## Part IA

## Differential Equations

Year
2023
2022
2021
2020
2019
2018
2017
2016
2015
2014
2013
2012
2011
2010
2009
2008
2007
2006
2005
2004
2003
2002
2001

## Paper 2, Section I

## 1A Differential Equations

Find the general solution $y(x)$ of the differential equation

$$
y^{\prime \prime \prime}-4 y^{\prime \prime}+4 y^{\prime}=x e^{2 x} .
$$

## Paper 2, Section I

2A Differential Equations
(a) Find the solution $y(x)$ of

$$
x^{2} y^{\prime}-\cos (2 y)=1
$$

subject to $y \rightarrow 9 \pi / 4$ as $x \rightarrow \infty$. [If your answer involves inverse trigonometric functions, then you should specify their range.]
(b) Find the general solution $u(x)$ of the equation

$$
x u^{\prime}=x+u \text {. }
$$

## Paper 2, Section II

## 5A Differential Equations

(a) Consider the linear differential equation

$$
\begin{equation*}
y^{\prime}+p(x) y=f(x), \tag{*}
\end{equation*}
$$

where $p(x)$ and $f(x)$ are given nonzero functions. Show how to express the general solution $y(x)$ in terms of two integrals involving $p(x)$ and $f(x)$, to be specified.

If $y_{1}(x)$ and $y_{2}(x)$ are distinct solutions of $(*)$, express the general solution of $(*)$ in terms of $y_{1}(x)$ and $y_{2}(x)$.
(b) Find the general solution $y(x)$ of the differential equation

$$
x y^{\prime}-\left(2 x^{2}+1\right) y=x^{2} .
$$

Show that there is only one solution of this equation with $y(x)$ bounded as $x \rightarrow \infty$, and determine its limiting value. Sketch this solution.

Paper 2, Section II
6A Differential Equations
The function $y(x, \mu)$ satisfies

$$
\begin{equation*}
\frac{\partial y}{\partial x}=y+\mu\left(x+y^{2}\right), \quad y(0, \mu)=1, \tag{*}
\end{equation*}
$$

and the function $u(x, \mu)$ is defined by $u=\partial y / \partial \mu$. Show that

$$
\frac{\partial u}{\partial x}=u+x+y^{2}+2 \mu y u, \quad u(0, \mu)=0
$$

Determine $y(x, 0)$ and then $u(x, 0)$.
For small $\mu$, the solution of ( $*$ ) can be approximated by a series

$$
y(x, \mu)=y_{0}(x)+\mu y_{1}(x)+\mu^{2} y_{2}(x)+\cdots .
$$

Specify the functions $y_{0}(x), y_{1}(x)$ and $y_{2}(x)$.

## Paper 2, Section II

## 7A Differential Equations

The Dirac $\delta$-function can be defined by the properties $\delta(t)=0$ for $t \neq 0$ and $\int_{a}^{b} f(t) \delta(t) d t=f(0)$ for any $a<0<b$ and function $f(t)$ that is continuous at $t=0$. The function $H(t)$ is defined by

$$
H(t)= \begin{cases}1 & \text { for } \quad t \geqslant 0 \\ 0 & \text { for } \quad t<0\end{cases}
$$

(a) Prove that
(i) $\delta(p t)=\delta(t) /|p|$ for any nonzero real constant $p$;
(ii) for any differentiable function $f(t)$

$$
\int_{-\infty}^{\infty} f(t) \delta^{\prime}(t) d t=-f^{\prime}(0)
$$

(iii) $H^{\prime}(t)=\delta(t)$.
(b) An electronic system has two time-dependent variables $x(t)$ and $y(t)$, and two inputs to which a constant unit signal is applied, each starting at a particular time. The differential equations governing the system take the form

$$
\begin{aligned}
\dot{x}+2 y & =H(t) \\
\dot{y}-2 x & =H(t-\pi)
\end{aligned}
$$

At $t=-\pi$, the system has $x=1$ and $y=0$. Find $x(t)$ for $t<0$. Show that $x(t)$ can be written for $t>0$ as

$$
x(t)=a \sin 2 t+b+q(t) \sin ^{2} t
$$

where the constants $a$ and $b$ and the function $q(t)$ are to be specified. Sketch $q(t)$ for $0<t<2 \pi$.

Paper 2, Section II

## 8A Differential Equations

(a) Classify the equilibrium point of the system

$$
\frac{d x}{d t}=4 x+2 y, \quad \frac{d y}{d t}=-x+y
$$

Sketch the phase portrait showing both the direction of any straight-line trajectories and the shapes of a representative selection of non-straight trajectories to indicate the direction of motion in each part of phase space.
(b) Consider the second-order differential equation for $x(t)$

$$
\ddot{x}+3 \dot{x}-4 \log \frac{x^{2}+1}{2}=0 .
$$

(i) Rewrite the equation as a system of two first-order equations for $x(t)$ and $y(t)$, where $y=\dot{x}$, and find the equilibrium points of that system.
(ii) Use linearisation to classify the equilibrium points.
(iii) On a sketch of the $(x, y)$-plane, show the regions where $\dot{x}$ and $\dot{y}$ are both positive, both negative, or one positive and one negative.
(iv) Using the information obtained in parts (i)-(iii), sketch the trajectories of the system, including the trajectories through $(1,0)$.

