Part IA

Probability

Year

Paper 2, Section I

3F Probability

(a) State and prove Markov's inequality.

(b) Let X be a standard normal random variable. Compute the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$.

(c) Prove that, for all $v > 0$,

$$\int_v^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leqslant e^{-v^2/2}.$$

Paper 2, Section I

4F Probability

A $2k$-*spalindrome* is a sequence of $2k$ digits that contains k distinct digits and reads the same backwards as forwards.

(a) What is the probability that a sequence of $2k$ digits, chosen independently and uniformly at random from $\{0, 1, \ldots, 9\}$, is a $2k$-spalindrome?

(b) Suppose now a sequence of $3k$ digits is chosen independently and uniformly at random from $\{0, 1, \ldots, 9\}$. What is the probability that this longer sequence contains a $2k$-spalindrome? [*Hint: Consider the event that the subsequence starting in position ℓ is a $2k$-spalindrome.*]

Paper 2, Section II

9F Probability

In a group of people, each pair are friends with probability 1/2, and friendships between different pairs of people are independent. Each person's birthday is distributed independently and uniformly among the 365 days of the year. Birthdays are independent of friendships.

The number of people in the group, N, has a Poisson distribution with mean 365.

(a) What is the expectation of the number of pairs of friends with the same birthday?

(b) Let Z_i be the number of people born on the ith day of the year. Find the joint probability mass function of $(Z_1, \ldots, Z_{365})$.

(c) What is the probability that no pair of friends have the same birthday? [You may express your answer in terms of the constant

$$C = \sum_{n=2}^{\infty} \frac{2^{-n(n-1)/2}}{n!} \approx 0.27 \,. \quad]$$

Paper 2, Section II

10F Probability

Let X be a random variable with probability density function

$$f(x) = \frac{x^{n-1}e^{-x}}{(n-1)!} \quad \text{for } x \geqslant 0,$$

where n is a positive integer.

(a) Find the moment generating function $M_X(t)$ for $t < 1$.

(b) Find the mean and variance of X.

(c) Prove that, for every $q \geqslant 0$,

$$\int_0^{n+q\sqrt{n}} \frac{x^{n-1}e^{-x}}{(n-1)!}\, dx \to \Phi(q) \quad \text{as } n \to \infty\,,$$

where Φ is the distribution function of a standard normal random variable. [You may cite any result from the course, provided that it is clearly stated.]

Paper 2, Section II

11F Probability

Let T_1 and T_2 be independent exponential random variables with means λ_1^{-1} and λ_2^{-1}, respectively. Let $V = \min(T_1, T_2)$ and $W = \max(T_1, T_2)$.

(a) Find the distribution of V. What is the probability that $V = T_1$?

(b) Find $\Pr(V \leqslant t \mid V > s)$ for $t > s > 0$.

From now on, suppose that $\lambda_1 = \lambda_2 = \lambda$.

(c) Prove that V and $W - V$ are independent. What is the distribution of $W - V$?

(d) Hence, find the distribution of $2V/(W + V)$.

Paper 2, Section II

12F Probability

Let X be a random variable taking values in $\{0, 1, 2, \dots\}$, with $\Pr(X \geqslant 2) > 0$.

(a) Define the *probability generating function* G_X of X. Show that the first and second derivatives of G_X are positive and non-decreasing on $(0, 1]$.

Now consider a branching process which starts with a population of 1. For each $n \geqslant 1$, each individual in generation n gives rise to an independent number of offspring, distributed as X, which together form generation $n + 1$.

(b) Let d be the probability that the population eventually becomes extinct. Prove that d is the smallest non-negative solution to $t = G_X(t)$.

(c) Let $\mathbb{E}(X) = \mu$. Show that if $\mu > 1$ then $d < 1$.

(d) Suppose that $\mu > 1$ and that X has variance σ^2. Show that for $t \in [0, 1]$,

$$G_X(t) \leqslant 1 - \mu(1 - t) + \frac{1}{2}(\sigma^2 + \mu^2 - \mu)(1 - t)^2.$$

Hence find an upper bound $d^* < 1$ for the extinction probability d, where d^* is given in terms of μ and σ^2.

Paper 2, Section I
3F Probability

What does it mean to say a function is *convex*? State Jensen's inequality for a convex function f and an integrable random variable X.

Let $x_1, \ldots, x_n$ be positive real numbers. Show that

$$\frac{\sum_{i=1}^n x_i \log x_i}{\sum_{i=1}^n x_i} \geqslant \log\left(\frac{\sum_{i=1}^n x_i}{n}\right).$$

[You may use without proof a standard sufficient condition for convexity if it is stated carefully.]

Paper 2, Section I
4F Probability

Let X be a random variable with mean μ and variance σ^2. Let

$$G(a) = \mathbb{E}\big[(X-a)^2\big].$$

Show that $G(a) \geqslant \sigma^2$ for all a. For what value of a is there equality?

Let

$$H(a) = \mathbb{E}\big[|X-a|\big].$$

Supposing that X is a continuous random variable with probability density function f, express $H(a)$ in terms of f. Show that H is minimised for a such that $\int_{-\infty}^a f(x)dx = 1/2$.

Paper 2, Section II
9F Probability

(a) Let U and V be two *bounded* random variables such that $\mathbb{E}[U^k] = \mathbb{E}[V^k]$ for all non-negative integers k. Show that U and V have the same moment generating function.

(b) Let X be a continuous random variable with probability density function

$$f(x) = Ae^{-x^2/2}$$

for all real x, where A is a normalising constant. Compute the moment generating function of X.

(c) Let Y be a discrete random variable with probability mass function

$$\mathbb{P}(Y = n) = Be^{-n^2/2}$$

for all integers n, where B is a normalising constant. Show that

$$\mathbb{E}[e^{kY}] = \mathbb{E}[e^{kX}]$$

for all integers k, where X is a standard normal random variable.

(d) Let U and V be *unbounded* random variables such that U^k and V^k are integrable and $\mathbb{E}[U^k] = \mathbb{E}[V^k]$ for all non-negative integers k. Does it follow that U and V have the same distribution?

Paper 2, Section II
10F Probability

(a) Let X be a random variable valued in $\{1, 2, \ldots\}$ and let G_X be its probability generating function. Show that

$$\mathbb{P}(X = n) = \frac{G_X^{(n)}(0)}{n!}$$

where $G_X^{(n)}$ denotes the nth derivative of G_X.

(b) Let Y be another random variable valued in $\{1, 2, \ldots\}$, independent of X. Prove that $G_{X+Y}(s) = G_X(s)G_Y(s)$ for all $0 \leqslant s \leqslant 1$.

(c) Compute G_X in the case where X is a geometric random variable taking values in $\{1, 2, \ldots\}$ with $\mathbb{P}(X = 1) = p$ for a given constant $0 < p \leqslant 1$.

(d) A jar contains n marbles. Initially, all of the marbles are red. Every minute, a marble is drawn at random from the jar, and then replaced with a blue marble. Let T be the number of minutes until the jar contains only blue marbles. Compute the probability generating function G_T.

Paper 2, Section II
11F Probability

Consider a coin that is biased such that when tossed the probability of heads is p and tails is $1-p$.

(a) Suppose that the coin was tossed n times. What is the probability that the coin came up heads exactly k times?

(b) Suppose that the coin was tossed n times. Given that the coin came up heads exactly k times, what is the probability that the coin came up heads k times in a row?

(c) Suppose that the coin was tossed repeatedly until heads came up k times. What is the probability that the total number of tosses was n?

(d) Suppose that the coin was tossed repeatedly until heads came up k times in a row. Find the expected number of tosses.

Paper 2, Section II
12F Probability

Let $A_1, A_2, \ldots$ be a collection of events. Let $N = \sum_{n \geqslant 1} \mathbf{1}_{A_n}$ be the random variable that counts how many of these events occur. Note that N takes values in $\{0, 1, \ldots\} \cup \{\infty\}$.

(a) By considering the quantity $\mathbb{E}(N)$, show that if $\sum_{n \geqslant 1} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(\text{an infinite number of the events occur}) = 0$.

(b) Suppose now that the events are independent. Show the inequality $\mathbb{E}(2^{-N}) \leqslant e^{-\frac{1}{2}\mathbb{E}(N)}$, with the convention that $2^{-\infty} = 0$. [*Hint: use the inequality* $1 - x \leqslant e^{-x}$ *for all* x.]

(c) Again suppose that the events are independent. Show that if $\sum_{n \geqslant 1} \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(\text{an infinite number of the events occur}) = 1$.

(d) A monkey types by randomly striking keys on a 26-letter keyboard, with each letter of the alphabet equally likely to be struck and the keystrokes independent. Show that with probability one, the word HELLO appears infinitely often.

Paper 2, Section I
3D Probability

A coin has probability p of landing heads. Let q_n be the probability that the number of heads after n tosses is even. Give an expression for q_{n+1} in terms of q_n. Hence, or otherwise, find q_n.

Paper 2, Section I
4F Probability

Let X be a continuous random variable taking values in $[0, \sqrt{3}]$. Let the probability density function of X be

$$f_X(x) = \frac{c}{1+x^2}\,, \quad \text{for } x \in [0, \sqrt{3}],$$

where c is a constant.

Find the value of c and calculate the mean, variance and median of X.

[*Recall that the median of X is the number m such that $\mathbb{P}(X \leqslant m) = \frac{1}{2}$.*]

Paper 2, Section II

9E Probability

(a) (i) Define the *conditional probability* $\mathbb{P}(A|B)$ of the event A given the event B. Let $\{B_j : 1 \leqslant j \leqslant n\}$ be a partition of the sample space such that $\mathbb{P}(B_j) > 0$ for all j. Show that, if $\mathbb{P}(A) > 0$,

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{k=1}^{n} \mathbb{P}(A|B_k)\mathbb{P}(B_k)}.$$

(ii) There are n urns, the rth of which contains $r-1$ red balls and $n-r$ blue balls. Alice picks an urn (uniformly) at random and removes two balls without replacement. Find the probability that the first ball is blue, and the conditional probability that the second ball is blue, given that the first is blue. [You may assume, if you wish, that $\sum_{i=1}^{n-1} i(i-1) = \frac{1}{3}n(n-1)(n-2)$.]

(b) (i) What is meant by saying that two events A and B are *independent*? Two fair (6-sided) dice are rolled. Let A_t be the event that the sum of the numbers shown is t, and let B_i be the event that the first die shows i. For what values of t and i are the two events A_t and B_i independent?

(ii) The casino at Monte Corona features the following game: three coins each show heads with probability $3/5$ and tails otherwise. The first counts 10 points for a head and 2 for a tail; the second counts 4 points for both a head and a tail; and the third counts 3 points for a head and 20 for a tail. You and your opponent each choose a coin. You cannot both choose the same coin. Each of you tosses your coin once and the person with the larger score wins the jackpot. Would you prefer to be the first or the second to choose a coin?

Paper 2, Section II
10E Probability

(a) Alanya repeatedly rolls a fair six-sided die. What is the probability that the first number she rolls is a 1, given that she rolls a 1 before she rolls a 6?

(b) Let $(X_n)_{n\geqslant 0}$ be a simple symmetric random walk on the integers starting at $x \in \mathbb{Z}$, that is,

$$X_n = \begin{cases} x & \text{if } n = 0 \\ x + \sum_{i=1}^{n} Y_i & \text{if } n \geqslant 1 \end{cases},$$

where $(Y_n)_{n\geqslant 1}$ is a sequence of IID random variables with $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = \frac{1}{2}$. Let $T = \min\{n \geqslant 0 : X_n = 0\}$ be the time that the walk first hits 0.

(i) Let n be a positive integer. For $0 < x < n$, calculate the probability that the walk hits 0 before it hits n.

(ii) Let $x = 1$ and let A be the event that the walk hits 0 before it hits 3. Find $\mathbb{P}(X_1 = 0|A)$. Hence find $\mathbb{E}(T|A)$.

(iii) Let $x = 1$ and let B be the event that the walk hits 0 before it hits 4. Find $\mathbb{E}(T|B)$.

Paper 2, Section II
11D Probability

Let Δ be the disc of radius 1 with centre at the origin O. Let P be a random point uniformly distributed in Δ. Let (R, Θ) be the polar coordinates of P. Show that R and Θ are independent and find their probability density functions f_R and f_Θ.

Let A, B and C be three random points selected independently and uniformly in Δ. Find the expected area of triangle OAB and hence find the probability that C lies in the interior of triangle OAB.

