Part IA

Vector Calculus

Paper 3, Section I

3B Vector Calculus

What does it mean for a vector field $\mathbf{F}$ in $\mathbb{R}^3$ to be *irrotational*?

Given a field $\mathbf{F}$ that is irrotational everywhere, and given a fixed point $\mathbf{x}_0$, write down the definition of a scalar potential $V(\mathbf{x})$ that satisfies $\mathbf{F} = -\boldsymbol{\nabla} V$ and $V(\mathbf{x}_0) = 0$. Show that this potential is well-defined.

Given vector fields $\mathbf{A}_0$ and $\mathbf{B}$ with $\boldsymbol{\nabla} \times \mathbf{A}_0 = \mathbf{B}$, write down the form of the general solution $\mathbf{A}$ to $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$. State a necessary condition on $\mathbf{B}$ for such an $\mathbf{A}_0$ to exist.

Paper 3, Section I

4B Vector Calculus

Cartesian coordinates x, y, z and cylindrical polar coordinates ρ, ϕ, z are related by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi\,.$$

Find scalars h_ρ, h_ϕ and unit vectors $\mathbf{e}_\rho, \mathbf{e}_\phi$ such that $\mathrm{d}\mathbf{x} = h_\rho \mathbf{e}_\rho\, \mathrm{d}\rho + h_\phi \mathbf{e}_\phi\, \mathrm{d}\phi + \mathbf{e}_z \mathrm{d}z\,.$

A region V is defined by

$$\rho_0 \leqslant \rho \leqslant \rho_0 + \Delta\rho\,, \qquad \phi_0 \leqslant \phi \leqslant \phi_0 + \Delta\phi\,, \qquad z_0 \leqslant z \leqslant z_0 + \Delta z\,,$$

where $\rho_0, \phi_0, z_0, \Delta\rho, \Delta\phi$ and Δz are positive constants. Write down, or calculate, the scalar areas of its six faces and its volume ΔV.

For a vector field $\mathbf{F}(\mathbf{x}) = F(\rho)\mathbf{e}_\rho$, calculate the value of

$$\lim_{\Delta\rho \to 0} \frac{1}{\Delta V} \int_{\partial V} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S\,,$$

where ∂V and $\mathbf{n}$ are the surface and outward normal of the region V.

Paper 3, Section II

9B Vector Calculus

The vector fields $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{w}(\mathbf{x},t)$ obey the evolution equations

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}P\,,$$
$$\frac{\partial \mathbf{w}}{\partial t} = (\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w}\,,$$

where P is a given scalar field. Show that the scalar field $h = \mathbf{u}\cdot\mathbf{w}$ obeys an evolution equation of the form

$$\frac{\partial h}{\partial t} = (\mathbf{w}\cdot\boldsymbol{\nabla})f + (\mathbf{u}\cdot\boldsymbol{\nabla})g\,,$$

where the scalar fields f and g should be identified.

Suppose that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $\mathbf{w} = \boldsymbol{\nabla}\times\mathbf{u}$. Show that, if $\mathbf{u}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} = 0$ on the surface S of a fixed volume V with outward normal $\mathbf{n}$, then

$$\frac{dH}{dt} = 0, \text{ where } H = \int_V h\,dV\,.$$

Suppose that $\mathbf{u} = (a^2-\rho^2)\rho\sin z\,\mathbf{e}_\phi + a\rho^2\sin z\,\mathbf{e}_z$ in cylindrical polar coordinates ρ,ϕ,z, where a is a constant, and that $\mathbf{w} = \boldsymbol{\nabla}\times\mathbf{u}$. Show that $h = -2a\rho^4\sin^2 z$, and calculate the value of H when V is the cylinder $0 \leqslant \rho \leqslant a$, $0 \leqslant z \leqslant \pi$.

$$\left[\textit{In cylindrical polar coordinates } \boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.\right]$$

Paper 3, Section II

10B Vector Calculus

Show that

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\,\boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b}\,\boldsymbol{\nabla} \cdot \mathbf{a} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} - (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b}\,.$$

State Stokes' theorem for a vector field in $\mathbb{R}^3$, specifiying the orientation of the integrals.

The vector fields $\mathbf{m}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ satisfy the conditions $\mathbf{m} = \mathbf{n}$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on an open surface S with unit normal $\mathbf{n}(\mathbf{x})$. By applying Stokes' theorem to the vector field $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S (\delta_{ij} - n_i n_j)\frac{\partial v_i}{\partial x_j} dS = \oint_C \left[\mathbf{v} \cdot (d\mathbf{x} \times \mathbf{n})\right], \qquad (*)$$

where C is the boundary of S. Describe the orientation of $d\mathbf{x} \times \mathbf{n}$ relative to S and C.

Verify $(*)$ when S is the hemisphere $r = R$, $z \geqslant 0$ and $\mathbf{v} = r \sin\theta\, \mathbf{e}_\theta$ in spherical polar coordinates r, θ, ϕ.

[*You may use the formulae* $(\mathbf{e}_r \cdot \boldsymbol{\nabla})\mathbf{e}_\theta = \mathbf{0}$ *and*

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\,,$$

and you may quote formulae for dS and $d\mathbf{x}$ in these coordinates without derivation.]

Paper 3, Section II
11B Vector Calculus

(a) Verify the identity

$$\boldsymbol{\nabla} \cdot (\kappa \psi \boldsymbol{\nabla} \phi) = \psi \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \phi) + \kappa \boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \phi \,,$$

where $\kappa(\mathbf{x})$, $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are differentiable scalar functions.

Let V be a region in $\mathbb{R}^3$ that is bounded by a closed surface S. The function $\phi(\mathbf{x})$ satisfies

$$\boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} \phi) = 0 \text{ in } V \text{ and } \phi = f(\mathbf{x}) \text{ on } S \,,$$

where κ and f are given functions and $\kappa > 0$. Show that ϕ is unique.

The function $w(\mathbf{x})$ also satisfies $w = f(\mathbf{x})$ on S. By writing $w = \phi + \psi$, show that

$$\int_V \kappa |\boldsymbol{\nabla} w|^2 \, dV \geqslant \int_V \kappa |\boldsymbol{\nabla} \phi|^2 \, dV \,.$$

(b) A steady temperature field $T(\mathbf{x})$ due to a distribution of heat sources $H(\mathbf{x})$ in a medium with spatially varying thermal diffusivity $\kappa(\mathbf{x})$ satisfies

$$\boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T) + H = 0 \,.$$

Show that the heat flux $\int_S \mathbf{q} \cdot d\mathbf{S}$ across a closed surface S, where $\mathbf{q} = -\kappa \boldsymbol{\nabla} T$, can be expressed as an integral of the heat sources within S.

By using this version of Gauss's law, or otherwise, find the temperature field $T(r)$ for the spherically symmetric case when

$$\kappa(r) = r^\alpha, \quad -1 < \alpha < 2, \qquad\qquad H(r) = \begin{cases} H_0 & \text{if } r \leqslant 1 \\ 0 & \text{if } r > 1 \end{cases}$$

subject to the condition that $T \to 0$ as $r \to \infty$. What goes wrong if $\alpha \leqslant -1$?

Deduce that if $w(r)$ satisfies $w(1) = 1$ and $w(r) \to 0$ as $r \to \infty$ (sufficiently rapidly for the integral to converge) then

$$\int_1^\infty r^{\alpha+2} \left(\frac{dw}{dr} \right)^2 dr \geqslant \alpha + 1 \,.$$

Paper 3, Section II

12B Vector Calculus

(a) State the transformation law for the components of an nth-rank tensor $T_{ij\ldots k}$ under a rotation of the basis vectors, being careful to specify how any rotation matrix relates the new basis $\{\mathbf{e}_i'\}$ to the original basis $\{\mathbf{e}_j\}$, $i, j = 1, 2, 3$.

If $\phi(\mathbf{x})$ is a scalar field, show that $\partial^2\phi/\partial x_i \partial x_j$ transforms as a second-rank tensor.

Define what it means for a tensor to be *isotropic*. Write down the most general isotropic tensors of rank k for $k = 0, 1, 2, 3$.

(b) Explain briefly why T_{ijkl}, defined by

$$T_{ijkl} = \int_{\mathbb{R}^3} x_i x_j e^{-r^2} \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV\,, \quad \text{where } r = |\mathbf{x}|,$$

is an isotropic fourth-rank tensor.

Assuming that

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\,,$$

use symmetry, contractions and a scalar integral to determine the constants α, β and γ.

[*Hint:* $\nabla^2(1/r) = 0$ for $r \neq 0$.]

Paper 3, Section I
3A Vector Calculus

Let D be the region in the positive quadrant of the xy plane defined by

$$y \leqslant x \leqslant \alpha y\,, \qquad \frac{1}{y} \leqslant x \leqslant \frac{\alpha}{y}\,,$$

where $\alpha > 1$ is a constant. By using the change of variables $u = x/y$, $v = xy$, or otherwise, evaluate

$$\int_D x^2\, dx\, dy\ .$$

Paper 3, Section I
4A Vector Calculus

Consider the curve in $\mathbb{R}^3$ defined by $y = \log x$, $z = 0$. Using a parametrization of your choice, find an expression for the unit tangent vector $\mathbf{t}$ at a general point on the curve. Calculate the curvature κ as a function of your chosen parameter. Hence find the maximum value of κ and the point on the curve at which it is attained.

[*You may assume that* $\kappa = |\,\mathbf{t} \times (d\mathbf{t}/ds)\,|$ *where* s *is the arc-length.*]

Paper 3, Section II
9A Vector Calculus

(a) Using Cartesian coordinates x_i in $\mathbb{R}^3$, write down an expression for $\partial r/\partial x_i$, where r is the radial coordinate ($r^2 = x_i x_i$), and deduce that

$$\nabla \cdot (\, g(r)\mathbf{x}\,) \;=\; r g'(r) + 3g(r)$$

for any differentiable function $g(r)$.

(b) For spherical polar coordinates r, θ, ϕ satisfying

$$x_1 = r\sin\theta\cos\phi\,, \quad x_2 = r\sin\theta\sin\phi\,, \quad x_3 = r\cos\theta\,,$$

find scalars h_r, h_θ, h_ϕ and unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ such that

$$d\mathbf{x} \;=\; h_r\,\mathbf{e}_r\,dr \;+\; h_\theta\,\mathbf{e}_\theta\,d\theta \;+\; h_\phi\,\mathbf{e}_\phi\,d\phi\;.$$

Hence, using the relation $df = d\mathbf{x}\cdot\nabla f$, find an expression for ∇f in spherical polars for any differentiable function $f(\mathbf{x})$.

(c) Consider the vector fields

$$\mathbf{A}^+ \;=\; \frac{1}{r}\tan\frac{\theta}{2}\,\mathbf{e}_\phi \quad (r\neq 0,\, \theta\neq\pi)\,, \qquad \mathbf{A}^- \;=\; -\frac{1}{r}\cot\frac{\theta}{2}\,\mathbf{e}_\phi \quad (r\neq 0,\, \theta\neq 0)\,.$$

Compute $\nabla\times\mathbf{A}^+$ and $\nabla\times\mathbf{A}^-$ and use the result in part (a) to check explicitly that your answers have zero divergence.

[*You may use without proof the formula* $\nabla\times\mathbf{A} \;=\; \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\mathbf{e}_r & h_\theta\mathbf{e}_\theta & h_\phi\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix}$.]

(d) From your answers in part (c), explain briefly on general grounds why

$$\mathbf{A}^+ \;-\; \mathbf{A}^- \;=\; \nabla f$$

for some function $f(\mathbf{x})$. Find a solution for f that is defined on the region $x_1 > 0$.

Paper 3, Section II
10A Vector Calculus

Let H be the unbounded surface defined by $x^2 + y^2 = z^2 + 1$, and S the bounded surface defined as the subset of H with $1 \leqslant z \leqslant \sqrt{2}$. Calculate the vector area element $d\mathbf{S}$ on S in terms of ρ and ϕ, where $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Sketch the surface and indicate the sense of the corresponding normal.

Compute directly

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

where $\mathbf{A} = (-yz^2, xz^2, 0)$. Now verify your answer using Stokes' Theorem.

What is the value of

$$\int_{S'} \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

where S' is defined as the subset of H with $-1 \leqslant z \leqslant \sqrt{2}$? Justify your answer.

Paper 3, Section II
11A Vector Calculus

Let V be a region in $\mathbb{R}^3$ with boundary a closed surface S. Consider a function ϕ defined in V that satisfies

$$\nabla^2\phi - m^2\phi = 0$$

for some constant $m \geqslant 0$.

(i) If $\partial\phi/\partial n = g$ on S, for some given function g, show that ϕ is unique provided that $m > 0$. Does this conclusion change if $m = 0$?

[*Recall:* $\partial/\partial n = \mathbf{n} \cdot \nabla$, *where* $\mathbf{n}$ *is the outward pointing unit normal on* S.]

(ii) Now suppose instead that $\phi = f$ on S, for some given function f. Show that for any function ψ with $\psi = f$ on S,

$$\int_V (\,|\nabla\psi|^2 + m^2\psi^2\,)\, dV \;\geqslant\; \int_V (\,|\nabla\phi|^2 + m^2\phi^2\,)\, dV\ .$$

What is the condition for equality to be achieved, and is this result sufficient to deduce that ϕ is unique? Justify your answers, distinguishing carefully between the cases $m > 0$ and $m = 0$.

Paper 3, Section II
12A Vector Calculus

Consider a rigid body B of uniform density ρ and total mass M rotating with constant angular velocity $\boldsymbol{\omega}$ relative to a point $\mathbf{a}$. The angular momentum $\mathbf{L}$ about $\mathbf{a}$ is defined by

$$\mathbf{L} = \int_B (\mathbf{x} - \mathbf{a}) \times [\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a})] \, \rho \, dV \,,$$

and the inertia tensor $I_{ij}(\mathbf{a})$ about $\mathbf{a}$ is defined by the relation

$$L_i = I_{ij}(\mathbf{a}) \, \omega_j \,.$$

(a) Given that $\mathbf{L}$ is a vector for any choice of the vector $\boldsymbol{\omega}$, show from first principles that $I_{ij}(\mathbf{a})$ is indeed a tensor, of rank 2.

Assuming that the centre of mass of B is located at the origin $\mathbf{0}$, so that

$$\int_B x_i \, dV \; = \; 0 \,,$$

show that

$$I_{ij}(\mathbf{a}) \; = \; I_{ij}(\mathbf{0}) \; + \; M(\, a_k a_k \delta_{ij} - a_i a_j \,) \,,$$

and find an explicit integral expression for $I_{ij}(\mathbf{0})$.

(b) Now suppose that B is a cube centred at $\mathbf{0}$ with edges of length ℓ parallel to the coordinate axes, *i.e.* B occupies the region $-\frac{1}{2}\ell \, \leqslant \, x_i \, \leqslant \, \frac{1}{2}\ell$. Using symmetry, explain in outline why $I_{ij}(\mathbf{0}) \; = \; \lambda \, \delta_{ij}$ for some constant λ.

Given that $\lambda = M\ell^2/6$, find $I_{ij}(\mathbf{a})$ when $\mathbf{a} = \frac{1}{2}\ell(1, 1, 0)$, writing the result in matrix form. Hence, or otherwise, show that if the cube is rotating relative to $\mathbf{a}$ with $|\boldsymbol{\omega}| = 1$ then, depending on the direction of the angular velocity, $|\mathbf{L}|$ has a maximum value that is four times larger than its minimum value.

