Part IA

Dynamics and Relativity

Paper 4, Section I
3C Dynamics and Relativity

A rocket moves vertically upwards in a uniform gravitational field and emits exhaust gas downwards with time-dependent speed $U(t)$ relative to the rocket. Derive the rocket equation

$$m(t)\frac{dv}{dt} + U(t)\frac{dm}{dt} = -m(t)g\,,$$

where $m(t)$ and $v(t)$ are respectively the rocket's mass and upward speed at time t.

Suppose now that $m(t) = m_0 - \alpha t$ and $U(t) = U_0 m_0/m(t)$, where m_0, U_0 and α are constants. What is the condition for the rocket to lift off from rest at $t = 0$? Assuming that this condition is satisfied, find $v(t)$.

State the dimensions of all the quantities involved in your expression for $v(t)$, and verify that the expression is dimensionally consistent.

[*You may neglect any relativistic effects.*]

Paper 4, Section I
4C Dynamics and Relativity

In two-dimensional space-time an inertial frame S' has velocity v relative to another inertial frame S. State the Lorentz transformation that relates coordinates (x', t') in S' to coordinates (x, t) in S, assuming that the frames coincide when $t = t' = 0$.

Show that if $x_\pm = x \pm ct$ and $x'_\pm = x' \pm ct'$ then the Lorentz transformation can be expressed in the form

$$x'_+ = \lambda(v)x_+ \quad \text{and} \quad x'_- = \lambda(-v)x_-\,, \quad \text{where} \quad \lambda(v) = \left(\frac{c-v}{c+v}\right)^{1/2}. \qquad (*)$$

Deduce that $x^2 - c^2t^2 = x'^2 - c^2t'^2$.

Use $(*)$ to verify that successive Lorentz transformations with velocities v_1 and v_2 result in another Lorentz transformation with velocity v_3, to be determined in terms of v_1 and v_2.

Paper 4, Section II
9C Dynamics and Relativity

Find the moment of inertia of a uniform-density sphere with mass M and radius a with respect to an axis passing through its centre.

Such a sphere is released from rest on a plane inclined at an angle α to the horizontal. Let t_s and t_r be the times taken for the sphere to travel a distance l along the plane assuming either sliding without friction or rolling without slipping, respectively. Discuss whether energy is conserved in each of the two cases. Show that $t_s/t_r = \sqrt{5/7}$.

The uniform-density sphere is replaced by a sphere of the same mass whose density varies radially such that its moment of inertia is $\gamma M a^2$ for some constant γ. Determine the new value for t_s/t_r.

Paper 4, Section II
10C Dynamics and Relativity

(a) Write down the 4-momentum of a particle of rest mass m and 3-velocity $\mathbf{v}$, and the 4-momentum of a photon of frequency ω (having zero rest mass) moving in the direction of the unit 3-vector $\mathbf{e}$.

Show that if P_1 and P_2 are timelike future-pointing 4-vectors then $P_1 \cdot P_2 \geqslant 0$ (where the dot denotes the Lorentz-invariant scalar product). Hence or otherwise show that the law of conservation of 4-momentum forbids a photon to spontaneously decay into an electron–positron pair. [Electrons and positrons have equal and non-zero rest masses.]

(b) In the laboratory frame an electron travelling with 3-velocity $\mathbf{u}$ collides with a positron at rest. They annihilate, producing two photons of frequencies ω_1 and ω_2 that move off at angles θ_1 and θ_2 to $\mathbf{u}$, respectively. Explain why the 3-momenta of the photons and $\mathbf{u}$ lie in a plane.

By considering energy and two components of 3-momentum in the laboratory frame, or otherwise, show that

$$\frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2} = \sqrt{\frac{\gamma-1}{\gamma+1}}$$

where $\gamma = 1/\sqrt{1-u^2/c^2}$.

Paper 4, Section II

11C Dynamics and Relativity

Consider a system of N particles with position vectors $\mathbf{r}_i(t)$ and masses m_i, where $i = 1, 2, \ldots, N$. Particle i experiences an external force $\mathbf{F}_i$ and an internal force $\mathbf{F}_{ij}$ from particle j, for each $j \neq i$. Stating clearly any assumptions you need, show that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{L}}{dt} = \mathbf{G}\,,$$

where $\mathbf{P}$ is the total momentum, $\mathbf{F}$ is the total external force, $\mathbf{L}$ is the total angular momentum about a fixed point $\mathbf{a}$, and $\mathbf{G}$ is the total external torque about $\mathbf{a}$.

Does the result $\dfrac{d\mathbf{L}}{dt} = \mathbf{G}$ still hold if the fixed point $\mathbf{a}$ is replaced by the moving centre of mass of the system? Justify your answer.

Suppose now that all the particles have the same mass m and that the external force on particle i is $-k\dfrac{d\mathbf{r}_i}{dt}$, where k is a constant. Show that

$$\mathbf{L}(t) = \mathbf{L}(0)e^{-kt/m}\,.$$

Paper 4, Section II

12C Dynamics and Relativity

A particle of mass m moves in a plane under an attractive force of magnitude $mf(r)$ towards the origin O. You may assume that the acceleration $\mathbf{a}$ in polar coordinates (r,θ) is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}\,,$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in the directions of increasing r and θ respectively, and the dot denotes d/dt.

(a) Show that $l = r^2\dot{\theta}$ is a constant of the motion. Introducing $u = 1/r$, show that

$$\dot{r} = -l\frac{du}{d\theta}$$

and derive the geometric orbit equation

$$l^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right)\,.$$

(b) Suppose now that

$$f(r) = \frac{3r+9}{r^3}\,,$$

and that initially the particle is at distance $r_0 = 1$ from O, and moving with speed $v_0 = 4$ in the direction of decreasing r and increasing θ that makes an angle $\pi/3$ with the radial vector pointing towards O.

Show that $l = 2\sqrt{3}$ and find u as a function of θ. Hence, or otherwise, show that the particle returns to its original position after one revolution about O and then flies off to infinity.

Paper 4, Section I
3C Dynamics and Relativity

A particle of mass m, charge q, and position vector $\mathbf{x}$ moves in a constant non-zero electric field $\mathbf{E}$ and a constant non-zero magnetic field $\mathbf{B}$, with $\mathbf{E}$ perpendicular to $\mathbf{B}$. The particle's motion is described by $m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B})$. At time $t = 0$ the particle is located at $\mathbf{x} = \mathbf{x}_0$ and has velocity $\dot{\mathbf{x}} = \mathbf{v}$, where $\mathbf{v}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$.

(a) Using vector methods, show that the motion lies in a plane and give the vector equation of that plane.

(b) Adopt a Cartesian coordinate system centred on $\mathbf{x}_0$ with the x-axis directed along $\mathbf{E}$ and the y-axis along $\mathbf{B}$. Assume $\mathbf{v} = \mathbf{0}$. Find an expression for $\mathbf{x}$ as a function of t.

Paper 4, Section I
4C Dynamics and Relativity

Consider space-time with only one spatial dimension, and two inertial frames S and S'. Frame S' moves relative to frame S with speed u, and their origins coincide when clocks in the two frames read $t = t' = 0$.

According to an observer at the origin of frame S, an event has coordinates (ct, x). According to an observer at the origin of frame S', its coordinates are (ct', x'), which are given by

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = A \begin{pmatrix} ct \\ x \end{pmatrix},$$

where c is the speed of light and A is a 2×2 matrix.

(a) Write down the matrix A in terms of $\beta = u/c$ when working in:

(i) Newtonian dynamics;

(ii) special relativity.

Show that the two transformations agree in an appropriate limit, assuming $|x| < c|t|$.

(b) Calculate the eigenvalues and eigenvectors of A in special relativity, and interpret the eigenvectors.

Paper 4, Section II
9C Dynamics and Relativity

Consider two particles of masses m_1 and m_2, and locations $\mathbf{x}_1$ and $\mathbf{x}_2$, that exert forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ upon each other. There are no external forces. The particles' equations of motion are $m_1\ddot{\mathbf{x}}_1 = \mathbf{F}_{12}$, and $m_2\ddot{\mathbf{x}}_2 = \mathbf{F}_{21}$.

(a) Define the centre of mass $\mathbf{R}$. Prove that the centre of mass moves at a constant velocity. If $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, show that $\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$, where you must give an expression for μ.

(b) For the remainder of the question, assume the force law

$$\mathbf{F}_{ij} = -km_i m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3},$$

with k a positive constant.

Let $m_1 = m_2 = m$. In a Cartesian coordinate system whose origin is at the centre of mass, verify that

$$\mathbf{x}_1 = a(\cos\omega t,\, \sin\omega t\,, 0), \qquad \mathbf{x}_2 = -a(\cos\omega t,\, \sin\omega t,\, 0), \qquad (\dagger)$$

is a solution to the equations of motion, where a is a fixed constant and ω is a frequency that you should find.

(c) A third particle is now placed upon the z-axis. Its mass m_3 is negligible compared to m, and its position vector $\mathbf{x}_3$ obeys $m_3\ddot{\mathbf{x}}_3 = \mathbf{F}_{31} + \mathbf{F}_{32}$, while the motion of particles 1 and 2 is given by ($\dagger$).

(i) If the initial velocity of the third particle is parallel to the z-axis, show that it remains on that axis and that its location $z(t)$ obeys

$$\ddot{z} = -\frac{2mkz}{(z^2 + a^2)^{3/2}}.$$

(ii) What is the effective potential governing the particle's motion? Describe the different kinds of behaviour possible.

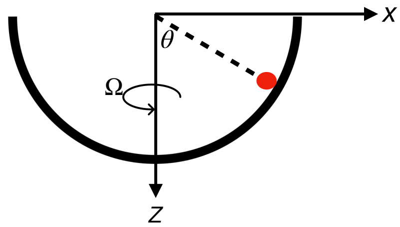
(iii) Assume that at $t = 0$, $z = z_0$, and $\dot{z} = 0$. If z_0 is very small, show that the motion is oscillatory and find the period of the oscillations.

(iv) Now assume that at $t = 0$, $z = 0$, and $\dot{z} = u$. What is the criterion for the particle to escape to infinity?

Paper 4, Section II
10C Dynamics and Relativity

Consider an infinitely long ramp with semi-circular cross-section of radius R, as shown in the figure. Adopt a Cartesian coordinate system with the y-axis directed along the ramp, pointing out of the page, and the z-axis directed vertically downwards. The ramp rotates about the z-axis with constant angular velocity $\mathbf{\Omega} = -\Omega\hat{\mathbf{z}}$ and the coordinate system rotates with the ramp.

A ball of mass m and negligible size slides along the surface of the ramp without any friction but experiences a constant gravitational acceleration $\mathbf{g} = g\hat{\mathbf{z}}$. A line from the ball to the origin projected on to the xz-plane makes an angle θ with the z-axis, as shown in the figure.



(a) If $\mathbf{x} = (x, y, z)$ is the ball's position vector, its equation of motion is

$$\ddot{\mathbf{x}} = -2\mathbf{\Omega} \times \dot{\mathbf{x}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) + \mathbf{g} + \frac{1}{m}\mathbf{N},$$

where $\mathbf{N}$ is the normal force due to the ramp. What do the first two terms on the right side correspond to? Write down the equation's three Cartesian components.

(b) Using your results from part (a), or otherwise, show that

$$\begin{aligned} R\ddot{\theta} &= -g\sin\theta + \Omega^2 R\cos\theta\sin\theta - 2\Omega\dot{y}\cos\theta, \\ \ddot{y} &= \Omega^2 y + 2\Omega R\dot{\theta}\cos\theta. \end{aligned}$$

Find all the solutions for which the ball is at rest in the rotating frame.

(c) Suppose that Ω is sufficiently small so that terms of order Ω^2 may be neglected and that at time $t = 0$, $\theta = \theta_0$, $\dot{\theta} = 0$, and $y = \dot{y} = 0$.

To linear order in small θ, show that the ball undergoes oscillations in θ and find their frequency. Determine the associated motion in y.

Paper 4, Section II
11C Dynamics and Relativity

(a) Define the moment of inertia of a rigid body V, of density ρ, rotating about a given axis.

A thin circular disc has radius r, thickness $\delta \ll r$, and uniform density ρ. Its centre of mass is at the origin of a Cartesian coordinate system whose z-axis is perpendicular to the disc's circular face. To leading order in small δ, find the disc's moment of inertia when it is rotating about:

(i) the x-axis,

(ii) a line of the form $y = 0$, $z = h$.

(b) Consider a cone with circular cross-section, base of radius R, height H, and uniform density ρ. The cone rotates about an axis that passes through its apex and which is perpendicular to its axis of symmetry.

(i) Using part (a)(ii), or otherwise, show that the cone's moment of inertia is $I = M(\alpha R^2 + \beta H^2)$, where M is the cone's mass, and α and β are constants you need to find.

[You may assume that the volume of the cone is $\frac{1}{3}\pi R^2 H$.]

(ii) If there is no friction and initially the cone has kinetic energy K_0, how long does it take to execute one full rotation?

(iii) Now suppose there is friction so that if the cone rotates by an angle $\Delta\theta$ the work done by the friction is equal to $W\Delta\theta$, where W is a constant.

If the cone initially has kinetic energy K_0, show that it comes to rest after a time

$$t = \sqrt{\frac{2K_0 I}{W^2}}.$$

Paper 4, Section II
12C Dynamics and Relativity

(a) State the definition of a four-vector U. Prove that $U \cdot U$ is the same in all inertial frames.

(b) Relative to an inertial reference frame S, a second inertial frame S' moves with constant three-velocity $\mathbf{V} = (V, 0, 0)$, and the two frames coincide when $t = t' = 0$.

A particle is travelling with a constant three-velocity $\mathbf{u} = (u_x, u_y, 0)$, as measured in frame S, and passes through the origin of S at $t = 0$.

(i) By considering the transformation of the particle's position vector in space-time, calculate $\mathbf{u}'$, the particle's three-velocity in S'.

(ii) Suppose that V/c is small. To leading order in V/c, show that

$$\mathbf{u}' = \mathbf{u} - \mathbf{V} + \frac{(\mathbf{V} \cdot \mathbf{u})}{c^2}\mathbf{u}.$$

(iii) A light source at the origin of frame S emits photons at an angle θ relative to the x-axis. According to an observer in frame S', the photons are emitted at an angle θ' relative to the x'-axis. Show

$$\theta' - \theta = \frac{V}{c}\sin\theta,$$

to leading order in small V/c.

(c) In the laboratory frame, a photon of wavelength λ collides with an electron of mass m, initially at rest. After the collision, the three-momenta of the photon and electron are collinear. Find the wavelength of the photon after the collision.

Paper 4, Section I
3C Dynamics and Relativity

A trolley travels with initial speed v_0 along a frictionless, horizontal, linear track. It slows down by ejecting gas in the direction of motion. The gas is emitted at a constant mass ejection rate α and with constant speed u relative to the trolley. The trolley and its supply of gas initially have a combined mass of m_0. How much time is spent ejecting gas before the trolley stops? [Assume that the trolley carries sufficient gas.]

Paper 4, Section I
4C Dynamics and Relativity

A rigid body composed of N particles with positions $\mathbf{x}_i$, and masses m_i $(i = 1, 2, \ldots, N)$, rotates about the z-axis with constant angular speed ω. Show that the body's kinetic energy is $T = \frac{1}{2} I \omega^2$, where you should give an expression for the moment of inertia I in terms of the particle masses and positions.

Consider a solid cuboid of uniform density, mass M, and dimensions $2a \times 2b \times 2c$. Choose coordinate axes so that the cuboid is described by the points (x, y, z) with $-a \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$, and $-c \leqslant z \leqslant c$. In terms of M, a, b, and c, find the cuboid's moment of inertia I for rotations about the z-axis.

Paper 4, Section II

9C Dynamics and Relativity

A particle of mass m follows a one-dimensional trajectory $x(t)$ in the presence of a variable force $F(x,t)$. Write down an expression for the work done by this force as the particle moves from $x(t_a) = a$ to $x(t_b) = b$. Assuming that this is the only force acting on the particle, show that the work done by the force is equal to the change in the particle's kinetic energy.

What does it mean if a force is said to be *conservative*?

A particle moves in a force field given by

$$F(x) = \begin{cases} -F_0\, e^{-x/\lambda} & x \geqslant 0 \\ F_0\, e^{x/\lambda} & x < 0 \end{cases}$$

where F_0 and λ are positive constants. The particle starts at the origin $x = 0$ with initial velocity $v_0 > 0$. Show that, as the particle's position increases from $x = 0$ to larger $x > 0$, the particle's velocity v at position x is given by

$$v(x) = \sqrt{v_0^2 + v_e^2\left(e^{-|x|/\lambda} - 1\right)}$$

where you should determine v_e. What determines whether the particle will escape to infinity or oscillate about the origin? Sketch $v(x)$ versus x for each of these cases, carefully identifying any significant velocities or positions.