## Paper 2, Section I

1A Differential Equations
Consider the integral

$$
I(x)=\int_{0}^{\pi} e^{x \cos \theta} d \theta
$$

Show, by differentiating under the integral sign, that

$$
\frac{d I}{d x}=\int_{0}^{\pi} x \sin ^{2} \theta e^{x \cos \theta} d \theta .
$$

Hence, or otherwise, show that

$$
\frac{d^{2} I}{d x^{2}}+\frac{1}{x} \frac{d I}{d x}-I=0 .
$$

## Paper 2, Section I

## 2B Differential Equations

Solve the difference equation

$$
x_{n+3}-6 x_{n+2}+12 x_{n+1}-8 x_{n}=0,
$$

given initial conditions $x_{0}=0, x_{1}=4, x_{2}=24$.

## Paper 2, Section II

## 5C Differential Equations

(a) What is meant by an ordinary point and a regular singular point of a linear second-order ordinary differential equation?

Consider

$$
x \frac{d^{2} y}{d x^{2}}+(1-x) \frac{d y}{d x}+\lambda y=0
$$

where $\lambda$ is a real constant.
Find a solution to $(\dagger)$ in the form of a series expansion around $x=0$. Obtain the general expression for the coefficients in the series.

For what values of $\lambda$ do you obtain polynomial solutions?
(b) Determine the Wronskian of the equation ( $\dagger$ ) as a function of $x$.

Let $\lambda=1$. Verify that $y_{1}=1-x$ is a solution to $(\dagger)$. Using the Wronskian, calculate a second solution $y_{2}$ in the form

$$
y_{2}=(1-x) \log x+b_{1} x+b_{2} x^{2}+\ldots,
$$

where $b_{1}$ and $b_{2}$ are constants you need to find.

## Paper 2, Section II

## 6A Differential Equations

(a) Let $f(x, y)$ be a real-valued function depending smoothly on real variables $x$ and $y$, and $g(t)=f(a+t \cos \gamma, b+t \sin \gamma)$, where $a, b$ and $\gamma$ are constants. Express $g^{\prime}(t)$ and $g^{\prime \prime}(t)$ in terms of partial derivatives of $f$.

Write down sufficient conditions for $g$ to have a local minimum at $t=0$ and deduce that a stationary point of $f$ at $(x, y)=(a, b)$ is a local minimum if

$$
\frac{\partial^{2} f}{\partial y^{2}}>0 \quad \text { and } \quad \frac{\partial^{2} f}{\partial x^{2}} \frac{\partial^{2} f}{\partial y^{2}}>\left(\frac{\partial^{2} f}{\partial x \partial y}\right)^{2}
$$

(b) Now let

$$
f(x, y)=x^{4}-3 x^{2}+2 x y+y^{2}
$$

Find all stationary points of $f$ and show that those at $(x, y) \neq(0,0)$ are local minima.
Show also that $g(t)$ with $a=b=0$ has either (i) a local minimum or (ii) a local maximum at $t=0$, depending on the value of $\gamma$. Determine carefully the ranges of values of $\tan \gamma$ for which cases (i) and (ii) occur and sketch the typical behaviour of $g(t)$ in each of these cases.

## Paper 2, Section II

## 7C Differential Equations

Consider the system of linear differential equations

$$
\frac{d \mathbf{z}}{d t}-A \mathbf{z}=\mathbf{f}, \quad \text { where } \quad A=\left(\begin{array}{ll}
3 & -6 \\
1 & -2
\end{array}\right)
$$

(a) Suppose $\mathbf{f}=\mathbf{0}$. Show that the general solution to $(\dagger)$ takes the form

$$
\mathbf{z}=\alpha \mathbf{u}_{1} e^{\lambda_{1} t}+\beta \mathbf{u}_{2} e^{\lambda_{2} t}
$$

where $\alpha$ and $\beta$ are arbitrary constants. Calculate $\mathbf{u}_{1}, \mathbf{u}_{2}, \lambda_{1}$, and $\lambda_{2}$.
(b) Suppose now that $\mathbf{f}=(1, a)^{T}$, where $a$ is a constant parameter.

By writing $\mathbf{f}$ as a linear combination of $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$, determine the value(s) of $a$ for which the particular integral depends on time.

Using matrix methods, find the general solution to ( $\dagger$ ).
(c) Consider

$$
\frac{d^{n} \mathbf{z}}{d t^{n}}-A \mathbf{z}=\mathbf{0}
$$

where $n>1$ is an integer.
Show that $(\star)$ is a solution to this system of equations. How many other linearly independent solutions must there be?

Paper 2, Section II

## 8B Differential Equations

(a) Consider the system

$$
\begin{equation*}
\dot{x}=8 x-2 x^{2}-2 x y^{2}, \quad \dot{y}=x y-y \tag{*}
\end{equation*}
$$

for $x(t) \geqslant 0, y(t) \geqslant 0$.
Find all the equilibrium points of $(*)$ and determine their type. Explain how solutions close to each equilibrium point will evolve, sketching their trajectories. [You may quote general results without proof.]
(b) Consider the system

$$
\begin{equation*}
\dot{x}=x(1-y), \quad \dot{y}=3 y(x-1), \tag{**}
\end{equation*}
$$

defined for $x>0, y>0$.
Show that it has precisely one equilibrium point in the given range. Obtain an equation for $d y / d x$. Show that this equation is separable and hence obtain a solution in the form $E(x, y)=C$, where $C$ is a constant and $E(x, y)$ is a nontrivial conserved quantity for solutions of $(* *)$. Show that $E(x, y)$ has a single stationary point in the quadrant $x>0, y>0$, and identify what type of stationary point it is. Hence show that solutions close to the equilibrium point at time $t=0$ remain close at all times.

