$ matrix with n orthonormal eigenvectors $\mathbf{e}_i$ and corresponding eigenvalues λ_i. Obtain coefficients a_i such that

$$\mathbf{x} = \sum_i a_i \mathbf{e}_i$$

is a solution to the equation

$$A\mathbf{x} - \mu\mathbf{x} = \mathbf{f},$$

where $\mathbf{f}$ is a given vector and μ is a given scalar that is not an eigenvalue of A.

How would your answer differ if $\mu = \lambda_1$?

Find a_i and hence $\mathbf{x}$ when

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{f} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

in the cases (i) $\mu = 2$ and (ii) $\mu = 1$.

1/II/8C **Vectors and Matrices**

Prove that the eigenvalues of a Hermitian matrix are real and that eigenvectors corresponding to distinct eigenvalues are orthogonal (i.e. $\mathbf{e}_i^* \cdot \mathbf{e}_j = 0$).

Let A be a real 3×3 non-zero antisymmetric matrix. Show that iA is Hermitian. Hence show that there exists a (complex) eigenvector $\mathbf{e}_1$ such $A\mathbf{e}_1 = \lambda\mathbf{e}_1$, where λ is imaginary.

Show further that there exist real vectors $\mathbf{u}$ and $\mathbf{v}$ and a real number θ such that

$$A\mathbf{u} = \theta\mathbf{v} \qquad \text{and} \qquad A\mathbf{v} = -\theta\mathbf{u}.$$

Show also that A has a real eigenvector $\mathbf{e}_3$ such that $A\mathbf{e}_3 = 0$.

Let $R = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$. By considering the action of R on $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{e_3}$, show that R is a rotation matrix.