Paper 3, Section I

3B Vector Calculus

(a) Prove that

$$\boldsymbol{\nabla} \times (\psi \mathbf{A}) = \psi\, \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla}\psi \times \mathbf{A}\,,$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B}\,,$$

where $\mathbf{A}$ and $\mathbf{B}$ are differentiable vector fields and ψ is a differentiable scalar field.

(b) Find the solution of $\nabla^2 u = 16r^2$ on the two-dimensional domain $\mathcal{D}$ when

(i) $\mathcal{D}$ is the unit disc $0 \leqslant r \leqslant 1$, and $u = 1$ on $r = 1$;

(ii) $\mathcal{D}$ is the annulus $1 \leqslant r \leqslant 2$, and $u = 1$ on both $r = 1$ and $r = 2$.

[*Hint: the Laplacian in plane polar coordinates is:*

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,. \quad]$$

Paper 3, Section I

4B Vector Calculus

(a) What is meant by an *antisymmetric* tensor of second rank? Show that if a second rank tensor is antisymmetric in one Cartesian coordinate system, it is antisymmetric in every Cartesian coordinate system.

(b) Consider the vector field $\mathbf{F} = (y,\, z,\, x)$ and the second rank tensor defined by $T_{ij} = \partial F_i / \partial x_j$. Calculate the components of the antisymmetric part of T_{ij} and verify that it equals $-(1/2)\epsilon_{ijk}B_k$, where ϵ_{ijk} is the alternating tensor and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{F}$.

Paper 3, Section II

9B Vector Calculus

(a) Given a space curve $\mathbf{r}(t) = (\,x(t),\, y(t),\, z(t)\,)$, with t a parameter (not necessarily arc-length), give mathematical expressions for the unit tangent, unit normal, and unit binormal vectors.

(b) Consider the closed curve given by

$$x = 2\cos^3 t, \qquad y = \sin^3 t, \qquad z = \sqrt{3}\sin^3 t, \qquad (*)$$

where $t \in [0,\, 2\pi)$.

Show that the unit tangent vector $\mathbf{T}$ may be written as

$$\mathbf{T} = \pm\frac{1}{2}\left(-2\cos t,\, \sin t,\, \sqrt{3}\sin t\right),$$

with each sign associated with a certain range of t, which you should specify.

Calculate the unit normal and the unit binormal vectors, and hence deduce that the curve lies in a plane.

(c) A closed space curve $\mathcal{C}$ lies in a plane with unit normal $\mathbf{n} = (a,\, b,\, c)$. Use Stokes' theorem to prove that the planar area enclosed by $\mathcal{C}$ is the absolute value of the line integral

$$\frac{1}{2}\int_{\mathcal{C}} (bz - cy)dx + (cx - az)dy + (ay - bx)dz.$$

Hence show that the planar area enclosed by the curve given by $(*)$ is $(3/2)\pi$.

Paper 3, Section II

10B Vector Calculus

(a) By considering an appropriate double integral, show that

$$\int_0^\infty e^{-ax^2}\, dx = \sqrt{\frac{\pi}{4a}}\,,$$

where $a > 0$.

(b) Calculate $\int_0^1 x^y dy$, treating x as a constant, and hence show that

$$\int_0^\infty \frac{(e^{-u} - e^{-2u})}{u}\, du = \log 2\,.$$

(c) Consider the region $\mathcal{D}$ in the x-y plane enclosed by $x^2 + y^2 = 4$, $y = 1$, and $y = \sqrt{3}x$ with $1 < y < \sqrt{3}x$.

Sketch $\mathcal{D}$, indicating any relevant polar angles.

A surface $\mathcal{S}$ is given by $z = xy/(x^2 + y^2)$. Calculate the volume below this surface and above $\mathcal{D}$.

Paper 3, Section II

11B Vector Calculus

(a) By a suitable change of variables, calculate the volume enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where a, b, and c are constants.

(b) Suppose T_{ij} is a second rank tensor. Use the divergence theorem to show that

$$\int_{\mathcal{S}} T_{ij} n_j \, dS = \int_{\mathcal{V}} \frac{\partial T_{ij}}{\partial x_j} \, dV \,, \qquad (*)$$

where $\mathcal{S}$ is a closed surface, with unit normal n_j, and $\mathcal{V}$ is the volume it encloses.

[*Hint: Consider* $e_i T_{ij}$ *for a constant vector* e_i.]

(c) A half-ellipsoidal membrane $\mathcal{S}$ is described by the *open* surface $4x^2+4y^2+z^2 = 4$, with $z \geqslant 0$. At a given instant, air flows beneath the membrane with velocity $\mathbf{u} = (-y,\, x,\, \alpha)$, where α is a constant. The flow exerts a force on the membrane given by

$$F_i = \int_{\mathcal{S}} \beta^2 u_i u_j n_j \, dS \,,$$

where β is a constant parameter.

Show the vector $a_i = \partial(u_i u_j)/\partial x_j$ can be rewritten as $\mathbf{a} = -(x,\, y,\, 0)$.

Hence use $(*)$ to calculate the force F_i on the membrane.

Paper 3, Section II

12B Vector Calculus

For a given charge distribution $\rho(\mathbf{x},t)$ and current distribution $\mathbf{J}(\mathbf{x},t)$ in $\mathbb{R}^3$, the electric and magnetic fields, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, satisfy Maxwell's equations, which in suitable units, read

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\rho\,,\qquad \boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0\,,\qquad \boldsymbol{\nabla}\times\mathbf{B}=\mathbf{J}+\frac{\partial\mathbf{E}}{\partial t}\,.$$

The Poynting vector $\mathbf{P}$ is defined as $\mathbf{P}=\mathbf{E}\times\mathbf{B}$.

(a) For a closed surface $\mathcal{S}$ around a volume $\mathcal{V}$, show that

$$\int_{\mathcal{S}}\mathbf{P}\cdot d\mathbf{S}=-\int_{\mathcal{V}}\mathbf{E}\cdot\mathbf{J}\,dV-\frac{\partial}{\partial t}\int_{\mathcal{V}}\frac{|\mathbf{E}|^2+|\mathbf{B}|^2}{2}\,dV\,. \qquad (*)$$

(b) Suppose $\mathbf{J}=\mathbf{0}$ and consider an electromagnetic wave

$$\mathbf{E}=E_0\,\hat{\mathbf{y}}\cos(kx-\omega t)\quad\text{and}\quad\mathbf{B}=B_0\,\hat{\mathbf{z}}\cos(kx-\omega t)\,,$$

where E_0, B_0, k and ω are positive constants. Show that these fields satisfy Maxwell's equations for appropriate E_0, ω, and ρ.

Confirm the wave satisfies the integral identity $(*)$ by considering its propagation through a box $\mathcal{V}$, defined by $0\leqslant x\leqslant\pi/(2k)$, $0\leqslant y\leqslant L$, and $0\leqslant z\leqslant L$.

Paper 2, Section I

3B Vector Calculus

(a) Evaluate the line integral

$$\int_{(0,1)}^{(1,2)} (x^2 - y)dx + (y^2 + x)dy$$

along

(i) a straight line from $(0, 1)$ to $(1, 2)$,

(ii) the parabola $x = t$, $y = 1 + t^2$.

(b) State Green's theorem. The curve C_1 is the circle of radius a centred on the origin and traversed anticlockwise and C_2 is another circle of radius $b < a$ traversed clockwise and completely contained within C_1 but may or may not be centred on the origin. Find

$$\int_{C_1 \cup C_2} y(xy - \lambda)dx + x^2y\,dy$$

as a function of λ.

Paper 2, Section II

9B Vector Calculus

Write down Stokes' theorem for a vector field $\mathbf{A}(\mathbf{x})$ on $\mathbb{R}^3$.

Let the surface S be the part of the inverted paraboloid

$$z = 5 - x^2 - y^2, \quad 1 < z < 4,$$

and the vector field $\mathbf{A}(\mathbf{x}) = (3y, -xz, yz^2)$.

(a) Sketch the surface S and directly calculate $I = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$.

(b) Now calculate I a different way by using Stokes' theorem.

Paper 2, Section II

10B Vector Calculus

(a) State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$ where $r = |\boldsymbol{x}|$.

(b) A vector field $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{r} + \frac{(\boldsymbol{a}\cdot\boldsymbol{x})\boldsymbol{x}}{r^3},$$

where $\boldsymbol{a}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$ using suffix notation and show how d_{ij} splits naturally into symmetric and antisymmetric parts. Show that

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \frac{2\boldsymbol{a}\times\boldsymbol{x}}{r^3}.$$

(c) Consider the equation

$$\nabla^2 u = f$$

on a bounded domain $V \subset \mathbb{R}^3$ subject to the mixed boundary condition

$$(1-\lambda)u + \lambda\frac{du}{dn} = 0$$

on the smooth boundary $S = \partial V$, where $\lambda \in [0,1)$ is a constant. Show that if a solution exists, it will be unique.

Find the spherically symmetric solution $u(r)$ for the choice $f = 6$ in the region $r = |\boldsymbol{x}| \leqslant b$ for $b > 0$, as a function of the constant $\lambda \in [0,1)$. Explain why a solution does not exist for $\lambda = 1$.

Paper 3, Section I

3B Vector Calculus

Apply the divergence theorem to the vector field $\mathbf{u}(\mathbf{x}) = \mathbf{a}\phi(\mathbf{x})$ where $\mathbf{a}$ is an arbitrary constant vector and ϕ is a scalar field, to show that

$$\int_V \boldsymbol{\nabla}\phi \, dV = \int_S \phi \, d\mathbf{S},$$

where V is a volume bounded by the surface S and $d\mathbf{S}$ is the outward pointing surface element.

Verify that this result holds when $\phi = x + y$ and V is the spherical volume $x^2 + y^2 + z^2 \leqslant a^2$. [You may use the result that $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where θ and ϕ are the usual angular coordinates in spherical polars and the components of $d\mathbf{S}$ are with respect to standard Cartesian axes.]

Paper 3, Section I

4B Vector Calculus

Let

$$\begin{aligned} u &= (2x + x^2 z + z^3)\,\exp((x+y)z) \\ v &= (x^2 z + z^3)\,\exp((x+y)z) \\ w &= (2z + x^3 + x^2 y + xz^2 + yz^2)\,\exp((x+y)z) \end{aligned}$$

Show that $u\,dx + v\,dy + w\,dz$ is an *exact differential*, clearly stating any criteria that you use.

Show that for any path between $(-1,0,1)$ and $(1,0,1)$

$$\int_{(-1,0,1)}^{(1,0,1)} (u\,dx + v\,dy + w\,dz) = 4\sinh 1.$$

Paper 3, Section II

9B Vector Calculus

Define the *Jacobian*, J, of the one-to-one transformation

$$(x, y, z) \to (u, v, w).$$

Give a careful explanation of the result

$$\int_D f(x, y, z)\, dx\, dy\, dz = \int_\Delta |J|\, \phi(u, v, w)\, du\, dv\, dw,$$

where

$$\phi(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w))$$

and the region D maps under the transformation to the region Δ.

Consider the region D defined by

$$x^2 + \frac{y^2}{k^2} + z^2 \leqslant 1$$

and

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2\alpha^2} - \frac{z^2}{\gamma^2} \geqslant 1,$$

where α, γ and k are positive constants.

Let $\tilde{D}$ be the intersection of D with the plane $y = 0$. Write down the conditions for $\tilde{D}$ to be non-empty. Sketch the geometry of $\tilde{D}$ in $x > 0$, clearly specifying the curves that define its boundaries and points that correspond to minimum and maximum values of x and of z on the boundaries.

Use a suitable change of variables to evaluate the volume of the region D, clearly explaining the steps in your calculation.

Paper 3, Section II

10B Vector Calculus

For a given set of coordinate axes the components of a 2nd rank tensor T are given by T_{ij}.

(a) Show that if λ is an eigenvalue of the matrix with elements T_{ij} then it is also an eigenvalue of the matrix of the components of T in any other coordinate frame.

Show that if T is a symmetric tensor then the multiplicity of the eigenvalues of the matrix of components of T is independent of coordinate frame.

A symmetric tensor T in three dimensions has eigenvalues λ, λ, μ, with $\mu \neq \lambda$.

Show that the components of T can be written in the form

$$T_{ij} = \alpha\delta_{ij} + \beta n_i n_j \tag{1}$$

where n_i are the components of a unit vector.

(b) The tensor T is defined by

$$T_{ij}(\mathbf{y}) = \int_S x_i\, x_j\, \exp(-c|\mathbf{y} - \mathbf{x}|^2)\, dA(\mathbf{x}),$$

where S is the surface of the unit sphere, $\mathbf{y}$ is the position vector of a point on S, and c is a constant.

Deduce, with brief reasoning, that the components of T can be written in the form (1) with $n_i = y_i$. [You may quote any results derived in part (a).]

Using suitable spherical polar coordinates evaluate T_{kk} and $T_{ij}y_i y_j$.

Explain how to deduce the values of α and β from T_{kk} and $T_{ij}y_i y_j$. [You do not need to write out the detailed formulae for these quantities.]

Paper 3, Section II

11B Vector Calculus

Show that for a vector field $\mathbf{A}$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

Hence find an $\mathbf{A}(\mathbf{x})$, with $\nabla \cdot \mathbf{A} = 0$, such that $\mathbf{u} = (y^2, z^2, x^2) = \nabla \times \mathbf{A}$. *[Hint: Note that $\mathbf{A}(\mathbf{x})$ is not defined uniquely. Choose your expression for $\mathbf{A}(\mathbf{x})$ to be as simple as possible.]*

Now consider the cone $x^2 + y^2 \leqslant z^2 \tan^2 \alpha$, $0 \leqslant z \leqslant h$. Let S_1 be the curved part of the surface of the cone ($x^2 + y^2 = z^2 \tan^2 \alpha$, $0 \leqslant z \leqslant h$) and S_2 be the flat part of the surface of the cone ($x^2 + y^2 \leqslant h^2 \tan^2 \alpha$, $z = h$).

Using the variables z and ϕ as used in cylindrical polars (r, ϕ, z) to describe points on S_1, give an expression for the surface element $d\mathbf{S}$ in terms of dz and $d\phi$.

Evaluate $\int_{S_1} \mathbf{u} \cdot d\mathbf{S}$.

What does the divergence theorem predict about the two surface integrals $\int_{S_1} \mathbf{u} \cdot d\mathbf{S}$ and $\int_{S_2} \mathbf{u} \cdot d\mathbf{S}$ where in each case the vector $d\mathbf{S}$ is taken outwards from the cone?

What does Stokes theorem predict about the integrals $\int_{S_1} \mathbf{u} \cdot d\mathbf{S}$ and $\int_{S_2} \mathbf{u} \cdot d\mathbf{S}$ (defined as in the previous paragraph) and the line integral $\int_C \mathbf{A} \cdot d\mathbf{l}$ where C is the circle $x^2 + y^2 = h^2 \tan^2 \alpha$, $z = h$ and the integral is taken in the anticlockwise sense, looking from the positive z direction?

Evaluate $\int_{S_2} \mathbf{u} \cdot d\mathbf{S}$ and $\int_C \mathbf{A} \cdot d\mathbf{l}$, making your method clear and verify that each of these predictions holds.

Paper 3, Section II

12B Vector Calculus

(a) The function u satisfies $\nabla^2 u = 0$ in the volume V and $u = 0$ on S, the surface bounding V.

Show that $u = 0$ everywhere in V.