## Paper 2, Section I

## 1A Differential Equations

Solve the difference equation

$$
y_{n+2}-4 y_{n+1}+4 y_{n}=n
$$

subject to the initial conditions $y_{0}=1$ and $y_{1}=0$.

## Paper 2, Section I

## 2A Differential Equations

Let $y_{1}$ and $y_{2}$ be two linearly independent solutions to the differential equation

$$
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}+p(x) \frac{\mathrm{d} y}{\mathrm{~d} x}+q(x) y=0
$$

Show that the Wronskian $W=y_{1} y_{2}^{\prime}-y_{2} y_{1}^{\prime}$ satisfies

$$
\frac{\mathrm{d} W}{\mathrm{~d} x}+p W=0
$$

Deduce that if $y_{2}\left(x_{0}\right)=0$ then

$$
y_{2}(x)=y_{1}(x) \int_{x_{0}}^{x} \frac{W(t)}{y_{1}(t)^{2}} \mathrm{~d} t
$$

Given that $y_{1}(x)=x^{3}$ satisfies the equation

$$
x^{2} \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}-x \frac{\mathrm{~d} y}{\mathrm{~d} x}-3 y=0
$$

find the solution which satisfies $y(1)=0$ and $y^{\prime}(1)=1$.

Paper 2, Section II

## 5A Differential Equations

For a linear, second order differential equation define the terms ordinary point, singular point and regular singular point.

For $a, b \in \mathbb{R}$ and $b \notin \mathbb{Z}$ consider the following differential equation

$$
\begin{equation*}
x \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}+(b-x) \frac{\mathrm{d} y}{\mathrm{~d} x}-a y=0 \tag{*}
\end{equation*}
$$

Find coefficients $c_{m}(a, b)$ such that the function $y_{1}=M(x, a, b)$, where

$$
M(x, a, b)=\sum_{m=0}^{\infty} c_{m}(a, b) x^{m}
$$

satisfies $(*)$. By making the substitution $y=x^{1-b} u(x)$, or otherwise, find a second linearly independent solution of the form $y_{2}=x^{1-b} M(x, \alpha, \beta)$ for suitable $\alpha, \beta$.

Suppose now that $b=1$. By considering a limit of the form

$$
\lim _{b \rightarrow 1} \frac{y_{2}-y_{1}}{b-1}
$$

or otherwise, obtain two linearly independent solutions to $(*)$ in terms of $M$ and derivatives thereof.

## Paper 2, Section II

## 6A Differential Equations

By means of the change of variables $\eta=x-t$ and $\xi=x+t$, show that the wave equation for $u=u(x, t)$

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}-\frac{\partial^{2} u}{\partial t^{2}}=0 \tag{*}
\end{equation*}
$$

is equivalent to the equation

$$
\frac{\partial^{2} U}{\partial \eta \partial \xi}=0
$$

where $U(\eta, \xi)=u(x, t)$. Hence show that the solution to $(*)$ on $x \in \mathbf{R}$ and $t>0$, subject to the initial conditions

$$
u(x, 0)=f(x), \quad \frac{\partial u}{\partial t}(x, 0)=g(x)
$$

is

$$
u(x, t)=\frac{1}{2}[f(x-t)+f(x+t)]+\frac{1}{2} \int_{x-t}^{x+t} g(y) \mathrm{d} y .
$$

Deduce that if $f(x)=0$ and $g(x)=0$ on the interval $\left|x-x_{0}\right|>r$ then $u(x, t)=0$ on $\left|x-x_{0}\right|>r+t$.

Suppose now that $y=y(x, t)$ is a solution to the wave equation $(*)$ on the finite interval $0<x<L$ and obeys the boundary conditions

$$
y(0, t)=y(L, t)=0
$$

for all $t$. The energy is defined by

$$
E(t)=\frac{1}{2} \int_{0}^{L}\left[\left(\frac{\partial y}{\partial x}\right)^{2}+\left(\frac{\partial y}{\partial t}\right)^{2}\right] \mathrm{d} x
$$

By considering $\mathrm{d} E / \mathrm{d} t$, or otherwise, show that the energy remains constant in time.

## Paper 2, Section II

## 7A Differential Equations

The function $\theta=\theta(t)$ takes values in the interval $(-\pi, \pi]$ and satisfies the differential equation

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \theta}{\mathrm{~d} t^{2}}+(\lambda-2 \mu) \sin \theta+\frac{2 \mu \sin \theta}{\sqrt{5+4 \cos \theta}}=0 \tag{*}
\end{equation*}
$$

where $\lambda$ and $\mu$ are positive constants.
Let $\omega=\dot{\theta}$. Express ( $*$ ) in terms of a pair of first order differential equations in $(\theta, \omega)$. Show that if $3 \lambda<4 \mu$ then there are three fixed points in the region $0 \leqslant \theta \leqslant \pi$.