In the case of oscillatory motion, find the period of oscillation in terms of v_0, v_e, and λ. [*Hint: You may use the fact that*

$$\int_w^1 \frac{du}{u\sqrt{u-w}} = \frac{2\cos^{-1}\sqrt{w}}{\sqrt{w}}$$

for $0 < w < 1$.]

Paper 4, Section II

10C Dynamics and Relativity

(a) A mass m is acted upon by a central force

$$\mathbf{F} = -\frac{km}{r^3}\mathbf{r}$$

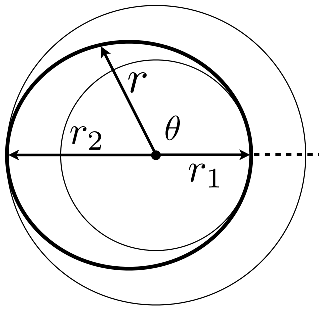
where k is a positive constant and $\mathbf{r}$ is the displacement of the mass from the origin. Show that the angular momentum and energy of the mass are conserved.

(b) Working in plane polar coordinates (r, θ), or otherwise, show that the distance $r = |\mathbf{r}|$ between the mass and the origin obeys the following differential equation

$$\ddot{r} = -\frac{k}{r^2} + \frac{h^2}{r^3}$$

where h is the angular momentum per unit mass.

(c) A satellite is initially in a *circular* orbit of radius r_1 and experiences the force described above. At $\theta = 0$ and time t_1, the satellite emits a short rocket burst putting it on an *elliptical* orbit with its closest distance to the centre r_1 and farthest distance r_2. When $\theta = \pi$ and the time is t_2, the satellite reaches the farthest distance and a second short rocket burst puts the rocket on a *circular* orbit of radius r_2. (See figure.) [Assume that the duration of the rocket bursts is negligible.]



(i) Show that the satellite's angular momentum per unit mass while in the elliptical orbit is

$$h = \sqrt{\frac{Ckr_1r_2}{r_1 + r_2}}$$

where C is a number you should determine.

(ii) What is the change in speed as a result of the rocket burst at time t_1? And what is the change in speed at t_2?

(iii) Given that the elliptical orbit can be described by

$$r = \frac{h^2}{k(1 + e\cos\theta)}$$

where e is the eccentricity of the orbit, find $t_2 - t_1$ in terms of r_1, r_2, and k. [*Hint: The area of an ellipse is equal to πab, where a and b are its semi-major and semi-minor axes; these are related to the eccentricity by $e = \sqrt{1 - \frac{b^2}{a^2}}$.*]

Paper 4, Section II

11C Dynamics and Relativity

Consider an inertial frame of reference S and a frame of reference S' which is rotating with constant angular velocity $\boldsymbol{\omega}$ relative to S. Assume that the two frames have a common origin O.

Let $\mathbf{A}$ be any vector. Explain why the derivative of $\mathbf{A}$ in frame S is related to its derivative in S' by the following equation

$$\left(\frac{d\mathbf{A}}{dt}\right)_S = \left(\frac{d\mathbf{A}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{A}\,.$$

[*Hint: It may be useful to use Cartesian basis vectors in both frames.*]

Let $\mathbf{r}(t)$ be the position vector of a particle, measured from O. Derive the expression relating the particle's acceleration as observed in S, $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S$, to the acceleration observed in S', $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'}$, written in terms of $\mathbf{r}$, $\boldsymbol{\omega}$ and $\left(\frac{d\mathbf{r}}{dt}\right)_{S'}$.

A small bead of mass m is threaded on a smooth, rigid, circular wire of radius R. At any given instant, the wire hangs in a vertical plane with respect to a downward gravitational acceleration $\mathbf{g}$. The wire is rotating with constant angular velocity $\boldsymbol{\omega}$ about its vertical diameter. Let $\theta(t)$ be the angle between the downward vertical and the radial line going from the centre of the hoop to the bead.

(i) Show that $\theta(t)$ satisfies the following equation of motion

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right) \sin\theta\,.$$

(ii) Find any equilibrium angles and determine their stability.

(iii) Find the force of the wire on the bead as a function of θ and $\dot{\theta}$.

Paper 4, Section II

12C Dynamics and Relativity

Write down the expression for the momentum of a particle of rest mass m, moving with velocity $\mathbf{v}$ where $v = |\mathbf{v}|$ is near the speed of light c. Write down the corresponding 4-momentum.

Such a particle experiences a force $\mathbf{F}$. Why is the following expression for the particle's acceleration,

$$\mathbf{a} = \frac{\mathbf{F}}{m},$$

not generally correct? Show that the force can be written as follows

$$\mathbf{F} = m\gamma \left(\frac{\gamma^2}{c^2}(\mathbf{v} \cdot \mathbf{a})\mathbf{v} + \mathbf{a} \right) .$$

Invert this expression to find the particle's acceleration as the sum of two vectors, one parallel to $\mathbf{F}$ and one parallel to $\mathbf{v}$.

A particle with rest mass m and charge q is in the presence of a constant electric field $\mathbf{E}$ which exerts a force $\mathbf{F} = q\mathbf{E}$ on the particle. If the particle is at rest at $t = 0$, its motion will be in the direction of $\mathbf{E}$ for $t > 0$. Determine the particle's speed for $t > 0$. How does the velocity behave as $t \to \infty$?

[*Hint: You may find that trigonometric substitution is helpful in evaluating an integral.*]

Paper 2, Section I

4C Dynamics and Relativity

A particle P with unit mass moves in a central potential $\Phi(r) = -k/r$ where $k > 0$. Initially P is a distance R away from the origin moving with speed u on a trajectory which, in the absence of any force, would be a straight line whose shortest distance from the origin is b. The shortest distance between P's actual trajectory and the origin is p, with $0 < p < b$, at which point it is moving with speed w.

(i) Assuming $u^2 \gg 2k/R$, find w^2/k in terms of b and p.

(ii) Assuming $u^2 < 2k/R$, find an expression for P's farthest distance from the origin q in the form

$$Aq^2 + Bq + C = 0$$

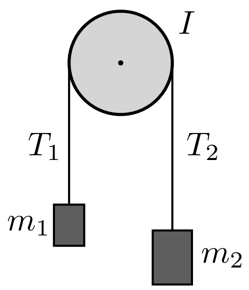
where A, B, and C depend only on R, b, k, and the angular momentum L.

[You do not need to prove that energy and angular momentum are conserved.]

Paper 2, Section II

11C Dynamics and Relativity

An axially symmetric pulley of mass M rotates about a fixed, horizontal axis, say the x-axis. A string of fixed length and negligible mass connects two blocks with masses $m_1 = M$ and $m_2 = 2M$. The string is hung over the pulley, with one mass on each side. The tensions in the string due to masses m_1 and m_2 can respectively be labelled T_1 and T_2. The moment of inertia of the pulley is $I = qMa^2$, where q is a number and a is the radius of the pulley at the points touching the string.



The motion of the pulley is opposed by a frictional torque of magnitude $\lambda M\omega$, where ω is the angular velocity of the pulley and λ is a real positive constant. Obtain a first-order differential equation for ω and, from it, find $\omega(t)$ given that the system is released from rest.

The surface of the pulley is defined by revolving the function $b(x)$ about the x-axis, with

$$b(x) = \begin{cases} a(1+|x|) & -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find a value for the constant q given that the pulley has uniform mass density ρ.

Paper 2, Section II

12C Dynamics and Relativity

(a) A moving particle with rest mass M decays into two particles (photons) with zero rest mass. Derive an expression for $\sin\frac{\theta}{2}$, where θ is the angle between the spatial momenta of the final state particles, and show that it depends only on Mc^2 and the energies of the massless particles. (c is the speed of light in vacuum.)

(b) A particle P with rest mass M decays into two particles: a particle R with rest mass $0 < m < M$ and another particle with zero rest mass. Using dimensional analysis explain why the speed v of R in the rest frame of P can be expressed as

$$v = cf(r), \quad \text{with} \quad r = \frac{m}{M},$$

and f a dimensionless function of r. Determine the function $f(r)$.

Choose coordinates in the rest frame of P such that R is emitted at $t = 0$ from the origin in the x-direction. The particle R decays after a time τ, measured in its own rest frame. Determine the spacetime coordinates (ct, x), in the rest frame of P, corresponding to this event.

Paper 4, Section I

3A Dynamics and Relativity

A rocket of mass $m(t)$ moving at speed $v(t)$ and ejecting fuel behind it at a constant speed u relative to the rocket, is subject to an external force F. Considering a small time interval δt, derive the rocket equation

$$m\frac{dv}{dt} + u\frac{dm}{dt} = F.$$

In deep space where $F = 0$, how much faster does the rocket go if it burns half of its mass in fuel?

Paper 4, Section I

4A Dynamics and Relativity

Galileo releases a cannonball of mass m from the top of the leaning tower of Pisa, a vertical height h above the ground. Ignoring the rotation of the Earth but assuming that the cannonball experiences a quadratic drag force whose magnitude is γv^2 (where v is the speed of the cannonball), find the time for it to hit the ground in terms of h, m, γ and g, the acceleration due to gravity. [You may assume that g is constant.]

Paper 4, Section II

9A Dynamics and Relativity

In an alien invasion, a flying saucer hovers at a fixed point S, a height l far above the White House, which is at point W. A wrecking ball of mass m is attached to one end of a light inextensible rod, also of length l. The other end of the rod is attached to the flying saucer. The wrecking ball is initially at rest at point B, and the angle WSB is θ_0. At W, the acceleration due to gravity is g. Assume that the rotation of the Earth can be neglected and that the only force acting is Earth's gravity.

(a) Under the approximations that gravity acts everywhere parallel to the line SW and that the acceleration due to Earth's gravity is constant throughout the space through which the wrecking ball is travelling, find the speed v_1 with which the wrecking ball hits the White House, in terms of the constants introduced above.

(b) Taking into account the fact that gravity is non-uniform and acts toward the centre of the Earth, find the speed v_2 with which the wrecking ball hits the White House in terms of the constants introduced above and R, where R is the radius of the Earth, which you may assume is exactly spherical.

(c) Finally, show that

$$v_2 = v_1 \left(1 + (A + B\cos\theta_0)\frac{l}{R} + O\left(\frac{l^2}{R^2}\right)\right),$$

where A and B are constants, which you should determine.

Paper 4, Section II

10A Dynamics and Relativity

(a) A particle of mass m and positive charge q moves with velocity $\dot{\mathbf{x}}$ in a region in which the magnetic field $\mathbf{B} = (0, 0, B)$ is constant and no other forces act, where $B > 0$. Initially, the particle is at position $\mathbf{x} = (1, 0, 0)$ and $\dot{\mathbf{x}} = (0, v, v)$. Write the equation of motion of the particle and then solve it to find $\mathbf{x}$ as a function of time t. Sketch its path in (x, y, z).

(b) For $B = 0$, three point particles, each of charge q, are fixed at $(0, a/\sqrt{3}, 0)$, $(a/2, -a/(2\sqrt{3}), 0)$ and $(-a/2, -a/(2\sqrt{3}), 0)$, respectively. Another point particle of mass m and charge q is constrained to move in the $z = 0$ plane and suffers Coulomb repulsion from each fixed charge. Neglecting any magnetic fields,

(i) Find the position of an equilibrium point.

(ii) By finding the form of the electric potential near this point, deduce that the equilibrium is stable.

(iii) Consider small displacements of the point particle from the equilibrium point. By resolving forces in the directions $(1, 0, 0)$ and $(0, 1, 0)$, show that the frequency of oscillation is

$$\omega = A \frac{|q|}{\sqrt{m\epsilon_0 a^3}},$$

where A is a constant which you should find.

[You may assume that if two identical charges q are separated by a distance d then the repulsive Coulomb force experienced by each of the charges is $q^2/(4\pi\epsilon_0 d^2)$, where ϵ_0 is a constant.]

Paper 4, Section II

11A Dynamics and Relativity

(a) Writing a mass dimension as M, a time dimension as T, a length dimension as L and a charge dimension as Q, write, using relations that you know, the dimensions of:
(i) force
(ii) electric field

(b) In the Large Hadron Collider at CERN, a proton of rest mass m and charge $q > 0$ is accelerated by a constant electric field $\mathbf{E} \neq \mathbf{0}$. At time $t = 0$, the particle is at rest at the origin.

Writing the proton's position as $\mathbf{x}(t)$ and including relativistic effects, calculate $\dot{\mathbf{x}}(t)$. Use your answers to part (a) to check that the dimensions in your expression are correct.

Sketch a graph of $|\dot{\mathbf{x}}(t)|$ versus t, commenting on the $t \to \infty$ limit.

Calculate $|\mathbf{x}(t)|$ as an explicit function of t and find the non-relativistic limit at small times t. What kind of motion is this?

(c) At a later time t_0, an observer in the laboratory frame sees a cosmic microwave photon of energy E_γ hit the accelerated proton, leaving only a Δ^+ particle of mass m_Δ in the final state. In its rest frame, the Δ^+ takes a time t_Δ to decay. How long does it take to decay in the laboratory frame as a function of $q, \mathbf{E}, t_0, m, E_\gamma, m_\Delta, t_\Delta$ and c, the speed of light in a vacuum?

Paper 4, Section II

12A Dynamics and Relativity

An inertial frame S and another reference frame S' have a common origin O, and S' rotates with angular velocity vector $\boldsymbol{\omega}(t)$ with respect to S. Derive the results (a) and (b) below, where dot denotes a derivative with respect to time t:

(a) The rates of change of an arbitrary vector $\mathbf{a}(t)$ in frames S and S' are related by

$$(\dot{\mathbf{a}})_S = (\dot{\mathbf{a}})_{S'} + \boldsymbol{\omega} \times \mathbf{a}.$$

(b) The accelerations in S and S' are related by

$$(\ddot{\mathbf{r}})_S = (\ddot{\mathbf{r}})_{S'} + 2\boldsymbol{\omega} \times (\dot{\mathbf{r}})_{S'} + (\dot{\boldsymbol{\omega}})_{S'} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),$$

where $\mathbf{r}(t)$ is the position vector relative to O.

Just after passing the South Pole, a ski-doo of mass m is travelling on a constant longitude with speed v. Find the magnitude and direction of the sideways component of apparent force experienced by the ski-doo. [The sideways component is locally along the surface of the Earth and perpendicular to the motion of the ski-doo.]

Paper 4, Section I

3A Dynamics and Relativity

(a) Define an *inertial frame.*

(b) Specify three different types of Galilean transformation on inertial frames whose space coordinates are $\mathbf{x}$ and whose time coordinate is t.

(c) State the *Principle of Galilean Relativity.*

(d) Write down the equation of motion for a particle in one dimension x in a potential $V(x)$. Prove that energy is conserved. A particle is at position x_0 at time t_0. Find an expression for time t as a function of x in terms of an integral involving V.

(e) Write down the x values of any equilibria and state (without justification) whether they are stable or unstable for:
(i) $V(x) = (x^2 - 4)^2$
(ii) $V(x) = e^{-1/x^2}$ for $x \neq 0$ and $V(0) = 0$.

Paper 4, Section I

4A Dynamics and Relativity

Explain what is meant by a *central force* acting on a particle moving in three dimensions.

Show that the angular momentum of a particle about the origin for a central force is conserved, and hence that its path lies in a plane.

Show that, in the approximation in which the Sun is regarded as fixed and only its gravitational field is considered, a straight line joining the Sun and an orbiting planet sweeps out equal areas in equal time (Kepler's second law).

Paper 4, Section II

9A Dynamics and Relativity

Consider a rigid body, whose shape and density distribution are rotationally symmetric about a horizontal axis. The body has mass M, radius a and moment of inertia I about its axis of rotational symmetry and is rolling down a non-slip slope inclined at an angle α to the horizontal. By considering its energy, calculate the acceleration of the disc down the slope in terms of the quantities introduced above and g, the acceleration due to gravity.

(a) A sphere with density proportional to r^c (where r is distance to the sphere's centre and c is a positive constant) is launched up a non-slip slope of constant incline at the same time, level and speed as a vertical disc of constant density. Find c such that the sphere and the disc return to their launch points at the same time.

(b) Two spherical glass marbles of equal radius are released from rest at time $t = 0$ on an inclined non-slip slope of constant incline from the same level. The glass in each marble is of constant and equal density, but the second marble has two spherical air bubbles in it whose radii are half the radius of the marbles, initially centred directly above and below the marble's centre, respectively. Each bubble is centred half-way between the centre of the marble and its surface. At a later time t, find the ratio of the distance travelled by the first marble and the second. [You may state without proof any theorems that you use and neglect the mass of air in the bubbles.]