The function v satisfies $\nabla^2 v = 0$ in V and v is specified on S. Show that for all functions w such that $w = v$ on S

$$\int_V \nabla v \cdot \nabla w \, dV = \int_V |\nabla v|^2 \, dV.$$

Hence show that

$$\int_V |\nabla w|^2 \, dV = \int_V \{|\nabla v|^2 + |\nabla(w-v)|^2\} \, dV \geqslant \int_V |\nabla v|^2 \, dV.$$

(b) The function ϕ satisfies $\nabla^2 \phi = \rho(\mathbf{x})$ in the spherical region $|\mathbf{x}| < a$, with $\phi = 0$ on $|\mathbf{x}| = a$. The function $\rho(\mathbf{x})$ is spherically symmetric, i.e. $\rho(\mathbf{x}) = \rho(|\mathbf{x}|) = \rho(r)$.

Suppose that the equation and boundary conditions are satisfied by a spherically symmetric function $\Phi(r)$. Show that

$$4\pi r^2 \Phi'(r) = 4\pi \int_0^r s^2 \rho(s) \, ds.$$

Hence find the function $\Phi(r)$ when $\rho(r)$ is given by $\rho(r) = \begin{cases} \rho_0 & \text{if } 0 \leqslant r \leqslant b \\ 0 & \text{if } b < r \leqslant a \end{cases}$, with ρ_0 constant.

Explain how the results obtained in part (a) of the question imply that $\Phi(r)$ is the only solution of $\nabla^2 \phi = \rho(r)$ which satisfies the specified boundary condition on $|\mathbf{x}| = a$.

Use your solution and the results obtained in part (a) of the question to show that, for any function w such that $w = 1$ on $r = b$ and $w = 0$ on $r = a$,

$$\int_{U(b,a)} |\nabla w|^2 \, dV \geqslant \frac{4\pi ab}{a-b},$$

where $U(b,a)$ is the region $b < r < a$.

Paper 3, Section I

3C Vector Calculus

Derive a formula for the curvature of the two-dimensional curve $\mathbf{x}(u) = (u, f(u))$.

Verify your result for the semicircle with radius a given by $f(u) = \sqrt{a^2 - u^2}$.

Paper 3, Section I

4C Vector Calculus

In plane polar coordinates (r, θ), the orthonormal basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ satisfy

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0}, \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \quad \text{and} \quad \boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\,.$$

Hence derive the expression $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$ for the Laplacian operator ∇^2.

Calculate the Laplacian of $\phi(r, \theta) = \alpha r^\beta \cos(\gamma\theta)$, where α, β and γ are constants. Hence find all solutions to the equation

$$\nabla^2\phi = 0 \quad \text{in} \quad 0 \leqslant r \leqslant a, \quad \text{with} \quad \partial\phi/\partial r = \cos(2\theta) \text{ on } r = a\,.$$

Explain briefly how you know that there are no other solutions.

Paper 3, Section II

9C Vector Calculus

Given a one-to-one mapping $u = u(x, y)$ and $v = v(x, y)$ between the region D in the (x, y)-plane and the region D' in the (u, v)-plane, state the formula for transforming the integral $\iint_D f(x, y)\, dx\, dy$ into an integral over D', with the Jacobian expressed explicitly in terms of the partial derivatives of u and v.

Let D be the region $x^2 + y^2 \leqslant 1$, $y \geqslant 0$ and consider the change of variables $u = x + y$ and $v = x^2 + y^2$. Sketch D, the curves of constant u and the curves of constant v in the (x, y)-plane. Find and sketch the image D' of D in the (u, v)-plane.

Calculate $I = \iint_D (x + y)\, dx\, dy$ using this change of variables. Check your answer by calculating I directly.

Paper 3, Section II
10C Vector Calculus
State the formula of Stokes's theorem, specifying any orientation where needed.

Let $\mathbf{F} = (y^2 z, xz + 2xyz, 0)$. Calculate $\boldsymbol{\nabla} \times \mathbf{F}$ and verify that $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{F} = 0$.

Sketch the surface S defined as the union of the surface $z = -1$, $1 \leqslant x^2 + y^2 \leqslant 4$ and the surface $x^2 + y^2 + z = 3$, $1 \leqslant x^2 + y^2 \leqslant 4$.

Verify Stokes's theorem for $\mathbf{F}$ on S.

Paper 3, Section II
11C Vector Calculus
Use Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t},$$

to derive expressions for $\dfrac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E}$ and $\dfrac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B}$ in terms of ρ and $\mathbf{J}$.

Now suppose that there exists a scalar potential ϕ such that $\mathbf{E} = -\boldsymbol{\nabla}\phi$, and $\phi \to 0$ as $r \to \infty$. If $\rho = \rho(r)$ is spherically symmetric, calculate $\mathbf{E}$ using Gauss's flux method, i.e. by integrating a suitable equation inside a sphere centred at the origin. Use your result to find $\mathbf{E}$ and ϕ in the case when $\rho = 1$ for $r < a$ and $\rho = 0$ otherwise.

For each integer $n \geqslant 0$, let S_n be the sphere of radius 4^{-n} centred at the point $(1 - 4^{-n}, 0, 0)$. Suppose that ρ vanishes outside S_0, and has the constant value 2^n in the volume between S_n and S_{n+1} for $n \geqslant 0$. Calculate $\mathbf{E}$ and ϕ at the point $(1, 0, 0)$.

Paper 3, Section II

12C Vector Calculus

(a) Suppose that a tensor T_{ij} can be decomposed as

$$T_{ij} = S_{ij} + \epsilon_{ijk} V_k \,, \qquad (*)$$

where S_{ij} is symmetric. Obtain expressions for S_{ij} and V_k in terms of T_{ij}, and check that $(*)$ is satisfied.

(b) State the most general form of an isotropic tensor of rank k for $k = 0, 1, 2, 3$, and verify that your answers are isotropic.

(c) The general form of an isotropic tensor of rank 4 is

$$T_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \,.$$

Suppose that A_{ij} and B_{ij} satisfy the linear relationship $A_{ij} = T_{ijkl} B_{kl}$, where T_{ijkl} is isotropic. Express B_{ij} in terms of A_{ij}, assuming that $\beta^2 \neq \gamma^2$ and $3\alpha + \beta + \gamma \neq 0$. If instead $\beta = -\gamma \neq 0$ and $\alpha \neq 0$, find all B_{ij} such that $A_{ij} = 0$.

(d) Suppose that C_{ij} and D_{ij} satisfy the quadratic relationship $C_{ij} = T_{ijklmn} D_{kl} D_{mn}$, where T_{ijklmn} is an isotropic tensor of rank 6. If C_{ij} is symmetric and D_{ij} is antisymmetric, find the most general non-zero form of $T_{ijklmn} D_{kl} D_{mn}$ and prove that there are only two independent terms. [*Hint: You do not need to use the general form of an isotropic tensor of rank* 6.]

Paper 3, Section I

3B Vector Calculus

Use the change of variables $x = r\cosh\theta$, $y = r\sinh\theta$ to evaluate

$$\int_A y\,dx\,dy\,,$$

where A is the region of the xy-plane bounded by the two line segments:

$y = 0, \quad 0 \leqslant x \leqslant 1\,;$

$5y = 3x, \quad 0 \leqslant x \leqslant \frac{5}{4}\,;$

and the curve

$x^2 - y^2 = 1, \quad x \geqslant 1\,.$

Paper 3, Section I

4B Vector Calculus

(a) The two sets of basis vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ (where $i = 1, 2, 3$) are related by

$$\mathbf{e}'_i = R_{ij}\mathbf{e}_j\,,$$

where R_{ij} are the entries of a rotation matrix. The components of a vector $\mathbf{v}$ with respect to the two bases are given by

$$\mathbf{v} = v_i\mathbf{e}_i = v'_i\mathbf{e}'_i\,.$$

Derive the relationship between v_i and v'_i.

(b) Let $\mathbf{T}$ be a 3×3 array defined in each (right-handed orthonormal) basis. Using part (a), state and prove the quotient theorem as applied to $\mathbf{T}$.

Paper 3, Section II
9B Vector Calculus

(a) The time-dependent vector field $\mathbf{F}$ is related to the vector field $\mathbf{B}$ by

$$\mathbf{F}(\mathbf{x},t) = \mathbf{B}(\mathbf{z})\,,$$

where $\mathbf{z} = t\mathbf{x}$. Show that

$$(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{F} = t\,\frac{\partial \mathbf{F}}{\partial t}\,.$$

(b) The vector fields $\mathbf{B}$ and $\mathbf{A}$ satisfy $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Show that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.

(c) The vector field $\mathbf{B}$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Show that

$$\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times\big(\mathbf{D}(\mathbf{x})\times\mathbf{x}\big)\,,$$

where

$$\mathbf{D}(\mathbf{x}) = \int_0^1 t\,\mathbf{B}(t\mathbf{x})\,dt\,.$$

Paper 3, Section II
10B Vector Calculus

By a suitable choice of $\mathbf{u}$ in the divergence theorem

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{u}\,dV = \int_S \mathbf{u}\cdot d\mathbf{S}\,,$$

show that

$$\int_V \boldsymbol{\nabla}\phi\,dV = \int_S \phi\,d\mathbf{S} \qquad (*)$$

for any continuously differentiable function ϕ.

For the curved surface of the cone

$$\mathbf{x} = (r\cos\theta, r\sin\theta, \sqrt{3}\,r), \qquad 0 \leqslant \sqrt{3}\,r \leqslant 1, \quad 0 \leqslant \theta \leqslant 2\pi,$$

show that $d\mathbf{S} = (\sqrt{3}\,\cos\theta, \sqrt{3}\,\sin\theta, -1)\,r\,dr\,d\theta$.

Verify that $(*)$ holds for this cone and $\phi(x,y,z) = z^2$.

Paper 3, Section II

11B Vector Calculus

(a) Let $\mathbf{x} = \mathbf{r}(s)$ be a smooth curve parametrised by arc length s. Explain the meaning of the terms in the equation

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}\,,$$

where $\kappa(s)$ is the curvature of the curve.

Now let $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Show that there is a scalar $\tau(s)$ (the torsion) such that

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}$$

and derive an expression involving κ and τ for $\dfrac{d\mathbf{n}}{ds}$.

(b) Given a (nowhere zero) vector field $\mathbf{F}$, the *field lines*, or *integral curves*, of $\mathbf{F}$ are the curves parallel to $\mathbf{F}(\mathbf{x})$ at each point $\mathbf{x}$. Show that the curvature κ of the field lines of $\mathbf{F}$ satisfies

$$\frac{\mathbf{F} \times (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}}{F^3} = \pm\kappa\mathbf{b}\,, \qquad (*)$$

where $F = |\mathbf{F}|$.

(c) Use $(*)$ to find an expression for the curvature at the point (x, y, z) of the field lines of $\mathbf{F}(x, y, z) = (x, y, -z)$.

Paper 3, Section II

12B Vector Calculus

Let S be a piecewise smooth closed surface in $\mathbb{R}^3$ which is the boundary of a volume V.

(a) The smooth functions ϕ and ϕ_1 defined on $\mathbb{R}^3$ satisfy

$$\nabla^2\phi = \nabla^2\phi_1 = 0$$

in V and $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) = f(\mathbf{x})$ on S. By considering an integral of $\boldsymbol{\nabla}\psi\boldsymbol{\cdot}\boldsymbol{\nabla}\psi$, where $\psi = \phi - \phi_1$, show that $\phi_1 = \phi$.

(b) The smooth function u defined on $\mathbb{R}^3$ satisfies $u(\mathbf{x}) = f(\mathbf{x}) + C$ on S, where f is the function in part (a) and C is constant. Show that

$$\int_V \boldsymbol{\nabla} u\boldsymbol{\cdot}\boldsymbol{\nabla} u \, dV \geqslant \int_V \boldsymbol{\nabla}\phi\boldsymbol{\cdot}\boldsymbol{\nabla}\phi \, dV$$

where ϕ is the function in part (a). When does equality hold?

(c) The smooth function $w(\mathbf{x}, t)$ satisfies

$$\nabla^2 w = \frac{\partial w}{\partial t}$$

in V and $\dfrac{\partial w}{\partial t} = 0$ on S for all t. Show that

$$\frac{d}{dt}\int_V \boldsymbol{\nabla} w\boldsymbol{\cdot}\boldsymbol{\nabla} w \, dV \leqslant 0$$

with equality only if $\nabla^2 w = 0$ in V.

Paper 3, Section I

3C Vector Calculus

State the chain rule for the derivative of a composition $t \mapsto f(\mathbf{X}(t))$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{X} : \mathbb{R} \to \mathbb{R}^n$ are smooth.

Consider parametrized curves given by

$$\mathbf{x}(t) = (x(t), y(t)) = (a\cos t, a\sin t)\,.$$

Calculate the tangent vector $\dfrac{d\mathbf{x}}{dt}$ in terms of $x(t)$ and $y(t)$. Given that $u(x,y)$ is a smooth function in the upper half-plane $\{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ satisfying

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = u\,,$$

deduce that

$$\frac{d}{dt}\,u\left(x(t), y(t)\right) \;=\; u\left(x(t), y(t)\right).$$

If $u(1,1) = 10$, find $u(-1,1)$.

Paper 3, Section I

4C Vector Calculus

If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, show that $T_{ij} = v_i w_j$ defines a rank 2 tensor. For which choices of the vectors $\mathbf{v}$ and $\mathbf{w}$ is T_{ij} isotropic?

Write down the most general isotropic tensor of rank 2.

Prove that ϵ_{ijk} defines an isotropic rank 3 tensor.

Paper 3, Section II

9C Vector Calculus

What is a *conservative* vector field on $\mathbb{R}^n$?

State Green's theorem in the plane $\mathbb{R}^2$.

(a) Consider a smooth vector field $\mathbf{V} = \big(P(x,y), Q(x,y)\big)$ defined on all of $\mathbb{R}^2$ which satisfies

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0\,.$$

By considering

$$F(x,y) = \int_0^x P(x',0)\,dx' \;+\; \int_0^y Q(x,y')\,dy'$$

or otherwise, show that $\mathbf{V}$ is conservative.

(b) Now let $\mathbf{V} = \big(1+\cos(2\pi x + 2\pi y), 2 + \cos(2\pi x + 2\pi y)\big)$. Show that there exists a smooth function $F(x,y)$ such that $\mathbf{V} = \nabla F$.

Calculate $\int_C \mathbf{V}\cdot d\mathbf{x}$, where C is a smooth curve running from $(0,0)$ to $(m,n) \in \mathbb{Z}^2$. Deduce that there does *not* exist a smooth function $F(x,y)$ which satisfies $\mathbf{V} = \nabla F$ and which is, in addition, periodic with period 1 in each coordinate direction, *i.e.* $F(x,y) = F(x+1,y) = F(x,y+1)$.

Paper 3, Section II

10C Vector Calculus

Define the *Jacobian* $J[\mathbf{u}]$ of a smooth mapping $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$. Show that if $\mathbf{V}$ is the vector field with components

$$V_i = \frac{1}{3!}\epsilon_{ijk}\epsilon_{abc}\frac{\partial u_a}{\partial x_j}\frac{\partial u_b}{\partial x_k}\,u_c\,,$$

then $J[\mathbf{u}] = \nabla \cdot \mathbf{V}$. If $\mathbf{v}$ is another such mapping, state the chain rule formula for the derivative of the composition $\mathbf{w}(\mathbf{x}) = \mathbf{u}(\mathbf{v}(\mathbf{x}))$, and hence give $J[\mathbf{w}]$ in terms of $J[\mathbf{u}]$ and $J[\mathbf{v}]$.