Classify all the fixed points of the system in the case $3 \lambda<4 \mu$. Sketch the phase portrait in the case $\lambda=1$ and $\mu=3 / 2$.

Comment briefly on the case when $3 \lambda>4 \mu$.

## Paper 2, Section II

## 8A Differential Equations

For an $n \times n$ matrix $A$, define the matrix exponential by

$$
\exp (A)=\sum_{m=0}^{\infty} \frac{A^{m}}{m!}
$$

where $A^{0} \equiv I$, with $I$ being the $n \times n$ identity matrix. [You may assume that $\exp ((s+t) A)=\exp (s A) \exp (t A)$ for real numbers $s, t$ and you do not need to consider issues of convergence.] Show that

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \exp (t A)=A \exp (t A)
$$

Deduce that the unique solution to the initial value problem

$$
\frac{\mathrm{d} \mathbf{y}}{\mathrm{~d} t}=A \mathbf{y}, \quad \mathbf{y}(0)=\mathbf{y}_{0}, \quad \text { where } \mathbf{y}(t)=\left(\begin{array}{c}
y_{1}(t) \\
\vdots \\
y_{n}(t)
\end{array}\right)
$$

is $\mathbf{y}(t)=\exp (t A) \mathbf{y}_{0}$.
Let $\mathbf{x}=\mathbf{x}(t)$ and $\mathbf{f}=\mathbf{f}(t)$ be vectors of length $n$ and $A$ a real $n \times n$ matrix. By considering a suitable integrating factor, show that the unique solution to

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{x}}{\mathrm{~d} t}-A \mathbf{x}=\mathbf{f}, \quad \mathbf{x}(0)=\mathbf{x}_{0} \tag{*}
\end{equation*}
$$

is given by

$$
\mathbf{x}(t)=\exp (t A) \mathbf{x}_{0}+\int_{0}^{t} \exp [(t-s) A] \mathbf{f}(s) \mathrm{d} s
$$

Hence, or otherwise, solve the system of differential equations (*) when

$$
A=\left(\begin{array}{lll}
2 & 2 & -2 \\
5 & 1 & -3 \\
1 & 5 & -3
\end{array}\right), \quad \mathbf{f}(t)=\left(\begin{array}{c}
\sin t \\
3 \sin t \\
0
\end{array}\right), \quad \mathbf{x}_{0}=\left(\begin{array}{l}
1 \\
1 \\
2
\end{array}\right)
$$

[Hint: Compute $A^{2}$ and show that $A^{3}=0$.]

## Paper 1, Section I

## 2A Differential Equations

Solve the differential equation

$$
\frac{d y}{d x}=\frac{1}{x+e^{2 y}},
$$

subject to the initial condition $y(1)=0$.

## Paper 1, Section II

## 7A Differential Equations

Show that for each $t>0$ and $x \in \mathbb{R}$ the function

$$
K(x, t)=\frac{1}{\sqrt{4 \pi t}} \exp \left(-\frac{x^{2}}{4 t}\right)
$$

satisfies the heat equation

$$
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}} .
$$

For $t>0$ and $x \in \mathbb{R}$ define the function $u=u(x, t)$ by the integral

$$
u(x, t)=\int_{-\infty}^{\infty} K(x-y, t) f(y) d y .
$$

Show that $u$ satisfies the heat equation and $\lim _{t \rightarrow 0^{+}} u(x, t)=f(x)$. [Hint: You may find it helpful to consider the substitution $Y=(x-y) / \sqrt{4 t}$.]

Burgers' equation is

$$
\frac{\partial w}{\partial t}+w \frac{\partial w}{\partial x}=\frac{\partial^{2} w}{\partial x^{2}} .
$$

By considering the transformation

$$
w(x, t)=-2 \frac{1}{u} \frac{\partial u}{\partial x},
$$

solve Burgers' equation with the initial condition $\lim _{t \rightarrow 0^{+}} w(x, t)=g(x)$.

## Paper 1, Section II

## 8A Differential Equations

Solve the system of differential equations for $x(t), y(t), z(t)$,

$$
\begin{aligned}
\dot{x} & =3 z-x, \\
\dot{y} & =3 x+2 y-3 z+\cos t-2 \sin t, \\
\dot{z} & =3 x-z,
\end{aligned}
$$

subject to the initial conditions $x(0)=y(0)=0, z(0)=1$.

## Paper 2, Section I

## 1C Differential Equations

The function $y(x)$ satisfies the inhomogeneous second-order linear differential equation

$$
y^{\prime \prime}-2 y^{\prime}-3 y=-16 x e^{-x}
$$

Find the solution that satisfies the conditions that $y(0)=1$ and $y(x)$ is bounded as $x \rightarrow \infty$.

## Paper 2, Section I

## 2C Differential Equations

Consider the first order system

$$
\begin{equation*}
\frac{d \boldsymbol{v}}{d t}-B \boldsymbol{v}=e^{\lambda t} \boldsymbol{x} \tag{1}
\end{equation*}
$$

to be solved for $\boldsymbol{v}(t)=\left(v_{1}(t), v_{2}(t), \ldots, v_{n}(t)\right) \in \mathbb{R}^{n}$, where the $n \times n$ matrix $B, \lambda \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^{n}$ are all independent of time. Show that if $\lambda$ is not an eigenvalue of $B$ then there is a solution of the form $\boldsymbol{v}(t)=e^{\lambda t} \boldsymbol{u}$, with $\boldsymbol{u}$ constant.