Paper 4, Section II

10A Dynamics and Relativity

Define the *4-momentum* P of a particle of rest mass m and velocity $\mathbf{u}$. Calculate the power series expansion of the component P^0 for small $|\mathbf{u}|/c$ (where c is the speed of light in vacuo) up to and including terms of order $|\mathbf{u}|^4$, and interpret the first two terms.

(a) At CERN, anti-protons are made by colliding a moving proton with another proton at rest in a fixed target. The collision in question produces three protons and an anti-proton. Assume that the rest mass of a proton is identical to the rest mass of an anti-proton. What is the smallest possible speed of the incoming proton (measured in the laboratory frame)?

(b) A moving particle of rest mass M decays into N particles with 4-momenta Q_i, and rest masses m_i, where $i = 1, 2, \ldots, N$. Show that

$$M = \frac{1}{c}\sqrt{\left(\sum_{i=1}^{N} Q_i\right) \cdot \left(\sum_{j=1}^{N} Q_j\right)}.$$

Thus, show that

$$M \geqslant \sum_{i=1}^{N} m_i.$$

(c) A particle A decays into particle B and a massless particle 1. Particle B subsequently decays into particle C and a massless particle 2. Show that

$$0 \leqslant (Q_1 + Q_2) \cdot (Q_1 + Q_2) \leqslant \frac{(m_A^2 - m_B^2)(m_B^2 - m_C^2)c^2}{m_B^2},$$

where Q_1 and Q_2 are the 4-momenta of particles 1 and 2 respectively and m_A, m_B, m_C are the masses of particles A, B and C respectively.

Paper 4, Section II

11A Dynamics and Relativity

Write down the Lorentz force law for a charge q travelling at velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

In a space station which is in an inertial frame, an experiment is performed in vacuo where a ball is released from rest a distance h from a wall. The ball has charge $q > 0$ and at time t, it is a distance $z(t)$ from the wall. A constant electric field of magnitude E points toward the wall in a perpendicular direction, but there is no magnetic field. Find the speed of the ball on its first impact.

Every time the ball bounces, its speed is reduced by a factor $\gamma < 1$. Show that the total distance travelled by the ball before it comes to rest is

$$L = h\frac{q_1(\gamma)}{q_2(\gamma)}$$

where q_1 and q_2 are quadratic functions which you should find explicitly.

A gas leak fills the apparatus with an atmosphere and the experiment is repeated. The ball now experiences an additional drag force $\mathbf{D} = -\alpha|\mathbf{v}|\mathbf{v}$, where $\mathbf{v}$ is the instantaneous velocity of the ball and $\alpha > 0$. Solve the system before the first bounce, finding an explicit solution for the distance $z(t)$ between the ball and the wall as a function of time of the form

$$z(t) = h - Af(Bt)$$

where f is a function and A and B are dimensional constants, all of which you should find explicitly.

Paper 4, Section II

12A Dynamics and Relativity

The position $\mathbf{x} = (x, y, z)$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. The particle is subject to a force $\mathbf{F} = -9m|\boldsymbol{\omega}|^2\mathbf{x}$. What is the equation of motion of the particle?

Find the trajectory of the particle in the coordinates (x, y, z) if $\boldsymbol{\omega} = (0, 0, \omega)$ and at $t = 0$, $\mathbf{x} = (1, 0, 0)$ and $\dot{\mathbf{x}} = (0, 0, 0)$.

Find the maximum value of the speed $|\dot{\mathbf{x}}|$ of the particle and the times at which it travels at this speed.

[*Hint: You may find using the variable* $\xi = x + iy$ *helpful.*]

Paper 4, Section I

3A Dynamics and Relativity

Consider a system of particles with masses m_i and position vectors $\mathbf{x}_i$. Write down the definition of the position of the *centre of mass* $\mathbf{R}$ of the system. Let T be the total kinetic energy of the system. Show that

$$T = \frac{1}{2}M\dot{\mathbf{R}}\cdot\dot{\mathbf{R}} + \frac{1}{2}\sum_i m_i\dot{\mathbf{y}}_i\cdot\dot{\mathbf{y}}_i\,,$$

where M is the total mass and $\mathbf{y}_i$ is the position vector of particle i with respect to $\mathbf{R}$.

The particles are connected to form a rigid body which rotates with angular speed ω about an axis $\mathbf{n}$ through $\mathbf{R}$, where $\mathbf{n}\cdot\mathbf{n} = 1$. Show that

$$T = \frac{1}{2}M\dot{\mathbf{R}}\cdot\dot{\mathbf{R}} + \frac{1}{2}I\omega^2\,,$$

where $I = \sum_i I_i$ and I_i is the moment of inertia of particle i about $\mathbf{n}$.

Paper 4, Section I

4A Dynamics and Relativity

A tennis ball of mass m is projected vertically upwards with initial speed u_0 and reaches its highest point at time T. In addition to uniform gravity, the ball experiences air resistance, which produces a frictional force of magnitude αv, where v is the ball's speed and α is a positive constant. Show by dimensional analysis that T can be written in the form

$$T = \frac{m}{\alpha}\,f(\lambda)$$

for some function f of a dimensionless quantity λ.

Use the equation of motion of the ball to find $f(\lambda)$.

Paper 4, Section II
9A Dynamics and Relativity

(a) A photon with energy E_1 in the laboratory frame collides with an electron of rest mass m that is initially at rest in the laboratory frame. As a result of the collision the photon is deflected through an angle θ as measured in the laboratory frame and its energy changes to E_2.

Derive an expression for $\dfrac{1}{E_2} - \dfrac{1}{E_1}$ in terms of θ, m and c.

(b) A deuterium atom with rest mass m_1 and energy E_1 in the laboratory frame collides with another deuterium atom that is initially at rest in the laboratory frame. The result of this collision is a proton of rest mass m_2 and energy E_2, and a tritium atom of rest mass m_3. Show that, if the proton is emitted perpendicular to the incoming trajectory of the deuterium atom as measured in the laboratory frame, then

$$m_3^2 = m_2^2 + 2\left(m_1 + \frac{E_1}{c^2}\right)\left(m_1 - \frac{E_2}{c^2}\right).$$

Paper 4, Section II

10A Dynamics and Relativity

A particle of unit mass moves under the influence of a central force. By considering the components of the acceleration in polar coordinates (r, θ) prove that the magnitude of the angular momentum is conserved. [You may use $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$.]

Now suppose that the central force is derived from the potential k/r, where k is a constant.

(a) Show that the total energy of the particle can be written in the form

$$E = \tfrac{1}{2}\dot{r}^2 + V_{\text{eff}}(r).$$

Sketch $V_{\text{eff}}(r)$ in the cases $k > 0$ and $k < 0$.

(b) The particle is projected from a very large distance from the origin with speed v and impact parameter b. [The *impact parameter* is the distance of closest approach to the origin in absence of any force.]

(i) In the case $k < 0$, sketch the particle's trajectory and find the shortest distance p between the particle and the origin, and the speed u of the particle when $r = p$.

(ii) In the case $k > 0$, sketch the particle's trajectory and find the corresponding shortest distance $\widetilde{p}$ between the particle and the origin, and the speed $\widetilde{u}$ of the particle when $r = \widetilde{p}$.

(iii) Find $p\widetilde{p}$ and $u\widetilde{u}$ in terms of b and v. [In answering part (iii) you should assume that $|k|$ is the same in parts (i) and (ii).]

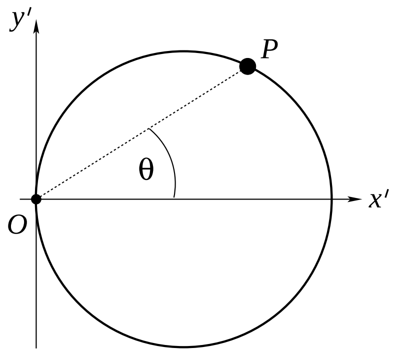
Paper 4, Section II
11A Dynamics and Relativity

(a) Consider an inertial frame S, and a frame S' which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to S. The two frames share a common origin. Identify each term in the equation

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_S - 2\,\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

(b) A small bead P of unit mass can slide without friction on a circular hoop of radius a. The hoop is horizontal and rotating with constant angular speed ω about a fixed vertical axis through a point O on its circumference.

(i) Using Cartesian axes in the rotating frame S', with origin at O and x'-axis along the diameter of the hoop through O, write down the position vector of P in terms of a and the angle θ shown in the diagram $(-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi)$.



(ii) Working again in the rotating frame, find, in terms of a, θ, $\dot{\theta}$ and ω, an expression for the horizontal component of the force exerted by the hoop on the bead.

(iii) For what value of θ is the bead in stable equilibrium? Find the frequency of small oscillations of the bead about that point.

Paper 4, Section II

12A Dynamics and Relativity

(a) A rocket moves in a straight line with speed $v(t)$ and is subject to no external forces. The rocket is composed of a body of mass M and fuel of mass $m(t)$, which is burnt at constant rate α and the exhaust is ejected with constant speed u relative to the rocket. Show that

$$(M+m)\frac{dv}{dt} - \alpha u = 0\,.$$

Show that the speed of the rocket when all its fuel is burnt is

$$v_0 + u \log\left(1 + \frac{m_0}{M}\right)\,,$$

where v_0 and m_0 are the speed of the rocket and the mass of the fuel at $t = 0$.

(b) A two-stage rocket moves in a straight line and is subject to no external forces. The rocket is initially at rest. The masses of the bodies of the two stages are kM and $(1-k)M$, with $0 \leqslant k \leqslant 1$, and they initially carry masses km_0 and $(1-k)m_0$ of fuel. Both stages burn fuel at a constant rate α when operating and the exhaust is ejected with constant speed u relative to the rocket. The first stage operates first, until all its fuel is burnt. The body of the first stage is then detached with negligible force and the second stage ignites.

Find the speed of the second stage when all its fuel is burnt. For $0 \leqslant k < 1$ compare it with the speed of the rocket in part (a) in the case $v_0 = 0$. Comment on the case $k = 1$.

Paper 4, Section I

3B Dynamics and Relativity

With the help of definitions or equations of your choice, determine the dimensions, in terms of mass (M), length (L), time (T) and charge (Q), of the following quantities:

(i) force;

(ii) moment of a force (*i.e.* torque);

(iii) energy;

(iv) Newton's gravitational constant G;

(v) electric field $\mathbf{E}$;

(vi) magnetic field $\mathbf{B}$;

(vii) the vacuum permittivity ϵ_0.

Paper 4, Section I

4B Dynamics and Relativity

The radial equation of motion of a particle moving under the influence of a central force is

$$\ddot{r} - \frac{h^2}{r^3} = -kr^n,$$

where h is the angular momentum per unit mass of the particle, n is a constant, and k is a positive constant.

Show that circular orbits with $r = a$ are possible for any positive value of a, and that they are stable to small perturbations that leave h unchanged if $n > -3$.

Paper 4, Section II
9B Dynamics and Relativity

(a) A rocket, moving non-relativistically, has speed $v(t)$ and mass $m(t)$ at a time t after it was fired. It ejects mass with constant speed u relative to the rocket. Let the total momentum, at time t, of the system (rocket and ejected mass) in the direction of the motion of the rocket be $P(t)$. Explain carefully why $P(t)$ can be written in the form

$$P(t) = m(t)\, v(t) - \int_0^t \big(v(\tau) - u\big) \frac{dm(\tau)}{d\tau} d\tau \,. \qquad (*)$$

If the rocket experiences no external force, show that

$$m\frac{dv}{dt} + u\frac{dm}{dt} = 0 \,. \qquad (\dagger)$$

Derive the expression corresponding to $(*)$ for the total kinetic energy of the system at time t. Show that kinetic energy is not necessarily conserved.

(b) Explain carefully how $(*)$ should be modified for a rocket moving relativistically, given that there are no external forces. Deduce that

$$\frac{d(m\gamma v)}{dt} = \left(\frac{v-u}{1-uv/c^2}\right)\frac{d(m\gamma)}{dt} \,,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ and hence that

$$m\gamma^2\frac{dv}{dt} + u\frac{dm}{dt} = 0 \,. \qquad (\ddagger)$$

(c) Show that $(\dagger)$ and $(\ddagger)$ agree in the limit $c \to \infty$. Briefly explain the fact that kinetic energy is not conserved for the non-relativistic rocket, but relativistic energy is conserved for the relativistic rocket.

Paper 4, Section II
10B Dynamics and Relativity

A particle of unit mass moves with angular momentum h in an attractive central force field of magnitude $\dfrac{k}{r^2}$, where r is the distance from the particle to the centre and k is a constant. *You may assume* that the equation of its orbit can be written in plane polar coordinates in the form

$$r = \frac{\ell}{1 + e\cos\theta}\,,$$

where $\ell = \dfrac{h^2}{k}$ and e is the eccentricity. Show that the energy of the particle is

$$\frac{h^2(e^2-1)}{2\ell^2}\,.$$

A comet moves in a parabolic orbit about the Sun. When it is at its perihelion, a distance d from the Sun, and moving with speed V, it receives an impulse which imparts an additional velocity of magnitude αV directly away from the Sun. Show that the eccentricity of its new orbit is $\sqrt{1+4\alpha^2}$, and sketch the two orbits on the same axes.

Paper 4, Section II
11B Dynamics and Relativity

(a) Alice travels at constant speed v to Alpha Centauri, which is at distance d from Earth. She then turns around (taking very little time to do so), and returns at speed v. Bob stays at home. By how much has Bob aged during the journey? By how much has Alice aged? [No justification is required.]

Briefly explain what is meant by the *twin paradox* in this context. Why is it not a paradox?

(b) Suppose instead that Alice's world line is given by

$$-c^2t^2 + x^2 = c^2t_0^2\,,$$

where t_0 is a positive constant. Bob stays at home, at $x = \alpha c t_0$, where $\alpha > 1$. Alice and Bob compare their ages on both occasions when they meet. By how much does Bob age? Show that Alice ages by $2t_0\cosh^{-1}\alpha$.

Paper 4, Section II

12B Dynamics and Relativity

State what the vectors $\mathbf{a}$, $\mathbf{r}$, $\mathbf{v}$ and $\boldsymbol{\omega}$ represent in the following equation:

$$\mathbf{a} = \mathbf{g} - 2\,\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,, \qquad (*)$$

where $\mathbf{g}$ is the acceleration due to gravity.

Assume that the radius of the Earth is 6×10^6 m, that $|\mathbf{g}| = 10\,\mathrm{ms}^{-2}$, and that there are 9×10^4 seconds in a day. Use these data to determine roughly the order of magnitude of each term on the right hand side of $(*)$ in the case of a particle dropped from a point at height 20 m above the surface of the Earth.

Taking again $|\mathbf{g}| = 10\,\mathrm{ms}^{-2}$, find the time T of the particle's fall in the absence of rotation.

Use a suitable approximation scheme to show that

$$\mathbf{R} \approx \mathbf{R}_0 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}\,T^3 - \frac{1}{2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}_0)\,T^2\,,$$

where $\mathbf{R}$ is the position vector of the point at which the particle lands, and $\mathbf{R}_0$ is the position vector of the point at which the particle would have landed in the absence of rotation.

The particle is dropped at latitude 45°. Find expressions for the approximate northerly and easterly displacements of $\mathbf{R}$ from $\mathbf{R}_0$ in terms of ω, g, R_0 (the magnitudes of $\boldsymbol{\omega}$, $\mathbf{g}$ and $\mathbf{R}_0$, respectively), and T. You should ignore the curvature of the Earth's surface.

Paper 4, Section I

3C Dynamics and Relativity

Find the moment of inertia of a uniform sphere of mass M and radius a about an axis through its centre.

The kinetic energy T of any rigid body with total mass M, centre of mass $\mathbf{R}$, moment of inertia I about an axis of rotation through $\mathbf{R}$, and angular velocity ω about that same axis, is given by $T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}I\omega^2$. What physical interpretation can be given to the two parts of this expression?

A spherical marble of uniform density and mass M rolls without slipping at speed V along a flat surface. Explaining any relationship that you use between its speed and angular velocity, show that the kinetic energy of the marble is $\frac{7}{10}MV^2$.

Paper 4, Section I

4C Dynamics and Relativity

Write down the 4-momentum of a particle with energy E and 3-momentum $\mathbf{p}$. State the relationship between the energy E and wavelength λ of a photon.

An electron of mass m is at rest at the origin of the laboratory frame: write down its 4-momentum. The electron is scattered by a photon of wavelength λ_1 travelling along the x-axis: write down the initial 4-momentum of the photon. Afterwards, the photon has wavelength λ_2 and has been deflected through an angle θ. Show that

$$\lambda_2 - \lambda_1 = \frac{2h}{mc}\sin^2(\tfrac{1}{2}\theta)$$

where c is the speed of light and h is Planck's constant.