Find the probability that O, A, B and C are the vertices of a convex quadrilateral.

Paper 2, Section II
12F Probability

State and prove Chebyshev's inequality.

Let $(X_i)_{i \geqslant 1}$ be a sequence of independent, identically distributed random variables such that

$$\mathbb{P}(X_i = 0) = p \text{ and } \mathbb{P}(X_i = 1) = 1 - p$$

for some $p \in [0, 1]$, and let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

(i) Prove that

$$B_n(p) := \mathbb{E}\left(f\left(\frac{X_1 + \cdots + X_n}{n}\right)\right)$$

is a polynomial function of p, for any natural number n.

(ii) Let $\delta > 0$. Prove that

$$\sum_{k \in K_\delta} \binom{n}{k} p^k (1 - p)^{n-k} \leqslant \frac{1}{4n\delta^2} ,$$

where K_δ is the set of natural numbers $0 \leqslant k \leqslant n$ such that $|k/n - p| > \delta$.

(iii) Show that

$$\sup_{p \in [0,1]} |f(p) - B_n(p)| \to 0$$

as $n \to \infty$. [You may use without proof that, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| \leqslant \epsilon$ for all $x, y \in [0, 1]$ with $|x - y| \leqslant \delta$.]

Paper 1, Section I

4F Probability

A robot factory begins with a single generation-0 robot. Each generation-n robot independently builds some number of generation-$(n+1)$ robots before breaking down. The number of generation-$(n+1)$ robots built by a generation-n robot is 0, 1, 2 or 3 with probabilities $\frac{1}{12}$, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{12}$ respectively. Find the expectation of the total number of generation-n robots produced by the factory. What is the probability that the factory continues producing robots forever?

[*Standard results about branching processes may be used without proof as long as they are carefully stated.*]

Paper 1, Section II

11F Probability

Let $A_1, A_2, \ldots, A_n$ be events in some probability space. State and prove the inclusion-exclusion formula for the probability $\mathbb{P}(\bigcup_{i=1}^{n} A_i)$. Show also that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \geqslant \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j).$$

Suppose now that $n \geqslant 2$ and that whenever $i \neq j$ we have $\mathbb{P}(A_i \cap A_j) \leqslant 1/n$. Show that there is a constant c independent of n such that $\sum_{i=1}^{n} \mathbb{P}(A_i) \leqslant c\sqrt{n}$.

Paper 1, Section II

12F Probability

(a) Let Z be a $N(0,1)$ random variable. Write down the probability density function (pdf) of Z, and verify that it is indeed a pdf. Find the moment generating function (mgf) $m_Z(\theta) = \mathbb{E}(e^{\theta Z})$ of Z and hence, or otherwise, verify that Z has mean 0 and variance 1.

(b) Let $(X_n)_{n\geqslant 1}$ be a sequence of IID $N(0,1)$ random variables. Let $S_n = \sum_{i=1}^{n} X_i$ and let $U_n = S_n/\sqrt{n}$. Find the distribution of U_n.

(c) Let $Y_n = X_n^2$. Find the mean μ and variance σ^2 of Y_1. Let $T_n = \sum_{i=1}^{n} Y_i$ and let $V_n = (T_n - n\mu)/\sigma\sqrt{n}$.

If $(W_n)_{n\geqslant 1}$ is a sequence of random variables and W is a random variable, what does it mean to say that $W_n \to W$ in distribution? State carefully the continuity theorem and use it to show that $V_n \to Z$ in distribution.

[You may **not** assume the central limit theorem.]

Paper 2, Section I
3F Probability

(a) Prove that $\log(n!) \sim n \log n$ as $n \to \infty$.

(b) State Stirling's approximation for $n!$.

(c) A school party of n boys and n girls travel on a red bus and a green bus. Each bus can hold n children. The children are distributed at random between the buses.

Let A_n be the event that the boys all travel on the red bus and the girls all travel on the green bus. Show that

$$\mathbb{P}(A_n) \sim \frac{\sqrt{\pi n}}{4^n} \text{ as } n \to \infty.$$

Paper 2, Section I
4F Probability

Let X and Y be independent exponential random variables each with parameter 1. Write down the joint density function of X and Y.

Let $U = 6X + 8Y$ and $V = 2X + 3Y$. Find the joint density function of U and V.

Are U and V independent? Briefly justify your answer.

Paper 2, Section II

9F Probability

(a) State the axioms that must be satisfied by a probability measure $\mathbb{P}$ on a probability space Ω.

Let A and B be events with $\mathbb{P}(B) > 0$. Define the conditional probability $\mathbb{P}(A|B)$.

Let B_1, B_2, … be pairwise disjoint events with $\mathbb{P}(B_i) > 0$ for all i and $\Omega = \cup_{i=1}^{\infty} B_i$. Starting from the axioms, show that

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

and deduce Bayes' theorem.

(b) Two identical urns contain white balls and black balls. Urn I contains 45 white balls and 30 black balls. Urn II contains 12 white balls and 36 black balls. You do not know which urn is which.

(i) Suppose you select an urn and draw one ball at random from it. The ball is white. What is the probability that you selected Urn I?

(ii) Suppose instead you draw one ball at random from each urn. One of the balls is white and one is black. What is the probability that the white ball came from Urn I?

(c) Now suppose there are n identical urns containing white balls and black balls, and again you do not know which urn is which. Each urn contains 1 white ball. The ith urn contains $2^i - 1$ black balls ($1 \leqslant i \leqslant n$). You select an urn and draw one ball at random from it. The ball is white. Let $p(n)$ be the probability that if you replace this ball and again draw a ball at random from the same urn then the ball drawn on the second occasion is also white. Show that $p(n) \to \frac{1}{3}$ as $n \to \infty$.

Paper 2, Section II

10F Probability

Let m and n be positive integers with $n > m > 0$ and let $p \in (0,1)$ be a real number. A random walk on the integers starts at m. At each step, the walk moves up 1 with probability p and down 1 with probability $q = 1-p$. Find, with proof, the probability that the walk hits n before it hits 0.

Patricia owes a very large sum £$2(N!)$ of money to a member of a violent criminal gang. She must return the money this evening to avoid terrible consequences but she only has £$N!$. She goes to a casino and plays a game with the probability of her winning being $\frac{18}{37}$. If she bets £a on the game and wins then her £a is returned along with a further £a; if she loses then her £a is lost.

The rules of the casino allow Patricia to play the game repeatedly until she runs out of money. She may choose the amount £a that she bets to be any integer a with $1 \leqslant a \leqslant N$, but it must be the same amount each time. What choice of a would be best and why?

What choice of a would be best, and why, if instead the probability of her winning the game is $\frac{19}{37}$?

Paper 2, Section II

11F Probability

Recall that a random variable X in $\mathbb{R}^2$ is *bivariate normal* or *Gaussian* if $u^T X$ is normal for all $u \in \mathbb{R}^2$. Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be bivariate normal.

(a) (i) Show that if A is a 2×2 real matrix then AX is bivariate normal.

(ii) Let $\mu = \mathbb{E}(X)$ and $V = \mathrm{Var}\,(X) = \mathbb{E}[(X-\mu)(X-\mu)^T]$. Find the moment generating function $M_X(\lambda) = \mathbb{E}(e^{\lambda^T X})$ of X and deduce that the distribution of a bivariate normal random variable X is uniquely determined by μ and V.

(iii) Let $\mu_i = \mathbb{E}(X_i)$ and $\sigma_i^2 = \mathrm{Var}\,(X_i)$ for $i = 1, 2$. Let $\rho = \frac{\mathrm{Cov}\,(X_1, X_2)}{\sigma_1 \sigma_2}$ be the correlation of X_1 and X_2. Write down V in terms of some or all of μ_1, μ_2, σ_1, σ_2 and ρ. If $\mathrm{Cov}\,(X_1, X_2) = 0$, why must X_1 and X_2 be independent?

For each $a \in \mathbb{R}$, find $\mathrm{Cov}\,(X_1, X_2 - aX_1)$. Hence show that $X_2 = aX_1 + Y$ for some normal random variable Y in $\mathbb{R}$ that is independent of X_1 and some $a \in \mathbb{R}$ that should be specified.

(b) A certain species of East Anglian goblin has left arm of mean length 100cm with standard deviation 1cm, and right arm of mean length 102cm with standard deviation 2cm. The correlation of left- and right-arm-length of a goblin is $\frac{1}{2}$. You may assume that the distribution of left- and right-arm-lengths can be modelled by a bivariate normal distribution. What is the probability that a randomly selected goblin has longer right arm than left arm?

[You may give your answer in terms of the distribution function Φ of a $N(0,1)$ random variable Z. That is, $\Phi(t) = \mathbb{P}(Z \leqslant t)$.]

Paper 2, Section II

12F Probability

Let A_1, A_2, $\ldots$, A_n be events in some probability space. Let X be the number of A_i that occur (so X is a random variable). Show that

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

and

$$\mathrm{Var}\,(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} \big(\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)\big).$$

[*Hint: Write* $X = \sum_{i=1}^{n} X_i$ *where* $X_i = \begin{cases} 1 & \textit{if } A_i \textit{ occurs} \\ 0 & \textit{if not} \end{cases}$.]

A collection of n lightbulbs are arranged in a circle. Each bulb is on independently with probability p. Let X be the number of bulbs such that both that bulb and the next bulb clockwise are on. Find $\mathbb{E}(X)$ and $\mathrm{Var}\,(X)$.

Let B be the event that there is at least one pair of adjacent bulbs that are both on.

Use Markov's inequality to show that if $p = n^{-0.6}$ then $\mathbb{P}(B) \to 0$ as $n \to \infty$.

Use Chebychev's inequality to show that if $p = n^{-0.4}$ then $\mathbb{P}(B) \to 1$ as $n \to \infty$.

Paper 2, Section I

3F Probability

Let X and Y be independent Poisson random variables with parameters λ and μ respectively.

(i) Show that $X + Y$ is Poisson with parameter $\lambda + \mu$.

(ii) Show that the conditional distribution of X given $X + Y = n$ is binomial, and find its parameters.

Paper 2, Section I

4F Probability

(a) State the Cauchy–Schwarz inequality and Markov's inequality. State and prove Jensen's inequality.

(b) For a discrete random variable X, show that $\mathrm{Var}(X) = 0$ implies that X is constant, i.e. there is $x \in \mathbb{R}$ such that $\mathbb{P}(X = x) = 1$.

Paper 2, Section II
9F Probability

(a) Let Y and Z be independent discrete random variables taking values in sets S_1 and S_2 respectively, and let $F : S_1 \times S_2 \to \mathbb{R}$ be a function.

Let $E(z) = \mathbb{E}F(Y, z)$. Show that

$$\mathbb{E}E(Z) = \mathbb{E}F(Y, Z) .$$

Let $V(z) = \mathbb{E}(F(Y, z)^2) - (\mathbb{E}F(Y, z))^2$. Show that

$$\mathrm{Var}F(Y, Z) = \mathbb{E}V(Z) + \mathrm{Var}E(Z) .$$

(b) Let $X_1, \ldots, X_n$ be independent Bernoulli(p) random variables. For any function $F : \{0, 1\} \to \mathbb{R}$, show that

$$\mathrm{Var}F(X_1) = p(1 - p)(F(1) - F(0))^2 .$$

Let $\{0, 1\}^n$ denote the set of all 0-1 sequences of length n. By induction, or otherwise, show that for any function $F : \{0, 1\}^n \to \mathbb{R}$,

$$\mathrm{Var}F(X) \leqslant p(1 - p) \sum_{i=1}^{n} \mathbb{E}((F(X) - F(X^i))^2)$$

where $X = (X_1, \ldots, X_n)$ and $X^i = (X_1, \ldots, X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$.

Paper 2, Section II
10F Probability

(a) Let X and Y be independent random variables taking values ± 1, each with probability $\frac{1}{2}$, and let $Z = XY$. Show that X, Y and Z are pairwise independent. Are they independent?

(b) Let X and Y be discrete random variables with mean 0, variance 1, covariance ρ. Show that $\mathbb{E}\max\{X^2, Y^2\} \leqslant 1 + \sqrt{1 - \rho^2}$.

(c) Let X_1, X_2, X_3 be discrete random variables. Writing $a_{ij} = \mathbb{P}(X_i > X_j)$, show that $\min\{a_{12}, a_{23}, a_{31}\} \leqslant \frac{2}{3}$.