For $n=2$, given

$$
B=\left(\begin{array}{ll}
0 & 3 \\
1 & 0
\end{array}\right) \quad \lambda=2 \quad \text { and } \boldsymbol{x}=\binom{0}{1},
$$

find the general solution to (1).

## Paper 2, Section II

## 5C Differential Equations

Consider the problem of solving

$$
\begin{equation*}
\frac{d^{2} y}{d t^{2}}=t \tag{1}
\end{equation*}
$$

subject to the initial conditions $y(0)=\frac{d y}{d t}(0)=0$ using a discrete approach where $y$ is computed at discrete times, $y_{n}=y\left(t_{n}\right)$ where $t_{n}=n h(n=-1,0,1, \ldots, N)$ and $0<h=1 / N \ll 1$.
(a) By using Taylor expansions around $t_{n}$, derive the centred-difference formula

$$
\frac{y_{n+1}-2 y_{n}+y_{n-1}}{h^{2}}=\left.\frac{d^{2} y}{d t^{2}}\right|_{t=t_{n}}+O\left(h^{\alpha}\right)
$$

where the value of $\alpha$ should be found.
(b) Find the general solution of $y_{n+1}-2 y_{n}+y_{n-1}=0$ and show that this is the discrete version of the corresponding general solution to $\frac{d^{2} y}{d t^{2}}=0$.
(c) The fully discretized version of the differential equation (1) is

$$
\begin{equation*}
\frac{y_{n+1}-2 y_{n}+y_{n-1}}{h^{2}}=n h \quad \text { for } \quad n=0, \ldots, N-1 . \tag{2}
\end{equation*}
$$

By finding a particular solution first, write down the general solution to the difference equation (2). For the solution which satisfies the discretized initial conditions $y_{0}=0$ and $y_{-1}=y_{1}$, find the error in $y_{N}$ in terms of $h$ only.

## Paper 2, Section II

## 6C Differential Equations

Find all power series solutions of the form $y=\sum_{n=0}^{\infty} a_{n} x^{n}$ to the equation

$$
\left(1-x^{2}\right) y^{\prime \prime}-x y^{\prime}+\lambda^{2} y=0
$$

for $\lambda$ a real constant. [It is sufficient to give a recurrence relationship between coefficients.]
Impose the condition $y^{\prime}(0)=0$ and determine those values of $\lambda$ for which your power series gives polynomial solutions (i.e., $a_{n}=0$ for $n$ sufficiently large). Give the values of $\lambda$ for which the corresponding polynomials have degree less than 6 , and compute these polynomials. Hence, or otherwise, find a polynomial solution of

$$
\left(1-x^{2}\right) y^{\prime \prime}-x y^{\prime}+y=8 x^{4}-3
$$

satisfying $y^{\prime}(0)=0$.

## Paper 2, Section II

## 7C Differential Equations

Two cups of tea at temperatures $T_{1}(t)$ and $T_{2}(t)$ cool in a room at ambient constant temperature $T_{\infty}$. Initially $T_{1}(0)=T_{2}(0)=T_{0}>T_{\infty}$.

Cup 1 has cool milk added instantaneously at $t=1$ and then hot water added at a constant rate after $t=2$ which is modelled as follows

$$
\frac{d T_{1}}{d t}=-a\left(T_{1}-T_{\infty}\right)-\delta(t-1)+H(t-2)
$$

whereas cup 2 is left undisturbed and evolves as follows

$$
\frac{d T_{2}}{d t}=-a\left(T_{2}-T_{\infty}\right)
$$

where $\delta(t)$ and $H(t)$ are the Dirac delta and Heaviside functions respectively, and $a$ is a positive constant.
(a) Derive expressions for $T_{1}(t)$ when $0<t \leqslant 1$ and for $T_{2}(t)$ when $t>0$.
(b) Show for $1<t<2$ that

$$
T_{1}(t)=T_{\infty}+\left(T_{0}-T_{\infty}-e^{a}\right) e^{-a t} .
$$

(c) Derive an expression for $T_{1}(t)$ for $t>2$.
(d) At what time $t^{*}$ is $T_{1}=T_{2}$ ?
(e) Find how $t^{*}$ behaves for $a \rightarrow 0$ and explain your result.

## Paper 2, Section II

## 8C Differential Equations

Consider the nonlinear system

$$
\begin{aligned}
& \dot{x}=y-2 y^{3}, \\
& \dot{y}=-x .
\end{aligned}
$$

(a) Show that $H=H(x, y)=x^{2}+y^{2}-y^{4}$ is a constant of the motion.
(b) Find all the critical points of the system and analyse their stability. Sketch the phase portrait including the special contours with value $H(x, y)=\frac{1}{4}$.
(c) Find an explicit expression for $y=y(t)$ in the solution which satisfies $(x, y)=\left(\frac{1}{2}, 0\right)$ at $t=0$. At what time does it reach the point $(x, y)=\left(\frac{1}{4},-\frac{1}{2}\right)$ ?