Paper 4, Section II

9C Dynamics and Relativity

A particle is projected vertically upwards at speed V from the surface of the Earth, which may be treated as a perfect sphere. The variation of gravity with height should not be ignored, but the rotation of the Earth should be. Show that the height $z(t)$ of the particle obeys

$$\ddot{z} = -\frac{gR^2}{(R+z)^2},$$

where R is the radius of the Earth and g is the acceleration due to gravity measured at the Earth's surface.

Using dimensional analysis, show that the maximum height H of the particle and the time T taken to reach that height are given by

$$H = RF(\lambda) \quad \text{and} \quad T = \frac{V}{g}G(\lambda),$$

where F and G are functions of $\lambda = V^2/gR$.

Write down the equation of conservation of energy and deduce that

$$T = \int_0^H \sqrt{\frac{R+z}{V^2R - (2gR - V^2)z}}\, dz.$$

Hence or otherwise show that

$$F(\lambda) = \frac{\lambda}{2-\lambda} \quad \text{and} \quad G(\lambda) = \int_0^1 \sqrt{\frac{2-\lambda+\lambda x}{(2-\lambda)^3(1-x)}}\, dx.$$

Paper 4, Section II

10C Dynamics and Relativity

A particle of mass m and charge q has position vector $\mathbf{r}(t)$ and moves in a constant, uniform magnetic field $\mathbf{B}$ so that its equation of motion is

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}.$$

Let $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ be the particle's angular momentum. Show that

$$\mathbf{L} \cdot \mathbf{B} + \tfrac{1}{2}q|\mathbf{r} \times \mathbf{B}|^2$$

is a constant of the motion. Explain why the kinetic energy T is also constant, and show that it may be written in the form

$$T = \tfrac{1}{2}m\mathbf{u} \cdot \big((\mathbf{u} \cdot \mathbf{v})\mathbf{v} - r^2\ddot{\mathbf{u}}\big),$$

where $\mathbf{v} = \dot{\mathbf{r}}$, $r = |\mathbf{r}|$ and $\mathbf{u} = \mathbf{r}/r$.

[*Hint: Consider* $\mathbf{u} \cdot \dot{\mathbf{u}}$.]

Paper 4, Section II

11C Dynamics and Relativity

Consider a particle with position vector $\mathbf{r}(t)$ moving in a plane described by polar coordinates (r, θ). Obtain expressions for the radial (r) and transverse (θ) components of the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$.

A charged particle of unit mass moves in the electric field of another charge that is fixed at the origin. The electrostatic force on the particle is $-p/r^2$ in the radial direction, where p is a positive constant. The motion takes place in an unusual medium that resists radial motion but not tangential motion, so there is an additional radial force $-k\dot{r}/r^2$ where k is a positive constant. Show that the particle's motion lies in a plane. Using polar coordinates in that plane, show also that its angular momentum $h = r^2\dot{\theta}$ is constant.

Obtain the equation of motion

$$\frac{d^2u}{d\theta^2} + \frac{k}{h}\frac{du}{d\theta} + u = \frac{p}{h^2},$$

where $u = r^{-1}$, and find its general solution assuming that $k/|h| < 2$. Show that so long as the motion remains bounded it eventually becomes circular with radius h^2/p.

Obtain the expression

$$E = \tfrac{1}{2}h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) - pu$$

for the particle's total energy, that is, its kinetic energy plus its electrostatic potential energy. Hence, or otherwise, show that the energy is a decreasing function of time.

Paper 4, Section II

12C Dynamics and Relativity

Write down the Lorentz transform relating the components of a 4-vector between two inertial frames.

A particle moves along the x-axis of an inertial frame. Its position at time t is $x(t)$, its velocity is $u = dx/dt$, and its 4-position is $X = (ct, x)$, where c is the speed of light. The particle's 4-velocity is given by $U = dX/d\tau$ and its 4-acceleration is $A = dU/d\tau$, where *proper time* τ is defined by $c^2 d\tau^2 = c^2 dt^2 - dx^2$. Show that

$$U = \gamma\,(c, u) \quad \text{and} \quad A = \gamma^4 \dot{u}\,(u/c, 1)$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$ and $\dot{u} = du/dt$.

The *proper 3-acceleration* a of the particle is defined to be the spatial component of its 4-acceleration measured in the particle's instantaneous rest frame. By transforming A to the rest frame, or otherwise, show that

$$a = \gamma^3 \dot{u} = \frac{d}{dt}(\gamma u).$$

Given that the particle moves with constant proper 3-acceleration starting from rest at the origin, show that

$$x(t) = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2 t^2}{c^2}} - 1\right),$$

and that, if $at \ll c$, then $x \approx \frac{1}{2} a t^2$.

Paper 4, Section I

3C Dynamics and Relativity

A particle of mass m has charge q and moves in a constant magnetic field $\mathbf{B}$. Show that the particle's path describes a helix. In which direction is the axis of the helix, and what is the particle's rotational angular frequency about that axis?

Paper 4, Section I

4C Dynamics and Relativity

What is a *4-vector*? Define the inner product of two 4-vectors and give the meanings of the terms *timelike*, *null* and *spacelike*. How do the four components of a 4-vector change under a Lorentz transformation of speed v? [Without loss of generality, you may take the velocity of the transformation to be along the positive x-axis.]

Show that a 4-vector that is timelike in one frame of reference is also timelike in a second frame of reference related by a Lorentz transformation. [Again, you may without loss of generality take the velocity of the transformation to be along the positive x-axis.]

Show that any null 4-vector may be written in the form $a(1, \hat{\mathbf{n}})$ where a is real and $\hat{\mathbf{n}}$ is a unit 3-vector. Given any two null 4-vectors that are *future-pointing*, that is, which have positive time-components, show that their sum is either null or timelike.

Paper 4, Section II

9C Dynamics and Relativity

A rocket of mass $m(t)$, which includes the mass of its fuel and everything on board, moves through free space in a straight line at speed $v(t)$. When its engines are operational, they burn fuel at a constant mass rate α and eject the waste gases behind the rocket at a constant speed u relative to the rocket. Obtain the rocket equation

$$m\frac{dv}{dt} - \alpha u = 0.$$

The rocket is initially at rest in a cloud of space dust which is also at rest. The engines are started and, as the rocket travels through the cloud, it collects dust which it stores on board for research purposes. The mass of dust collected in a time δt is given by $\beta\,\delta x$, where δx is the distance travelled in that time and β is a constant. Obtain the new equations

$$\begin{aligned}\frac{dm}{dt} &= \beta v - \alpha,\\ m\frac{dv}{dt} &= \alpha u - \beta v^2.\end{aligned}$$

By eliminating t, or otherwise, obtain the relationship

$$m = \lambda m_0 u\sqrt{\frac{(\lambda u - v)^{\lambda-1}}{(\lambda u + v)^{\lambda+1}}}\,,$$

where m_0 is the initial mass of the rocket and $\lambda = \sqrt{\alpha/\beta u}$.

If $\lambda > 1$, show that the fuel will be exhausted before the speed of the rocket can reach λu. Comment on the case when $\lambda < 1$, giving a physical interpretation of your answer.

Paper 4, Section II

10C Dynamics and Relativity

A reference frame S' rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame S that has the same origin as S'. A particle of mass m at position vector $\mathbf{x}$ is subject to a force $\mathbf{F}$. Derive the equation of motion for the particle in S'.

A marble moves on a smooth plane which is inclined at an angle θ to the horizontal. The whole plane rotates at constant angular speed ω about a vertical axis through a point O fixed in the plane. Coordinates (ξ, η) are defined with respect to axes fixed in the plane: $O\xi$ horizontal and $O\eta$ up the line of greatest slope in the plane. Ensuring that you account for the normal reaction force, show that the motion of the marble obeys

$$\begin{aligned}\ddot{\xi} &= \omega^2 \xi + 2\omega \dot{\eta} \cos\theta, \\ \ddot{\eta} &= \omega^2 \eta \cos^2\theta - 2\omega \dot{\xi} \cos\theta - g \sin\theta.\end{aligned}$$

By considering the marble's kinetic energy as measured on the plane in the rotating frame, or otherwise, find a constant of the motion.

[You may assume that the marble never leaves the plane.]

Paper 4, Section II

11C Dynamics and Relativity

A thin flat disc of radius a has density (mass per unit area) $\rho(r,\theta) = \rho_0(a-r)$ where (r,θ) are plane polar coordinates on the disc and ρ_0 is a constant. The disc is free to rotate about a light, thin rod that is rigidly fixed in space, passing through the centre of the disc orthogonal to it. Find the moment of inertia of the disc about the rod.

The section of the disc lying in $r \geqslant \frac{1}{2}a$, $-\frac{\pi}{13} \leqslant \theta \leqslant \frac{\pi}{13}$ is cut out and removed. Starting from rest, a constant torque τ is applied to the remaining part of the disc until its angular speed about the axis reaches Ω. Show that this takes a time

$$\frac{3\pi\rho_0 a^5 \Omega}{32\tau}.$$

After this time, no further torque is applied and the partial disc continues to rotate at constant angular speed Ω. Given that the total mass of the partial disc is $k\rho_0 a^3$, where k is a constant that you need not determine, find the position of the centre of mass, and hence its acceleration. From where does the force required to produce this acceleration arise?

Paper 4, Section II

12C Dynamics and Relativity

Define the *4-momentum* of a particle and describe briefly the principle of conservation of 4-momentum.

A photon of angular frequency ω is absorbed by a particle of rest mass m that is stationary in the laboratory frame of reference. The particle then splits into two equal particles, each of rest mass αm.

Find the maximum possible value of α as a function of $\mu = \hbar\omega/mc^2$. Verify that as $\mu \to 0$, this maximum value tends to $\frac{1}{2}$. For general μ, show that when the maximum value of α is achieved, the resulting particles are each travelling at speed $c/(1+\mu^{-1})$ in the laboratory frame.

Paper 4, Section I

3B Dynamics and Relativity

A hot air balloon of mass M is equipped with a bag of sand of mass $m = m(t)$ which decreases in time as the sand is gradually released. In addition to gravity the balloon experiences a constant upwards buoyancy force T and we neglect air resistance effects. Show that if $v(t)$ is the upward speed of the balloon then

$$(M+m)\frac{dv}{dt} = T - (M+m)g.$$

Initially at $t = 0$ the mass of sand is $m(0) = m_0$ and the balloon is at rest in equilibrium. Subsequently the sand is released at a constant rate and is depleted in a time t_0. Show that the speed of the balloon at time t_0 is

$$gt_0 \left(\left(1 + \frac{M}{m_0}\right) \ln\left(1 + \frac{m_0}{M}\right) - 1 \right).$$

[You may use without proof the indefinite integral $\int t/(A-t)\,dt = -t - A\ln|A-t| + C$.]

Paper 4, Section I

4B Dynamics and Relativity

A frame S' moves with constant velocity v along the x axis of an inertial frame S of Minkowski space. A particle P moves with constant velocity u' along the x' axis of S'. Find the velocity u of P in S.

The rapidity φ of any velocity w is defined by $\tanh\varphi = w/c$. Find a relation between the rapidities of u, u' and v.

Suppose now that P is initially at rest in S and is subsequently given n successive velocity increments of $c/2$ (each delivered in the instantaneous rest frame of the particle). Show that the resulting velocity of P in S is

$$c\left(\frac{e^{2n\alpha}-1}{e^{2n\alpha}+1}\right)$$

where $\tanh\alpha = 1/2$.

[You may use without proof the addition formulae $\sinh(a+b) = \sinh a\cosh b + \cosh a \sinh b$ and $\cosh(a+b) = \cosh a\cosh b + \sinh a\sinh b$.]

Paper 4, Section II

9B Dynamics and Relativity

(a) A particle P of unit mass moves in a plane with polar coordinates (r, θ). You may assume that the radial and angular components of the acceleration are given by $(\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$, where the dot denotes d/dt. The particle experiences a central force corresponding to a potential $V = V(r)$.

(i) Prove that $l = r^2\dot{\theta}$ is constant in time and show that the time dependence of the radial coordinate $r(t)$ is equivalent to the motion of a particle in one dimension x in a potential V_{eff} given by

$$V_{\text{eff}} = V(x) + \frac{l^2}{2x^2}.$$

(ii) Now suppose that $V(r) = -e^{-r}$. Show that if $l^2 < 27/e^3$ then two circular orbits are possible with radii $r_1 < 3$ and $r_2 > 3$. Determine whether each orbit is stable or unstable.

(b) Kepler's first and second laws for planetary motion are the following statements:
K1: the planet moves on an ellipse with a focus at the Sun;
K2: the line between the planet and the Sun sweeps out equal areas in equal times.
Show that **K2** implies that the force acting on the planet is a central force.
Show that **K2** together with **K1** implies that the force is given by the inverse square law.
[You may assume that an ellipse with a focus at the origin has polar equation $\frac{A}{r} = 1 + \varepsilon \cos\theta$ with $A > 0$ and $0 < \varepsilon < 1$.]

Paper 4, Section II
10B Dynamics and Relativity

(a) A rigid body Q is made up of N particles of masses m_i at positions $\mathbf{r}_i(t)$. Let $\mathbf{R}(t)$ denote the position of its centre of mass. Show that the total kinetic energy of Q may be decomposed into T_1, the kinetic energy of the centre of mass, plus a term T_2 representing the kinetic energy about the centre of mass.
Suppose now that Q is rotating with angular velocity $\boldsymbol{\omega}$ about its centre of mass. Define the moment of inertia I of Q (about the axis defined by $\boldsymbol{\omega}$) and derive an expression for T_2 in terms of I and $\omega = |\boldsymbol{\omega}|$.

(b) Consider a uniform rod AB of length $2l$ and mass M. Two such rods AB and BC are freely hinged together at B. The end A is attached to a fixed point O on a perfectly smooth horizontal floor and AB is able to rotate freely about O. The rods are initially at rest, lying in a vertical plane with C resting on the floor and each rod making angle α with the horizontal. The rods subsequently move under gravity in their vertical plane.
Find an expression for the angular velocity of rod AB when it makes angle θ with the floor. Determine the speed at which the hinge strikes the floor.

Paper 4, Section II
11B Dynamics and Relativity

(i) An inertial frame S has orthonormal coordinate basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. A second frame S' rotates with angular velocity $\boldsymbol{\omega}$ relative to S and has coordinate basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. The motion of S' is characterised by the equations $d\mathbf{e}'_i/dt = \boldsymbol{\omega}\times\mathbf{e}'_i$ and at $t=0$ the two coordinate frames coincide.
If a particle P has position vector $\mathbf{r}$ show that $\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r}$ where $\mathbf{v}$ and $\mathbf{v}'$ are the velocity vectors of P as seen by observers fixed respectively in S and S'.

(ii) For the remainder of this question you may assume that $\mathbf{a} = \mathbf{a}' + 2\,\boldsymbol{\omega}\times\mathbf{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ where $\mathbf{a}$ and $\mathbf{a}'$ are the acceleration vectors of P as seen by observers fixed respectively in S and S', and that $\boldsymbol{\omega}$ is constant.

Consider again the frames S and S' in (i). Suppose that $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$ with ω constant. A particle of mass m moves under a force $\mathbf{F} = -4m\omega^2\mathbf{r}$. When viewed in S' its position and velocity at time $t=0$ are $(x', y', z') = (1, 0, 0)$ and $(\dot{x}', \dot{y}', \dot{z}') = (0, 0, 0)$. Find the motion of the particle in the coordinates of S'. Show that for an observer fixed in S', the particle achieves its maximum speed at time $t = \pi/(4\omega)$ and determine that speed. [*Hint: you may find it useful to consider the combination* $\zeta = x' + iy'$.]

Paper 4, Section II

12B Dynamics and Relativity

(a) Let S with coordinates (ct, x, y) and S' with coordinates (ct', x', y') be inertial frames in Minkowski space with two spatial dimensions. S' moves with velocity v along the x-axis of S and they are related by the standard Lorentz transformation:

$$\begin{pmatrix} ct \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c & 0 \\ \gamma v/c & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

A photon is emitted at the spacetime origin. In S' it has frequency ν' and propagates at angle θ' to the x'-axis.
Write down the 4-momentum of the photon in the frame S'.
Hence or otherwise find the frequency of the photon as seen in S. Show that it propagates at angle θ to the x-axis in S, where

$$\tan\theta = \frac{\tan\theta'}{\gamma\left(1 + \dfrac{v}{c}\sec\theta'\right)}.$$

A light source in S' emits photons uniformly in all directions in the $x'y'$-plane. Show that for large v, in S half of the light is concentrated into a narrow cone whose semi-angle α is given by $\cos\alpha = v/c$.