Paper 2, Section II
11F Probability

(a) Consider a Galton–Watson process (X_n). Prove that the extinction probability q is the smallest non-negative solution of the equation $q = F(q)$ where $F(t) = \mathbb{E}(t^{X_1})$. [You should prove any properties of Galton–Watson processes that you use.]

In the case of a Galton–Watson process with

$$\mathbb{P}(X_1 = 1) = 1/4, \quad \mathbb{P}(X_1 = 3) = 3/4,$$

find the mean population size and compute the extinction probability.

(b) For each $n \in \mathbb{N}$, let Y_n be a random variable with distribution Poisson(n). Show that

$$\frac{Y_n - n}{\sqrt{n}} \to Z$$

in distribution, where Z is a standard normal random variable.

Deduce that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Paper 2, Section II
12F Probability

For a symmetric simple random walk (X_n) on $\mathbb{Z}$ starting at 0, let $M_n = \max_{i \leqslant n} X_i$.

(i) For $m \geqslant 0$ and $x \in \mathbb{Z}$, show that

$$\mathbb{P}(M_n \geqslant m, X_n = x) = \begin{cases} \mathbb{P}(X_n = x) & \text{if } x \geqslant m \\ \mathbb{P}(X_n = 2m - x) & \text{if } x < m. \end{cases}$$

(ii) For $m \geqslant 0$, show that $\mathbb{P}(M_n \geqslant m) = \mathbb{P}(X_n = m) + 2\sum_{x>m} \mathbb{P}(X_n = x)$ and that

$$\mathbb{P}(M_n = m) = \mathbb{P}(X_n = m) + \mathbb{P}(X_n = m + 1).$$

(iii) Prove that $\mathbb{E}(M_n^2) < \mathbb{E}(X_n^2)$.

Paper 2, Section I

3F Probability

Let X be a non-negative integer-valued random variable such that $0 < \mathbb{E}(X^2) < \infty$. Prove that

$$\frac{\mathbb{E}(X)^2}{\mathbb{E}(X^2)} \leqslant \mathbb{P}(X > 0) \leqslant \mathbb{E}(X).$$

[You may use any standard inequality.]

Paper 2, Section I

4F Probability

Let X and Y be real-valued random variables with joint density function

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & \text{if } x \geqslant 0 \text{ and } y \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the conditional probability density function of Y given X.

(ii) Find the expectation of Y given X.

Paper 2, Section II

9F Probability

For a positive integer N, $p \in [0,1]$, and $k \in \{0,1,\dots,N\}$, let

$$p_k(N,p) = \binom{N}{k} p^k (1-p)^{N-k}.$$

(a) For fixed N and p, show that $p_k(N,p)$ is a probability mass function on $\{0,1,\dots,N\}$ and that the corresponding probability distribution has mean Np and variance $Np(1-p)$.

(b) Let $\lambda > 0$. Show that, for any $k \in \{0,1,2,\dots\}$,

$$\lim_{N\to\infty} p_k(N,\lambda/N) = \frac{e^{-\lambda}\lambda^k}{k!}. \qquad (*)$$

Show that the right-hand side of $(*)$ is a probability mass function on $\{0,1,2,\dots\}$.

(c) Let $p \in (0,1)$ and let $a, b \in \mathbb{R}$ with $a < b$. For all N, find integers $k_a(N)$ and $k_b(N)$ such that

$$\sum_{k=k_a(N)}^{k_b(N)} p_k(N,p) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\,dx \qquad \text{as } N \to \infty.$$

[You may use the Central Limit Theorem.]

Paper 2, Section II

10F Probability

(a) For any random variable X and $\lambda > 0$ and $t > 0$, show that

$$\mathbb{P}(X > t) \leqslant \mathbb{E}(e^{\lambda X})e^{-\lambda t}.$$

For a standard normal random variable X, compute $\mathbb{E}(e^{\lambda X})$ and deduce that

$$\mathbb{P}(X > t) \leqslant e^{-\frac{1}{2}t^2}.$$

(b) Let $\mu, \lambda > 0$, $\mu \neq \lambda$. For independent random variables X and Y with distributions $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\mu)$, respectively, compute the probability density functions of $X + Y$ and $\min\{X, Y\}$.

Paper 2, Section II

11F Probability

Let $\beta > 0$. The *Curie–Weiss Model* of ferromagnetism is the probability distribution defined as follows. For $n \in \mathbb{N}$, define random variables $S_1, \ldots, S_n$ with values in $\{\pm 1\}$ such that the probabilities are given by

$$\mathbb{P}(S_1 = s_1, \ldots, S_n = s_n) = \frac{1}{Z_{n,\beta}} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j\right)$$

where $Z_{n,\beta}$ is the normalisation constant

$$Z_{n,\beta} = \sum_{s_1 \in \{\pm 1\}} \cdots \sum_{s_n \in \{\pm 1\}} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j\right).$$

(a) Show that $\mathbb{E}(S_i) = 0$ for any i.

(b) Show that $\mathbb{P}(S_2 = +1 | S_1 = +1) \geqslant \mathbb{P}(S_2 = +1)$. [You may use $\mathbb{E}(S_i S_j) \geqslant 0$ for all i, j without proof.]

(c) Let $M = \frac{1}{n} \sum_{i=1}^{n} S_i$. Show that M takes values in $E_n = \{-1 + \frac{2k}{n} : k = 0, \ldots, n\}$, and that for each $m \in E_n$ the number of possible values of $(S_1, \ldots, S_n)$ such that $M = m$ is

$$\frac{n!}{\left(\frac{1+m}{2} n\right)! \left(\frac{1-m}{2} n\right)!}.$$

Find $\mathbb{P}(M = m)$ for any $m \in E_n$.

Paper 2, Section II
12F Probability

(a) Let $k \in \{1, 2, \dots\}$. For $j \in \{0, \dots, k+1\}$, let D_j be the first time at which a simple symmetric random walk on $\mathbb{Z}$ with initial position j at time 0 hits 0 or $k+1$. Show $\mathbb{E}(D_j) = j(k+1-j)$. [If you use a recursion relation, you do not need to prove that its solution is unique.]

(b) Let (S_n) be a simple symmetric random walk on $\mathbb{Z}$ starting at 0 at time $n = 0$. For $k \in \{1, 2, \dots\}$, let T_k be the first time at which (S_n) has visited k distinct vertices. In particular, $T_1 = 0$. Show $\mathbb{E}(T_{k+1} - T_k) = k$ for $k \geqslant 1$. [You may use without proof that, conditional on $S_{T_k} = i$, the random variables $(S_{T_k+n})_{n \geqslant 0}$ have the distribution of a simple symmetric random walk starting at i.]

(c) For $n \geqslant 3$, let $\mathbb{Z}_n$ be the circle graph consisting of vertices $0, \dots, n-1$ and edges between k and $k+1$ where n is identified with 0. Let (Y_i) be a simple random walk on $\mathbb{Z}_n$ starting at time 0 from 0. Thus $Y_0 = 0$ and conditional on Y_i the random variable Y_{i+1} is $Y_i \pm 1$ with equal probability (identifying $k+n$ with k).

The *cover time* T of the simple random walk on $\mathbb{Z}_n$ is the first time at which the random walk has visited all vertices. Show that $\mathbb{E}(T) = n(n-1)/2$.

Paper 2, Section I

3F Probability

Let $X_1, \ldots, X_n$ be independent random variables, all with uniform distribution on $[0, 1]$. What is the probability of the event $\{X_1 > X_2 > \cdots > X_{n-1} > X_n\}$?

Paper 2, Section I

4F Probability

Define the *moment-generating function* m_Z of a random variable Z. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with distribution $\mathcal{N}(0, 1)$, and let $Z = X_1^2 + \cdots + X_n^2$. For $\theta < 1/2$, show that

$$m_Z(\theta) = (1 - 2\theta)^{-n/2}\,.$$

Paper 2, Section II

9F Probability

For any positive integer n and positive real number θ, the Gamma distribution $\Gamma(n, \theta)$ has density f_Γ defined on $(0, \infty)$ by

$$f_\Gamma(x) = \frac{\theta^n}{(n-1)!} x^{n-1} e^{-\theta x}\,.$$

For any positive integers a and b, the Beta distribution $B(a, b)$ has density f_B defined on $(0, 1)$ by

$$f_B(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1} (1-x)^{b-1}\,.$$

Let X and Y be independent random variables with respective distributions $\Gamma(n, \theta)$ and $\Gamma(m, \theta)$. Show that the random variables $X/(X+Y)$ and $X+Y$ are independent and give their distributions.

Paper 2, Section II

10F Probability

We randomly place n balls in m bins independently and uniformly. For each i with $1 \leqslant i \leqslant m$, let B_i be the number of balls in bin i.

(a) What is the distribution of B_i? For $i \neq j$, are B_i and B_j independent?

(b) Let E be the number of empty bins, C the number of bins with two or more balls, and S the number of bins with exactly one ball. What are the expectations of E, C and S?

(c) Let $m = an$, for an integer $a \geqslant 2$. What is $\mathbb{P}(E = 0)$? What is the limit of $\mathbb{E}[E]/m$ when $n \to \infty$?

(d) Instead, let $n = dm$, for an integer $d \geqslant 2$. What is $\mathbb{P}(C = 0)$? What is the limit of $\mathbb{E}[C]/m$ when $n \to \infty$?

Paper 2, Section II

11F Probability

Let X be a non-negative random variable such that $\mathbb{E}[X^2] > 0$ is finite, and let $\theta \in [0, 1]$.

(a) Show that

$$\mathbb{E}[X\,\mathbb{I}[\{X > \theta\mathbb{E}[X]\}]] \geqslant (1 - \theta)\mathbb{E}[X]\,.$$

(b) Let Y_1 and Y_2 be random variables such that $\mathbb{E}[Y_1^2]$ and $\mathbb{E}[Y_2^2]$ are finite. State and prove the Cauchy–Schwarz inequality for these two variables.

(c) Show that

$$\mathbb{P}(X > \theta\,\mathbb{E}[X]) \geqslant (1 - \theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}\,.$$

Paper 2, Section II

12F Probability

A random graph with n nodes $v_1, \ldots, v_n$ is drawn by placing an edge with probability p between v_i and v_j for all distinct i and j, independently. A triangle is a set of three distinct nodes v_i, v_j, v_k that are all connected: there are edges between v_i and v_j, between v_j and v_k and between v_i and v_k.

(a) Let T be the number of triangles in this random graph. Compute the maximum value and the expectation of T.

(b) State the Markov inequality. Show that if $p = 1/n^{\alpha}$, for some $\alpha > 1$, then $\mathbb{P}(T = 0) \to 1$ when $n \to \infty$.

(c) State the Chebyshev inequality. Show that if p is such that $\mathrm{Var}[T]/\mathbb{E}[T]^2 \to 0$ when $n \to \infty$, then $\mathbb{P}(T = 0) \to 0$ when $n \to \infty$.

Paper 2, Section I

3F Probability

Let U be a uniform random variable on $(0,1)$, and let $\lambda > 0$.

(a) Find the distribution of the random variable $-(\log U)/\lambda$.

(b) Define a new random variable X as follows: suppose a fair coin is tossed, and if it lands heads we set $X = U^2$ whereas if it lands tails we set $X = 1 - U^2$. Find the probability density function of X.

Paper 2, Section I

4F Probability

Let A, B be events in the sample space Ω such that $0 < P(A) < 1$ and $0 < P(B) < 1$. The event B is said to *attract* A if the conditional probability $P(A|B)$ is greater than $P(A)$, otherwise it is said that A *repels* B. Show that if B attracts A, then A attracts B. Does $B^c = \Omega \setminus B$ repel A?

Paper 2, Section II

9F Probability

Lionel and Cristiana have a and b million pounds, respectively, where $a, b \in \mathbb{N}$. They play a series of independent football games in each of which the winner receives one million pounds from the loser (a draw cannot occur). They stop when one player has lost his or her entire fortune. Lionel wins each game with probability $0 < p < 1$ and Cristiana wins with probability $q = 1 - p$, where $p \neq q$. Find the expected number of games before they stop playing.

Paper 2, Section II

10F Probability

Consider the function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Show that ϕ defines a probability density function. If a random variable X has probability density function ϕ, find the moment generating function of X, and find all moments $E[X^k]$, $k \in \mathbb{N}$.