## Paper 2, Section I

## 1B Differential Equations

Consider the following difference equation for real $u_{n}$ :

$$
u_{n+1}=a u_{n}\left(1-u_{n}^{2}\right)
$$

where $a$ is a real constant.
For $-\infty<a<\infty$ find the steady-state solutions, i.e. those with $u_{n+1}=u_{n}$ for all $n$, and determine their stability, making it clear how the number of solutions and the stability properties vary with $a$. [You need not consider in detail particular values of $a$ which separate intervals with different stability properties.]

## Paper 2, Section I

## 2B Differential Equations

Show that for given $P(x, y), Q(x, y)$ there is a function $F(x, y)$ such that, for any function $y(x)$,

$$
P(x, y)+Q(x, y) \frac{d y}{d x}=\frac{d}{d x} F(x, y)
$$

if and only if

$$
\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}
$$

Now solve the equation

$$
(2 y+3 x) \frac{d y}{d x}+4 x^{3}+3 y=0
$$

## Paper 2, Section II

## 5B Differential Equations

By choosing a suitable basis, solve the equation

$$
\left(\begin{array}{ll}
1 & 2 \\
1 & 0
\end{array}\right)\binom{\dot{x}}{\dot{y}}+\left(\begin{array}{cc}
-2 & 5 \\
2 & -1
\end{array}\right)\binom{x}{y}=e^{-4 t}\binom{3 b}{2}+e^{-t}\binom{-3}{c-1}
$$

subject to the initial conditions $x(0)=0, y(0)=0$.
Explain briefly what happens in the cases $b=2$ or $c=2$.

## Paper 2, Section II

## 6B Differential Equations

The function $u(x, y)$ satisfies the partial differential equation

$$
a \frac{\partial^{2} u}{\partial x^{2}}+b \frac{\partial^{2} u}{\partial x \partial y}+c \frac{\partial^{2} u}{\partial y^{2}}=0
$$

where $a, b$ and $c$ are non-zero constants.
Defining the variables $\xi=\alpha x+y$ and $\eta=\beta x+y$, where $\alpha$ and $\beta$ are constants, and writing $v(\xi, \eta)=u(x, y)$ show that

$$
a \frac{\partial^{2} u}{\partial x^{2}}+b \frac{\partial^{2} u}{\partial x \partial y}+c \frac{\partial^{2} u}{\partial y^{2}}=A(\alpha, \beta) \frac{\partial^{2} v}{\partial \xi^{2}}+B(\alpha, \beta) \frac{\partial^{2} v}{\partial \xi \partial \eta}+C(\alpha, \beta) \frac{\partial^{2} v}{\partial \eta^{2}},
$$

where you should determine the functions $A(\alpha, \beta), B(\alpha, \beta)$ and $C(\alpha, \beta)$.
If the quadratic $a s^{2}+b s+c=0$ has distinct real roots then show that $\alpha$ and $\beta$ can be chosen such that $A(\alpha, \beta)=C(\alpha, \beta)=0$ and $B(\alpha, \beta) \neq 0$.

If the quadratic $a s^{2}+b s+c=0$ has a repeated root then show that $\alpha$ and $\beta$ can be chosen such that $A(\alpha, \beta)=B(\alpha, \beta)=0$ and $C(\alpha, \beta) \neq 0$.

Hence find the general solutions of the equations

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}+3 \frac{\partial^{2} u}{\partial x \partial y}+2 \frac{\partial^{2} u}{\partial y^{2}}=0 \tag{i}
\end{equation*}
$$

and
(ii)

$$
\frac{\partial^{2} u}{\partial x^{2}}+2 \frac{\partial^{2} u}{\partial x \partial y}+\frac{\partial^{2} u}{\partial y^{2}}=0 .
$$

## Paper 2, Section II

## 7B Differential Equations

Consider the differential equation

$$
x^{2} \frac{d^{2} y}{d x^{2}}+x \frac{d y}{d x}-\left(x^{2}+\alpha^{2}\right) y=0 .
$$

What values of $x$ are ordinary points of the differential equation? What values of $x$ are singular points of the differential equation, and are they regular singular points or irregular singular points? Give clear definitions of these terms to support your answers.

For $\alpha$ not equal to an integer there are two linearly independent power series solutions about $x=0$. Give the forms of the two power series and the recurrence relations that specify the relation between successive coefficients. Give explicitly the first three terms in each power series.

For $\alpha$ equal to an integer explain carefully why the forms you have specified do not give two linearly independent power series solutions. Show that for such values of $\alpha$ there is (up to multiplication by a constant) one power series solution, and give the recurrence relation between coefficients. Give explicitly the first three terms.

If $y_{1}(x)$ is a solution of the above second-order differential equation then

$$
y_{2}(x)=y_{1}(x) \int_{c}^{x} \frac{1}{s\left[y_{1}(s)\right]^{2}} d s
$$

where $c$ is an arbitrarily chosen constant, is a second solution that is linearly independent of $y_{1}(x)$. For the case $\alpha=1$, taking $y_{1}(x)$ to be a power series, explain why the second solution $y_{2}(x)$ is not a power series.
[You may assume that any power series you use are convergent.]