(b) The centre-of-mass frame for a system of relativistic particles in Minkowski space is the frame in which the total relativistic 3-momentum is zero.
Two particles A_1 and A_2 of rest masses m_1 and m_2 move collinearly with uniform velocities u_1 and u_2 respectively, along the x-axis of a frame S. They collide, coalescing to form a single particle A_3.
Determine the velocity of the centre-of-mass frame of the system comprising A_1 and A_2.
Find the speed of A_3 in S and show that its rest mass m_3 is given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right),$$

where $\gamma_i = (1 - u_i^2/c^2)^{-1/2}$.

Paper 4, Section I

3B Dynamics and Relativity

Two particles of masses m_1 and m_2 have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Particle 2 exerts a force $\mathbf{F}_{12}(\mathbf{r})$ on particle 1 (where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$) and there are no external forces.

Prove that the centre of mass of the two-particle system will move at constant speed along a straight line.

Explain how the two-body problem of determining the motion of the system may be reduced to that of a single particle moving under the force $\mathbf{F}_{12}$.

Suppose now that $m_1 = m_2 = m$ and that

$$\mathbf{F}_{12} = -\frac{Gm^2}{r^3}\mathbf{r}$$

is gravitational attraction. Let C be a circle fixed in space. Is it possible (by suitable choice of initial conditions) for the two particles to be traversing C at the same constant angular speed? Give a brief reason for your answer.

Paper 4, Section I

4B Dynamics and Relativity

Let S and S' be inertial frames in 2-dimensional spacetime with coordinate systems (t, x) and (t', x') respectively. Suppose that S' moves with positive velocity v relative to S and the spacetime origins of S and S' coincide. Write down the Lorentz transformation relating the coordinates of any event relative to the two frames.

Show that events which occur simultaneously in S are not generally seen to be simultaneous when viewed in S'.

In S two light sources A and B are at rest and placed a distance d apart. They simultaneously each emit a photon in the positive x direction. Show that in S' the photons are separated by a constant distance $d\sqrt{\dfrac{c+v}{c-v}}$.

Paper 4, Section II

9B Dynamics and Relativity

Let (r, θ) be polar coordinates in the plane. A particle of mass m moves in the plane under an attractive force of $mf(r)$ towards the origin O. You may assume that the acceleration $\mathbf{a}$ is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}$$

where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are the unit vectors in the directions of increasing r and θ respectively, and the dot denotes d/dt.

(a) Show that $l = r^2\dot{\theta}$ is a constant of the motion. Introducing $u = 1/r$ show that $\dot{r} = -l\dfrac{du}{d\theta}$ and derive the geometric orbit equation

$$l^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right).$$

(b) Suppose now that

$$f(r) = \frac{3r+9}{r^3}$$

and that initially the particle is at distance $r_0 = 1$ from O, moving with speed $v_0 = 4$ in a direction making angle $\pi/3$ with the radial vector pointing towards O.

Show that $l = 2\sqrt{3}$ and find u as a function of θ. Hence or otherwise show that the particle returns to its original position after one revolution about O and then flies off to infinity.

Paper 4, Section II

10B Dynamics and Relativity

For any frame S and vector $\mathbf{A}$, let $\left[\dfrac{d\mathbf{A}}{dt}\right]_S$ denote the derivative of $\mathbf{A}$ relative to S. A frame of reference S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S and the two frames have a common origin O. [You may assume that for any vector $\mathbf{A}$, $\left[\dfrac{d\mathbf{A}}{dt}\right]_S = \left[\dfrac{d\mathbf{A}}{dt}\right]_{S'} + \boldsymbol{\omega} \times \mathbf{A}$.]

(a) If $\mathbf{r}(t)$ is the position vector of a point P from O, show that

$$\left[\frac{d^2\mathbf{r}}{dt^2}\right]_S = \left[\frac{d^2\mathbf{r}}{dt^2}\right]_{S'} + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{v}' = \left[\dfrac{d\mathbf{r}}{dt}\right]_{S'}$ is the velocity in S'.

Suppose now that $\mathbf{r}(t)$ is the position vector of a particle of mass m moving under a conservative force $\mathbf{F} = -\nabla\phi$ and a force $\mathbf{G}$ that is always orthogonal to the velocity $\mathbf{v}'$ in S'. Show that the quantity

$$E = \frac{1}{2}m\mathbf{v}'.\mathbf{v}' + \phi - \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r})$$

is a constant of the motion. [You may assume that $\nabla\left[(\boldsymbol{\omega} \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r})\right] = -2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.]

(b) A bead slides on a frictionless circular hoop of radius a which is forced to rotate with constant angular speed ω about a vertical diameter. Let $\theta(t)$ denote the angle between the line from the centre of the hoop to the bead and the downward vertical. Using the results of (a), or otherwise, show that

$$\ddot{\theta} + \left(\frac{g}{a} - \omega^2\cos\theta\right)\sin\theta = 0.$$

Deduce that if $\omega^2 > g/a$ there are two equilibrium positions $\theta = \theta_0$ off the axis of rotation, and show that these are stable equilibria.

Paper 4, Section II

11B Dynamics and Relativity

(a) State the parallel axis theorem for moments of inertia.

(b) A uniform circular disc D of radius a and total mass m can turn frictionlessly about a fixed horizontal axis that passes through a point A on its circumference and is perpendicular to its plane. Initially the disc hangs at rest (in constant gravity g) with its centre O being vertically below A. Suppose the disc is disturbed and executes free oscillations. Show that the period of small oscillations is $2\pi\sqrt{\dfrac{3a}{2g}}$.

(c) Suppose now that the disc is released from rest when the radius OA is vertical with O directly above A. Find the angular velocity and angular acceleration of O about A when the disc has turned through angle θ. Let $\mathbf{R}$ denote the reaction force at A on the disc. Find the acceleration of the centre of mass of the disc. Hence, or otherwise, show that the component of $\mathbf{R}$ parallel to OA is $mg(7\cos\theta - 4)/3$.

Paper 4, Section II

12B Dynamics and Relativity

(a) Define the 4-momentum $\mathbf{P}$ of a particle of rest mass m and 3-velocity $\mathbf{v}$, and the 4-momentum of a photon of frequency ν (having zero rest mass) moving in the direction of the unit vector $\mathbf{e}$.

Show that if $\mathbf{P}_1$ and $\mathbf{P}_2$ are timelike future-pointing 4-vectors then $\mathbf{P}_1.\mathbf{P}_2 \geqslant 0$ (where the dot denotes the Lorentz-invariant scalar product). Hence or otherwise show that the law of conservation of 4-momentum forbids a photon to spontaneously decay into an electron-positron pair. [Electrons and positrons have equal rest masses $m > 0$.]

(b) In the laboratory frame an electron travelling with velocity $\mathbf{u}$ collides with a positron at rest. They annihilate, producing two photons of frequencies ν_1 and ν_2 that move off at angles θ_1 and θ_2 to $\mathbf{u}$, in the directions of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. By considering 4-momenta in the laboratory frame, or otherwise, show that

$$\frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2} = \sqrt{\frac{\gamma-1}{\gamma+1}}$$

where $\gamma = \left(1 - \dfrac{u^2}{c^2}\right)^{-1/2}$.

Paper 4, Section I

3B Dynamics and Relativity

The motion of a planet in the gravitational field of a star of mass M obeys

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2}\ , \qquad r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = h\ ,$$

where $r(t)$ and $\theta(t)$ are polar coordinates in a plane and h is a constant. Explain one of Kepler's Laws by giving a geometrical interpretation of h.

Show that circular orbits are possible, and derive another of Kepler's Laws relating the radius a and the period T of such an orbit. Show that any circular orbit is stable under small perturbations that leave h unchanged.

Paper 4, Section I

4B Dynamics and Relativity

Inertial frames S and S' in two-dimensional space-time have coordinates (x, t) and (x', t'), respectively. These coordinates are related by a Lorentz transformation with v the velocity of S' relative to S. Show that if $x_\pm = x \pm ct$ and $x'_\pm = x' \pm ct'$ then the Lorentz transformation can be expressed in the form

$$x'_+ = \lambda(v) x_+ \quad \text{and} \quad x'_- = \lambda(-v) x_-\ , \quad \text{where} \quad \lambda(v) = \left(\frac{c-v}{c+v}\right)^{1/2} . \qquad (*)$$

Deduce that $x^2 - c^2 t^2 = x'^{\,2} - c^2 t'^{\,2}$.

Use the form $(*)$ to verify that successive Lorentz transformations with velocities v_1 and v_2 result in another Lorentz transformation with velocity v_3, to be determined in terms of v_1 and v_2.

Paper 4, Section II

9B Dynamics and Relativity

A particle with mass m and position $\mathbf{r}(t)$ is subject to a force

$$\mathbf{F} = \mathbf{A}(\mathbf{r}) + \dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}) .$$

(a) Suppose that $\mathbf{A} = -\boldsymbol{\nabla}\phi$. Show that

$$E = \frac{1}{2} m \dot{\mathbf{r}}^2 + \phi(\mathbf{r})$$

is constant, and interpret this result, explaining why the field $\mathbf{B}$ plays no role.

(b) Suppose, in addition, that $\mathbf{B} = -\boldsymbol{\nabla}\psi$ and that both ϕ and ψ depend only on $r = |\mathbf{r}|$. Show that

$$\mathbf{L} = m \mathbf{r} \times \dot{\mathbf{r}} - \psi \mathbf{r}$$

is independent of time if $\psi(r) = \mu/r$, for any constant μ.

(c) Now specialise further to the case $\psi = 0$. Explain why the result in (b) implies that the motion of the particle is confined to a plane. Show also that

$$\mathbf{K} = \mathbf{L} \times \dot{\mathbf{r}} - \phi \mathbf{r}$$

is constant provided $\phi(r)$ takes a certain form, to be determined.

[*Recall that* $\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}$ *and that if* f *depends only on* $r = |\mathbf{r}|$ *then* $\boldsymbol{\nabla} f = f'(r)\hat{\mathbf{r}}$.]

Paper 4, Section II

10B Dynamics and Relativity

The trajectory of a particle $\mathbf{r}(t)$ is observed in a frame S which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame I. Given that the time derivative in I of any vector $\mathbf{u}$ is

$$\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right)_I = \dot{\mathbf{u}} + \boldsymbol{\omega}\times\mathbf{u} \,,$$

where a dot denotes a time derivative in S, show that

$$m\,\ddot{\mathbf{r}} = \mathbf{F} - 2m\,\boldsymbol{\omega}\times\dot{\mathbf{r}} - m\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \,,$$

where $\mathbf{F}$ is the force on the particle and m is its mass.

Let S be the frame that rotates with the Earth. Assume that the Earth is a sphere of radius R. Let P be a point on its surface at latitude $\pi/2-\theta$, and define vertical to be the direction normal to the Earth's surface at P.

(a) A particle at P is released from rest in S and is acted on only by gravity. Show that its initial acceleration makes an angle with the vertical of approximately

$$\frac{\omega^2 R}{g}\sin\theta\cos\theta \,,$$

working to lowest non-trivial order in ω.

(b) Now consider a particle fired vertically upwards from P with speed v. Assuming that terms of order ω^2 and higher can be neglected, show that it falls back to Earth under gravity at a distance

$$\frac{4}{3}\frac{\omega v^3}{g^2}\sin\theta$$

from P. [*You may neglect the curvature of the Earth's surface and the vertical variation of gravity.*]

Paper 4, Section II
11B Dynamics and Relativity

A rocket carries equipment to collect samples from a stationary cloud of cosmic dust. The rocket moves in a straight line, burning fuel and ejecting gas at constant speed u relative to itself. Let $v(t)$ be the speed of the rocket, $M(t)$ its total mass, including fuel and any dust collected, and $m(t)$ the total mass of gas that has been ejected. Show that

$$M\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}M}{\mathrm{d}t} + (v-u)\frac{\mathrm{d}m}{\mathrm{d}t} = 0\ ,$$

assuming that all external forces are negligible.

(a) If no dust is collected and the rocket starts from rest with mass M_0, deduce that

$$v = u\log(M_0/M)\ .$$

(b) If cosmic dust is collected at a constant rate of α units of mass per unit time and fuel is consumed at a constant rate $\mathrm{d}m/\mathrm{d}t = \beta$, show that, with the same initial conditions as in (a),

$$v = \frac{u\beta}{\alpha}\Big(1 - (M/M_0)^{\alpha/(\beta-\alpha)}\Big)\ .$$

Verify that the solution in (a) is recovered in the limit $\alpha \to 0$.

Paper 4, Section II
12B Dynamics and Relativity

(a) Write down the relativistic energy E of a particle of rest mass m and speed v. Find the approximate form for E when v is small compared to c, keeping all terms up to order $(v/c)^2$. What new physical idea (when compared to Newtonian Dynamics) is revealed in this approximation?

(b) A particle of rest mass m is fired at an identical particle which is at rest in the laboratory frame. Let E be the relativistic energy and v the speed of the incident particle in this frame. After the collision, there are N particles in total, each with rest mass m. Assuming that four-momentum is conserved, find a lower bound on E and hence show that

$$v \geqslant \frac{N(N^2-4)^{1/2}}{N^2-2}\,c\ .$$

Paper 4, Section I

3B Dynamics and Relativity

A particle of mass m and charge q moves with trajectory $\mathbf{r}(t)$ in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Write down the Lorentz force on the particle and use Newton's Second Law to deduce that

$$\dot{\mathbf{r}} - \omega\,\mathbf{r} \times \hat{\mathbf{z}} = \mathbf{c}\,,$$

where $\mathbf{c}$ is a constant vector and ω is to be determined. Find $\mathbf{c}$ and hence $\mathbf{r}(t)$ for the initial conditions

$$\mathbf{r}(0) = a\hat{\mathbf{x}} \qquad \text{and} \qquad \dot{\mathbf{r}}(0) = u\hat{\mathbf{y}} + v\hat{\mathbf{z}}$$

where a, u and v are constants. Sketch the particle's trajectory in the case $a\omega + u = 0$.

[*Unit vectors* $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ *correspond to a set of Cartesian coordinates.*]

Paper 4, Section I

4B Dynamics and Relativity

Let S be an inertial frame with coordinates (t, x) in two-dimensional spacetime. Write down the Lorentz transformation giving the coordinates (t', x') in a second inertial frame S' moving with velocity v relative to S. If a particle has constant velocity u in S, find its velocity u' in S'. Given that $|u| < c$ and $|v| < c$, show that $|u'| < c$.

Paper 4, Section II

9B Dynamics and Relativity

A sphere of uniform density has mass m and radius a. Find its moment of inertia about an axis through its centre.

A marble of uniform density is released from rest on a plane inclined at an angle α to the horizontal. Let the time taken for the marble to travel a distance ℓ down the plane be: (i) t_1 if the plane is perfectly smooth; or (ii) t_2 if the plane is rough and the marble rolls without slipping.

Explain, with a clear discussion of the forces acting on the marble, whether or not its energy is conserved in each of the cases (i) and (ii). Show that $t_1/t_2 = \sqrt{5/7}$.

Suppose that the original marble is replaced by a new one with the same mass and radius but with a hollow centre, so that its moment of inertia is $\lambda m a^2$ for some constant λ. What is the new value for t_1/t_2?

Paper 4, Section II

10B Dynamics and Relativity

A particle of unit mass moves in a plane with polar coordinates (r, θ) and components of acceleration $(\ddot{r} - r\dot{\theta}^2,\ r\ddot{\theta} + 2\dot{r}\dot{\theta})$. The particle experiences a force corresponding to a potential $-Q/r$. Show that

$$E = \frac{1}{2}\dot{r}^2 + U(r) \qquad \text{and} \qquad h = r^2\dot{\theta}$$

are constants of the motion, where

$$U(r) = \frac{h^2}{2r^2} - \frac{Q}{r}.$$

Sketch the graph of $U(r)$ in the cases $Q > 0$ and $Q < 0$.

(a) Assuming $Q > 0$ and $h > 0$, for what range of values of E do bounded orbits exist? Find the minimum and maximum distances from the origin, $r_{\min}$ and $r_{\max}$, on such an orbit and show that

$$r_{\min} + r_{\max} = \frac{Q}{|E|}.$$

Prove that the minimum and maximum values of the particle's speed, $v_{\min}$ and $v_{\max}$, obey

$$v_{\min} + v_{\max} = \frac{2Q}{h}.$$

(b) Now consider trajectories with $E > 0$ and Q of either sign. Find the distance of closest approach, $r_{\min}$, in terms of the impact parameter, b, and v_∞, the limiting value of the speed as $r \to \infty$. Deduce that if $b \ll |Q|/v_\infty^2$ then, to leading order,

$$r_{\min} \approx \frac{2|Q|}{v_\infty^2} \quad \text{for} \quad Q < 0, \qquad\qquad r_{\min} \approx \frac{b^2 v_\infty^2}{2Q} \quad \text{for} \quad Q > 0.$$

Paper 4, Section II
11B Dynamics and Relativity

Consider a set of particles with position vectors $\mathbf{r}_i(t)$ and masses m_i, where $i = 1, 2, \ldots, N$. Particle i experiences an external force $\mathbf{F}_i$ and an internal force $\mathbf{F}_{ij}$ from particle j, for each $j \neq i$. Stating clearly any assumptions you need, show that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \qquad \text{and} \qquad \frac{d\mathbf{L}}{dt} = \mathbf{G},$$

where $\mathbf{P}$ is the total momentum, $\mathbf{F}$ is the total external force, $\mathbf{L}$ is the total angular momentum about a fixed point $\mathbf{a}$, and $\mathbf{G}$ is the total external torque about $\mathbf{a}$.