Now define

$$r(x) = \frac{P(X > x)}{\phi(x)}.$$

Show that for every $x > 0$,

$$\frac{1}{x} - \frac{1}{x^3} < r(x) < \frac{1}{x}.$$

Paper 2, Section II

11F Probability

State and prove Markov's inequality and Chebyshev's inequality, and deduce the weak law of large numbers.

If X is a random variable with mean zero and finite variance σ^2, prove that for any $a > 0$,

$$P(X \geqslant a) \leqslant \frac{\sigma^2}{\sigma^2 + a^2}.$$

[*Hint: Show first that* $P(X \geqslant a) \leqslant P\left((X+b)^2 \geqslant (a+b)^2\right)$ *for every* $b > 0$.]

Paper 2, Section II

12F Probability

When coin A is tossed it comes up heads with probability $\frac{1}{4}$, whereas coin B comes up heads with probability $\frac{3}{4}$. Suppose one of these coins is randomly chosen and is tossed twice. If both tosses come up heads, what is the probability that coin B was tossed? Justify your answer.

In each draw of a lottery, an integer is picked independently at random from the first n integers $1, 2, \ldots, n$, with replacement. What is the probability that in a sample of r successive draws the numbers are drawn in a non-decreasing sequence? Justify your answer.

Paper 2, Section I

3F Probability

Consider a particle situated at the origin $(0,0)$ of $\mathbb{R}^2$. At successive times a direction is chosen independently by picking an angle uniformly at random in the interval $[0,2\pi]$, and the particle then moves an Euclidean unit length in this direction. Find the expected squared Euclidean distance of the particle from the origin after n such movements.

Paper 2, Section I

4F Probability

Consider independent discrete random variables $X_1,\ldots,X_n$ and assume $E[X_i]$ exists for all $i=1,\ldots,n$.

Show that

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i].$$

If the $X_1,\ldots,X_n$ are also positive, show that

$$\prod_{i=1}^{n}\sum_{m=0}^{\infty} P(X_i > m) = \sum_{m=0}^{\infty} P\left(\prod_{i=1}^{n} X_i > m\right).$$

Paper 2, Section II

9F Probability

State the axioms of probability.

State and prove Boole's inequality.

Suppose you toss a sequence of coins, the i-th of which comes up heads with probability p_i, where $\sum_{i=1}^{\infty} p_i < \infty$. Calculate the probability of the event that infinitely many heads occur.

Suppose you repeatedly and independently roll a pair of fair dice and each time record the sum of the dice. What is the probability that an outcome of 5 appears before an outcome of 7? Justify your answer.

Paper 2, Section II

10F Probability

Define what it means for a random variable X to have a Poisson distribution, and find its moment generating function.

Suppose X, Y are independent Poisson random variables with parameters λ, μ. Find the distribution of $X+Y$.

If $X_1, \ldots, X_n$ are independent Poisson random variables with parameter $\lambda = 1$, find the distribution of $\sum_{i=1}^{n} X_i$. Hence or otherwise, find the limit of the real sequence

$$a_n = e^{-n} \sum_{j=0}^{n} \frac{n^j}{j!}, \quad n \in \mathbb{N}.$$

[Standard results may be used without proof provided they are clearly stated.]

Paper 2, Section II

11F Probability

For any function $g\colon \mathbb{R} \to \mathbb{R}$ and random variables X, Y, the "tower property" of conditional expectations is

$$E[g(X)] = E\big[E[g(X)|Y]\big].$$

Provide a proof of this property when both X, Y are discrete.

Let $U_1, U_2, \ldots$ be a sequence of independent uniform $U(0,1)$-random variables. For $x \in [0,1]$ find the expected number of U_i's needed such that their sum exceeds x, that is, find $E[N(x)]$ where

$$N(x) = \min \left\{ n : \sum_{i=1}^{n} U_i > x \right\}.$$

[*Hint: Write* $E[N(x)] = E\big[E[N(x)|U_1]\big]$.]

Paper 2, Section II

12F Probability

Give the definition of an exponential random variable X with parameter λ. Show that X is memoryless.

Now let X, Y be independent exponential random variables, each with parameter λ. Find the probability density function of the random variable $Z = \min(X, Y)$ and the probability $P(X > Y)$.

Suppose the random variables G_1, G_2 are independent and each has probability density function given by

$$f(y) = C^{-1} e^{-y} y^{-1/2}, \quad y > 0, \quad \text{where } C = \int_0^\infty e^{-y} y^{-1/2} dy.$$

Find the probability density function of $G_1 + G_2$. [You may use standard results without proof provided they are clearly stated.]

Paper 2, Section I
3F Probability

Let X be a random variable with mean μ and variance σ^2. Let

$$G(a) = \mathbb{E}\big[(X-a)^2\big].$$

Show that $G(a) \geqslant \sigma^2$ for all a. For what value of a is there equality?

Let

$$H(a) = \mathbb{E}\big[|X-a|\big].$$

Supposing that X has probability density function f, express $H(a)$ in terms of f. Show that H is minimised when a is such that $\int_{-\infty}^{a} f(x)dx = 1/2$.

Paper 2, Section I
4F Probability

(i) Let X be a random variable. Use Markov's inequality to show that

$$\mathbb{P}(X \geqslant k) \leqslant \mathbb{E}(e^{tX})e^{-kt}$$

for all $t \geqslant 0$ and real k.

(ii) Calculate $\mathbb{E}(e^{tX})$ in the case where X is a Poisson random variable with parameter $\lambda = 1$. Using the inequality from part (i) with a suitable choice of t, prove that

$$\frac{1}{k!} + \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \ldots \leqslant \left(\frac{e}{k}\right)^k$$

for all $k \geqslant 1$.

Paper 2, Section II

9F Probability

Let Z be an exponential random variable with parameter $\lambda = 1$. Show that

$$\mathbb{P}(Z > s + t \mid Z > s) = \mathbb{P}(Z > t)$$

for any $s, t \geqslant 0$.

Let $Z_{\text{int}} = \lfloor Z \rfloor$ be the greatest integer less than or equal to Z. What is the probability mass function of Z_{int}? Show that $\mathbb{E}(Z_{\text{int}}) = \frac{1}{e-1}$.

Let $Z_{\text{frac}} = Z - Z_{\text{int}}$ be the fractional part of Z. What is the density of Z_{frac}?

Show that Z_{int} and Z_{frac} are independent.

Paper 2, Section II

10F Probability

Let X be a random variable taking values in the non-negative integers, and let G be the probability generating function of X. Assuming G is everywhere finite, show that

$$G'(1) = \mu \text{ and } G''(1) = \sigma^2 + \mu^2 - \mu$$

where μ is the mean of X and σ^2 is its variance. [You may interchange differentiation and expectation without justification.]

Consider a branching process where individuals produce independent random numbers of offspring with the same distribution as X. Let X_n be the number of individuals in the n-th generation, and let G_n be the probability generating function of X_n. Explain carefully why

$$G_{n+1}(t) = G_n(G(t))$$

Assuming $X_0 = 1$, compute the mean of X_n. Show that

$$\text{Var}(X_n) = \sigma^2 \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1}.$$

Suppose $\mathbb{P}(X = 0) = 3/7$ and $\mathbb{P}(X = 3) = 4/7$. Compute the probability that the population will eventually become extinct. You may use standard results on branching processes as long as they are clearly stated.

Paper 2, Section II

11F Probability

Let X be a geometric random variable with $\mathbb{P}(X = 1) = p$. Derive formulae for $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ in terms of p.

A jar contains n balls. Initially, all of the balls are red. Every minute, a ball is drawn at random from the jar, and then replaced with a green ball. Let T be the number of minutes until the jar contains only green balls. Show that the expected value of T is $n\sum_{i=1}^{n} 1/i$. What is the variance of T?

Paper 2, Section II

12F Probability

Let Ω be the sample space of a probabilistic experiment, and suppose that the sets $B_1, B_2, \ldots, B_k$ are a partition of Ω into events of positive probability. Show that

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_{j=1}^{k} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}$$

for any event A of positive probability.

A drawer contains two coins. One is an unbiased coin, which when tossed, is equally likely to turn up heads or tails. The other is a biased coin, which will turn up heads with probability p and tails with probability $1-p$. One coin is selected (uniformly) at random from the drawer. Two experiments are performed:

(a) The selected coin is tossed n times. Given that the coin turns up heads k times and tails $n-k$ times, what is the probability that the coin is biased?

(b) The selected coin is tossed repeatedly until it turns up heads k times. Given that the coin is tossed n times in total, what is the probability that the coin is biased?

Paper 2, Section I

3F Probability

Given two events A and B with $P(A) > 0$ and $P(B) > 0$, define the conditional probability $P(A \mid B)$.

Show that

$$P(B \mid A) = P(A \mid B)\frac{P(B)}{P(A)}\,.$$

A random number N of fair coins are tossed, and the total number of heads is denoted by H. If $P(N = n) = 2^{-n}$ for $n = 1, 2, \ldots$, find $P(N = n \mid H = 1)$.

Paper 2, Section I

4F Probability

Define the *probability generating function* $G(s)$ of a random variable X taking values in the non-negative integers.

A coin shows heads with probability $p \in (0, 1)$ on each toss. Let N be the number of tosses up to and including the first appearance of heads, and let $k \geqslant 1$. Find the probability generating function of $X = \min\{N, k\}$.

Show that $E(X) = p^{-1}(1 - q^k)$ where $q = 1 - p$.

Paper 2, Section II

9F Probability

(i) Define the *moment generating function* $M_X(t)$ of a random variable X. If X, Y are independent and $a, b \in \mathbb{R}$, show that the moment generating function of $Z = aX + bY$ is $M_X(at)M_Y(bt)$.

(ii) Assume $T > 0$, and $M_X(t) < \infty$ for $|t| < T$. Explain the expansion

$$M_X(t) = 1 + \mu t + \tfrac{1}{2}s^2t^2 + \mathrm{o}(t^2)$$

where $\mu = E(X)$ and $s^2 = E(X^2)$. [You may assume the validity of interchanging expectation and differentiation.]

(iii) Let X, Y be independent, identically distributed random variables with mean 0 and variance 1, and assume their moment generating function M satisfies the condition of part (ii) with $T = \infty$.

Suppose that $X+Y$ and $X-Y$ are independent. Show that $M(2t) = M(t)^3M(-t)$, and deduce that $\psi(t) = M(t)/M(-t)$ satisfies $\psi(t) = \psi(t/2)^2$.

Show that $\psi(h) = 1 + \mathrm{o}(h^2)$ as $h \to 0$, and deduce that $\psi(t) = 1$ for all t.

Show that X and Y are normally distributed.

Paper 2, Section II

10F Probability

(i) Define the distribution function F of a random variable X, and also its density function f assuming F is differentiable. Show that

$$f(x) = -\frac{d}{dx}P(X > x)\,.$$

(ii) Let U, V be independent random variables each with the uniform distribution on $[0,1]$. Show that

$$P(V^2 > U > x) = \tfrac{1}{3} - x + \tfrac{2}{3}x^{3/2}, \qquad x \in (0,1)\,.$$

What is the probability that the random quadratic equation $x^2 + 2Vx + U = 0$ has real roots?

Given that the two roots R_1, R_2 of the above quadratic are real, what is the probability that both $|R_1| \leqslant 1$ and $|R_2| \leqslant 1$?

Paper 2, Section II

11F Probability

(i) Let X_n be the size of the n^{th} generation of a branching process with family-size probability generating function $G(s)$, and let $X_0 = 1$. Show that the probability generating function $G_n(s)$ of X_n satisfies $G_{n+1}(s) = G(G_n(s))$ for $n \geqslant 0$.

(ii) Suppose the family-size mass function is $P(X_1 = k) = 2^{-k-1}$, $k = 0, 1, 2, \ldots$. Find $G(s)$, and show that

$$G_n(s) = \frac{n-(n-1)s}{n+1-ns} \qquad \text{for } |s| < 1 + \frac{1}{n}\,.$$

Deduce the value of $P(X_n = 0)$.

(iii) Write down the moment generating function of X_n/n. Hence or otherwise show that, for $x \geqslant 0$,

$$P(X_n/n > x \mid X_n > 0) \to e^{-x} \qquad \text{as } n \to \infty\,.$$

[You may use the continuity theorem but, if so, should give a clear statement of it.]

Paper 2, Section II

12F Probability

Let X, Y be independent random variables with distribution functions F_X, F_Y. Show that $U = \min\{X, Y\}$, $V = \max\{X, Y\}$ have distribution functions

$$F_U(u) = 1 - (1 - F_X(u))(1 - F_Y(u))\,, \quad F_V(v) = F_X(v)F_Y(v)\,.$$

Now let X, Y be independent random variables, each having the exponential distribution with parameter 1. Show that U has the exponential distribution with parameter 2, and that $V - U$ is independent of U.