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Let there be given, for each $t \in \mathbb{R}$, a smooth mapping $\mathbf{u}_t : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{u}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{F}(\mathbf{x}) + o(t)$ as $t \to 0$. Show that

$$J[\mathbf{u}_t] \;=\; 1 + tQ(x) + o(t)$$

for some $Q(x)$, and express Q in terms of $\mathbf{F}$. Assuming now that $\mathbf{u}_{t+s}(\mathbf{x}) = \mathbf{u}_t(\mathbf{u}_s(\mathbf{x}))$, deduce that if $\nabla \cdot \mathbf{F} = 0$ then $J[\mathbf{u}_t] = 1$ for all $t \in \mathbb{R}$. What geometric property of the mapping $\mathbf{x} \mapsto \mathbf{u}_t(\mathbf{x})$ does this correspond to?

Paper 3, Section II
11C Vector Calculus

(a) For smooth scalar fields u and v, derive the identity

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

and deduce that

$$\int_{\rho \leqslant |\mathbf{x}| \leqslant r} \left(v\nabla^2 u - u\nabla^2 v\right)\, dV = \int_{|\mathbf{x}|=r} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS - \int_{|\mathbf{x}|=\rho} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS\,.$$

Here ∇^2 is the Laplacian, $\dfrac{\partial}{\partial n} = \mathbf{n}\cdot\nabla$ where $\mathbf{n}$ is the unit outward normal, and dS is the scalar area element.

(b) Give the expression for $\big(\nabla \times \mathbf{V}\big)_i$ in terms of ϵ_{ijk}. Hence show that

$$\nabla \times \big(\nabla \times \mathbf{V}\big) \;=\; \nabla(\nabla \cdot \mathbf{V}) \;-\; \nabla^2\mathbf{V}\,.$$

(c) Assume that if $\nabla^2\varphi = -\rho$, where $\varphi(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ and $\nabla\varphi(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, then

$$\varphi(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\, dV\,.$$

The vector fields $\mathbf{B}$ and $\mathbf{J}$ satisfy

$$\nabla \times \mathbf{B} \;=\; \mathbf{J}\,.$$

Show that $\nabla \cdot \mathbf{J} = 0$. In the case that $\mathbf{B} = \nabla \times \mathbf{A}$, with $\nabla \cdot\, \mathbf{A} = 0$, show that

$$\mathbf{A}(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\, dV\,, \qquad (*)$$

and hence that

$$\mathbf{B}(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{y}) \times (\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\, dV\,.$$

Verify that $\mathbf{A}$ given by $(*)$ does indeed satisfy $\nabla \cdot\, \mathbf{A} = 0$. [It may be useful to make a change of variables in the right hand side of $(*)$.]

Paper 3, Section II
12C Vector Calculus

(a) Let

$$\mathbf{F} = (z, x, y)$$

and let C be a circle of radius R lying in a plane with unit normal vector (a, b, c). Calculate $\nabla \times \mathbf{F}$ and use this to compute $\oint_C \mathbf{F} \cdot d\mathbf{x}$. Explain any orientation conventions which you use.

(b) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field such that the matrix with entries $\dfrac{\partial F_j}{\partial x_i}$ is symmetric. Prove that $\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$ for every circle $C \subset \mathbb{R}^3$.

(c) Let $\mathbf{F} = \dfrac{1}{r}(x, y, z)$, where $r = \sqrt{x^2 + y^2 + z^2}$ and let C be the circle which is the intersection of the sphere $(x-5)^2 + (y-3)^2 + (z-2)^2 = 1$ with the plane $3x - 5y - z = 2$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

(d) Let $\mathbf{F}$ be the vector field defined, for $x^2 + y^2 > 0$, by

$$\mathbf{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right).$$

Show that $\nabla \times \mathbf{F} = \mathbf{0}$. Let C be the curve which is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = x + 200$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

Paper 3, Section I

3A Vector Calculus

(i) For $r = |\mathbf{x}|$ with $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad (i = 1, 2, 3).$$

(ii) Consider the vector fields $\mathbf{F}(\mathbf{x}) = r^2\mathbf{x}$, $\mathbf{G}(\mathbf{x}) = (\mathbf{a}\cdot\mathbf{x})\mathbf{x}$ and $\mathbf{H}(\mathbf{x}) = \mathbf{a}\times\hat{\mathbf{x}}$, where $\mathbf{a}$ is a constant vector in $\mathbb{R}^3$ and $\hat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{x}$. Using suffix notation, or otherwise, find the divergence and the curl of each of $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

Paper 3, Section I

4A Vector Calculus

The smooth curve $\mathcal{C}$ in $\mathbb{R}^3$ is given in parametrised form by the function $\mathbf{x}(u)$. Let s denote arc length measured along the curve.

(a) Express the tangent $\mathbf{t}$ in terms of the derivative $\mathbf{x}' = d\mathbf{x}/du$, and show that $du/ds = |\mathbf{x}'|^{-1}$.

(b) Find an expression for $d\mathbf{t}/ds$ in terms of derivatives of $\mathbf{x}$ with respect to u, and show that the curvature κ is given by

$$\kappa = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3}.$$

[*Hint: You may find the identity* $(\mathbf{x}'\cdot\mathbf{x}'')\mathbf{x}' - (\mathbf{x}'\cdot\mathbf{x}')\mathbf{x}'' = \mathbf{x}'\times(\mathbf{x}'\times\mathbf{x}'')$ *helpful.*]

(c) For the curve

$$\mathbf{x}(u) = \begin{pmatrix} u\cos u \\ u\sin u \\ 0 \end{pmatrix},$$

with $u \geqslant 0$, find the curvature as a function of u.

Paper 3, Section II
9A Vector Calculus

The vector field $\mathbf{F}(\mathbf{x})$ is given in terms of cylindrical polar coordinates (ρ, ϕ, z) by

$$\mathbf{F}(\mathbf{x}) = f(\rho)\mathbf{e}_\rho,$$

where f is a differentiable function of ρ, and $\mathbf{e}_\rho = \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y$ is the unit basis vector with respect to the coordinate ρ. Compute the partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$, $\partial F_3/\partial z$ and hence find the divergence $\nabla \cdot \mathbf{F}$ in terms of ρ and ϕ.

The domain V is bounded by the surface $z = (x^2+y^2)^{-1}$, by the cylinder $x^2+y^2 = 1$, and by the planes $z = \frac{1}{4}$ and $z = 1$. Sketch V and compute its volume.

Find the most general function $f(\rho)$ such that $\nabla \cdot \mathbf{F} = 0$, and verify the divergence theorem for the corresponding vector field $\mathbf{F}(\mathbf{x})$ in V.

Paper 3, Section II
10A Vector Calculus

State Stokes' theorem.

Let S be the surface in $\mathbb{R}^3$ given by $z^2 = x^2 + y^2 + 1 - \lambda$, where $0 \leqslant z \leqslant 1$ and λ is a positive constant. Sketch the surface S for representative values of λ and find the surface element $\mathbf{dS}$ with respect to the Cartesian coordinates x and y.

Compute $\nabla \times \mathbf{F}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

and verify Stokes' theorem for $\mathbf{F}$ on the surface S for every value of λ.

Now compute $\nabla \times \mathbf{G}$ for the vector field

$$\mathbf{G}(\mathbf{x}) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

and find the line integral $\int_{\partial S} \mathbf{G} \cdot \mathbf{dx}$ for the boundary ∂S of the surface S. Is it possible to obtain this result using Stokes' theorem? Justify your answer.

Paper 3, Section II

11A Vector Calculus

(i) Starting with the divergence theorem, derive Green's first theorem

$$\int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi)\, dV = \int_{\partial V} \psi \frac{\partial \phi}{\partial n}\, dS.$$

(ii) The function $\phi(\mathbf{x})$ satisfies Laplace's equation $\nabla^2 \phi = 0$ in the volume V with given boundary conditions $\phi(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \partial V$. Show that $\phi(\mathbf{x})$ is the only such function. Deduce that if $\phi(\mathbf{x})$ is constant on ∂V then it is constant in the whole volume V.

(iii) Suppose that $\phi(\mathbf{x})$ satisfies Laplace's equation in the volume V. Let V_r be the sphere of radius r centred at the origin and contained in V. The function $f(r)$ is defined by

$$f(r) = \frac{1}{4\pi r^2} \int_{\partial V_r} \phi(\mathbf{x})\, dS.$$

By considering the derivative df/dr, and by introducing the Jacobian in spherical polar coordinates and using the divergence theorem, or otherwise, show that $f(r)$ is constant and that $f(r) = \phi(\mathbf{0})$.

(iv) Let M denote the maximum of ϕ on ∂V_r and m the minimum of ϕ on ∂V_r. By using the result from (iii), or otherwise, show that $m \leqslant \phi(\mathbf{0}) \leqslant M$.

Paper 3, Section II

12A Vector Calculus

(a) Let t_{ij} be a rank 2 tensor whose components are invariant under rotations through an angle π about each of the three coordinate axes. Show that t_{ij} is diagonal.

(b) An array of numbers a_{ij} is given in one orthonormal basis as $\delta_{ij} + \epsilon_{1ij}$ and in another rotated basis as δ_{ij}. By using the invariance of the determinant of any rank 2 tensor, or otherwise, prove that a_{ij} is not a tensor.

(c) Let a_{ij} be an array of numbers and b_{ij} a tensor. Determine whether the following statements are true or false. Justify your answers.

(i) If $a_{ij} b_{ij}$ is a scalar for any rank 2 tensor b_{ij}, then a_{ij} is a rank 2 tensor.

(ii) If $a_{ij} b_{ij}$ is a scalar for any symmetric rank 2 tensor b_{ij}, then a_{ij} is a rank 2 tensor.

(iii) If a_{ij} is antisymmetric and $a_{ij} b_{ij}$ is a scalar for any symmetric rank 2 tensor b_{ij}, then a_{ij} is an antisymmetric rank 2 tensor.

(iv) If a_{ij} is antisymmetric and $a_{ij} b_{ij}$ is a scalar for any antisymmetric rank 2 tensor b_{ij}, then a_{ij} is an antisymmetric rank 2 tensor.

Paper 3, Section I

3A Vector Calculus

(a) For $\mathbf{x} \in \mathbb{R}^n$ and $r = |\mathbf{x}|$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.$$

(b) Use index notation and your result in (a), or otherwise, to compute

(i) $\nabla \cdot \big(f(r)\mathbf{x}\big)$, and

(ii) $\nabla \times \big(f(r)\mathbf{x}\big)$ for $n = 3$.

(c) Show that for each $n \in \mathbb{N}$ there is, up to an arbitrary constant, just one vector field $\mathbf{F}(\mathbf{x})$ of the form $f(r)\mathbf{x}$ such that $\nabla \cdot \mathbf{F}(\mathbf{x}) = 0$ everywhere on $\mathbb{R}^n \backslash \{\mathbf{0}\}$, and determine $\mathbf{F}$.

Paper 3, Section I

4A Vector Calculus

Let $\mathbf{F}(\mathbf{x})$ be a vector field defined everywhere on the domain $G \subset \mathbb{R}^3$.

(a) Suppose that $\mathbf{F}(\mathbf{x})$ has a potential $\phi(\mathbf{x})$ such that $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ for $\mathbf{x} \in G$. Show that

$$\int_\gamma \mathbf{F} \cdot \mathbf{dx} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

for any smooth path γ from $\mathbf{a}$ to $\mathbf{b}$ in G. Show further that necessarily $\nabla \times \mathbf{F} = \mathbf{0}$ on G.

(b) State a condition for G which ensures that $\nabla \times \mathbf{F} = \mathbf{0}$ implies $\int_\gamma \mathbf{F} \cdot \mathbf{dx}$ is path-independent.

(c) Compute the line integral $\oint_\gamma \mathbf{F} \cdot \mathbf{dx}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix},$$

where γ denotes the anti-clockwise path around the unit circle in the (x, y)-plane. Compute $\nabla \times \mathbf{F}$ and comment on your result in the light of (b).

Paper 3, Section II
9A Vector Calculus

The surface C in $\mathbb{R}^3$ is given by $z^2 = x^2 + y^2$.

(a) Show that the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is tangent to the surface C everywhere.

(b) Show that the surface integral $\int_S \mathbf{F} \cdot \mathbf{dS}$ is a constant independent of S for any surface S which is a subset of C, and determine this constant.

(c) The volume V in $\mathbb{R}^3$ is bounded by the surface C and by the cylinder $x^2 + y^2 = 1$. Sketch V and compute the volume integral

$$\int_V \nabla \cdot \mathbf{F} \, dV$$

directly by integrating over V.

(d) Use the Divergence Theorem to verify the result you obtained in part (b) for the integral $\int_S \mathbf{F} \cdot \mathbf{dS}$, where S is the portion of C lying in $-1 \leqslant z \leqslant 1$.

Paper 3, Section II
10A Vector Calculus

(a) State Stokes' Theorem for a surface S with boundary ∂S.

(b) Let S be the surface in $\mathbb{R}^3$ given by $z^2 = 1 + x^2 + y^2$ where $\sqrt{2} \leqslant z \leqslant \sqrt{5}$. Sketch the surface S and find the surface element $\mathbf{dS}$ with respect to the Cartesian coordinates x and y.

(c) Compute $\nabla \times \mathbf{F}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y \\ x \\ xy(x+y) \end{pmatrix}$$

and verify Stokes' Theorem for $\mathbf{F}$ on the surface S.

Paper 3, Section II

11A Vector Calculus

(i) Starting with Poisson's equation in $\mathbb{R}^3$,

$$\nabla^2 \phi(\mathbf{x}) = f(\mathbf{x}),$$

derive Gauss' flux theorem

$$\int_V f(\mathbf{x})\, dV = \int_{\partial V} \mathbf{F}(\mathbf{x}) \cdot \mathbf{dS}$$

for $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ and for any volume $V \subseteq \mathbb{R}^3$.

(ii) Let

$$I = \int_S \frac{\mathbf{x} \cdot \mathbf{dS}}{|\mathbf{x}|^3}.$$

Show that $I = 4\pi$ if S is the sphere $|\mathbf{x}| = R$, and that $I = 0$ if S bounds a volume that does not contain the origin.

(iii) Show that the electric field defined by

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^3}, \quad \mathbf{x} \neq \mathbf{a},$$

satisfies

$$\int_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \begin{cases} 0 & \text{if } \mathbf{a} \notin V \\ \dfrac{q}{\epsilon_0} & \text{if } \mathbf{a} \in V \end{cases}$$

where ∂V is a surface bounding a closed volume V and $\mathbf{a} \notin \partial V$, and where the electric charge q and permittivity of free space ϵ_0 are constants. This is Gauss' law for a point electric charge.

(iv) Assume that $f(\mathbf{x})$ is spherically symmetric around the origin, i.e., it is a function only of $|\mathbf{x}|$. Assume that $\mathbf{F}(\mathbf{x})$ is also spherically symmetric. Show that $\mathbf{F}(\mathbf{x})$ depends only on the values of f inside the sphere with radius $|\mathbf{x}|$ but not on the values of f outside this sphere.

Paper 3, Section II

12A Vector Calculus

(a) Show that any rank 2 tensor t_{ij} can be written uniquely as a sum of two rank 2 tensors s_{ij} and a_{ij} where s_{ij} is symmetric and a_{ij} is antisymmetric.