Does the result $\dfrac{d\mathbf{L}}{dt} = \mathbf{G}$ still hold if the fixed point $\mathbf{a}$ is replaced by the centre of mass of the system? Justify your answer.

Suppose now that the external force on particle i is $-k\dfrac{d\mathbf{r}_i}{dt}$ and that all the particles have the same mass m. Show that

$$\mathbf{L}(t) = \mathbf{L}(0)\, e^{-kt/m}\,.$$

Paper 4, Section II
12B Dynamics and Relativity

A particle A of rest mass m is fired at an identical particle B which is stationary in the laboratory. On impact, A and B annihilate and produce two massless photons whose energies are equal. Assuming conservation of four-momentum, show that the angle θ between the photon trajectories is given by

$$\cos\theta = \frac{E - 3mc^2}{E + mc^2}$$

where E is the relativistic energy of A.

Let v be the speed of the incident particle A. For what value of v/c will the photons move in perpendicular directions? If v is very small compared with c, show that

$$\theta \approx \pi - v/c\,.$$

[*All quantities referred to are measured in the laboratory frame.*]

Paper 4, Section I

3A Dynamics and Relativity

A rocket moves vertically upwards in a uniform gravitational field and emits exhaust gas downwards with time-dependent speed $U(t)$ relative to the rocket. Derive the rocket equation

$$m(t)\frac{\mathrm{d}v}{\mathrm{d}t} + U(t)\frac{\mathrm{d}m}{\mathrm{d}t} = -m(t)g\,,$$

where $m(t)$ and $v(t)$ are respectively the rocket's mass and upward vertical speed at time t. Suppose now that $m(t) = m_0 - \alpha t$, $U(t) = U_0 m_0/m(t)$ and $v(0) = 0$. What is the condition for the rocket to lift off at $t = 0$? Assuming that this condition is satisfied, find $v(t)$.

State the dimensions of all the quantities involved in your expression for $v(t)$, and verify that the expression is dimensionally consistent.

[*You may assume that all speeds are small compared with the speed of light and neglect any relativistic effects.*]

Paper 4, Section I

4A Dynamics and Relativity

(a) Explain what is meant by a *central force* acting on a particle moving in three dimensions.

(b) Show that the orbit of a particle experiencing a central force lies in a plane.

(c) Show that, in the approximation in which the Sun is regarded as fixed and only its gravitational field is considered, a straight line joining the Sun and an orbiting planet sweeps out equal areas in equal times (Kepler's second law).

[*With respect to the basis vectors* $(\mathbf{e}_r, \mathbf{e}_\theta)$ *of plane polar coordinates, the velocity* $\dot{\mathbf{x}}$ *and acceleration* $\ddot{\mathbf{x}}$ *of a particle are given by* $\dot{\mathbf{x}} = (\dot r, r\dot\theta)$ *and* $\ddot{\mathbf{x}} = (\ddot r - r\dot\theta^2, r\ddot\theta + 2\dot r\dot\theta)$.]

Paper 4, Section II

9A Dynamics and Relativity

Davros departs on a rocket voyage from the planet Skaro, travelling at speed u (where $0 < u < c$) in the positive x direction in Skaro's rest frame. After travelling a distance L in Skaro's rest frame, he jumps onto another rocket travelling at speed v' (where $0 < v' < c$) in the positive x direction in the first rocket's rest frame. After travelling a further distance L in Skaro's rest frame, he jumps onto a third rocket, travelling at speed w'' (where $0 < w'' < c$) in the negative x direction in the second rocket's rest frame.

Let v and w be Davros' speed on the second and third rockets, respectively, in Skaro's rest frame. Show that

$$v = (u + v')\left(1 + \frac{uv'}{c^2}\right)^{-1} .$$

Express w in terms of u, v', w'' and c.

How large must w'' be, expressed in terms of u, v' and c, to ensure that Davros eventually returns to Skaro?

Supposing that w'' satisfies this condition, draw a spacetime diagram illustrating Davros' journey. Label clearly each point where he boards a rocket and the point of his return to Skaro, and give the coordinates of each point in Skaro's rest frame, expressed in terms of u, v, w, c and L.

Hence, or otherwise, calculate how much older Davros will be on his return, and how much time will have elapsed on Skaro during his voyage, giving your answers in terms of u, v, w, c and L.

[*You may neglect any effects due to gravity and any corrections arising from Davros' brief accelerations when getting onto or leaving rockets.*]

Paper 4, Section II

10A Dynamics and Relativity

(a) Write down expressions for the relativistic 3-momentum $\mathbf{p}$ and energy E of a particle of rest mass m and velocity $\mathbf{v}$. Show that these expressions are consistent with

$$E^2 = \mathbf{p}.\mathbf{p}\, c^2 + m^2 c^4 \,. \qquad (*)$$

Define the 4-momentum $\mathbf{P}$ for such a particle and obtain $(*)$ by considering the invariance properties of $\mathbf{P}$.

(b) Two particles, each with rest mass m and energy E, moving in opposite directions, collide head on. Show that it is consistent with the conservation of 4-momentum for the collision to result in a set of n particles of rest masses μ_i (for $1 \leqslant i \leqslant n$) only if

$$E \geqslant \frac{1}{2}\left(\sum_{i=1}^{n} \mu_i\right) c^2 \,.$$

(c) A particle of rest mass m_1 and energy E_1 is fired at a stationary particle of rest mass m_2. Show that it is consistent with the conservation of 4-momentum for the collision to result in a set of n particles of rest masses μ_i (for $1 \leqslant i \leqslant n$) only if

$$E_1 \geqslant \frac{(\sum_{i=1}^{n} \mu_i)^2 - m_1^2 - m_2^2}{2m_2}\, c^2 \,.$$

Deduce the minimum frequency required for a photon fired at a stationary particle of rest mass m_2 to result in the same set of n particles, assuming that the conservation of 4-momentum is the only relevant constraint.

Paper 4, Section II

11A Dynamics and Relativity

Obtain the moment of inertia of a uniform disc of radius a and mass M about its axis of rotational symmetry. A uniform rigid body of mass $3M/4$ takes the form of a disc of radius a with a concentric circular hole of radius $a/2$ cut out. Calculate the body's moment of inertia about its axis of rotational symmetry.

The body rolls without slipping, with its axis of symmetry horizontal, down a plane inclined at angle α to the horizontal. Determine its acceleration and the frictional and normal-reaction forces resulting from contact with the plane.

Paper 4, Section II

12A Dynamics and Relativity

(a) A particle of charge q moves with velocity $\mathbf{v}$ in a constant magnetic field $\mathbf{B}$. Give an expression for the Lorentz force $\mathbf{F}$ experienced by the particle. If no other forces act on the particle, show that its kinetic energy is independent of time.

(b) Four point particles, each of positive charge Q, are fixed at the four corners of a square with sides of length $2a$. Another point particle, of positive charge q, is constrained to move in the plane of the square but is otherwise free.

By considering the form of the electrostatic potential near the centre of the square, show that the state in which the particle of charge q is stationary at the centre of the square is a stable equilibrium. Obtain the frequency of small oscillations about this equilibrium.

[*The Coulomb potential for two point particles of charges Q and q separated by distance r is* $Qq/4\pi\epsilon_0 r$.]

4/I/3B **Dynamics**

Two particles of masses m_1 and m_2 have position vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at time t. The particle of mass m_1 experiences a force $\mathbf{f}$ and the particle of mass m_2 experiences a force $-\mathbf{f}$. Show that the centre of mass moves at a constant velocity, and derive an equation of motion for the relative separation $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

Now suppose that $\mathbf{f} = -k\mathbf{r}$, where k is a positive constant. The particles are initially at rest a distance d apart. Calculate how long it takes before they collide.

4/I/4B **Dynamics**

A damped pendulum is described by the equation

$$\ddot{x} + 2k\dot{x} + \omega^2 \sin x = 0\,,$$

where k and ω are real positive constants. Determine the location of all the equilibrium points of the system. Classify the equilibrium points in the two cases $k > \omega$ and $k < \omega$.

4/II/9B **Dynamics**

An octopus of mass m_o swims horizontally in a straight line by jet propulsion. At time $t = 0$ the octopus is at rest, and its internal cavity contains a mass m_w of water (so that the mass of the octopus plus water is $m_o + m_w$). It then starts to move by ejecting the water backwards at a constant rate Q units of mass per unit time and at a constant speed V relative to itself. The speed of the octopus at time t is $u(t)$, and the mass of the octopus plus remaining water is $m(t)$. The drag force exerted by the surrounding water on the octopus is αu^2, where α is a positive constant.

Show that, during ejection of water, the equation of motion is

$$m\frac{du}{dt} = QV - \alpha u^2\,. \tag{1}$$

Once all the water has been ejected, at time $t = t_c$, the octopus has attained a velocity u_c. Use dimensional analysis to show that

$$u_c = Vf(\lambda, \mu)\,, \tag{2}$$

where λ and μ are two dimensionless quantities and f is an unknown function. Solve equation (1) to find an explicit expression for u_c, and verify that your answer is of the form given in equation (2).

4/II/10B **Dynamics**

A body of mass m moves in the gravitational field of a much larger spherical object of mass M located at the origin. Starting from the equations of motion

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2},$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0,$$

show that:

(i) the body moves in an orbit of the form

$$\frac{h^2 u}{GM} = 1 + e\cos(\theta - \theta_0)\,, \qquad (*)$$

where $u = 1/r$, h is the constant angular momentum per unit mass, and e and θ_0 are constants;

(ii) the total energy of the body is

$$E = \frac{mG^2M^2}{2h^2}\left(e^2 - 1\right)\,.$$

A meteorite is moving very far from the Earth with speed V, and in the absence of the effect of the Earth's gravitational field would miss the Earth by a shortest distance b (measured from the Earth's centre). Show that in the subsequent motion

$$h = bV,$$

and

$$e = \left[1 + \frac{b^2V^4}{G^2M^2}\right]^{\frac{1}{2}}\,.$$

Use equation $(*)$ to find the distance of closest approach, and show that the meteorite will collide with the Earth if

$$b < \left[R^2 + \frac{2GMR}{V^2}\right]^{\frac{1}{2}}\,,$$

where R is the radius of the Earth.

4/II/11B **Dynamics**

An inertial reference frame S and another reference frame S' have a common origin O, and S' rotates with angular velocity $\boldsymbol{\omega}(t)$ with respect to S. Show the following:

(i) the rates of change of an arbitrary vector $\mathbf{a}(t)$ in frames S and S' are related by

$$\left(\frac{d\mathbf{a}}{dt}\right)_S = \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}\,;$$

(ii) the accelerations in S and S' are related by

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{S'} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,,$$

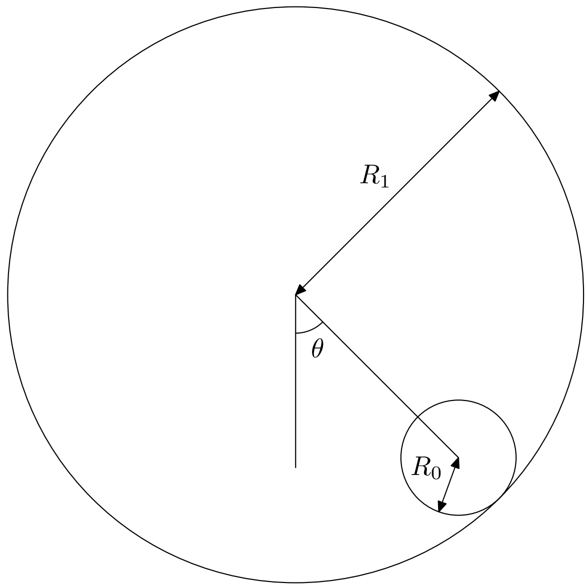
where $\mathbf{r}(t)$ is the position vector relative to O.

A train of mass m at latitude λ in the Northern hemisphere travels North with constant speed V along a track which runs North–South. Find the magnitude and direction of the sideways force exerted on the train by the track.

4/II/12B **Dynamics**

A uniform solid sphere has mass m and radius R_0. Calculate the moment of inertia of the sphere about an axis through its centre.

A long hollow circular cylinder of radius R_1 (where $R_1 > 2R_0$) is held fixed with its axis horizontal. The sphere is held initially at rest in contact with the inner surface of the cylinder at $\theta = \alpha$, where $\alpha < \pi/2$ and θ is the angle between the line joining the centre of the sphere to the cylinder axis and the downward vertical, as shown in the figure.



The sphere is then released, and rolls without slipping. Show that the angular velocity of the sphere is

$$\left(\frac{R_1 - R_0}{R_0}\right)\dot{\theta}.$$

Show further that the time, T_R, it takes the sphere to reach $\theta = 0$ is

$$T_R = \sqrt{\frac{7\,(R_1 - R_0)}{10g}} \quad \int_0^{\alpha} \frac{d\theta}{(\cos\theta - \cos\alpha)^{\frac{1}{2}}}\,.$$

If, instead, the cylinder and sphere surfaces are highly polished, so that the sphere now slides without rolling, find the time, T_S, it takes to reach $\theta = 0$.

Without further calculation, explain qualitatively how your answers for T_R and T_S would be affected if the solid sphere were replaced by a hollow spherical shell of the same radius and mass.

4/I/3C **Dynamics**

A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive, giving careful explanations, the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm\,,$$

where v and m are the speed and total mass of the rocket (including fuel) at time t.

If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}\,.$$

4/I/4C **Dynamics**

Sketch the graph of $y = 3x^2 - 2x^3$.

A particle of unit mass moves along the x axis in the potential $V(x) = 3x^2 - 2x^3$. Sketch the phase plane, and describe briefly the motion of the particle on the different trajectories.

4/II/9C **Dynamics**

A small ring of mass m is threaded on a smooth rigid wire in the shape of a parabola given by $x^2 = 4az$, where x measures horizontal distance and z measures distance vertically upwards. The ring is held at height $z = h$, then released.

(i) Show by dimensional analysis that the period of oscillations, T, can be written in the form

$$T = (a/g)^{1/2}G(h/a)$$

for some function G.

(ii) Show that G is given by

$$G(\beta) = 2\sqrt{2}\int_{-1}^{1}\left(\frac{1+\beta u^2}{1-u^2}\right)^{\frac{1}{2}}du\,,$$

and find, to first order in h/a, the period of small oscillations.

4/II/10C **Dynamics**

A particle of mass m experiences, at the point with position vector $\mathbf{r}$, a force $\mathbf{F}$ given by

$$\mathbf{F} = -k\mathbf{r} - e\dot{\mathbf{r}} \times \mathbf{B}\,,$$

where k and e are positive constants and $\mathbf{B}$ is a constant, uniform, vector field.

(i) Show that $m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + k\mathbf{r} \cdot \mathbf{r}$ is constant. Give a physical interpretation of each term and a physical explanation of the fact that $\mathbf{B}$ does not arise in this expression.

(ii) Show that $m(\dot{\mathbf{r}} \times \mathbf{r}) \cdot \mathbf{B} + \frac{1}{2}e(\mathbf{r} \times \mathbf{B}) \cdot (\mathbf{r} \times \mathbf{B})$ is constant.

(iii) Given that the particle was initially at rest at $\mathbf{r}_0$, derive an expression for $\mathbf{r} \cdot \mathbf{B}$ at time t.

4/II/11C **Dynamics**

A particle moves in the gravitational field of the Sun. The angular momentum per unit mass of the particle is h and the mass of the Sun is M. Assuming that the particle moves in a plane, write down the equations of motion in polar coordinates, and derive the equation

$$\frac{d^2u}{d\theta^2} + u = k\,,$$

where $u = 1/r$ and $k = GM/h^2$.

Write down the equation of the orbit (u as a function of θ), given that the particle moves with the escape velocity and is at the perihelion of its orbit, a distance r_0 from the Sun, when $\theta = 0$. Show that

$$\sec^4(\theta/2)\frac{d\theta}{dt} = \frac{h}{r_0^2}$$

and hence that the particle reaches a distance $2r_0$ from the Sun at time $8r_0^2/(3h)$.