Hence or otherwise show that V has the same distribution as $X + \frac{1}{2}Y$, and deduce the mean and variance of V.

[You may use without proof that X has mean 1 and variance 1.]

Paper 2, Section I
3F Probability

Let X be a random variable taking non-negative integer values and let Y be a random variable taking real values.

(a) Define the probability-generating function $G_X(s)$. Calculate it explicitly for a Poisson random variable with mean $\lambda > 0$.

(b) Define the moment-generating function $M_Y(t)$. Calculate it explicitly for a normal random variable $\mathrm{N}(0,1)$.

(c) By considering a random sum of independent copies of Y, prove that, for general X and Y, $G_X\big(M_Y(t)\big)$ is the moment-generating function of some random variable.

Paper 2, Section I
4F Probability

What does it mean to say that events $A_1, \ldots, A_n$ are (i) *pairwise independent*, (ii) *independent*?

Consider pairwise disjoint events B_1, B_2, B_3 and C, with

$$\mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(B_3) = p \text{ and } \mathbb{P}(C) = q, \text{ where } 3p + q \leqslant 1.$$

Let $0 \leqslant q \leqslant 1/16$. Prove that the events $B_1 \cup C$, $B_2 \cup C$ and $B_3 \cup C$ are pairwise independent if and only if

$$p = -q + \sqrt{q}\ .$$

Prove or disprove that there exist $p > 0$ and $q > 0$ such that these three events are independent.

Paper 2, Section II

9F Probability

(a) Let $B_1, \ldots, B_n$ be pairwise disjoint events such that their union $B_1 \cup B_2 \cup \ldots \cup B_n$ gives the whole set of outcomes, with $\mathbb{P}(B_i) > 0$ for $1 \leqslant i \leqslant n$. Prove that for any event A with $\mathbb{P}(A) > 0$ and for any i

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum\limits_{1 \leqslant j \leqslant n} \mathbb{P}(A|B_j)\mathbb{P}(B_j)} \, .$$

(b) A prince is equally likely to sleep on any number of mattresses from six to eight; on half the nights a pea is placed beneath the lowest mattress. With only six mattresses his sleep is always disturbed by the presence of a pea; with seven a pea, if present, is unnoticed in one night out of five; and with eight his sleep is undisturbed despite an offending pea in two nights out of five.

What is the probability that, on a given night, the prince's sleep was undisturbed?

On the morning of his wedding day, he announces that he has just spent the most peaceful and undisturbed of nights. What is the expected number of mattresses on which he slept the previous night?

Paper 2, Section II

10F Probability

(a) State Markov's inequality.

(b) Let r be a given positive integer. You toss an unbiased coin repeatedly until the first head appears, which occurs on the H_1th toss. Next, I toss the same coin until I get my first tail, which occurs on my T_1th toss. Then you continue until you get your second head with a further H_2 tosses; then I continue with a further T_2 tosses until my second tail. We continue for r turns like this, and generate a sequence H_1, T_1, H_2, $T_2, \ldots, H_r$, T_r of random variables. The total number of tosses made is Y_r. (For example, for $r = 2$, a sequence of outcomes $t\,t\,h\,|\,t\,|\,t\,t\,t\,h\,|\,h\,h\,t$ gives $H_1 = 3$, $T_1 = 1$, $H_2 = 4$, $T_2 = 3$ and $Y_2 = 11$.)

Find the probability-generating functions of the random variables H_j and T_j. Hence or otherwise obtain the mean values $\mathbb{E}H_j$ and $\mathbb{E}T_j$.

Obtain the probability-generating function of the random variable Y_r, and find the mean value $\mathbb{E}\,Y_r$.

Prove that, for $n \geqslant 2r$,

$$\mathbb{P}(Y_r = n) = \frac{1}{2^n}\binom{n-1}{2r-1}.$$

For $r = 1$, calculate $\mathbb{P}(Y_1 \geqslant 5)$, and confirm that it satisfies Markov's inequality.

Paper 2, Section II

11F Probability

I was given a clockwork orange for my birthday. Initially, I place it at the centre of my dining table, which happens to be exactly 20 units long. One minute after I place it on the table it moves one unit towards the left end of the table or one unit towards the right, each with probability 1/2. It continues in this manner at one minute intervals, with the direction of each move being independent of what has gone before, until it reaches either end of the table where it promptly falls off. If it falls off the left end it will break my Ming vase. If it falls off the right end it will land in a bucket of sand leaving the vase intact.

(a) Derive the difference equation for the probability that the Ming vase will survive, in terms of the current distance k from the orange to the left end, where $k = 1, \ldots, 19$.

(b) Derive the corresponding difference equation for the expected time when the orange falls off the table.

(c) Write down the general formula for the solution of each of the difference equations from (a) and (b). [*No proof is required.*]

(d) Based on parts (a)–(c), calculate the probability that the Ming vase will survive if, instead of placing the orange at the centre of the table, I place it initially 3 units from the right end of the table. Calculate the expected time until the orange falls off.

(e) Suppose I place the orange 3 units from the left end of the table. Calculate the probability that the orange will fall off the right end before it reaches a distance 1 unit from the left end of the table.

Paper 2, Section II

12F Probability

A circular island has a volcano at its central point. During an eruption, lava flows from the mouth of the volcano and covers a sector with random angle Φ (measured in radians), whose line of symmetry makes a random angle Θ with some fixed compass bearing.

The variables Θ and Φ are independent. The probability density function of Θ is constant on $(0, 2\pi)$ and the probability density function of Φ is of the form $A(\pi - \phi/2)$ where $0 < \phi < 2\pi$, and A is a constant.

(a) Find the value of A. Calculate the expected value and the variance of the sector angle Φ. Explain briefly how you would simulate the random variable Φ using a uniformly distributed random variable U.

(b) H_1 and H_2 are two houses on the island which are collinear with the mouth of the volcano, but on different sides of it. Find

(i) the probability that H_1 is hit by the lava;

(ii) the probability that both H_1 and H_2 are hit by the lava;

(iii) the probability that H_2 is not hit by the lava given that H_1 is hit.

Paper 2, Section I
3F Probability

Jensen's inequality states that for a convex function f and a random variable X with a finite mean, $\mathbb{E}f(X) \geqslant f\big(\mathbb{E}X\big)$.

(a) Suppose that $f(x) = x^m$ where m is a positive integer, and X is a random variable taking values $x_1, \ldots, x_N \geqslant 0$ with equal probabilities, and where the sum $x_1 + \ldots + x_N = 1$. Deduce from Jensen's inequality that

$$\sum_{i=1}^{N} f(x_i) \geqslant N f\left(\frac{1}{N}\right). \tag{1}$$

(b) N horses take part in m races. The results of different races are independent. The probability for horse i to win any given race is $p_i \geqslant 0$, with $p_1 + \ldots + p_N = 1$.

Let Q be the probability that a single horse wins all m races. Express Q as a polynomial of degree m in the variables $p_1, \ldots, p_N$.

By using (1) or otherwise, prove that $Q \geqslant N^{1-m}$.

Paper 2, Section I
4F Probability

Let X and Y be two non-constant random variables with finite variances. The correlation coefficient $\rho(X,Y)$ is defined by

$$\rho(X,Y) = \frac{\mathbb{E}\big[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\big]}{\big(\mathrm{Var}\ X\big)^{1/2}\big(\mathrm{Var}\ Y\big)^{1/2}}.$$

(a) Using the Cauchy–Schwarz inequality or otherwise, prove that

$$-1 \leqslant \rho(X,Y) \leqslant 1\,.$$

(b) What can be said about the relationship between X and Y when either (i) $\rho(X,Y) = 0$ or (ii) $|\rho(X,Y)| = 1$. [*Proofs are not required.*]

(c) Take $0 \leqslant r \leqslant 1$ and let X, X' be independent random variables taking values ± 1 with probabilities $1/2$. Set

$$Y = \begin{cases} X, & \text{with probability } r, \\ X', & \text{with probability } 1-r. \end{cases}$$

Find $\rho(X,Y)$.

Paper 2, Section II
9F Probability

(a) What does it mean to say that a random variable X with values $n = 1, 2, \ldots$ has a geometric distribution with a parameter p where $p \in (0,1)$?

An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks for breeding. Assume yaks are loners and roam about the Himalayas at random. The probability $p \in (0,1)$ that a given trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a breeding pair is obtained.

(b) Find the expected value of N.

(c) Find the variance of N.

Paper 2, Section II
10F Probability

The yearly levels of water in the river Camse are independent random variables X_1, X_2, …, with a given continuous distribution function $F(x) = \mathbb{P}(X_i \leqslant x)$, $x \geqslant 0$ and $F(0) = 0$. The levels have been observed in years 1, …, n and their values X_1, …, X_n recorded. The local council has decided to construct a dam of height

$$Y_n = \max\ \big[X_1, \ldots, X_n\big].$$

Let τ be the subsequent time that elapses before the dam overflows:

$$\tau = \min\ \big[t \geqslant 1 :\ X_{n+t} > Y_n\big].$$

(a) Find the distribution function $\mathbb{P}(Y_n \leqslant z)$, $z > 0$, and show that the mean value $\mathbb{E}Y_n = \displaystyle\int_0^\infty [1 - F(z)^n]\mathrm{d}z$.

(b) Express the conditional probability $\mathbb{P}(\tau = k \,|\, Y_n = z)$, where $k = 1, 2, \ldots$ and $z > 0$, in terms of F.

(c) Show that the unconditional probability

$$\mathbb{P}(\tau = k) = \frac{n}{(k+n-1)(k+n)}, \quad k = 1, 2, \ldots.$$

(d) Determine the mean value $\mathbb{E}\,\tau$.

Paper 2, Section II

11F Probability

In a branching process every individual has probability p_k of producing exactly k offspring, $k = 0, 1, \ldots$, and the individuals of each generation produce offspring independently of each other and of individuals in preceding generations. Let X_n represent the size of the nth generation. Assume that $X_0 = 1$ and $p_0 > 0$ and let $F_n(s)$ be the generating function of X_n. Thus

$$F_1(s) = \mathbb{E}s^{X_1} = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leqslant 1.$$

(a) Prove that

$$F_{n+1}(s) = F_n(F_1(s)).$$

(b) State a result in terms of $F_1(s)$ about the probability of eventual extinction. [*No proofs are required.*]

(c) Suppose the probability that an individual leaves k descendants in the next generation is $p_k = 1/2^{k+1}$, for $k \geqslant 0$. Show from the result you state in (b) that extinction is certain. Prove further that in this case

$$F_n(s) = \frac{n - (n-1)s}{(n+1) - ns}, \quad n \geqslant 1,$$

and deduce the probability that the nth generation is empty.

Paper 2, Section II
12F Probability

Let X_1, X_2 be bivariate normal random variables, with the joint probability density function

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\varphi(x_1,x_2)}{2(1-\rho^2)}\right],$$

where

$$\varphi(x_1,x_2) = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2$$

and $x_1, x_2 \in \mathbb{R}$.

(a) Deduce that the marginal probability density function

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right].$$

(b) Write down the moment-generating function of X_2 in terms of μ_2 and σ_2. [*No proofs are required.*]

(c) By considering the ratio $f_{X_1,X_2}(x_1,x_2)/f_{X_2}(x_2)$ prove that, conditional on $X_2 = x_2$, the distribution of X_1 is normal, with mean and variance $\mu_1 + \rho\sigma_1(x_2-\mu_2)/\sigma_2$ and $\sigma_1^2(1-\rho^2)$, respectively.

Paper 2, Section I

3F Probability

Consider a pair of jointly normal random variables X_1, X_2, with mean values μ_1, μ_2, variances σ_1^2, σ_2^2 and correlation coefficient ρ with $|\rho| < 1$.

(a) Write down the joint probability density function for (X_1, X_2).

(b) Prove that X_1, X_2 are independent if and only if $\rho = 0$.

Paper 2, Section I

4F Probability

Prove the law of total probability: if $A_1, \ldots, A_n$ are pairwise disjoint events with $\mathbb{P}(A_i) > 0$, and $B \subseteq A_1 \cup \ldots \cup A_n$ then $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$.