(b) Assume that the rank 2 tensor t_{ij} is invariant under any rotation about the z-axis, as well as under a rotation of angle π about any axis in the (x, y)-plane through the origin.

(i) Show that there exist $\alpha, \beta \in \mathbb{R}$ such that t_{ij} can be written as

$$t_{ij} = \alpha\delta_{ij} + \beta\delta_{i3}\delta_{j3}. \qquad (*)$$

(ii) Is there some proper subgroup of the rotations specified above for which the result $(*)$ still holds if the invariance of t_{ij} is restricted to this subgroup? If so, specify the smallest such subgroup.

(c) The array of numbers d_{ijk} is such that $d_{ijk}s_{ij}$ is a vector for any symmetric matrix s_{ij}.

(i) By writing d_{ijk} as a sum of d^s_{ijk} and d^a_{ijk} with $d^s_{ijk} = d^s_{jik}$ and $d^a_{ijk} = -d^a_{jik}$, show that d^s_{ijk} is a rank 3 tensor. [You may assume without proof the Quotient Theorem for tensors.]

(ii) Does d^a_{ijk} necessarily have to be a tensor? Justify your answer.

Paper 3, Section I

3C Vector Calculus

The curve C is given by

$$\mathbf{r}(t) = \left(\sqrt{2}e^t,\, -e^t \sin t,\, e^t \cos t\right), \quad -\infty < t < \infty .$$

(i) Compute the arc length of C between the points with $t = 0$ and $t = 1$.

(ii) Derive an expression for the curvature of C as a function of arc length s measured from the point with $t = 0$.

Paper 3, Section I

4C Vector Calculus

State a necessary and sufficient condition for a vector field $\mathbf{F}$ on $\mathbb{R}^3$ to be conservative.

Check that the field

$$\mathbf{F} = (2x\cos y - 2z^3,\, 3 + 2ye^z - x^2 \sin y,\, y^2 e^z - 6xz^2)$$

is conservative and find a scalar potential for $\mathbf{F}$.

Paper 3, Section II

9C Vector Calculus

Give an explicit formula for $\mathcal{J}$ which makes the following result hold:

$$\int_D f \; dx\; dy\; dz = \int_{D'} \phi\; |\mathcal{J}|\; du\; dv\; dw\,,$$

where the region D, with coordinates x, y, z, and the region D', with coordinates u, v, w, are in one-to-one correspondence, and

$$\phi(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))\,.$$

Explain, in outline, why this result holds.

Let D be the region in $\mathbb{R}^3$ defined by $4 \leqslant x^2 + y^2 + z^2 \leqslant 9$ and $z \geqslant 0$. Sketch the region and employ a suitable transformation to evaluate the integral

$$\int_D (x^2 + y^2)\; dx\; dy\; dz\,.$$

Paper 3, Section II
10C Vector Calculus

Consider the bounded surface S that is the union of $x^2+y^2=4$ for $-2 \leqslant z \leqslant 2$ and $(4-z)^2 = x^2+y^2$ for $2 \leqslant z \leqslant 4$. Sketch the surface.

Using suitable parametrisations for the two parts of S, calculate the integral

$$\int_S (\nabla \times \mathbf{F}\ ) \cdot d\mathbf{S}$$

for $\mathbf{F} = yz^2\mathbf{i}$.

Check your result using Stokes's Theorem.

Paper 3, Section II
11C Vector Calculus

If $\mathbf{E}$ and $\mathbf{B}$ are vectors in $\mathbb{R}^3$, show that

$$T_{ij} = E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(E_kE_k + B_kB_k\right)$$

is a second rank tensor.

Now assume that $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ obey Maxwell's equations, which in suitable units read

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\,,
\end{aligned}$$

where ρ is the charge density and $\mathbf{J}$ the current density. Show that

$$\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) = \mathbf{M} - \rho\mathbf{E} - \mathbf{J}\times\mathbf{B} \quad \text{where} \quad M_i = \frac{\partial T_{ij}}{\partial x_j}.$$

Paper 3, Section II
12C Vector Calculus

(a) Prove that

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}\,.$$

(b) State the divergence theorem for a vector field $\mathbf{F}$ in a closed region $\Omega \subset \mathbb{R}^3$ bounded by $\partial\Omega$.

For a smooth vector field $\mathbf{F}$ and a smooth scalar function g prove that

$$\int_{\Omega} \mathbf{F} \cdot \nabla g + g\nabla \cdot \mathbf{F}\; dV = \int_{\partial\Omega} g\mathbf{F} \cdot \mathbf{n}\; dS\,,$$

where $\mathbf{n}$ is the outward unit normal on the surface $\partial\Omega$.

Use this identity to prove that the solution u to the Laplace equation $\nabla^2 u = 0$ in Ω with $u = f$ on $\partial\Omega$ is unique, provided it exists.

Paper 3, Section I

3C Vector Calculus

Define what it means for a differential $P\,dx + Q\,dy$ to be exact, and derive a necessary condition on $P(x, y)$ and $Q(x, y)$ for this to hold. Show that one of the following two differentials is exact and the other is not:

$$y^2\,dx + 2xy\,dy\,,$$
$$y^2\,dx + xy^2\,dy\,.$$

Show that the differential which is not exact can be written in the form $g\,df$ for functions $f(x, y)$ and $g(y)$, to be determined.

Paper 3, Section I

4C Vector Calculus

What does it mean for a second-rank tensor T_{ij} to be *isotropic*? Show that δ_{ij} is isotropic. By considering rotations through $\pi/2$ about the coordinate axes, or otherwise, show that the most general isotropic second-rank tensor in $\mathbb{R}^3$ has the form $T_{ij} = \lambda\delta_{ij}$, for some scalar λ.

Paper 3, Section II

9C Vector Calculus

State Stokes' Theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

Consider the surface S defined by

$$z = x^2 + y^2, \qquad \frac{1}{9} \leqslant z \leqslant 1.$$

Sketch the surface and calculate the area element $d\mathbf{S}$ in terms of suitable coordinates or parameters. For the vector field

$$\mathbf{B} = (-y^3, x^3, z^3)$$

compute $\nabla \times \mathbf{B}$ and calculate $I = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$.

Use Stokes' Theorem to express I as an integral over ∂S and verify that this gives the same result.

Paper 3, Section II

10C Vector Calculus

Consider the transformation of variables

$$x = 1 - u, \quad y = \frac{1-v}{1-uv}.$$

Show that the interior of the unit square in the uv plane

$$\{(u,v) : \ 0 < u < 1, \ 0 < v < 1\}$$

is mapped to the interior of the unit square in the xy plane,

$$R = \{(x,y) : \ 0 < x < 1, \ 0 < y < 1\}.$$

[*Hint: Consider the relation between v and y when $u = \alpha$, for $0 < \alpha < 1$ constant.*]

Show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{(1-(1-x)y)^2}{x}.$$

Now let

$$u = \frac{1-t}{1-wt}, \quad v = 1 - w.$$

By calculating

$$\frac{\partial(x,y)}{\partial(t,w)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(t,w)}$$

as a function of x and y, or otherwise, show that

$$\int_R \frac{x(1-y)}{(1-(1-x)y)(1-(1-x^2)y)^2}\,dx\,dy = 1.$$

Paper 3, Section II

11C Vector Calculus

(a) Prove the identity

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}).$$

(b) If $\mathbf{E}$ is an irrotational vector field (i.e. $\nabla\times\mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\nabla\phi$.

Show that the vector field

$$(xy^2ze^{-x^2z}, -ye^{-x^2z}, \tfrac{1}{2}x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

Paper 3, Section II

12C Vector Calculus

(i) Let V be a bounded region in $\mathbb{R}^3$ with smooth boundary $S = \partial V$. Show that Poisson's equation in V

$$\nabla^2 u = \rho$$

has at most one solution satisfying $u = f$ on S, where ρ and f are given functions.

Consider the alternative boundary condition $\partial u/\partial n = g$ on S, for some given function g, where n is the outward pointing normal on S. Derive a necessary condition in terms of ρ and g for a solution u of Poisson's equation to exist. Is such a solution unique?

(ii) Find the most general spherically symmetric function $u(r)$ satisfying

$$\nabla^2 u = 1$$

in the region $r = |\mathbf{r}| \leqslant a$ for $a > 0$. Hence in each of the following cases find all possible solutions satisfying the given boundary condition at $r = a$:

(a) $u = 0\,,$

(b) $\frac{\partial u}{\partial n} = 0\,.$

Compare these with your results in part (i).

Paper 3, Section I

3C Vector Calculus

Cartesian coordinates x, y, z and spherical polar coordinates r, θ, ϕ are related by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \ .$$

Find scalars h_r, h_θ, h_ϕ and unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ such that

$$\mathrm{d}\mathbf{x} = h_r\mathbf{e}_r\,\mathrm{d}r + h_\theta\mathbf{e}_\theta\,\mathrm{d}\theta + h_\phi\mathbf{e}_\phi\,\mathrm{d}\phi \ .$$

Verify that the unit vectors are mutually orthogonal.

Hence calculate the area of the open surface defined by $\theta = \alpha$, $0 \leqslant r \leqslant R$, $0 \leqslant \phi \leqslant 2\pi$, where α and R are constants.

Paper 3, Section I

4C Vector Calculus

State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$, where $r = |\mathbf{x}|$.

Vector fields $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are given by $\mathbf{u} = r^\alpha\mathbf{x}$ and $\mathbf{v} = \mathbf{k}\times\mathbf{u}$, where α is a constant and $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$, and deduce that $\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$ and $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. When $\alpha = -3$, show that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and

$$\boldsymbol{\nabla}\times\mathbf{v} = \frac{3(\mathbf{k}\cdot\mathbf{x})\mathbf{x} - \mathbf{k}r^2}{r^5} \ .$$

Paper 3, Section II
9C Vector Calculus

Write down the most general isotropic tensors of rank 2 and 3. Use the tensor transformation law to show that they are, indeed, isotropic.

Let V be the sphere $0 \leqslant r \leqslant a$. Explain briefly why

$$T_{i_1 \ldots i_n} = \int_V x_{i_1} \ldots x_{i_n} \, \mathrm{d}V$$

is an isotropic tensor for any n. Hence show that

$$\int_V x_i x_j \, \mathrm{d}V = \alpha \delta_{ij}, \quad \int_V x_i x_j x_k \, \mathrm{d}V = 0 \quad \text{and} \int_V x_i x_j x_k x_l \, \mathrm{d}V = \beta(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

for some scalars α and β, which should be determined using suitable contractions of the indices or otherwise. Deduce the value of

$$\int_V \mathbf{x} \times (\boldsymbol{\Omega} \times \mathbf{x}) \, \mathrm{d}V \ ,$$

where $\boldsymbol{\Omega}$ is a constant vector.

[*You may assume that the most general isotropic tensor of rank 4 is*

$$\lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk} \ ,$$

where λ, μ and ν are scalars.]

Paper 3, Section II
10C Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{x})$ in a region V bounded by a piecewise smooth surface S with outward normal $\mathbf{n}$.

Show, by suitable choice of $\mathbf{u}$, that

$$\int_V \boldsymbol{\nabla} f \, \mathrm{d}V = \int_S f \, \mathrm{d}\mathbf{S} \qquad (*)$$

for a scalar field $f(\mathbf{x})$.

Let V be the paraboloidal region given by $z \geqslant 0$ and $x^2 + y^2 + cz \leqslant a^2$, where a and c are positive constants. Verify that $(*)$ holds for the scalar field $f = xz$.

Paper 3, Section II

11C Vector Calculus

The electric field $\mathbf{E}(\mathbf{x})$ due to a static charge distribution with density $\rho(\mathbf{x})$ satisfies

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \ , \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \ , \tag{1}$$

where $\phi(\mathbf{x})$ is the corresponding electrostatic potential and ε_0 is a constant.

(a) Show that the total charge Q contained within a closed surface S is given by Gauss' Law

$$Q = \varepsilon_0 \int_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} \ .$$

Assuming spherical symmetry, deduce the electric field and potential due to a point charge q at the origin i.e. for $\rho(\mathbf{x}) = q\,\delta(\mathbf{x})$.

(b) Let $\mathbf{E}_1$ and $\mathbf{E}_2$, with potentials ϕ_1 and ϕ_2 respectively, be the solutions to (1) arising from two different charge distributions with densities ρ_1 and ρ_2. Show that

$$\frac{1}{\varepsilon_0}\int_V \phi_1\rho_2 \,\mathrm{d}V + \int_{\partial V} \phi_1 \boldsymbol{\nabla}\phi_2 \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \phi_2\rho_1 \,\mathrm{d}V + \int_{\partial V} \phi_2 \boldsymbol{\nabla}\phi_1 \cdot \mathrm{d}\mathbf{S} \tag{2}$$

for any region V with boundary ∂V, where $\mathrm{d}\mathbf{S}$ points out of V.

(c) Suppose that $\rho_1(\mathbf{x}) = 0$ for $|\mathbf{x}| \leqslant a$ and that $\phi_1(\mathbf{x}) = \Phi$, a constant, on $|\mathbf{x}| = a$. Use the results of (a) and (b) to show that

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_{r>a} \frac{\rho_1(\mathbf{x})}{r}\,\mathrm{d}V \ .$$

[*You may assume that* $\phi_1 \to 0$ *as* $|\mathbf{x}| \to \infty$ *sufficiently rapidly that any integrals over the 'sphere at infinity' in* (2) *are zero.*]

Paper 3, Section II

12C Vector Calculus

The vector fields $\mathbf{A}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ obey the evolution equations

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \boldsymbol{\nabla}\psi \ , \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{B} \ , \tag{2}$$

where $\mathbf{u}$ is a given vector field and ψ is a given scalar field. Use suffix notation to show that the scalar field $h = \mathbf{A} \cdot \mathbf{B}$ obeys an evolution equation of the form

$$\frac{\partial h}{\partial t} = \mathbf{B} \cdot \boldsymbol{\nabla} f - \mathbf{u} \cdot \boldsymbol{\nabla} h \ ,$$

where the scalar field f should be identified.

Suppose that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Show that, if $\mathbf{u} \cdot \mathbf{n} = \mathbf{B} \cdot \mathbf{n} = 0$ on the surface S of a fixed volume V with outward normal $\mathbf{n}$, then

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \ , \quad \text{where } H = \int_V h \, \mathrm{d}V \ .$$

Suppose that $\mathbf{A} = ar^2 \sin\theta \, \mathbf{e}_\theta + r(a^2 - r^2)\sin\theta \, \mathbf{e}_\phi$ with respect to spherical polar coordinates, and that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Show that

$$h = ar^2(a^2 + r^2)\sin^2\theta \ ,$$

and calculate the value of H when V is the sphere $r \leqslant a$.

$$\left[\textit{In spherical polar coordinates } \boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta \mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} . \right]$$

Paper 3, Section I
3C Vector Calculus

Consider the vector field

$$\mathbf{F} = (-y/(x^2+y^2),\, x/(x^2+y^2),\, 0)$$

defined on all of $\mathbb{R}^3$ except the z axis. Compute $\boldsymbol{\nabla} \times \mathbf{F}$ on the region where it is defined.

Let γ_1 be the closed curve defined by the circle in the xy-plane with centre $(2,2,0)$ and radius 1, and γ_2 be the closed curve defined by the circle in the xy-plane with centre $(0,0,0)$ and radius 1.

By using your earlier result, or otherwise, evaluate the line integral $\oint_{\gamma_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$.