4/II/12C **Dynamics**

The ith particle of a system of N particles has mass m_i and, at time t, position vector $\mathbf{r}_i$ with respect to an origin O. It experiences an external force $\mathbf{F}_i^e$, and also an internal force $\mathbf{F}_{ij}$ due to the jth particle (for each $j = 1, ..., N$, $j \neq i$), where $\mathbf{F}_{ij}$ is parallel to $\mathbf{r}_i - \mathbf{r}_j$ and Newton's third law applies.

(i) Show that the position of the centre of mass, $\mathbf{X}$, satisfies

$$M\frac{d^2\mathbf{X}}{dt^2} = \mathbf{F}^e \,,$$

where M is the total mass of the system and $\mathbf{F}^e$ is the sum of the external forces.

(ii) Show that the total angular momentum of the system about the origin, $\mathbf{L}$, satisfies

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \,,$$

where $\mathbf{N}$ is the total moment about the origin of the external forces.

(iii) Show that $\mathbf{L}$ can be expressed in the form

$$\mathbf{L} = M\mathbf{X} \times \mathbf{V} + \sum_i m_i \mathbf{r}_i' \times \mathbf{v}_i' \,,$$

where $\mathbf{V}$ is the velocity of the centre of mass, $\mathbf{r}_i'$ is the position vector of the ith particle relative to the centre of mass, and $\mathbf{v}_i'$ is the velocity of the ith particle relative to the centre of mass.

(iv) In the case $N = 2$ when the internal forces are derived from a potential $U(|\mathbf{r}|)$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and there are no external forces, show that

$$\frac{dT}{dt} + \frac{dU}{dt} = 0 \,,$$

where T is the total kinetic energy of the system.

4/I/3C **Dynamics**

A car is at rest on a horizontal surface. The engine is switched on and suddenly sets the wheels spinning at a constant angular velocity Ω. The wheels have radius r and the coefficient of friction between the ground and the surface of the wheels is μ. Calculate the time T when the wheels start rolling without slipping. If the car is started on an upward slope in a similar manner, explain whether T is increased or decreased relative to the case where the car starts on a horizontal surface.

4/I/4C **Dynamics**

For the dynamical system

$$\ddot{x} = -\sin x,$$

find the stable and unstable fixed points and the equation determining the separatrix. Sketch the phase diagram. If the system starts on the separatrix at $x = 0$, write down an integral determining the time taken for the velocity $\dot{x}$ to reach zero. Show that the integral is infinite.

4/II/9C **Dynamics**

A motorcycle of mass M moves on a bowl-shaped surface specified by its height $h(r)$ where $r = \sqrt{x^2+y^2}$ is the radius in cylindrical polar coordinates (r,ϕ,z). The torque exerted by the motorcycle engine on the rear wheel results in a force $\mathbf{F}(t)$ pushing the motorcycle forward. Assuming $\mathbf{F}(t)$ is directed along the motorcycle's velocity and that the motorcycle's vertical velocity and acceleration are small, show that the motion is described by

$$\ddot{r} - r\dot{\phi}^2 = -g\frac{dh}{dr} + \frac{F(t)}{M}\frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}}\,,$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{F(t)}{M}\frac{r\dot{\phi}}{\sqrt{\dot{r}^2 + r^2\dot{\phi}^2}}\,,$$

where dots denote time derivatives, $F(t) = |\mathbf{F}(t)|$ and g is the acceleration due to gravity.

The motorcycle rider can adjust $F(t)$ to produce the desired trajectory. If the rider wants to move on a curve $r(\phi)$, show that $\phi(t)$ must obey

$$\dot{\phi}^2 = g\,\frac{dh}{dr}\Bigg/\left(r + \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - \frac{d^2r}{d\phi^2}\right).$$

Now assume that $h(r) = r^2/\ell$, with ℓ a constant, and $r(\phi) = \epsilon\phi$ with ϵ a positive constant, and $0 \leqslant \phi < \infty$ so that the desired trajectory is a spiral curve. Assuming that $\phi(t)$ tends to infinity as t tends to infinity, show that $\dot{\phi}(t)$ tends to $\sqrt{2g/\ell}$ and $F(t)$ tends to $4\epsilon Mg/\ell$ as t tends to infinity.

4/II/10C **Dynamics**

A particle of mass m bounces back and forth between two walls of mass M moving towards each other in one dimension. The walls are separated by a distance $\ell(t)$. The wall on the left has velocity $+V(t)$ and the wall on the right has velocity $-V(t)$. The particle has speed $v(t)$. Friction is negligible and the particle–wall collisions are elastic.

Consider a collision between the particle and the wall on the right. Show that the centre–of–mass velocity of the particle–wall system is $v_{\rm cm} = (mv - MV)/(m+M)$. Calculate the particle's speed following the collision.

Assume that the particle is much lighter than the walls, i.e., $m \ll M$. Show that the particle's speed increases by approximately $2V$ every time it collides with a wall.

Assume also that $v \gg V$ (so that particle–wall collisions are frequent) and that the velocities of the two walls remain nearly equal and opposite. Show that in a time interval Δt, over which the change in V is negligible, the wall separation changes by $\Delta\ell \approx -2V\Delta t$. Show that the number of particle–wall collisions during Δt is approximately $v\Delta t/\ell$ and that the particle's speed increases by $\Delta v \approx -(\Delta\ell/\ell)v$ during this time interval.

Hence show that under the given conditions the particle speed v is approximately proportional to ℓ^{-1}.

4/II/11C **Dynamics**

Two light, rigid rods of length 2ℓ have a mass m attached to each end. Both are free to move in two dimensions. The two rods are placed so that their two ends are located at $(-d,+2\ell)$, $(-d,0)$, and $(+d,0)$, $(+d,-2\ell)$ respectively, where d is positive. They are set in motion with no rotation, with centre–of–mass velocities $(+V,0)$ and $(-V,0)$, so that the lower mass on the first rod collides head on with the upper mass on the second rod at the origin $(0,0)$. [*You may assume that the impulse is directed along the x-axis.*]

Assuming the collision is elastic, calculate the centre–of–mass velocity $\boldsymbol{v}$ and the angular velocity $\boldsymbol{\omega}$ of each rod immediately after the collision.

Assuming a coefficient of restitution e, compute $\boldsymbol{v}$ and $\boldsymbol{\omega}$ for each rod after the collision.

4/II/12C **Dynamics**

A particle of mass m and charge $q > 0$ moves in a time-dependent magnetic field $\mathbf{B} = (0, 0, B_z(t))$.

Write down the equations of motion governing the particle's x, y and z coordinates.

Show that the speed of the particle in the (x, y) plane, $V = \sqrt{\dot{x}^2 + \dot{y}^2}$, is a constant.

Show that the general solution of the equations of motion is

$$\begin{aligned} x(t) &= x_0 + V \int_0^t dt' \cos\left(-\int_0^{t'} dt'' q \frac{B_z(t'')}{m} + \phi\right), \\ y(t) &= y_0 + V \int_0^t dt' \sin\left(-\int_0^{t'} dt'' q \frac{B_z(t'')}{m} + \phi\right), \\ z(t) &= z_0 + v_z t, \end{aligned}$$

and interpret each of the six constants of integration, x_0, y_0, z_0, v_z, V and ϕ. [*Hint: Solve the equations for the particle's* **velocity** *in cylindrical polars.*]

Let $B_z(t) = \beta t$, where β is a positive constant. Assuming that $x_0 = y_0 = z_0 = v_z = \phi = 0$ and $V = 1$, calculate the position of the particle in the limit $t \to \infty$ (you may assume this limit exists). [*Hint: You may use the results* $\int_0^\infty dx \cos(x^2) = \int_0^\infty dx \sin(x^2) = \sqrt{\pi/8}$.]

4/I/3C **Dynamics**

Planetary Explorers Ltd. want to put a communications satellite of mass m into geostationary orbit around the spherical planet Zog (*i.e.* with the satellite always above the same point on the surface of Zog). The mass of Zog is M, the length of its day is T and G is the gravitational constant.

Write down the equations of motion for a general orbit of the satellite and determine the radius and speed of the geostationary orbit.

Describe briefly how the orbit is modified if the satellite is released at the correct radius and on the correct trajectory for a geostationary orbit, but with a little too much speed. Comment on how the satellite's speed varies around such an orbit.

4/I/4C **Dynamics**

A car of mass M travelling at speed U on a smooth, horizontal road attempts an emergency stop. The car skids in a straight line with none of its wheels able to rotate.

Calculate the stopping distance and time on a dry road where the dry friction coefficient between the tyres and the road is μ.

At high speed on a wet road the grip of each of the four tyres changes from dry friction to a lubricated drag equal to $\frac{1}{4}\lambda u$ for each tyre, where λ is the drag coefficient and u the instantaneous speed of the car. However, the tyres regain their dry-weather grip when the speed falls below $\frac{1}{4}U$. Calculate the stopping distance and time under these conditions.

4/II/9C **Dynamics**

A particle of mass m and charge q moving in a vacuum through a magnetic field $\mathbf{B}$ and subject to no other forces obeys

$$m\,\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}(t)$ is the location of the particle.

For $\mathbf{B} = (0, 0, B)$ with constant B, and using cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$, or otherwise, determine the motion of the particle in the $z = 0$ plane if its initial speed is u_0 with $\dot{z} = 0$. [*Hint: Choose the origin so that* $\dot{r} = 0$ *and* $\ddot{r} = 0$ *at* $t = 0$.]

Due to a leak, a small amount of gas enters the system, causing the particle to experience a drag force $\mathbf{D} = -\mu\dot{\mathbf{r}}$, where $\mu \ll qB$. Write down the new governing equations and show that the speed of the particle decays exponentially. Sketch the path followed by the particle. [*Hint: Consider the equations for the velocity in Cartesian coordinates; you need not apply any initial conditions.*]

4/II/10C **Dynamics**

A keen cyclist wishes to analyse her performance on training rollers. She decides that the key components are her bicycle's rear wheel and the roller on which the wheel sits. The wheel, of radius R, has its mass M entirely at its outer edge. The roller, which is driven by the wheel without any slippage, is a solid cylinder of radius S and mass $M/2$. The angular velocities of the wheel and roller are ω and σ, respectively.

Determine I and J, the moments of inertia of the wheel and roller, respectively. Find the ratio of the angular velocities of the wheel and roller. Show that the combined total kinetic energy of the wheel and roller is $\frac{1}{2}K\omega^2$, where

$$K = \frac{5}{4}MR^2$$

is the effective combined moment of inertia of the wheel and roller.

Why should K be used instead of just I or J in the equation connecting torque with angular acceleration? The cyclist believes the torque she can produce at the back wheel is $T = Q(1 - \omega/\Omega)$ where Q and Ω are dimensional constants. Determine the angular velocity of the wheel, starting from rest, as a function of time.

In an attempt to make the ride more realistic, the cyclist adds a fan (of negligible mass) to the roller. The fan imposes a frictional torque $-\gamma\sigma^2$ on the roller, where γ is a dimensional constant. Determine the new maximum speed for the wheel.

4/II/11C **Dynamics**

A puck of mass m located at $\mathbf{r} = (x, y)$ slides without friction under the influence of gravity on a surface of height $z = h(x, y)$. Show that the equations of motion can be approximated by

$$\ddot{\mathbf{r}} = -g\nabla h\,,$$

where g is the gravitational acceleration and the small slope approximation $\sin\phi \approx \tan\phi$ is used.

Determine the motion of the puck when $h(x, y) = \alpha x^2$.

Sketch the surface

$$h(x, y) = h(r) = \frac{1}{r^2} - \frac{1}{r}$$

as a function of r, where $r^2 = x^2 + y^2$. Write down the equations of motion of the puck on this surface in polar coordinates $\mathbf{r} = (r, \theta)$ under the assumption that the small slope approximation can be used. Show that L, the angular momentum per unit mass about the origin, is conserved. Show also that the initial kinetic energy per unit mass of the puck is $E_0 = \frac{1}{2}L^2/r_0^2$ if the puck is released at radius r_0 with negligible radial velocity. Determine and sketch $\dot{r}^2$ as a function of r for this release condition. What condition relating L, r_0 and g must be satisfied for the orbit to be bounded?

4/II/12C **Dynamics**

In an experiment a ball of mass m is released from a height h_0 above a flat, horizontal plate. Assuming the gravitational acceleration g is constant and the ball falls through a vacuum, find the speed u_0 of the ball on impact.

Determine the speed u_1 at which the ball rebounds if the coefficient of restitution for the collision is γ. What fraction of the impact energy is dissipated during the collision? Determine also the maximum height h_n the ball reaches after the n^{th} bounce, and the time T_n between the n^{th} and $(n+1)^{th}$ bounce. What is the total distance travelled by the ball before it comes to rest if $\gamma < 1$?

If the experiment is repeated in an atmosphere then the ball experiences a drag force $D = -\alpha\,|u|\,u$, where α is a dimensional constant and u the instantaneous velocity of the ball. Write down and solve the modified equation for $u(t)$ before the ball first hits the plate.

4/I/3A **Dynamics**

A lecturer driving his car of mass m_1 along the flat at speed U_1 accidentally collides with a stationary vehicle of mass m_2. As both vehicles are old and very solidly built, neither suffers damage in the collision: they simply bounce elastically off each other in a straight line. Determine how both vehicles are moving after the collision if neither driver applied their brakes. State any assumptions made and consider all possible values of the mass ratio $R = m_1/m_2$. You may neglect friction and other such losses.

An undergraduate drives into a rigid rock wall at speed V. The undergraduate's car of mass M is modern and has a crumple zone of length L at its front. As this zone crumples upon impact, it exerts a net force $F = (L - y)^{-1/2}$ on the car, where y is the amount the zone has crumpled. Determine the value of y at the point the car stops moving forwards as a function of V, where $V < 2L^{\frac{1}{4}}/M^{\frac{1}{2}}$.

4/I/4A **Dynamics**

A small spherical bubble of radius a containing carbon dioxide rises in water due to a buoyancy force $\rho g V$, where ρ is the density of water, g is gravitational attraction and V is the volume of the bubble. The drag on a bubble moving at speed u is $6\pi\mu a u$, where μ is the dynamic viscosity of water, and an accelerating bubble acts like a particle of mass $\alpha\rho V$, for some constant α. Find the location at time t of a bubble released from rest at $t = 0$ and show the bubble approaches a steady rise speed

$$U = \frac{2}{9}\frac{\rho g}{\mu}a^2. \qquad (*)$$

Under some circumstances the carbon dioxide gradually dissolves in the water, which leads to the bubble radius varying as $a^2 = a_0^2 - \beta t$, where a_0 is the bubble radius at $t = 0$ and β is a constant. Under the assumption that the bubble rises at speed given by $(*)$, determine the height to which it rises before it disappears.

4/II/9A **Dynamics**

A horizontal table oscillates with a displacement $\mathbf{A}\sin\omega t$, where $\mathbf{A} = (A_x, 0, A_z)$ is the amplitude vector and ω the angular frequency in an inertial frame of reference with the z axis vertically upwards, normal to the table. A block sitting on the table has mass m and linear friction that results in a force $\mathbf{F} = -\lambda\mathbf{u}$, where λ is a constant and $\mathbf{u}$ is the velocity difference between the block and the table. Derive the equations of motion for this block in the frame of reference of the table using axes (ξ, η, ζ) on the table parallel to the axes (x, y, z) in the inertial frame.

For the case where $A_z = 0$, show that at late time the block will approach the steady orbit

$$\xi = \xi_0 - A_x \sin\theta \cos(\omega t - \theta),$$

where

$$\sin^2\theta = \frac{m^2\omega^2}{\lambda^2 + m^2\omega^2}$$

and ξ_0 is a constant.

Given that there are no attractive forces between block and table, show that the block will only remain in contact with the table if $\omega^2 A_z < g$.

4/II/10A **Dynamics**

A small probe of mass m is in low orbit about a planet of mass M. If there is no drag on the probe then its orbit is governed by

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^3}\,\mathbf{r},$$

where $\mathbf{r}$ is the location of the probe relative to the centre of the planet and G is the gravitational constant. Show that the basic orbital trajectory is elliptical. Determine the orbital period for the probe if it is in a circular orbit at a distance r_0 from the centre of the planet.

Data returned by the probe shows that the planet has a very extensive but diffuse atmosphere. This atmosphere induces a drag on the probe that may be approximated by the linear law $\mathbf{D} = -A\dot{\mathbf{r}}$, where $\mathbf{D}$ is the drag force and A is a constant. Show that the angular momentum of the probe about the planet decays exponentially.