There are n people in a lecture room. Their birthdays are independent random variables, and each person's birthday is equally likely to be any of the 365 days of the year. By using the bound $1 - x \leqslant e^{-x}$ for $0 \leqslant x \leqslant 1$, prove that if $n \geqslant 29$ then the probability that at least two people have the same birthday is at least $2/3$.

[*In calculations, you may take* $\sqrt{1 + 8 \times 365 \ln 3} = 56.6$.]

Paper 2, Section II

9F Probability

I throw two dice and record the scores S_1 and S_2. Let X be the sum $S_1 + S_2$ and Y the difference $S_1 - S_2$.

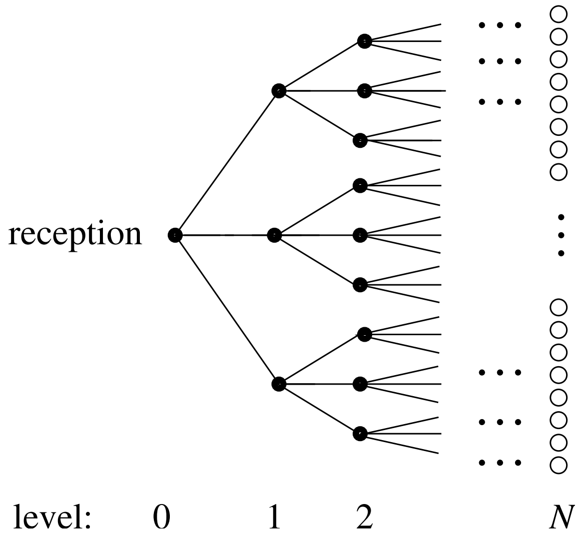
(a) Suppose that the dice are fair, so the values $1, \ldots, 6$ are equally likely. Calculate the mean and variance of both X and Y. Find all the values of x and y at which the probabilities $\mathbb{P}(X = x)$, $\mathbb{P}(Y = y)$ are each either greatest or least. Determine whether the random variables X and Y are independent.

(b) Now suppose that the dice are unfair, and that they give the values $1, \ldots, 6$ with probabilities $p_1, \ldots, p_6$ and $q_1, \ldots, q_6$, respectively. Write down the values of $\mathbb{P}(X = 2)$, $\mathbb{P}(X = 7)$ and $\mathbb{P}(X = 12)$. By comparing $\mathbb{P}(X = 7)$ with $\sqrt{\mathbb{P}(X = 2)\,\mathbb{P}(X = 12)}$ and applying the arithmetic-mean–geometric-mean inequality, or otherwise, show that the probabilities $\mathbb{P}(X = 2)$, $\mathbb{P}(X = 3)$, $\ldots$, $\mathbb{P}(X = 12)$ cannot all be equal.

Paper 2, Section II

10F Probability

No-one in their right mind would wish to be a guest at the Virtual Reality Hotel. See the diagram below showing a part of the floor plan of the hotel where rooms are represented by black or white circles. The hotel is built in a shape of a tree: there is one room (reception) situated at level 0, three rooms at level 1, nine at level 2, and so on. The rooms are joined by corridors to their neighbours: each room has four neighbours, apart from the reception, which has three neighbours. Each corridor is blocked with probability $1/3$ and open for passage in both directions with probability $2/3$, independently for different corridors. Every room at level N, where N is a given very large number, has an open window through which a guest can (and should) escape into the street. An arriving guest is placed in the reception and then wanders freely, insofar as the blocked corridors allow.



(a) Prove that the probability that the guest will not escape is close to a solution of the equation $\phi(t) = t$, where $\phi(t)$ is a probability-generating function that you should specify.

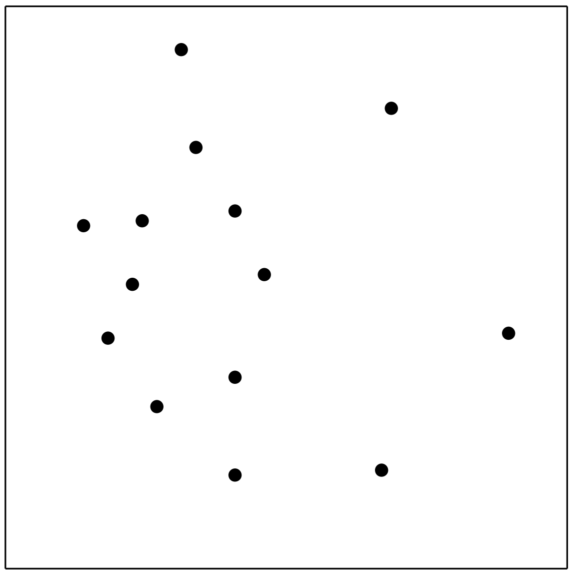
(b) Hence show that the guest's chance of escape is approximately $(9 - 3\sqrt{3})/4$.

Paper 2, Section II

11F Probability

Let X and Y be two independent uniformly distributed random variables on $[0,1]$. Prove that $\mathbb{E}X^k = \dfrac{1}{k+1}$ and $\mathbb{E}(XY)^k = \dfrac{1}{(k+1)^2}$, and find $\mathbb{E}(1-XY)^k$, where k is a non-negative integer.

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be n independent random points of the unit square $\mathcal{S} = \{(x,y) : 0 \leqslant x,y \leqslant 1\}$. We say that (X_i,Y_i) is a *maximal external point* if, for each $j=1,\ldots,n$, either $X_j \leqslant X_i$ or $Y_j \leqslant Y_i$. (For example, in the figure below there are three maximal external points.) Determine the expected number of maximal external points.



Paper 2, Section II

12F Probability

Let A_1, A_2 and A_3 be three pairwise disjoint events such that the union $A_1 \cup A_2 \cup A_3$ is the full event and $\mathbb{P}(A_1), \mathbb{P}(A_2), \mathbb{P}(A_3) > 0$. Let E be any event with $\mathbb{P}(E) > 0$. Prove the formula

$$\mathbb{P}(A_i|E) = \frac{\mathbb{P}(A_i)\mathbb{P}(E|A_i)}{\sum\limits_{j=1,2,3} \mathbb{P}(A_j)\mathbb{P}(E|A_j)}\,.$$

A Royal Navy speedboat has intercepted an abandoned cargo of packets of the deadly narcotic spitamin. This sophisticated chemical can be manufactured in only three places in the world: a plant in Authoristan (A), a factory in Bolimbia (B) and the ultramodern laboratory on board of a pirate submarine Crash (C) cruising ocean waters. The investigators wish to determine where this particular cargo comes from, but in the absence of prior knowledge they have to assume that each of the possibilities A, B and C is equally likely.

It is known that a packet from A contains pure spitamin in 95% of cases and is contaminated in 5% of cases. For B the corresponding figures are 97% and 3%, and for C they are 99% and 1%.

Analysis of the captured cargo showed that out of 10000 packets checked, 9800 contained the pure drug and the remaining 200 were contaminated. On the basis of this analysis, the Royal Navy captain estimated that 98% of the packets contain pure spitamin and reported his opinion that with probability roughly 0.5 the cargo was produced in B and with probability roughly 0.5 it was produced in C.

Assume that the number of contaminated packets follows the binomial distribution $\mathrm{Bin}(10000, \delta/100)$ where δ equals 5 for A, 3 for B and 1 for C. Prove that the captain's opinion is wrong: there is an overwhelming chance that the cargo comes from B.

[**Hint:** *Let E be the event that 200 out of 10000 packets are contaminated. Compare the ratios of the conditional probabilities $\mathbb{P}(E|A)$, $\mathbb{P}(E|B)$ and $\mathbb{P}(E|C)$. You may find it helpful that $\ln 3 \approx 1.09861$ and $\ln 5 \approx 1.60944$. You may also take $\ln(1-\delta/100) \approx -\delta/100$.*]

2/I/3F Probability

There are n socks in a drawer, three of which are red and the rest black. John chooses his socks by selecting two at random from the drawer and puts them on. He is three times more likely to wear socks of different colours than to wear matching red socks. Find n.

For this value of n, what is the probability that John wears matching black socks?

2/I/4F Probability

A standard six-sided die is thrown. Calculate the mean and variance of the number shown.

The die is thrown n times. By using Chebyshev's inequality, find an n such that

$$\mathbb{P}\left(\left|\frac{Y_n}{n} - 3.5\right| > 1.5\right) \leqslant 0.1$$

where Y_n is the total of the numbers shown over the n throws.

2/II/9F Probability

A population evolves in generations. Let Z_n be the number of members in the nth generation, with $Z_0 = 1$. Each member of the nth generation gives birth to a family, possibly empty, of members of the $(n+1)$th generation; the size of this family is a random variable and we assume that the family sizes of all individuals form a collection of independent identically distributed random variables each with generating function G.

Let G_n be the generating function of Z_n. State and prove a formula for G_n in terms of G. Determine the mean of Z_n in terms of the mean of Z_1.

Suppose that Z_1 has a Poisson distribution with mean λ. Find an expression for x_{n+1} in terms of x_n, where $x_n = \mathbb{P}\{Z_n = 0\}$ is the probability that the population becomes extinct by the nth generation.

2/II/10F **Probability**

A and B play a series of games. The games are independent, and each is won by A with probability p and by B with probability $1-p$. The players stop when the number of wins by one player is three greater than the number of wins by the other player. The player with the greater number of wins is then declared overall winner.

(i) Find the probability that exactly 5 games are played.

(ii) Find the probability that A is the overall winner.

2/II/11F **Probability**

Let X and Y have the bivariate normal density function

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}, \qquad x, y \in \mathbb{R},$$

for fixed $\rho \in (-1,1)$. Let $Z = (Y - \rho X)/\sqrt{1-\rho^2}$. Show that X and Z are independent $N(0,1)$ variables. Hence, or otherwise, determine

$$\mathbb{P}(X > 0, Y > 0).$$

2/II/12F **Probability**

The discrete random variable Y has distribution given by

$$\mathbb{P}(Y = k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

where $p \in (0,1)$. Determine the mean and variance of Y.

A fair die is rolled until all 6 scores have occurred. Find the mean and standard deviation of the number of rolls required.

[Hint: $\displaystyle\sum_{i=1}^{6} \left(\frac{6}{i}\right)^2 = 53.7$]

2/I/3F **Probability**

Let X and Y be independent random variables, each uniformly distributed on $[0,1]$. Let $U = \min(X,Y)$ and $V = \max(X,Y)$. Show that $\mathbb{E}U = \frac{1}{3}$, and hence find the covariance of U and V.

2/I/4F **Probability**

Let X be a normally distributed random variable with mean 0 and variance 1. Define, and determine, the moment generating function of X. Compute $\mathbb{E}X^r$ for $r = 0,1,2,3,4$.

Let Y be a normally distributed random variable with mean μ and variance σ^2. Determine the moment generating function of Y.

2/II/9F **Probability**

Let N be a non-negative integer-valued random variable with

$$P\{N = r\} = p_r, \quad r = 0,1,2,\ldots$$

Define $\mathbb{E}N$, and show that

$$\mathbb{E}N = \sum_{n=1}^{\infty} P\{N \geqslant n\}\,.$$

Let X_1, X_2, ... be a sequence of independent and identically distributed continuous random variables. Let the random variable N mark the point at which the sequence stops decreasing: that is, $N \geqslant 2$ is such that

$$X_1 \geqslant X_2 \geqslant \ldots \geqslant X_{N-1} < X_N\,,$$

where, if there is no such finite value of N, we set $N = \infty$. Compute $P\{N = r\}$, and show that $P\{N = \infty\} = 0$. Determine $\mathbb{E}N$.

2/II/10F **Probability**

Let X and Y be independent non-negative random variables, with densities f and g respectively. Find the joint density of $U = X$ and $V = X + aY$, where a is a positive constant.

Let X and Y be independent and exponentially distributed random variables, each with density

$$f(x) = \lambda e^{-\lambda x}, \quad x \geqslant 0\,.$$

Find the density of $X + \frac{1}{2}Y$. Is it the same as the density of the random variable $\max(X, Y)$?

2/II/11F **Probability**

Let A_1, A_2, …, A_n $(n \geqslant 2)$ be events in a sample space. For each of the following statements, either prove the statement or provide a counterexample.