By explicit computation, evaluate the line integral $\oint_{\gamma_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$. Is your result consistent with Stokes' theorem? Explain your answer briefly.

Paper 3, Section I
4C Vector Calculus

A curve in two dimensions is defined by the parameterised Cartesian coordinates

$$x(u) = ae^{bu}\cos u\,, \qquad y(u) = ae^{bu}\sin u\,,$$

where the constants $a, b > 0$. Sketch the curve segment corresponding to the range $0 \leqslant u \leqslant 3\pi$. What is the length of the curve segment between the points $(x(0), y(0))$ and $(x(U), y(U))$, as a function of U?

A geometrically sensitive ant walks along the curve with varying speed $\kappa(u)^{-1}$, where $\kappa(u)$ is the curvature at the point corresponding to parameter u. Find the time taken by the ant to walk from $(\,x(2n\pi),\, y(2n\pi)\,)$ to $(\,x(2(n{+}1)\pi),\, y(2(n{+}1)\pi)\,)$, where n is a positive integer, and hence verify that this time is independent of n.

[*You may quote without proof the formula* $\kappa(u) = \dfrac{|\,x'(u)y''(u) - y'(u)x''(u)\,|}{(\,(x'(u))^2 + (y'(u))^2\,)^{3/2}}$.]

Paper 3, Section II
9C Vector Calculus

(a) Define a rank two tensor and show that if two rank two tensors A_{ij} and B_{ij} are the same in one Cartesian coordinate system, then they are the same in all Cartesian coordinate systems.

The quantity C_{ij} has the property that, for every rank two tensor A_{ij}, the quantity $C_{ij}A_{ij}$ is a scalar. Is C_{ij} necessarily a rank two tensor? Justify your answer with a proof from first principles, or give a counterexample.

(b) Show that, if a tensor T_{ij} is invariant under rotations about the x_3-axis, then it has the form

$$\begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} .$$

(c) The *inertia tensor* about the origin of a rigid body occupying volume V and with variable mass density $\rho(\mathbf{x})$ is defined to be

$$I_{ij} = \int_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\, \mathrm{d}V .$$

The rigid body B has uniform density ρ and occupies the cylinder

$$\{\, (x_1, x_2, x_3) \, : \, -2 \leqslant x_3 \leqslant 2, \, x_1^2 + x_2^2 \leqslant 1 \,\} .$$

Show that the inertia tensor of B about the origin is diagonal in the (x_1, x_2, x_3) coordinate system, and calculate its diagonal elements.

Paper 3, Section II
10C Vector Calculus

Let $f(x,y)$ be a function of two variables, and R a region in the xy-plane. State the rule for evaluating $\int_R f(x,y)\,\mathrm{d}x\mathrm{d}y$ as an integral with respect to new variables $u(x,y)$ and $v(x,y)$.

Sketch the region R in the xy-plane defined by

$$R = \{\,(x,y) : x^2+y^2 \leqslant 2\,,\ x^2-y^2 \geqslant 1\,,\ x \geqslant 0\,,\ y \geqslant 0\,\}\,.$$

Sketch the corresponding region in the uv-plane, where

$$u = x^2+y^2\,, \qquad v = x^2-y^2\,.$$

Express the integral

$$I = \int_R (x^5 y - x y^5)\exp(4x^2y^2)\,\mathrm{d}x\,\mathrm{d}y$$

as an integral with respect to u and v. Hence, or otherwise, calculate I.

Paper 3, Section II
11C Vector Calculus

State the divergence theorem (also known as Gauss' theorem) relating the surface and volume integrals of appropriate fields.

The surface S_1 is defined by the equation $z = 3 - 2x^2 - 2y^2$ for $1 \leqslant z \leqslant 3$; the surface S_2 is defined by the equation $x^2+y^2=1$ for $0 \leqslant z \leqslant 1$; the surface S_3 is defined by the equation $z=0$ for x, y satisfying $x^2+y^2 \leqslant 1$. The surface S is defined to be the union of the surfaces S_1, S_2 and S_3. Sketch the surfaces S_1, S_2, S_3 and (hence) S.

The vector field $\mathbf{F}$ is defined by

$$\mathbf{F}(x,y,z) = (\,xy + x^6,\ -\tfrac{1}{2}y^2 + y^8,\ z\,)\,.$$

Evaluate the integral

$$\oint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}\,,$$

where the surface element $\mathrm{d}\mathbf{S}$ points in the direction of the outward normal to S.

Paper 3, Section II

12C Vector Calculus

Given a spherically symmetric mass distribution with density ρ, explain how to obtain the gravitational field $\mathbf{g} = -\boldsymbol{\nabla}\phi$, where the potential ϕ satisfies Poisson's equation

$$\nabla^2\phi = 4\pi G\rho\,.$$

The remarkable planet Geometria has radius 1 and is composed of an infinite number of stratified spherical shells S_n labelled by integers $n \geqslant 1$. The shell S_n has uniform density $2^{n-1}\rho_0$, where ρ_0 is a constant, and occupies the volume between radius 2^{-n+1} and 2^{-n}.

Obtain a closed form expression for the mass of Geometria.

Obtain a closed form expression for the gravitational field $\mathbf{g}$ due to Geometria at a distance $r = 2^{-N}$ from its centre of mass, for each positive integer $N \geqslant 1$. What is the potential $\phi(r)$ due to Geometria for $r > 1$?

Paper 3, Section I

3B Vector Calculus

What does it mean for a vector field $\mathbf{F}$ to be *irrotational* ?

The field $\mathbf{F}$ is irrotational and $\mathbf{x}_0$ is a given point. Write down a scalar potential $V(\mathbf{x})$ with $\mathbf{F} = -\boldsymbol{\nabla} V$ and $V(\mathbf{x}_0) = 0$. Show that this potential is well defined.

For what value of m is the field $\dfrac{\cos\theta\cos\phi}{r}\,\mathbf{e}_\theta + \dfrac{m\sin\phi}{r}\,\mathbf{e}_\phi$ irrotational, where (r,θ,ϕ) are spherical polar coordinates? What is the corresponding potential $V(\mathbf{x})$ when $\mathbf{x}_0$ is the point $r = 1, \theta = 0$?

$$\left[\textit{In spherical polar coordinates } \boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} . \right]$$

Paper 3, Section I

4B Vector Calculus

State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$, where $r = |\mathbf{x}|$.

A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{\mathbf{k}}{r} + \frac{(\mathbf{k}\cdot\mathbf{x})\mathbf{x}}{r^3},$$

where $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$ using suffix notation, and show that d_{ij} splits naturally into symmetric and antisymmetric parts. Deduce that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and that

$$\boldsymbol{\nabla}\times\mathbf{u} = \frac{2\mathbf{k}\times\mathbf{x}}{r^3} .$$

Paper 3, Section II
9B Vector Calculus

Let S be a bounded region of $\mathbb{R}^2$ and ∂S be its boundary. Let u be the unique solution to Laplace's equation in S, subject to the boundary condition $u = f$ on ∂S, where f is a specified function. Let w be any smooth function with $w = f$ on ∂S. By writing $w = u + \delta$, or otherwise, show that

$$\int_S |\nabla w|^2 \, \mathrm{d}A \geqslant \int_S |\nabla u|^2 \, \mathrm{d}A \; . \tag{$*$}$$

Let S be the unit disc in $\mathbb{R}^2$. By considering functions of the form $g(r)\cos\theta$ on both sides of $(*)$, where r and θ are polar coordinates, deduce that

$$\int_0^1 \left(r\left(\frac{\mathrm{d}g}{\mathrm{d}r}\right)^2 + \frac{g^2}{r} \right) \mathrm{d}r \geqslant 1$$

for any differentiable function $g(r)$ satisfying $g(1) = 1$ and for which the integral converges at $r = 0$.

$$\left[\nabla f(r,\theta) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta} \right), \qquad \nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \, . \right]$$

Paper 3, Section II
10B Vector Calculus

Give a necessary condition for a given vector field $\mathbf{J}$ to be the curl of another vector field $\mathbf{B}$. Is the vector field $\mathbf{B}$ unique? If not, explain why not.

State Stokes' theorem and use it to evaluate the area integral

$$\int_S \left(y^2, z^2, x^2 \right) \cdot \mathbf{dA} \; ,$$

where S is the half of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

that lies in $z \geqslant 0$, and the area element $\mathbf{dA}$ points out of the ellipsoid.

Paper 3, Section II
11B Vector Calculus

A second-rank tensor $T(\mathbf{y})$ is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j)|\mathbf{y} - \mathbf{x}|^{2n-2}\,\mathrm{d}A(\mathbf{x})\,,$$

where $\mathbf{y}$ is a fixed vector with $|\mathbf{y}| = a$, $n > -1$, and the integration is over all points $\mathbf{x}$ lying on the surface S of the sphere of radius a, centred on the origin. Explain briefly why T might be expected to have the form

$$T_{ij} \;=\; \alpha\delta_{ij} + \beta y_i\, y_j\,,$$

where α and β are scalar constants.

Show that $\mathbf{y}\cdot(\mathbf{y} - \mathbf{x}) = a^2(1 - \cos\theta)\,$, where θ is the angle between $\mathbf{y}$ and $\mathbf{x}$, and find a similar expression for $|\mathbf{y} - \mathbf{x}|^2$. Using suitably chosen spherical polar coordinates, show that

$$y_i T_{ij}\, y_j \;=\; \frac{\pi a^2 (2a)^{2n+2}}{n+2}\,.$$

Hence, by evaluating another scalar integral, determine α and β, and find the value of n for which T is isotropic.

Paper 3, Section II
12B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{x})$ in a region V of $\mathbb{R}^3$ bounded by a smooth surface S.

Let $f(x, y, z)$ be a homogeneous function of degree n, that is, $f(kx, ky, kz) = k^n f(x, y, z)$ for any real number k. By differentiating with respect to k, show that

$$\mathbf{x}\cdot\boldsymbol{\nabla} f = nf\,.$$

Deduce that

$$\int_V f\,\mathrm{d}V = \frac{1}{n+3}\int_S f\,\mathbf{x}\cdot\mathbf{dA}\,. \qquad (\dagger)$$

Let V be the cone $0 \leqslant z \leqslant \alpha$, $\alpha\sqrt{x^2+y^2} \leqslant z$, where α is a positive constant. Verify that ($\dagger$) holds for the case $f = z^4 + \alpha^4(x^2+y^2)^2$.

3/I/3C **Vector Calculus**

A curve is given in terms of a parameter t by

$$\mathbf{x}(t) = (t - \tfrac{1}{3}t^3\,,\, t^2\,,\, t + \tfrac{1}{3}t^3)\,.$$

(i) Find the arc length of the curve between the points with $t = 0$ and $t = 1$.

(ii) Find the unit tangent vector at the point with parameter t, and show that the principal normal is orthogonal to the z direction at each point on the curve.

3/I/4C **Vector Calculus**

What does it mean to say that T_{ij} transforms as a second rank tensor?

If T_{ij} transforms as a second rank tensor, show that $\dfrac{\partial T_{ij}}{\partial x_j}$ transforms as a vector.

3/II/9C **Vector Calculus**

Let $\mathbf{F} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$, where $\mathbf{x}$ is the position vector and $\boldsymbol{\omega}$ is a uniform vector field.

(i) Use the divergence theorem to evaluate the surface integral $\displaystyle\int_S \mathbf{F} \cdot d\mathbf{S}$, where S is the closed surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

(ii) Show that $\boldsymbol{\nabla} \times \mathbf{F} = 0$. Show further that the scalar field ϕ given by

$$\phi = \tfrac{1}{2}\,(\boldsymbol{\omega} \cdot \mathbf{x})^2 - \tfrac{1}{2}\,(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{x} \cdot \mathbf{x})$$

satisfies $\mathbf{F} = \boldsymbol{\nabla}\phi$. Describe geometrically the surfaces of constant ϕ.

3/II/10C **Vector Calculus**

Find the effect of a rotation by $\pi/2$ about the z-axis on the tensor

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}.$$

Hence show that the most general isotropic tensor of rank 2 is $\lambda\delta_{ij}$, where λ is an arbitrary scalar.

Prove that there is no non-zero isotropic vector, and write down without proof the most general isotropic tensor of rank 3.

Deduce that if T_{ijkl} is an isotropic tensor then the following results hold, for some scalars μ and ν:

(i) $\epsilon_{ijk}\, T_{ijkl} \;\;= 0\,;$

(ii) $\delta_{ij}\, T_{ijkl} \;\;= \mu\, \delta_{kl}\,;$

(iii) $\epsilon_{ijm}\, T_{ijkl} = \nu\, \epsilon_{klm}\,.$

Verify these three results in the case $T_{ijkl} = \alpha\, \delta_{ij}\, \delta_{kl} + \beta\, \delta_{ik}\, \delta_{jl} + \gamma\, \delta_{il}\, \delta_{jk}$, expressing μ and ν in terms of α, β and γ.

3/II/11C **Vector Calculus**

Let V be a volume in $\mathbb{R}^3$ bounded by a closed surface S.

(a) Let f and g be twice differentiable scalar fields such that $f = 1$ on S and $\nabla^2 g = 0$ in V. Show that

$$\int_V \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \, dV = 0\,.$$

(b) Let V be the sphere $|\mathbf{x}| \leqslant a$. Evaluate the integral

$$\int_V \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, dV$$

in the cases where u and v are given in spherical polar coordinates by:

(i) $u = r\,, \quad\;\; v = r\cos\theta\,;$

(ii) $u = r/a\,, \quad v = r^2\cos^2\theta\,;$

(iii) $u = r/a\,, \quad v = 1/r\,.$

Comment on your results in the light of part (a).

3/II/12C **Vector Calculus**

Let A be the closed planar region given by

$$y \leqslant x \leqslant 2y, \quad \frac{1}{y} \leqslant x \leqslant \frac{2}{y}\,.$$

(i) Evaluate by means of a suitable change of variables the integral

$$\int_A \frac{x}{y}\, dx\, dy\,.$$

(ii) Let C be the boundary of A. Evaluate the line integral

$$\oint_C \frac{x^2}{2y} dy - dx$$

by integrating along each section of the boundary.

(iii) Comment on your results.

3/I/3A Vector Calculus

(i) Give definitions for the unit tangent vector $\hat{\mathbf{T}}$ and the curvature κ of a parametrised curve $\mathbf{x}(t)$ in $\mathbb{R}^3$. Calculate $\hat{\mathbf{T}}$ and κ for the circular helix

$$\mathbf{x}(t) = (a\cos t\,,\; a\sin t\,,\; bt)\,,$$

where a and b are constants.

(ii) Find the normal vector and the equation of the tangent plane to the surface S in $\mathbb{R}^3$ given by

$$z = x^2y^3 - y + 1$$

at the point $x = 1$, $y = 1$, $z = 1$.

3/I/4A Vector Calculus

By using suffix notation, prove the following identities for the vector fields $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})\,;$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})\,.$$

3/II/9A Vector Calculus

(i) Define what is meant by a conservative vector field. Given a vector field $\mathbf{A} = (A_1(x,y), A_2(x,y))$ and a function $\psi(x,y)$ defined in $\mathbb{R}^2$, show that, if $\psi\mathbf{A}$ is a conservative vector field, then

$$\psi\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}\right) = A_2\frac{\partial \psi}{\partial x} - A_1\frac{\partial \psi}{\partial y}\,.$$

(ii) Given two functions $P(x,y)$ and $Q(x,y)$ defined in $\mathbb{R}^2$, prove Green's theorem,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,,$$

where C is a simple closed curve bounding a region R in $\mathbb{R}^2$.