4/II/11A Dynamics

A particle of mass m and charge q moves through a magnetic field $\mathbf{B}$. There is no electric field or external force so that the particle obeys

$$m\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}$ is the location of the particle. Prove that the kinetic energy of the particle is preserved.

Consider an axisymmetric magnetic field described by $\mathbf{B} = (0, 0, B(r))$ in cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$. Determine the angular velocity of a circular orbit centred on $\mathbf{r} = \mathbf{0}$.

For a general orbit when $B(r) = B_0/r$, show that the angular momentum about the z-axis varies as $L = L_0 - qB_0(r - r_0)$, where L_0 is the angular momentum at radius r_0. Determine and sketch the relationship between $\dot{r}^2$ and r. [Hint: Use conservation of energy.] What is the escape velocity for the particle?

4/II/12A Dynamics

A circular cylinder of radius a, length L and mass m is rolling along a surface. Show that its moment of inertia is given by $\frac{1}{2}ma^2$.

At $t = 0$ the cylinder is at the bottom of a slope making an angle α to the horizontal, and is rolling with velocity V and angular velocity V/a. Assuming slippage does not occur, determine the position of the cylinder as a function of time. What is the maximum height that the cylinder reaches?

The frictional force between the cylinder and surface is given by $\mu mg \cos\alpha$, where μ is the friction coefficient. Show that the cylinder begins to slip rather than roll if $\tan\alpha > 3\mu$. Determine as a function of time the location, speed and angular velocity of the cylinder on the slope if this condition is satisfied. Show that slipping continues as the cylinder ascends and descends the slope. Find also the maximum height the cylinder reaches, and its speed and angular velocity when it returns to the bottom of the slope.

4/I/3E **Dynamics**

Because of an accident on launching, a rocket of unladen mass M lies horizontally on the ground. It initially contains fuel of mass m_0, which ignites and is emitted horizontally at a constant rate and at uniform speed u relative to the rocket. The rocket is initially at rest. If the coefficient of friction between the rocket and the ground is μ, and the fuel is completely burnt in a total time T, show that the final speed of the rocket is

$$u \log\left(\frac{M+m_0}{M}\right) - \mu g T.$$

4/I/4E **Dynamics**

Write down an expression for the total momentum $\mathbf{P}$ and angular momentum $\mathbf{L}$ with respect to an origin O of a system of n point particles of masses m_i, position vectors (with respect to O) $\mathbf{x}_i$, and velocities $\mathbf{v}_i$, $i = 1, \ldots, n$.

Show that with respect to a new origin O' the total momentum $\mathbf{P}'$ and total angular momentum $\mathbf{L}'$ are given by

$$\mathbf{P}' = \mathbf{P}, \qquad \mathbf{L}' = \mathbf{L} - \mathbf{b} \times \mathbf{P},$$

and hence

$$\mathbf{L}' \cdot \mathbf{P}' = \mathbf{L} \cdot \mathbf{P},$$

where $\mathbf{b}$ is the constant vector displacement of O' with respect to O. How does $\mathbf{L} \times \mathbf{P}$ change under change of origin?

Hence show that **either**

(1) the total momentum vanishes and the total angular momentum is independent of origin, **or**

(2) by choosing $\mathbf{b}$ in a way that should be specified, the total angular momentum with respect to O' can be made parallel to the total momentum.

4/II/9E **Dynamics**

Write down the equation of motion for a point particle with mass m, charge e, and position vector $\mathbf{x}(t)$ moving in a time-dependent magnetic field $\mathbf{B}(\mathbf{x}, t)$ with vanishing electric field, and show that the kinetic energy of the particle is constant. If the magnetic field is constant in direction, show that the component of velocity in the direction of $\mathbf{B}$ is constant. Show that, in general, the angular momentum of the particle is not conserved.

Suppose that the magnetic field is independent of time and space and takes the form $\mathbf{B} = (0, 0, B)$ and that $\dot{A}$ is the rate of change of area swept out by a radius vector joining the origin to the projection of the particle's path on the (x, y) plane. Obtain the equation

$$\frac{d}{dt}\left(m\dot{A} + \frac{eBr^2}{4}\right) = 0\ , \qquad (*)$$

where (r, θ) are plane polar coordinates. Hence obtain an equation replacing the equation of conservation of angular momentum.

Show further, using energy conservation and $(*)$, that the equations of motion in plane polar coordinates may be reduced to the first order non-linear system

$$\dot{r} = \sqrt{v^2 - \left(\frac{2c}{mr} - \frac{erB}{2m}\right)^2},$$

$$\dot{\theta} = \frac{2c}{mr^2} - \frac{eB}{2m}\ ,$$

where v and c are constants.

4/II/10E **Dynamics**

Write down the equations of motion for a system of n gravitating particles with masses m_i, and position vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n$.

The particles undergo a motion for which $\mathbf{x}_i(t) = a(t)\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Show that the equations of motion will be satisfied as long as the function $a(t)$ satisfies

$$\ddot{a} = -\frac{\Lambda}{a^2} \ , \qquad (*)$$

where Λ is a constant and the vectors $\mathbf{a}_i$ satisfy

$$\Lambda m_i \mathbf{a}_i = \mathbf{G}_i = \sum_{j \neq i} \frac{G m_i m_j \big(\mathbf{a}_i - \mathbf{a}_j\big)}{|\mathbf{a}_i - \mathbf{a}_j|^3} \ . \qquad (**)$$

Show that $(*)$ has as first integral

$$\frac{\dot{a}^2}{2} - \frac{\Lambda}{a} = \frac{k}{2} \ ,$$

where k is another constant. Show that

$$\mathbf{G}_i = \boldsymbol{\nabla}_i W \ ,$$

where $\boldsymbol{\nabla}_i$ is the gradient operator with respect to $\mathbf{a}_i$ and

$$W = -\sum_i \sum_{j<i} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} \ .$$

Using Euler's theorem for homogeneous functions (see below), or otherwise, deduce that

$$\sum_i \mathbf{a}_i \cdot \mathbf{G}_i = -W \ .$$

Hence show that all solutions of $(**)$ satisfy

$$\Lambda I = -W$$

where

$$I = \sum_i m_i \mathbf{a}_i^2 \ .$$

Deduce that Λ must be positive and that the total kinetic energy plus potential energy of the system of particles is equal to $\frac{k}{2} I$.

[*Euler's theorem states that if*

$$f(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^p f(x, y, z, \ldots) \ ,$$

then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \ldots = pf \ .]$$

4/II/11E **Dynamics**

State the parallel axis theorem and use it to calculate the moment of inertia of a uniform hemisphere of mass m and radius a about an axis through its centre of mass and parallel to the base.

[*You may assume that the centre of mass is located at a distance* $\frac{3}{8}a$ *from the flat face of the hemisphere, and that the moment of inertia of a full sphere about its centre is* $\frac{2}{5}Ma^2$, *with* $M = 2m$.]

The hemisphere initially rests on a rough horizontal plane with its base vertical. It is then released from rest and subsequently rolls on the plane without slipping. Let θ be the angle that the base makes with the horizontal at time t. Express the instantaneous speed of the centre of mass in terms of b and the rate of change of θ, where b is the instantaneous distance from the centre of mass to the point of contact with the plane. Hence write down expressions for the kinetic energy and potential energy of the hemisphere and deduce that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{(28 - 15\cos\theta)a}\,.$$

4/II/12E **Dynamics**

Let (r,θ) be plane polar coordinates and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ unit vectors in the direction of increasing r and θ respectively. Show that the velocity of a particle moving in the plane with polar coordinates $\big(r(t),\ \theta(t)\big)$ is given by

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\,,$$

and that the unit normal $\mathbf{n}$ to the particle path is parallel to

$$r\dot{\theta}\mathbf{e}_r - \dot{r}\mathbf{e}_\theta\,.$$

Deduce that the perpendicular distance p from the origin to the tangent of the curve $r = r(\theta)$ is given by

$$\frac{r^2}{p^2} = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

The particle, whose mass is m, moves under the influence of a central force with potential $V(r)$. Use the conservation of energy E and angular momentum h to obtain the equation

$$\frac{1}{p^2} = \frac{2m\big(E - V(r)\big)}{h^2}\,.$$

Hence express θ as a function of r as the integral

$$\theta = \int \frac{hr^{-2}dr}{\sqrt{2m\big(E - V_{\text{eff}}(r)\big)}}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{h^2}{2mr^2}\,.$$

Evaluate the integral and describe the orbit when $V(r) = \dfrac{c}{r^2}$, with c a positive constant.

4/I/3E **Dynamics**

The position x of the leading edge of an avalanche moving down a mountain side making a positive angle α to the horizontal satisfies the equation

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right) = gx\sin\alpha,$$

where g is the acceleration due to gravity.

By multiplying the equation by $x\frac{dx}{dt}$, obtain the first integral

$$x^2\dot{x}^2 = \frac{2g}{3}x^3\sin\alpha + c,$$

where c is an arbitrary constant of integration and the dot denotes differentiation with respect to time.

Sketch the positive quadrant of the $(x,\dot{x})$ phase plane. Show that all solutions approach the trajectory

$$\dot{x} = \left(\frac{2g\sin\alpha}{3}\right)^{\frac{1}{2}} x^{\frac{1}{2}}.$$

Hence show that, independent of initial conditions, the avalanche ultimately has acceleration $\frac{1}{3}g\sin\alpha$.

4/I/4E **Dynamics**

An inertial reference frame S and another reference frame S' have a common origin O. S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to S. Assuming the result that

$$\left(\frac{d\mathbf{a}}{dt}\right)_S = \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \boldsymbol{\omega}\times\mathbf{a}$$

for an arbitrary vector $\mathbf{a}(t)$, show that

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}} = \left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}'} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{x}}{dt}\right)_{\mathcal{S}'} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x}),$$

where $\mathbf{x}$ is the position vector of a point P measured from the origin.

A system of electrically charged particles, all with equal masses m and charges e, moves under the influence of mutual central forces $\mathbf{F}_{ij}$ of the form

$$\mathbf{F}_{ij} = (\mathbf{x}_i - \mathbf{x}_j)f(|\mathbf{x}_i - \mathbf{x}_j|).$$

In addition each particle experiences a Lorentz force due to a constant weak magnetic field $\mathbf{B}$ given by

$$e\frac{d\mathbf{x}_i}{dt}\times\mathbf{B}.$$

Transform the equations of motion to the rotating frame $\mathcal{S}'$. Show that if the angular velocity is chosen to satisfy

$$\boldsymbol{\omega} = -\frac{e}{2m}\mathbf{B},$$

and if terms of second order in $\mathbf{B}$ are neglected, then the equations of motion in the rotating frame are identical to those in the non-rotating frame in the absence of the magnetic field $\mathbf{B}$.

4/II/9E **Dynamics**

Write down the equations of motion for a system of n gravitating point particles with masses m_i and position vectors $\mathbf{x}_i = \mathbf{x}_i(t)$, $i = 1, 2, \ldots, n$.

Assume that $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Obtain a system of equations for the vectors $\mathbf{a}_i$ which does not involve the time variable t.

Show that the constant vectors $\mathbf{a}_i$ must be located at stationary points of the function

$$\sum_i \frac{1}{9} m_i\, \mathbf{a}_i\cdot\mathbf{a}_i \;+\; \frac{1}{2}\sum_j \sum_{i\neq j} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|}\;.$$

Show that for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other point?

4/II/10E **Dynamics**

Derive the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(u)}{mh^2u^2}\ ,$$

for the orbit $r^{-1} = u(\theta)$ of a particle of mass m and angular momentum hm moving under a central force $f(u)$ directed towards a fixed point O. Give an interpretation of h in terms of the area swept out by a radius vector.

If the orbits are found to be circles passing through O, then deduce that the force varies inversely as the fifth power of the distance, $f = cu^5$, where c is a constant. Is the force attractive or repulsive?

Show that, for fixed mass, the radius R of the circle varies inversely as the angular momentum of the particle, and hence that the time taken to traverse a complete circle is proportional to R^3.

[*You may assume, if you wish, the expressions for radial and transverse acceleration in the forms* $\ddot{r} - r\dot{\theta}^2,\ 2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

4/II/11E **Dynamics**

An electron of mass m moving with velocity $\dot{\mathbf{x}}$ in the vicinity of the North Pole experiences a force

$$\mathbf{F} = a\dot{\mathbf{x}} \times \frac{\mathbf{x}}{|\mathbf{x}|^3}\ ,$$

where a is a constant and the position vector $\mathbf{x}$ of the particle is with respect to an origin located at the North Pole. Write down the equation of motion of the electron, neglecting gravity. By taking the dot product of the equation with $\dot{\mathbf{x}}$ show that the speed of the electron is constant. By taking the cross product of the equation with $\mathbf{x}$ show that

$$m\mathbf{x} \times \dot{\mathbf{x}} - a\frac{\mathbf{x}}{|\mathbf{x}|} = \mathbf{L}\ ,$$

where $\mathbf{L}$ is a constant vector. By taking the dot product of this equation with $\mathbf{x}$, show that the electron moves on a cone centred on the North Pole.

4/II/12E **Dynamics**

Calculate the moment of inertia of a uniform rod of length $2l$ and mass M about an axis through its centre and perpendicular to its length. Assuming it moves in a plane, give an expression for the kinetic energy of the rod in terms of the speed of the centre and the angle that it makes with a fixed direction.

Two such rods are freely hinged together at one end and the other two ends slide on a perfectly smooth horizontal floor. The rods are initially at rest and lie in a vertical plane, each making an angle α to the horizontal. The rods subsequently move under gravity. Calculate the speed with which the hinge strikes the ground.

4/I/3A **Dynamics**

Derive the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(u)}{mh^2u^2}$$

for the motion of a particle of mass m under an attractive central force f, where $u = 1/r$ and r is the distance of the particle from the centre of force, and where mh is the angular momentum of the particle about the centre of force.

[*Hint: you may assume the expressions for the radial and transverse accelerations in the form* $\ddot{r} - r\dot{\theta}^2, 2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

4/I/4A **Dynamics**

Two particles of masses m_1 and m_2 at positions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are subject to forces $\mathbf{F}_1 = -\mathbf{F}_2 = \mathbf{f}(\mathbf{x}_1 - \mathbf{x}_2)$. Show that the centre of mass moves at a constant velocity. Obtain the equation of motion for the relative position of the particles. How does the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

of the system enter?

4/II/9A **Dynamics**

The position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. Write down the equation of motion of the particle under a force $\mathbf{F} = -4m\omega^2\mathbf{x}$.

Find the motion of the particle in (x, y, z) coordinates with initial condition

$$\mathbf{x} = (1, 0, 0) \quad \text{and} \quad \dot{\mathbf{x}} = (0, 0, 0) \qquad \text{at } t = 0,$$

where $\boldsymbol{\omega} = (0, 0, \omega)$. Show that the particle has a maximum speed at $t = (2n + 1)\pi/4\omega$, and find this speed.

[*Hint: you may find it useful to consider the combination* $\zeta = x + iy$.]

4/II/10A Dynamics

A spherical raindrop of radius $a(t) > 0$ and density ρ falls down at a velocity $v(t) > 0$ through a fine stationary mist. As the raindrop falls its volume grows at the rate $c\pi a^2 v$ with constant c. The raindrop is subject to the gravitational force and a resistive force $-k\rho\pi a^2 v^2$ with k a positive constant. Show a and v satisfy

$$\dot{a} = \tfrac{1}{4}cv,$$
$$\dot{v} = g - \tfrac{3}{4}(c+k)\frac{v^2}{a}.$$

Find an expression for $\frac{d}{dt}(v^2/a)$, and deduce that as time increases v^2/a tends to the constant value $g/(\frac{7}{8}c + \frac{3}{4}k)$, and thence the raindrop tends to a constant acceleration which is less than $\frac{1}{7}g$.

4/II/11A Dynamics

A spacecraft of mass m moves under the gravitational influence of the Sun of mass M and with universal gravitation constant G. After a disastrous manoeuvre, the unfortunate spacecraft finds itself exactly in a parabolic orbit about the Sun: the orbit with zero total energy. Using the conservation of energy and angular momentum, or otherwise, show that in the subsequent motion the distance of the spacecraft from the Sun $r(t)$ satisfies

$$(r - r_0)(r + 2r_0)^2 = \tfrac{9}{2}GM(t - t_0)^2,$$

with constants r_0 and t_0.

4/II/12A Dynamics

Find the moment of inertia of a uniform solid cylinder of radius a, length l and total mass M about its axis.

The cylinder is released from rest at the top of an inclined plane of length L and inclination θ to the horizontal. The first time the plane is perfectly smooth and the cylinder slips down the plane without rotating. The experiment is then repeated after the plane has been roughened, so that the cylinder now rolls without slipping at the point of contact. Show that the time taken to roll down the roughened plane is $\sqrt{\frac{3}{2}}$ times the time taken to slip down the smooth plane.