(i)

$$P\left(\bigcap_{k=2}^{n} A_k \,\middle|\, A_1\right) = \prod_{k=2}^{n} P\left(A_k \,\middle|\, \bigcap_{r=1}^{k-1} A_r\right), \quad \text{provided} \quad P\left(\bigcap_{k=1}^{n-1} A_k\right) > 0\,.$$

(ii)

$$\text{If} \quad \sum_{k=1}^{n} P\left(A_k\right) > n-1 \quad \text{then} \quad P\left(\bigcap_{k=1}^{n} A_k\right) > 0\,.$$

(iii)

$$\text{If} \quad \sum_{i<j} P\left(A_i \bigcap A_j\right) > \binom{n}{2} - 1 \quad \text{then} \quad P\left(\bigcap_{k=1}^{n} A_k\right) > 0\,.$$

(iv) If B is an event and if, for each k, $\{B, A_k\}$ is a pair of independent events, then $\{B, \cup_{k=1}^{n} A_k\}$ is also a pair of independent events.

2/II/12F **Probability**

Let A, B and C be three random points on a sphere with centre O. The positions of A, B and C are independent, and each is uniformly distributed over the surface of the sphere. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Calculate the probability that all three of the angles $\angle AOB$, $\angle AOC$ and $\angle BOC$ are acute. [**Hint:** *Condition on the value of the angle $\angle AOB$.*]

2/I/3F **Probability**

What is a convex function? State Jensen's inequality for a convex function of a random variable which takes finitely many values.

Let $p \geqslant 1$. By using Jensen's inequality, or otherwise, find the smallest constant c_p so that

$$(a+b)^p \leqslant c_p\,(a^p + b^p) \quad \text{for all } a, b \geqslant 0.$$

[*You may assume that $x \mapsto |x|^p$ is convex for $p \geqslant 1$.*]

2/I/4F **Probability**

Let K be a fixed positive integer and X a discrete random variable with values in $\{1, 2, \ldots, K\}$. Define the *probability generating function* of X. Express the mean of X in terms of its probability generating function. The *Dirichlet probability generating function* of X is defined as

$$q\,(z) = \sum_{n=1}^{K} \frac{1}{n^z} P\,(X = n)\,.$$

Express the mean of X and the mean of $\log X$ in terms of $q\,(z)$.

2/II/9F **Probability**

Suppose that a population evolves in generations. Let Z_n be the number of members in the n-th generation and $Z_0 \equiv 1$. Each member of the n-th generation gives birth to a family, possibly empty, of members of the $(n+1)$-th generation; the size of this family is a random variable and we assume that the family sizes of all individuals form a collection of independent identically distributed random variables with the same generating function G.

Let G_n be the generating function of Z_n. State and prove a formula for G_n in terms of G. Use this to compute the variance of Z_n.

Now consider the *total* number of individuals in the first n generations; this number is a random variable and we write H_n for its generating function. Find a formula that expresses $H_{n+1}(s)$ in terms of $H_n(s)$, $G(s)$ and s.

2/II/10F **Probability**

Let X, Y be independent random variables with values in $(0, \infty)$ and the same probability density $\frac{2}{\sqrt{\pi}} e^{-x^2}$. Let $U = X^2 + Y^2$, $V = Y/X$. Compute the joint probability density of U, V and the marginal densities of U and V respectively. Are U and V independent?

2/II/11F Probability

A normal deck of playing cards contains 52 cards, four each with face values in the set $\mathcal{F} = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$. Suppose the deck is well shuffled so that each arrangement is equally likely. Write down the probability that the top and bottom cards have the same face value.

Consider the following algorithm for shuffling:

S1: Permute the deck randomly so that each arrangement is equally likely.

S2: If the top and bottom cards do *not* have the same face value, toss a biased coin that comes up heads with probability p and go back to step S1 if head turns up. Otherwise stop.

All coin tosses and all permutations are assumed to be independent. When the algorithm stops, let X and Y denote the respective face values of the top and bottom cards and compute the probability that $X = Y$. Write down the probability that $X = x$ for some $x \in \mathcal{F}$ and the probability that $Y = y$ for some $y \in \mathcal{F}$. What value of p will make X and Y independent random variables? Justify your answer.

2/II/12F Probability

Let $\gamma > 0$ and define

$$f(x) = \gamma \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Find γ such that f is a probability density function. Let $\{X_i : i \geqslant 1\}$ be a sequence of independent, identically distributed random variables, each having f with the correct choice of γ as probability density. Compute the probability density function of $X_1 + \cdots + X_n$. [*You may use the identity*

$$m \int_{-\infty}^{\infty} \left\{ \left(1+y^2\right) \left[m^2 + (x-y)^2\right] \right\}^{-1} dy = \pi (m+1) \left\{ (m+1)^2 + x^2 \right\}^{-1},$$

valid for all $x \in \mathbb{R}$ *and* $m \in \mathbb{N}$.]

Deduce the probability density function of

$$\frac{X_1 + \cdots + X_n}{n}.$$

Explain why your result does not contradict the weak law of large numbers.

2/I/3F **Probability**

Suppose $c \geqslant 1$ and X_c is a positive real-valued random variable with probability density

$$f_c(t) = A_c t^{c-1} e^{-t^c},$$

for $t > 0$, where A_c is a constant.

Find the constant A_c and show that, if $c > 1$ and $s, t > 0$,

$$\mathbb{P}[X_c \geqslant s + t \mid X_c \geqslant t] < \mathbb{P}[X_c \geqslant s].$$

[*You may assume the inequality* $(1 + x)^c > 1 + x^c$ *for all* $x > 0$, $c > 1$.]

2/I/4F **Probability**

Describe the Poisson distribution characterised by parameter $\lambda > 0$. Calculate the mean and variance of this distribution in terms of λ.

Show that the sum of n independent random variables, each having the Poisson distribution with $\lambda = 1$, has a Poisson distribution with $\lambda = n$.

Use the central limit theorem to prove that

$$e^{-n}\left(1 + \frac{n}{1!} + \frac{n^2}{2!} + \ldots + \frac{n^n}{n!}\right) \to 1/2 \quad \text{as} \quad n \to \infty.$$

2/II/9F Probability

Given a real-valued random variable X, we define $\mathbb{E}[e^{iX}]$ by

$$\mathbb{E}\left[e^{iX}\right] \equiv \mathbb{E}\left[\cos X\right] + i\,\mathbb{E}\left[\sin X\right] .$$

Consider a second real-valued random variable Y, independent of X. Show that

$$\mathbb{E}\left[e^{i(X+Y)}\right] = \mathbb{E}\left[e^{iX}\right]\mathbb{E}\left[e^{iY}\right] .$$

You gamble in a fair casino that offers you unlimited credit despite your initial wealth of 0. At every game your wealth increases or decreases by £1 with equal probability 1/2. Let W_n denote your wealth after the n^{th} game. For a fixed real number u, compute $\phi(u)$ defined by

$$\phi\left(u\right) = \mathbb{E}\left[e^{iuW_n}\right] .$$

Verify that the result is real-valued.

Show that for n even,

$$\mathbb{P}\left[W_n = 0\right] = \gamma \int_0^{\pi/2} \left[\cos u\right]^n du\,,$$

for some constant γ, which you should determine. What is $\mathbb{P}\left[W_n = 0\right]$ for n odd?

2/II/10F Probability

Alice and Bill fight a paint-ball duel. Nobody has been hit so far and they are both left with one shot. Being exhausted, they need to take a breath before firing their last shot. This takes A seconds for Alice and B seconds for Bill. Assume these times are exponential random variables with means $1/\alpha$ and $1/\beta$, respectively.

Find the distribution of the (random) time that passes by before the next shot is fired. What is its standard deviation? What is the probability that Alice fires the next shot?

Assume Alice has probability 1/2 of hitting whenever she fires whereas Bill never misses his target. If the next shot is a hit, what is the probability that it was fired by Alice?

2/II/11F Probability

Let (S,T) be uniformly distributed on $[-1,1]^2$ and define $R=\sqrt{S^2+T^2}$. Show that, conditionally on

$$R \leqslant 1\,,$$

the vector (S,T) is uniformly distributed on the unit disc. Let (R,Θ) denote the point (S,T) in polar coordinates and find its probability density function $f\,(r,\theta)$ for $r\in[0,1]$, $\theta\in[0,2\pi)$. Deduce that R and Θ are independent.

Introduce the new random variables

$$X=\frac{S}{R}\sqrt{-2\log\left(R^2\right)},\;\; Y=\frac{T}{R}\sqrt{-2\log\left(R^2\right)}\,,$$

noting that under the above conditioning, (S,T) are uniformly distributed on the unit disc. The pair (X,Y) may be viewed as a (random) point in $\mathbb{R}^2$ with polar coordinates (Q,Ψ). Express Q as a function of R and deduce its density. Find the joint density of (Q,Ψ). Hence deduce that X and Y are independent normal random variables with zero mean and unit variance.

2/II/12F Probability

Let $a_1,a_2,...,a_n$ be a ranking of the yearly rainfalls in Cambridge over the next n years: assume $a_1,a_2,...,a_n$ is a random permutation of $1,2,...,n$. Year k is called a record year if $a_i>a_k$ for all $i<k$ (thus the first year is always a record year). Let $Y_i=1$ if year i is a record year and 0 otherwise.

Find the distribution of Y_i and show that $Y_1,Y_2,...,Y_n$ are independent and calculate the mean and variance of the number of record years in the next n years.

Find the probability that the second record year occurs at year i. What is the expected number of years until the second record year occurs?

2/I/3F Probability

Define the covariance, $\operatorname{cov}(X,Y)$, of two random variables X and Y.

Prove, or give a counterexample to, each of the following statements.

(a) For any random variables X, Y, Z

$$\operatorname{cov}(X+Y,Z)=\operatorname{cov}(X,Z)+\operatorname{cov}(Y,Z).$$

(b) If X and Y are identically distributed, not necessarily independent, random variables then

$$\operatorname{cov}(X+Y,X-Y)=0.$$

2/I/4F Probability

The random variable X has probability density function

$$f(x)=\begin{cases} cx(1-x) & \text{if } 0 \leqslant x \leqslant 1\\ 0 & \text{otherwise.}\end{cases}$$

Determine c, and the mean and variance of X.

2/II/9F Probability

Let X be a positive-integer valued random variable. Define its *probability generating function* p_X. Show that if X and Y are independent positive-integer valued random variables, then $p_{X+Y}=p_X p_Y$.

A non-standard pair of dice is a pair of six-sided unbiased dice whose faces are numbered with strictly positive integers in a non-standard way (for example, $(2,2,2,3,5,7)$ and $(1,1,5,6,7,8)$). Show that there exists a non-standard pair of dice A and B such that when thrown

$$P\{\text{total shown by } A \text{ and } B \text{ is } n\}=P\{\text{total shown by pair of ordinary dice is } n\}$$

for all $2 \leqslant n \leqslant 12$.

[*Hint:* $(x+x^2+x^3+x^4+x^5+x^6)=x(1+x)(1+x^2+x^4)=x(1+x+x^2)(1+x^3)$.]

2/II/10F **Probability**

Define the *conditional probability* $P(A \mid B)$ of the event A given the event B.

A bag contains four coins, each of which when tossed is equally likely to land on either of its two faces. One of the coins shows a head on each of its two sides, while each of the other three coins shows a head on only one side. A coin is chosen at random, and tossed three times in succession. If heads turn up each time, what is the probability that if the coin is tossed once more it will turn up heads again? Describe the sample space you use and explain carefully your calculations.

2/II/11F **Probability**

The random variables X_1 and X_2 are independent, and each has an exponential distribution with parameter λ. Find the joint density function of

$$Y_1 = X_1 + X_2\,, \quad Y_2 = X_1/X_2\,,$$

and show that Y_1 and Y_2 are independent. What is the density of Y_2?

2/II/12F **Probability**

Let $A_1, A_2, \ldots, A_r$ be events such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Show that the number N of events that occur satisfies

$$P(N = 0) \quad = \quad 1 - \sum_{i=1}^{r} P(A_i)\,.$$

Planet Zog is a sphere with centre O. A number N of spaceships land at random on its surface, their positions being independent, each uniformly distributed over the surface. A spaceship at A is in direct radio contact with another point B on the surface if $\angle AOB < \frac{\pi}{2}$. Calculate the probability that every point on the surface of the planet is in direct radio contact with at least one of the N spaceships.

[*Hint*: The intersection of the surface of a sphere with a plane through the centre of the sphere is called a *great circle*. You may find it helpful to use the fact that N random great circles partition the surface of a sphere into $N(N-1)+2$ disjoint regions with probability one.]

2/I/3F **Probability**

(a) Define the *probability generating function* of a random variable. Calculate the probability generating function of a binomial random variable with parameters n and p, and use it to find the mean and variance of the random variable.