## Paper 2, Section II

## 8B Differential Equations

The temperature $T$ in an oven is controlled by a heater which provides heat at rate $Q(t)$. The temperature of a pizza in the oven is $U$. Room temperature is the constant value $T_{r}$.
$T$ and $U$ satisfy the coupled differential equations

$$
\begin{aligned}
\frac{d T}{d t} & =-a\left(T-T_{r}\right)+Q(t) \\
\frac{d U}{d t} & =-b(U-T)
\end{aligned}
$$

where $a$ and $b$ are positive constants. Briefly explain the various terms appearing in the above equations.

Heating may be provided by a short-lived pulse at $t=0$, with $Q(t)=Q_{1}(t)=\delta(t)$ or by constant heating over a finite period $0<t<\tau$, with $Q(t)=Q_{2}(t)=\tau^{-1}(H(t)-H(t-$ $\tau)$ ), where $\delta(t)$ and $H(t)$ are respectively the Dirac delta function and the Heaviside step function. Again briefly, explain how the given formulae for $Q_{1}(t)$ and $Q_{2}(t)$ are consistent with their description and why the total heat supplied by the two heating protocols is the same.

For $t<0, T=U=T_{r}$. Find the solutions for $T(t)$ and $U(t)$ for $t>0$, for each of $Q(t)=Q_{1}(t)$ and $Q(t)=Q_{2}(t)$, denoted respectively by $T_{1}(t)$ and $U_{1}(t)$, and $T_{2}(t)$ and $U_{2}(t)$. Explain clearly any assumptions that you make about continuity of the solutions in time.

Show that the solutions $T_{2}(t)$ and $U_{2}(t)$ tend respectively to $T_{1}(t)$ and $U_{1}(t)$ in the limit as $\tau \rightarrow 0$ and explain why.

## Paper 2, Section I

## 1C Differential Equations

(a) The numbers $z_{1}, z_{2}, \ldots$ satisfy

$$
z_{n+1}=z_{n}+c_{n} \quad(n \geqslant 1),
$$

where $c_{1}, c_{2}, \ldots$ are given constants. Find $z_{n+1}$ in terms of $c_{1}, c_{2}, \ldots, c_{n}$ and $z_{1}$.
(b) The numbers $x_{1}, x_{2}, \ldots$ satisfy

$$
x_{n+1}=a_{n} x_{n}+b_{n} \quad(n \geqslant 1),
$$

where $a_{1}, a_{2}, \ldots$ are given non-zero constants and $b_{1}, b_{2}, \ldots$ are given constants. Let $z_{1}=x_{1}$ and $z_{n+1}=x_{n+1} / U_{n}$, where $U_{n}=a_{1} a_{2} \cdots a_{n}$. Calculate $z_{n+1}-z_{n}$, and hence find $x_{n+1}$ in terms of $x_{1}, b_{1}, \ldots, b_{n}$ and $U_{1}, \ldots, U_{n}$.

## Paper 2, Section I

## 2C Differential Equations

Consider the function

$$
f(x, y)=\frac{x}{y}+\frac{y}{x}-\frac{(x-y)^{2}}{a^{2}}
$$

defined for $x>0$ and $y>0$, where $a$ is a non-zero real constant. Show that $(\lambda, \lambda)$ is a stationary point of $f$ for each $\lambda>0$. Compute the Hessian and its eigenvalues at $(\lambda, \lambda)$.

## Paper 2, Section II

## 5C Differential Equations

The current $I(t)$ at time $t$ in an electrical circuit subject to an applied voltage $V(t)$ obeys the equation

$$
L \frac{d^{2} I}{d t^{2}}+R \frac{d I}{d t}+\frac{1}{C} I=\frac{d V}{d t}
$$

where $R, L$ and $C$ are the constant resistance, inductance and capacitance of the circuit with $R \geqslant 0, L>0$ and $C>0$.
(a) In the case $R=0$ and $V(t)=0$, show that there exist time-periodic solutions of frequency $\omega_{0}$, which you should find.
(b) In the case $V(t)=H(t)$, the Heaviside function, calculate, subject to the condition

$$
R^{2}>\frac{4 L}{C}
$$

the current for $t \geqslant 0$, assuming it is zero for $t<0$.
(c) If $R>0$ and $V(t)=\sin \omega_{0} t$, where $\omega_{0}$ is as in part (a), show that there is a timeperiodic solution $I_{0}(t)$ of period $T=2 \pi / \omega_{0}$ and calculate its maximum value $I_{M}$.
(i) Calculate the energy dissipated in each period, i.e., the quantity

$$
D=\int_{0}^{T} R I_{0}(t)^{2} d t
$$

Show that the quantity defined by

$$
Q=\frac{2 \pi}{D} \times \frac{L I_{M}^{2}}{2}
$$

satisfies $Q \omega_{0} R C=1$.
(ii) Write down explicitly the general solution $I(t)$ for all $R>0$, and discuss the relevance of $I_{0}(t)$ to the large time behaviour of $I(t)$.