Through an appropriate choice for P and Q, find an expression for the area of the region R, and apply this to evaluate the area of the ellipse bounded by the curve

$$x = a\cos\theta\,, \qquad y = b\sin\theta\,, \qquad 0 \le \theta \le 2\pi\,.$$

3/II/10A **Vector Calculus**

For a given charge distribution $\rho(x, y, z)$ and divergence-free current distribution $\mathbf{J}(x, y, z)$ (i.e. $\nabla \cdot \mathbf{J} = 0$) in $\mathbb{R}^3$, the electric and magnetic fields $\mathbf{E}(x, y, z)$ and $\mathbf{B}(x, y, z)$ satisfy the equations

$$\nabla \times \mathbf{E} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \cdot \mathbf{E} = \rho\,, \qquad \nabla \times \mathbf{B} = \mathbf{J}\,.$$

The radiation flux vector $\mathbf{P}$ is defined by $\mathbf{P} = \mathbf{E} \times \mathbf{B}$.

For a closed surface S around a region V, show using Gauss' theorem that the flux of the vector $\mathbf{P}$ through S can be expressed as

$$\iint_S \mathbf{P} \cdot \mathbf{dS} = -\iiint_V \mathbf{E} \cdot \mathbf{J}\, dV\,. \qquad (*)$$

For electric and magnetic fields given by

$$\mathbf{E}(x, y, z) = (z,\ 0,\ x)\,, \qquad \mathbf{B}(x, y, z) = (0,\ -xy,\ xz)\,,$$

find the radiation flux through the quadrant of the unit spherical shell given by

$$x^2 + y^2 + z^2 = 1\,, \quad \text{with} \quad 0 \le x \le 1\,, \quad 0 \le y \le 1\,, \quad -1 \le z \le 1\,.$$

[*If you use* $(*)$, *note that an open surface has been specified.*]

3/II/11A **Vector Calculus**

The function $\phi(x,y,z)$ satisfies $\nabla^2\phi = 0$ in V and $\phi = 0$ on S, where V is a region of $\mathbb{R}^3$ which is bounded by the surface S. Prove that $\phi = 0$ everywhere in V.

Deduce that there is at most one function $\psi(x,y,z)$ satisfying $\nabla^2\psi = \rho$ in V and $\psi = f$ on S, where $\rho(x,y,z)$ and $f(x,y,z)$ are given functions.

Given that the function $\psi = \psi(r)$ depends only on the radial coordinate $r = |\mathbf{x}|$, use Cartesian coordinates to show that

$$\nabla\psi = \frac{1}{r}\frac{d\psi}{dr}\mathbf{x}\,, \qquad \nabla^2\psi = \frac{1}{r}\frac{d^2(r\psi)}{dr^2}\,.$$

Find the general solution in this radial case for $\nabla^2\psi = c$ where c is a constant.

Find solutions $\psi(r)$ for a solid sphere of radius $r = 2$ with a central cavity of radius $r = 1$ in the following three regions:

(i) $0 \leqslant r \leqslant 1$ where $\nabla^2\psi = 0$ and $\psi(1) = 1$ and ψ bounded as $r \to 0$;

(ii) $1 \leqslant r \leqslant 2$ where $\nabla^2\psi = 1$ and $\psi(1) = \psi(2) = 1$;

(iii) $r \geqslant 2$ where $\nabla^2\psi = 0$ and $\psi(2) = 1$ and $\psi \to 0$ as $r \to \infty$.

3/II/12A **Vector Calculus**

Show that any second rank Cartesian tensor P_{ij} in $\mathbb{R}^3$ can be written as a sum of a symmetric tensor and an antisymmetric tensor. Further, show that P_{ij} can be decomposed into the following terms

$$P_{ij} = P\delta_{ij} + S_{ij} + \epsilon_{ijk}A_k\,, \qquad (\dagger)$$

where S_{ij} is symmetric and traceless. Give expressions for P, S_{ij} and A_k explicitly in terms of P_{ij}.

For an isotropic material, the stress P_{ij} can be related to the strain T_{ij} through the stress–strain relation, $P_{ij} = c_{ijkl}\,T_{kl}$, where the elasticity tensor is given by

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

and α, β and γ are scalars. As in $(\dagger)$, the strain T_{ij} can be decomposed into its trace T, a symmetric traceless tensor W_{ij} and a vector V_k. Use the stress–strain relation to express each of T, W_{ij} and V_k in terms of P, S_{ij} and A_k.

Hence, or otherwise, show that if T_{ij} is symmetric then so is P_{ij}. Show also that the stress-strain relation can be written in the form

$$P_{ij} = \lambda\,\delta_{ij}T_{kk} + \mu\,T_{ij}\,,$$

where μ and λ are scalars.

3/I/3A Vector Calculus

Consider the vector field $\mathbf{F}(\mathbf{x}) = \big((3x^3 - x^2)y,\ (y^3 - 2y^2 + y)x,\ z^2 - 1\big)$ and let S be the surface of a unit cube with one corner at (0, 0, 0), another corner at (1, 1, 1) and aligned with edges along the x-, y- and z-axes. Use the divergence theorem to evaluate

$$I = \int_S \mathbf{F} \cdot d\mathbf{S}\,.$$

Verify your result by calculating the integral directly.

3/I/4A Vector Calculus

Use suffix notation in Cartesian coordinates to establish the following two identities for the vector field $\mathbf{v}$:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0\,, \qquad (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla(\tfrac{1}{2}|\mathbf{v}|^2) - \mathbf{v} \times (\nabla \times \mathbf{v})\,.$$

3/II/9A Vector Calculus

Evaluate the line integral

$$\int \alpha(x^2 + xy)dx + \beta(x^2 + y^2)dy,$$

with α and β constants, along each of the following paths between the points $A = (1, 0)$ and $B = (0, 1)$:

(i) the straight line between A and B;

(ii) the x-axis from A to the origin $(0, 0)$ followed by the y-axis to B;

(iii) anti-clockwise from A to B around the circular path centred at the origin $(0, 0)$.

You should obtain the same answer for the three paths when $\alpha = 2\beta$. Show that when $\alpha = 2\beta$, the integral takes the same value along *any* path between A and B.

3/II/10A Vector Calculus

State Stokes' theorem for a vector field $\mathbf{A}$.

By applying Stokes' theorem to the vector field $\mathbf{A} = \phi\,\mathbf{k}$, where $\mathbf{k}$ is an arbitrary constant vector in $\mathbb{R}^3$ and ϕ is a scalar field defined on a surface S bounded by a curve ∂S, show that

$$\int_S d\mathbf{S} \times \nabla\phi = \int_{\partial S} \phi \; d\mathbf{x}\,.$$

For the vector field $\mathbf{A} = x^2 y^4 (1,\,1,\,1)$ in Cartesian coordinates, evaluate the line integral

$$I = \int \mathbf{A} \cdot d\mathbf{x}\,,$$

around the boundary of the quadrant of the unit circle lying between the x- and y-axes, that is, along the straight line from $(0,\,0,\,0)$ to $(1,\,0,\,0)$, then the circular arc $x^2 + y^2 = 1,\ z = 0$ from $(1,\,0,\,0)$ to $(0,\,1,\,0)$ and finally the straight line from $(0,\,1,\,0)$ back to $(0,\,0,\,0)$.

3/II/11A Vector Calculus

In a region R of $\mathbb{R}^3$ bounded by a closed surface S, suppose that ϕ_1 and ϕ_2 are both solutions of $\nabla^2\phi = 0$, satisfying boundary conditions on S given by $\phi = f$ on S, where f is a given function. Prove that $\phi_1 = \phi_2$.

In $\mathbb{R}^2$ show that

$$\phi(x,y) = (a_1 \cosh \lambda x + a_2 \sinh \lambda x)(b_1 \cos \lambda y + b_2 \sin \lambda y)$$

is a solution of $\nabla^2\phi = 0$, for any constants a_1, a_2, b_1, b_2 and λ. Hence, or otherwise, find a solution $\phi(x,y)$ in the region $x \geqslant 0$ and $0 \leqslant y \leqslant a$ which satisfies:

$$\begin{aligned} &\phi(x,0) = 0\,, \quad \phi(x,a) = 0, \quad x \geqslant 0\,, \\ &\phi(0,y) = \sin\frac{n\pi y}{a}\,, \quad \phi(x,y) \to 0 \ \text{ as } \ x \to \infty\,, \quad 0 \leqslant y \leqslant a\,, \end{aligned}$$

where a is a real constant and n is an integer.

3/II/12A **Vector Calculus**

Define what is meant by an isotropic tensor. By considering a rotation of a second rank isotropic tensor B_{ij} by 90° about the z-axis, show that its components must satisfy $B_{11} = B_{22}$ and $B_{13} = B_{31} = B_{23} = B_{32} = 0$. Now consider a second and different rotation to show that B_{ij} must be a multiple of the Kronecker delta, δ_{ij}.

Suppose that a homogeneous but anisotropic crystal has the conductivity tensor

$$\sigma_{ij} = \alpha\delta_{ij} + \gamma n_i n_j\,,$$

where α, γ are real constants and the n_i are the components of a constant unit vector $\mathbf{n}$ ($\mathbf{n} \cdot \mathbf{n} = 1$). The electric current density $\mathbf{J}$ is then given in components by

$$J_i = \sigma_{ij} E_j,$$

where E_j are the components of the electric field $\mathbf{E}$. Show that

(i) if $\alpha \neq 0$ and $\gamma \neq 0$, then there is a plane such that if $\mathbf{E}$ lies in this plane, then $\mathbf{E}$ and $\mathbf{J}$ must be parallel, and

(ii) if $\gamma \neq -\alpha$ and $\alpha \neq 0$, then $\mathbf{E} \neq 0$ implies $\mathbf{J} \neq 0$.

If $D_{ij} = \epsilon_{ijk} n_k$, find the value of γ such that

$$\sigma_{ij} D_{jk} D_{km} = -\sigma_{im}\,.$$

3/I/3A **Vector Calculus**

Let $\mathbf{A}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ be time-dependent, continuously differentiable vector fields on $\mathbb{R}^3$ satisfying

$$\frac{\partial \mathbf{A}}{\partial t} = \nabla \times \mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{A}\,.$$

Show that for any bounded region V,

$$\frac{d}{dt}\left[\frac{1}{2}\int_V (\mathbf{A}^2 + \mathbf{B}^2) dV\right] = -\int_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{S}\,,$$

where S is the boundary of V.

3/I/4A **Vector Calculus**

Given a curve $\boldsymbol{\gamma}(s)$ in $\mathbb{R}^3$, parameterised such that $\|\boldsymbol{\gamma}'(s)\| = 1$ and with $\boldsymbol{\gamma}''(s) \neq 0$, define the tangent $\mathbf{t}(s)$, the principal normal $\mathbf{p}(s)$, the curvature $\kappa(s)$ and the binormal $\mathbf{b}(s)$.

The torsion $\tau(s)$ is defined by

$$\tau = -\mathbf{b}' \cdot \mathbf{p}\,.$$

Sketch a circular helix showing $\mathbf{t}, \mathbf{p}, \mathbf{b}$ and $\mathbf{b}'$ at a chosen point. What is the sign of the torsion for your helix? Sketch a second helix with torsion of the opposite sign.

3/II/9A **Vector Calculus**

Let V be a bounded region of $\mathbb{R}^3$ and S be its boundary. Let ϕ be the unique solution to $\nabla^2\phi = 0$ in V, with $\phi = f(\mathbf{x})$ on S, where f is a given function. Consider any smooth function w also equal to $f(\mathbf{x})$ on S. Show, by using Green's first theorem or otherwise, that

$$\int_V |\nabla w|^2 \, dV \quad \geqslant \quad \int_V |\nabla \phi|^2 \, dV \, .$$

[*Hint: Set* $w = \phi + \delta$.]

Consider the partial differential equation

$$\frac{\partial}{\partial t} w = \nabla^2 w \, ,$$

for $w(t, \mathbf{x})$, with initial condition $w(0, \mathbf{x}) = w_0(\mathbf{x})$ in V, and boundary condition $w(t, \mathbf{x}) = f(\mathbf{x})$ on S for all $t \geqslant 0$. Show that

$$\frac{\partial}{\partial t} \int_V |\nabla w|^2 \, dV \quad \leqslant 0 \, , \qquad (*)$$

with equality holding only when $w(t, \mathbf{x}) = \phi(\mathbf{x})$.

Show that $(*)$ remains true with the boundary condition

$$\frac{\partial w}{\partial t} + \alpha(\mathbf{x}) \frac{\partial w}{\partial n} = 0$$

on S, provided $\alpha(\mathbf{x}) \geqslant 0$.

3/II/10A **Vector Calculus**

Write down Stokes' theorem for a vector field $\mathbf{B}(\mathbf{x})$ on $\mathbb{R}^3$.

Consider the bounded surface S defined by

$$z = x^2 + y^2, \quad \frac{1}{4} \leqslant z \leqslant 1 \, .$$

Sketch the surface and calculate the surface element $d\mathbf{S}$. For the vector field

$$\mathbf{B} = (-y^3, x^3, z^3) \, ,$$

calculate $I = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$ directly.

Show using Stokes' theorem that I may be rewritten as a line integral and verify this yields the same result.

3/II/11A **Vector Calculus**

Explain, with justification, the significance of the eigenvalues of the Hessian in classifying the critical points of a function $f : \mathbb{R}^n \to \mathbb{R}$. In what circumstances are the eigenvalues inconclusive in establishing the character of a critical point?

Consider the function on $\mathbb{R}^2$,

$$f(x, y) = xye^{-\alpha(x^2+y^2)}\,.$$

Find and classify all of its critical points, for all real α. How do the locations of the critical points change as $\alpha \to 0$?

3/II/12A **Vector Calculus**

Express the integral

$$I = \int_0^\infty dx \int_0^1 dy \int_0^x dz\ xe^{-Ax/y-Bxy-Cyz}$$

in terms of the new variables $\alpha = x/y$, $\beta = xy$, and $\gamma = yz$. Hence show that

$$I = \frac{1}{2A(A+B)(A+B+C)}\,.$$

You may assume A, B and C are positive. [*Hint: Remember to calculate the limits of the integral.*]

3/I/3C **Vector Calculus**

If $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields, show that

(i) $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}\,,$

(ii) $\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F})\,.$

3/I/4C **Vector Calculus**

Define the curvature, κ, of a curve in $\mathbb{R}^3$.

The curve C is parametrised by

$$\mathbf{x}(t) = \left(\frac{1}{2}e^t \cos t,\ \frac{1}{2}e^t \sin t,\ \frac{1}{\sqrt{2}}e^t\right) \quad \text{for } -\infty < t < \infty\,.$$

Obtain a parametrisation of the curve in terms of its arc length, s, measured from the origin. Hence obtain its curvature, $\kappa(s)$, as a function of s.

3/II/9C **Vector Calculus**

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ state if the following implications are true or false. (No justification is required.)

(i) f is differentiable $\Rightarrow$ f is continuous.

(ii) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist $\Rightarrow$ f is continuous.

(iii) directional derivatives $\dfrac{\partial f}{\partial \mathbf{n}}$ exist for all unit vectors $\mathbf{n} \in \mathbb{R}^2 \Rightarrow f$ is differentiable.

(iv) f is differentiable $\Rightarrow$ $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous.