(b) X is a binomial random variable with parameters n and p, Y is a binomial random variable with parameters m and p, and X and Y are independent. Find the distribution of $X + Y$; that is, determine $P\{X + Y = k\}$ for all possible values of k.

2/I/4F **Probability**

The random variable X is uniformly distributed on the interval $[0, 1]$. Find the distribution function and the probability density function of Y, where

$$Y = \frac{3X}{1 - X} \, .$$

2/II/9F **Probability**

State the inclusion-exclusion formula for the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs.

After a party the n guests take coats randomly from a pile of their n coats. Calculate the probability that no-one goes home with the correct coat.

Let $p(m, n)$ be the probability that exactly m guests go home with the correct coats. By relating $p(m, n)$ to $p(0, n-m)$, or otherwise, determine $p(m, n)$ and deduce that

$$\lim_{n \to \infty} p(m, n) = \frac{1}{em!} \, .$$

2/II/10F **Probability**

The random variables X and Y each take values in $\{0, 1\}$, and their joint distribution $p(x, y) = P\{X = x, \ Y = y\}$ is given by

$$p(0,0) = a, \qquad p(0,1) = b, \qquad p(1,0) = c, \qquad p(1,1) = d \, .$$

Find necessary and sufficient conditions for X and Y to be

(i) uncorrelated;

(ii) independent.

Are the conditions established in (i) and (ii) equivalent?

2/II/11F Probability

A laboratory keeps a population of aphids. The probability of an aphid passing a day uneventfully is $q < 1$. Given that a day is not uneventful, there is probability r that the aphid will have one offspring, probability s that it will have two offspring and probability t that it will die, where $r + s + t = 1$. Offspring are ready to reproduce the next day. The fates of different aphids are independent, as are the events of different days. The laboratory starts out with one aphid.

Let X_1 be the number of aphids at the end of the first day. What is the expected value of X_1? Determine an expression for the probability generating function of X_1.

Show that the probability of extinction does not depend on q, and that if $2r+3s \leqslant 1$ then the aphids will certainly die out. Find the probability of extinction if $r = 1/5$, $s = 2/5$ and $t = 2/5$.

[*Standard results on branching processes may be used without proof, provided that they are clearly stated.*]

2/II/12F Probability

Planet Zog is a ball with centre O. Three spaceships A, B and C land at random on its surface, their positions being independent and each uniformly distributed on its surface. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Spaceships A and B can communicate directly by radio if $\angle AOB < \pi/2$, and similarly for spaceships B and C and spaceships A and C. Given angle $\angle AOB = \gamma < \pi/2$, calculate the probability that C can communicate directly with *either* A or B. Given angle $\angle AOB = \gamma > \pi/2$, calculate the probability that C can communicate directly with *both* A and B. Hence, or otherwise, show that the probability that all three spaceships can keep in in touch (with, for example, A communicating with B via C if necessary) is $(\pi + 2)/(4\pi)$.

2/I/3F Probability

Define the *indicator function* I_A of an event A.

Let I_i be the indicator function of the event A_i, $1 \le i \le n$, and let $N = \sum_1^n I_i$ be the number of values of i such that A_i occurs. Show that $E(N) = \sum_i p_i$ where $p_i = P(A_i)$, and find $\operatorname{var}(N)$ in terms of the quantities $p_{ij} = P(A_i \cap A_j)$.

Using Chebyshev's inequality or otherwise, show that

$$P(N = 0) \le \frac{\operatorname{var}(N)}{\{E(N)\}^2}.$$

2/I/4F Probability

A coin shows heads with probability p on each toss. Let π_n be the probability that the number of heads after n tosses is even. Show carefully that $\pi_{n+1} = (1-p)\pi_n + p(1-\pi_n)$, $n \ge 1$, and hence find π_n. [The number 0 is even.]

2/II/9F Probability

(a) Define the *conditional probability* $P(A \mid B)$ of the event A given the event B. Let $\{B_i : 1 \le i \le n\}$ be a partition of the sample space Ω such that $P(B_i) > 0$ for all i. Show that, if $P(A) > 0$,

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_j P(A \mid B_j)P(B_j)}.$$

(b) There are n urns, the rth of which contains $r-1$ red balls and $n-r$ blue balls. You pick an urn (uniformly) at random and remove two balls without replacement. Find the probability that the first ball is blue, and the conditional probability that the second ball is blue given that the first is blue. [You may assume that $\sum_{i=1}^{n-1} i(i-1) = \frac{1}{3}n(n-1)(n-2)$.]

(c) What is meant by saying that two events A and B are independent?

(d) Two fair dice are rolled. Let A_s be the event that the sum of the numbers shown is s, and let B_i be the event that the first die shows i. For what values of s and i are the two events A_s, B_i independent?

2/II/10F Probability

There is a random number N of foreign objects in my soup, with mean μ and finite variance. Each object is a fly with probability p, and otherwise is a spider; different objects have independent types. Let F be the number of flies and S the number of spiders.

(a) Show that $G_F(s) = G_N(ps + 1 - p)$. [G_X denotes the probability generating function of a random variable X. You should present a clear statement of any general result used.]

(b) Suppose N has the Poisson distribution with parameter μ. Show that F has the Poisson distribution with parameter μp, and that F and S are independent.

(c) Let $p = \frac{1}{2}$ and suppose that F and S are independent. [You are given nothing about the distribution of N.] Show that $G_N(s) = G_N(\frac{1}{2}(1+s))^2$. By working with the function $H(s) = G_N(1-s)$ or otherwise, deduce that N has the Poisson distribution. [You may assume that $\left(1 + \frac{x}{n} + \mathrm{o}(n^{-1})\right)^n \to e^x$ as $n \to \infty$.]

2/II/11F Probability

Let X, Y, Z be independent random variables each with the uniform distribution on the interval $[0,1]$.

(a) Show that $X+Y$ has density function

$$f_{X+Y}(u) = \begin{cases} u & \text{if } 0 \le u \le 1, \\ 2-u & \text{if } 1 \le u \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that $P(Z > X+Y) = \frac{1}{6}$.

(c) You are provided with three rods of respective lengths X, Y, Z. Show that the probability that these rods may be used to form the sides of a triangle is $\frac{1}{2}$.

(d) Find the density function $f_{X+Y+Z}(s)$ of $X+Y+Z$ for $0 \leqslant s \leqslant 1$. Let W be uniformly distributed on $[0,1]$, and independent of X, Y, Z. Show that the probability that rods of lengths W, X, Y, Z may be used to form the sides of a quadrilateral is $\frac{5}{6}$.

2/II/12F Probability

(a) Explain what is meant by the term 'branching process'.

(b) Let X_n be the size of the nth generation of a branching process in which each family size has probability generating function G, and assume that $X_0 = 1$. Show that the probability generating function G_n of X_n satisfies $G_{n+1}(s) = G_n(G(s))$ for $n \ge 1$.

(c) Show that $G(s) = 1 - \alpha(1-s)^\beta$ is the probability generating function of a non-negative integer-valued random variable when $\alpha, \beta \in (0,1)$, and find G_n explicitly when G is thus given.

(d) Find the probability that $X_n = 0$, and show that it converges as $n \to \infty$ to $1 - \alpha^{1/(1-\beta)}$. Explain carefully why this implies that the probability of ultimate extinction equals $1 - \alpha^{1/(1-\beta)}$.

2/I/3F **Probability**

The following problem is known as Bertrand's paradox. A chord has been chosen at random in a circle of radius r. Find the probability that it is longer than the side of the equilateral triangle inscribed in the circle. Consider three different cases:

a) the middle point of the chord is distributed uniformly inside the circle,

b) the two endpoints of the chord are independent and uniformly distributed over the circumference,

c) the distance between the middle point of the chord and the centre of the circle is uniformly distributed over the interval $[0, r]$.

[*Hint: drawing diagrams may help considerably.*]

2/I/4F **Probability**

The Ruritanian authorities decided to pardon and release one out of three remaining inmates, A, B and C, kept in strict isolation in the notorious Alkazaf prison. The inmates know this, but can't guess who among them is the lucky one; the waiting is agonising. A sympathetic, but corrupted, prison guard approaches A and offers to name, in exchange for a fee, another inmate (not A) who is doomed to stay. He says: "This reduces your chances to remain here from 2/3 to 1/2: will it make you feel better?" A hesitates but then accepts the offer; the guard names B.

Assume that indeed B will not be released. Determine the conditional probability

$$P\big(A \text{ remains } \big| \; B \text{ named}\big) \; = \; \tfrac{P(A \& B \text{ remain})}{P(B \text{ named})}$$

and thus check the guard's claim, in three cases:

a) when the guard is completely unbiased (i.e., names any of B and C with probability 1/2 if the pair B, C is to remain jailed),

b) if he hates B and would certainly name him if B is to remain jailed,

c) if he hates C and would certainly name him if C is to remain jailed.

2/II/9F **Probability**

I play tennis with my parents; the chances for me to win a game against Mum (M) are p and against Dad (D) q, where $0 < q < p < 1$. We agreed to have three games, and their order can be DMD (where I play against Dad, then Mum then again Dad) or MDM. The results of games are independent.

Calculate under each of the two orders the probabilities of the following events:

a) that I win at least one game,

b) that I win at least two games,

c) that I win at least two games in succession (i.e., games 1 and 2 or 2 and 3, or 1, 2 and 3),

d) that I win exactly two games in succession (i.e., games 1 and 2 or 2 and 3, but not 1, 2 and 3),

e) that I win exactly two games (i.e., 1 and 2 or 2 and 3 or 1 and 3, but not 1, 2 and 3).

In each case a)– e) determine which order of games maximizes the probability of the event. In case e) assume in addition that $p + q > 3pq$.

2/II/10F **Probability**

A random point is distributed uniformly in a unit circle $\mathcal{D}$ so that the probability that it falls within a subset $\mathcal{A} \subseteq \mathcal{D}$ is proportional to the area of $\mathcal{A}$. Let R denote the distance between the point and the centre of the circle. Find the distribution function $F_R(x) = P(R < x)$, the expected value ER and the variance $\operatorname{Var} R = ER^2 - (ER)^2$.

Let Θ be the angle formed by the radius through the random point and the horizontal line. Prove that R and Θ are independent random variables.

Consider a coordinate system where the origin is placed at the centre of $\mathcal{D}$. Let X and Y denote the horizontal and vertical coordinates of the random point. Find the covariance $\operatorname{Cov}(X, Y) = E(XY) - EXEY$ and determine whether X and Y are independent.

Calculate the sum of expected values $E\frac{X}{R} + iE\frac{Y}{R}$. Show that it can be written as the expected value $Ee^{i\xi}$ and determine the random variable ξ.

2/II/11F Probability

Dipkomsky, a desperado in the wild West, is surrounded by an enemy gang and fighting tooth and nail for his survival. He has m guns, $m > 1$, pointing in different directions and tries to use them in succession to give an impression that there are several defenders. When he turns to a subsequent gun and discovers that the gun is loaded he fires it with probability $1/2$ and moves to the next one. Otherwise, i.e. when the gun is unloaded, he loads it with probability $3/4$ or simply moves to the next gun with complementary probability $1/4$. If he decides to load the gun he then fires it or not with probability $1/2$ and after that moves to the next gun anyway.

Initially, each gun had been loaded independently with probability p. Show that if after each move this distribution is preserved, then $p = 3/7$. Calculate the expected value EN and variance Var N of the number N of loaded guns under this distribution.

[*Hint: it may be helpful to represent* N *as a sum* $\sum_{1 \leq j \leq m} X_j$ *of random variables taking values* 0 *and* 1.]

2/II/12F Probability

A taxi travels between four villages, W, X, Y, Z, situated at the corners of a rectangle. The four roads connecting the villages follow the sides of the rectangle; the distance from W to X and Y to Z is 5 miles and from W to Z and Y to X 10 miles. After delivering a customer the taxi waits until the next call then goes to pick up the new customer and takes him to his destination. The calls may come from any of the villages with probability $1/4$ and each customer goes to any other village with probability $1/3$. Naturally, when travelling between a pair of adjacent corners of the rectangle, the taxi takes the straight route, otherwise (when it travels from W to Y or X to Z or vice versa) it does not matter. Distances within a given village are negligible. Let D be the distance travelled to pick up and deliver a single customer. Find the probabilitites that D takes each of its possible values. Find the expected value ED and the variance Var D.