(v) all second order partial derivatives of f exist $\Rightarrow$ $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is continuous at $(0,0)$ and find the partial derivatives $\dfrac{\partial f}{\partial x}(0,y)$ and $\dfrac{\partial f}{\partial y}(x,0)$. Then show that f is differentiable at $(0,0)$ and find its derivative. Investigate whether the second order partial derivatives $\dfrac{\partial^2 f}{\partial x\, \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\, \partial x}(0,0)$ are the same. Are the second order partial derivatives of f at $(0,0)$ continuous? Justify your answer.

3/II/10C **Vector Calculus**

Explain what is meant by an exact differential. The three-dimensional vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = \left(e^x z^3 + 3x^2(e^y - e^z),\, e^y(x^3 - z^3),\, 3z^2(e^x - e^y) - e^z x^3\right).$$

Find the most general function that has $\mathbf{F} \cdot \mathbf{dx}$ as its differential.

Hence show that the line integral

$$\int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{dx}$$

along any path in $\mathbb{R}^3$ between points $P_1 = (0, a, 0)$ and $P_2 = (b, b, b)$ vanishes for any values of a and b.

The two-dimensional vector field $\mathbf{G}$ is defined at all points in $\mathbb{R}^2$ except $(0,0)$ by

$$\mathbf{G} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).$$

($\mathbf{G}$ is not defined at $(0,0)$.) Show that

$$\oint_C \mathbf{G} \cdot \mathbf{dx} = 2\pi$$

for any closed curve C in $\mathbb{R}^2$ that goes around $(0,0)$ anticlockwise precisely once without passing through $(0,0)$.

3/II/11C **Vector Calculus**

Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0,0,0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, 0, 0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(\frac{-1}{4}, 0, 0)$. The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1. Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

3/II/12C **Vector Calculus**

State (without proof) the divergence theorem for a vector field $\mathbf{F}$ with continuous first-order partial derivatives throughout a volume V enclosed by a bounded oriented piecewise-smooth non-self-intersecting surface S.

By calculating the relevant volume and surface integrals explicitly, verify the divergence theorem for the vector field

$$\mathbf{F} = \left(x^3 + 2xy^2,\, y^3 + 2yz^2,\, z^3 + 2zx^2\right),$$

defined within a sphere of radius R centred at the origin.

Suppose that functions ϕ, ψ are continuous and that their first and second partial derivatives are all also continuous in a region V bounded by a smooth surface S.

Show that

$$(1) \qquad \int_V (\phi \nabla^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS}\,.$$

$$(2) \qquad \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS} - \int_S \psi \boldsymbol{\nabla}\phi \cdot \mathbf{dS}\,.$$

Hence show that if $\rho(\mathbf{x})$ is a continuous function on V and $g(\mathbf{x})$ a continuous function on S and ϕ_1 and ϕ_2 are two continuous functions such that

$$\nabla^2 \phi_1(\mathbf{x}) = \nabla^2 \phi_2(\mathbf{x}) = \rho(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ in } V, \text{ and}$$
$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ on } S,$$

then $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x}$ in V.

3/I/3A **Vector Calculus**

Sketch the curve $y^2 = x^2 + 1$. By finding a parametric representation, or otherwise, determine the points on the curve where the radius of curvature is least, and compute its value there.

[*Hint: you may use the fact that the radius of curvature of a parametrized curve* $(x(t), y(t))$ *is* $(\dot{x}^2 + \dot{y}^2)^{3/2} / |\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.]

3/I/4A **Vector Calculus**

Suppose V is a region in $\mathbb{R}^3$, bounded by a piecewise smooth closed surface S, and $\phi(\mathbf{x})$ is a scalar field satisfying

$$\begin{aligned} \boldsymbol{\nabla}^2 \phi &= 0 \quad \text{in } V, \\ \text{and} \quad \phi &= f(\mathbf{x}) \quad \text{on } S\,. \end{aligned}$$

Prove that ϕ is determined uniquely in V.

How does the situation change if the normal derivative of ϕ rather than ϕ itself is specified on S?

3/II/9A **Vector Calculus**

Let C be the closed curve that is the boundary of the triangle T with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Specify a direction along C and consider the integral

$$\oint_C \mathbf{A} \cdot d\mathbf{x}\ ,$$

where $\mathbf{A} = (z^2 - y^2,\ x^2 - z^2,\ y^2 - x^2)$. Explain why the contribution to the integral is the same from each edge of C, and evaluate the integral.

State Stokes's theorem and use it to evaluate the surface integral

$$\int_T (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S}\ ,$$

the components of the normal to T being positive.

Show that $d\mathbf{S}$ in the above surface integral can be written in the form $(1,1,1)\, dy\, dz$. Use this to verify your result by a direct calculation of the surface integral.

3/II/10A **Vector Calculus**

Write down an expression for the Jacobian J of a transformation

$$(x, y, z) \to (u, v, w)\,.$$

Use it to show that

$$\int_D f\;dx\,dy\,dz = \int_\Delta \phi\;|J|\;du\,dv\,dw$$

where D is mapped one-to-one onto Δ, and

$$\phi(u,v,w) \;=\; f\big(x(u,v,w),\;y(u,v,w),\;z(u,v,w)\big)\,.$$

Find a transformation that maps the ellipsoid D,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\,,$$

onto a sphere. Hence evaluate

$$\int_D x^2\,dx\,dy\,dz\,.$$

3/II/11A **Vector Calculus**

(a) Prove the identity

$$\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) \;=\; (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G} \;+\; (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} \;+\; \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G}) \;+\; \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})\,.$$

(b) If $\mathbf{E}$ is an irrotational vector field ($\boldsymbol{\nabla}\times\mathbf{E}=\mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E}=-\boldsymbol{\nabla}\phi$.

Show that

$$(2xy^2ze^{-x^2z},\;-2ye^{-x^2z},\;x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

3/II/12A **Vector Calculus**

State the divergence theorem. By applying this to $f(\mathbf{x})\mathbf{k}$, where $f(\mathbf{x})$ is a scalar field in a closed region V in $\mathbb{R}^3$ bounded by a piecewise smooth surface S, and $\mathbf{k}$ an arbitrary constant vector, show that

$$\int_V \boldsymbol{\nabla} f \, dV = \int_S f \, d\mathbf{S} \,. \qquad (*)$$

A vector field $\mathbf{G}$ satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \rho(\mathbf{x})$$
$$\text{with} \quad \rho(\mathbf{x}) = \begin{cases} \rho_0 & |\mathbf{x}| \leqslant a \\ 0 & |\mathbf{x}| > a \,. \end{cases}$$

By applying the divergence theorem to $\displaystyle\int_V \boldsymbol{\nabla} \cdot \mathbf{G} \, dV$, prove Gauss's law

$$\int_S \mathbf{G} \cdot d\mathbf{S} = \int_V \rho(\mathbf{x}) \, dV,$$

where S is the piecewise smooth surface bounding the volume V.

Consider the spherically symmetric solution

$$\mathbf{G}(\mathbf{x}) = G(r) \, \frac{\mathbf{x}}{r} \ ,$$

where $r = |\mathbf{x}|$. By using Gauss's law with S a sphere of radius r, centre $\mathbf{0}$, in the two cases $0 < r \leqslant a$ and $r > a$, show that

$$\mathbf{G}(\mathbf{x}) = \begin{cases} \dfrac{\rho_0}{3} \mathbf{x} & r \leqslant a \\ \dfrac{\rho_0}{3} \left(\dfrac{a}{r}\right)^3 \mathbf{x} & r > a \,. \end{cases}$$

The scalar field $f(\mathbf{x})$ satisfies $\mathbf{G} = \boldsymbol{\nabla} f$. Assuming that $f \to 0$ as $r \to \infty$, and that f is continuous at $r = a$, find f everywhere.

By using a symmetry argument, explain why $(*)$ is clearly satisfied for this f if S is any sphere centred at the origin.

3/I/3A **Vector Calculus**

Determine whether each of the following is the exact differential of a function, and if so, find such a function:

(a) $(\cosh\theta + \sinh\theta\cos\phi)d\theta + (\cosh\theta\sin\phi + \cos\phi)d\phi$,

(b) $3x^2(y^2+1)dx + 2(yx^3 - z^2)dy - 4yzdz$.

3/I/4A **Vector Calculus**

State the divergence theorem.

Consider the integral

$$I = \int_S r^n \mathbf{r} \cdot d\mathbf{S} \ ,$$

where $n > 0$ and S is the sphere of radius R centred at the origin. Evaluate I directly, and by means of the divergence theorem.

3/II/9A **Vector Calculus**

Two independent variables x_1 and x_2 are related to a third variable t by

$$x_1 = a + \alpha t \ , \quad x_2 = b + \beta t \ ,$$

where a, b, α and β are constants. Let f be a smooth function of x_1 and x_2, and let $F(t) = f(x_1, x_2)$. Show, by using the Taylor series for $F(t)$ about $t = 0$, that

$$f(x_1,x_2) = f(a,b) + (x_1 - a)\frac{\partial f}{\partial x_1} + (x_2 - b)\frac{\partial f}{\partial x_2}$$
$$+\frac{1}{2}\left((x_1-a)^2\frac{\partial^2 f}{\partial x_1^2} + 2(x_1-a)(x_2-b)\frac{\partial^2 f}{\partial x_1 \partial x_2} + (x_2-b)^2\frac{\partial^2 f}{\partial x_2^2}\right) + \ldots,$$

where all derivatives are evaluated at $x_1 = a, \ x_2 = b$.

Hence show that a stationary point (a,b) of $f(x_1,x_2)$ is a local minimum if

$$H_{11} > 0, \quad \det H_{ij} > 0 \ ,$$

where $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ is the Hessian matrix evaluated at (a,b).

Find two local minima of

$$f(x_1,x_2) = x_1^4 - x_1^2 + 2x_1x_2 + x_2^2 \ .$$

3/II/10A **Vector Calculus**

The domain S in the (x,y) plane is bounded by $y=x$, $y=ax\,(0\leqslant a\leqslant 1)$ and $xy^2=1\,(x,y\geqslant 0)$. Find a transformation

$$u=f(x,y),\quad v=g(x,y)\ ,$$

such that S is transformed into a rectangle in the (u,v) plane.

Evaluate

$$\int_D \frac{y^2z^2}{x}\,dx\,dy\,dz\ ,$$

where D is the region bounded by

$$y=x\,,\qquad y=zx,\qquad xy^2=1\qquad (x,y\geqslant 0)$$

and the planes

$$z=0\,,\qquad z=1\,.$$

3/II/11A **Vector Calculus**

Prove that

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}\nabla\cdot\mathbf{b}-\mathbf{b}\nabla\cdot\mathbf{a}+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}.$$

S is an open orientable surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ is any continuously differentiable vector field such that $\mathbf{n}\cdot\mathbf{v}=0$ on S. Let $\mathbf{m}$ be a continuously differentiable unit vector field which coincides with $\mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m}\times\mathbf{v}$, show that

$$\int_S \left(\delta_{ij}-n_in_j\right)\frac{\partial v_i}{\partial x_j}\,dS=\oint_C \mathbf{u}\cdot\mathbf{v}ds\ ,$$

where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}ds=d\mathbf{s}\times\mathbf{n}$. Verify this result by taking $\mathbf{v}=\mathbf{r}$, and S to be the disc $|\mathbf{r}|\leqslant R$ in the $z=0$ plane.

3/II/12A **Vector Calculus**

(a) Show, using Cartesian coordinates, that $\psi = 1/r$ satisfies Laplace's equation, $\nabla^2\psi = 0$, on $\mathbb{R}^3\backslash\{0\}$.

(b) ϕ and ψ are smooth functions defined in a 3-dimensional domain V bounded by a smooth surface S. Show that

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}.$$

(c) Let $\psi = 1/|\mathbf{r}-\mathbf{r}_0|$, and let V_ε be a domain bounded by a smooth outer surface S and an inner surface S_ε, where S_ε is a sphere of radius ε, centre $\mathbf{r}_0$. The function ϕ satisfies

$$\nabla^2\phi = -\rho(\mathbf{r}).$$

Use parts (a) and (b) to show, taking the limit $\varepsilon \to 0$, that ϕ at $\mathbf{r}_0$ is given by

$$4\pi\phi(\mathbf{r}_0) = \int_V \frac{\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}dV + \int_S \left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\frac{\partial\phi}{\partial n} - \phi(\mathbf{r})\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\right)dS,$$

where V is the domain bounded by S.

3/I/3C Vector Calculus

For a real function $f(x, y)$ with $x = x(t)$ and $y = y(t)$ state the chain rule for the derivative $\frac{d}{dt} f(x(t), y(t))$.

By changing variables to u and v, where $u = \alpha(x)y$ and $v = y/x$ with a suitable function $\alpha(x)$ to be determined, find the general solution of the equation

$$x \frac{\partial f}{\partial x} - 2y \frac{\partial f}{\partial y} = 6f \quad .$$

3/I/4A Vector Calculus

Suppose that

$$u = y^2 \sin(xz) + xy^2 z \cos(xz), \quad v = 2xy \sin(xz), \quad w = x^2 y^2 \cos(xz).$$

Show that $u\,dx + v\,dy + w\,dz$ is an exact differential.

Show that

$$\int_{(0,0,0)}^{(\pi/2,1,1)} u\,dx + v\,dy + w\,dz = \frac{\pi}{2}.$$

3/II/9C Vector Calculus

Explain, with justification, how the nature of a critical (stationary) point of a function $f(\mathbf{x})$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$ if H is non-singular. What happens if H is singular?

Let $f(x, y) = (y - x^2)(y - 2x^2) + \alpha x^2$. Find the critical points of f and determine their nature in the different cases that arise according to the values of the parameter $\alpha \in \mathbb{R}$.

3/II/10A Vector Calculus

State the rule for changing variables in a double integral.

Let D be the region defined by

$$\begin{cases} 1/x \le y \le 4x & \text{when } \frac{1}{2} \le x \le 1, \\ x \le y \le 4/x & \text{when } 1 \le x \le 2. \end{cases}$$

Using the transformation $u = y/x$ and $v = xy$, show that

$$\int_D \frac{4xy^3}{x^2+y^2}\, dxdy \;=\; \frac{15}{2} \ln \frac{17}{2}.$$

3/II/11B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \nabla\Omega\, dv \;=\; \int_S \Omega\, d\mathbf{S}\,. \qquad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x, y, z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \le r \le \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$ where a is a constant.

3/II/12B Vector Calculus

In $\mathbb{R}^3$ show that, within a closed surface S, there is at most one solution of Poisson's equation, $\nabla^2\phi = \rho$, satisfying the boundary condition on S

$$\alpha\frac{\partial\phi}{\partial n} + \phi = \gamma\,,$$

where α and γ are functions of position on S, and α is everywhere non-negative.

Show that

$$\phi(x, y) = e^{\pm lx}\sin ly$$

are solutions of Laplace's equation $\nabla^2\phi = 0$ on $\mathbb{R}^2$.

Find a solution $\phi(x, y)$ of Laplace's equation in the region $0 < x < \pi$, $0 < y < \pi$ that satisfies the boundary conditions

$$\begin{array}{cccc} \phi = 0 & \text{on} & 0 < x < \pi & y = 0 \\ \phi = 0 & \text{on} & 0 < x < \pi & y = \pi \\ \phi + \partial\phi/\partial n = 0 & \text{on} & x = 0 & 0 < y < \pi \\ \phi = \sin(ky) & \text{on} & x = \pi & 0 < y < \pi \end{array}$$

where k is a positive integer. Is your solution the only possible solution?