## Paper 2, Section II

## 6C Differential Equations

(a) Consider the system

$$
\begin{aligned}
& \frac{d x}{d t}=x(1-x)-x y \\
& \frac{d y}{d t}=\frac{1}{8} y(4 x-1)
\end{aligned}
$$

for $x(t) \geqslant 0, y(t) \geqslant 0$. Find the critical points, determine their type and explain, with the help of a diagram, the behaviour of solutions for large positive times $t$.
(b) Consider the system

$$
\begin{aligned}
& \frac{d x}{d t}=y+\left(1-x^{2}-y^{2}\right) x \\
& \frac{d y}{d t}=-x+\left(1-x^{2}-y^{2}\right) y
\end{aligned}
$$

for $(x(t), y(t)) \in \mathbb{R}^{2}$. Rewrite the system in polar coordinates by setting $x(t)=$ $r(t) \cos \theta(t)$ and $y(t)=r(t) \sin \theta(t)$, and hence describe the behaviour of solutions for large positive and large negative times.

## Paper 2, Section II

## 7C Differential Equations

Let $y_{1}$ and $y_{2}$ be two solutions of the differential equation

$$
y^{\prime \prime}(x)+p(x) y^{\prime}(x)+q(x) y(x)=0, \quad-\infty<x<\infty,
$$

where $p$ and $q$ are given. Show, using the Wronskian, that

- either there exist $\alpha$ and $\beta$, not both zero, such that $\alpha y_{1}(x)+\beta y_{2}(x)$ vanishes for all $x$,
- or given $x_{0}, A$ and $B$, there exist $a$ and $b$ such that $y(x)=a y_{1}(x)+b y_{2}(x)$ satisfies the conditions $y\left(x_{0}\right)=A$ and $y^{\prime}\left(x_{0}\right)=B$.

Find power series $y_{1}$ and $y_{2}$ such that an arbitrary solution of the equation

$$
y^{\prime \prime}(x)=x y(x)
$$

can be written as a linear combination of $y_{1}$ and $y_{2}$.

## Paper 2, Section II

## 8C Differential Equations

(a) Solve $\frac{d z}{d t}=z^{2}$ subject to $z(0)=z_{0}$. For which $z_{0}$ is the solution finite for all $t \in \mathbb{R}$ ?

Let $a$ be a positive constant. By considering the lines $y=a\left(x-x_{0}\right)$ for constant $x_{0}$, or otherwise, show that any solution of the equation

$$
\frac{\partial f}{\partial x}+a \frac{\partial f}{\partial y}=0
$$

is of the form $f(x, y)=F(y-a x)$ for some function $F$.
Solve the equation

$$
\frac{\partial f}{\partial x}+a \frac{\partial f}{\partial y}=f^{2}
$$

subject to $f(0, y)=g(y)$ for a given function $g$. For which $g$ is the solution bounded on $\mathbb{R}^{2}$ ?
(b) By means of the change of variables $X=\alpha x+\beta y$ and $T=\gamma x+\delta y$ for appropriate real numbers $\alpha, \beta, \gamma, \delta$, show that the equation

$$
\begin{equation*}
\frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial x \partial y}=0 \tag{*}
\end{equation*}
$$

can be transformed into the wave equation

$$
\frac{1}{c^{2}} \frac{\partial^{2} F}{\partial T^{2}}-\frac{\partial^{2} F}{\partial X^{2}}=0
$$

where $F$ is defined by $f(x, y)=F(\alpha x+\beta y, \gamma x+\delta y)$. Hence write down the general solution of $(*)$.

## Paper 2, Section I

## 1A Differential Equations

(a) Find the solution of the differential equation

$$
y^{\prime \prime}-y^{\prime}-6 y=0
$$

that is bounded as $x \rightarrow \infty$ and satisfies $y=1$ when $x=0$.
(b) Solve the difference equation

$$
\left(y_{n+1}-2 y_{n}+y_{n-1}\right)-\frac{h}{2}\left(y_{n+1}-y_{n-1}\right)-6 h^{2} y_{n}=0 .
$$

Show that if $0<h \ll 1$, the solution that is bounded as $n \rightarrow \infty$ and satisfies $y_{0}=1$ is approximately $(1-2 h)^{n}$.
(c) By setting $x=n h$, explain the relation between parts (a) and (b).

## Paper 2, Section I

## 2A Differential Equations

(a) For each non-negative integer $n$ and positive constant $\lambda$, let

$$
I_{n}(\lambda)=\int_{0}^{\infty} x^{n} e^{-\lambda x} d x
$$

By differentiating $I_{n}$ with respect to $\lambda$, find its value in terms of $n$ and $\lambda$.
(b) By making the change of variables $x=u+v, y=u-v$, transform the differential equation

$$
\frac{\partial^{2} f}{\partial x \partial y}=1
$$

into a differential equation for $g$, where $g(u, v)=f(x, y)$.

## Paper 2, Section II

## 5A Differential Equations

(a) Find and sketch the solution of

$$
y^{\prime \prime}+y=\delta(x-\pi / 2),
$$

where $\delta$ is the Dirac delta function, subject to $y(0)=1$ and $y^{\prime}(0)=0$.
(b) A bowl of soup, which Sam has just warmed up, cools down at a rate equal to the product of a constant $k$ and the difference between its temperature $T(t)$ and the temperature $T_{0}$ of its surroundings. Initially the soup is at temperature $T(0)=\alpha T_{0}$, where $\alpha>2$.
(i) Write down and solve the differential equation satisfied by $T(t)$.
(ii) At time $t_{1}$, when the temperature reaches half of its initial value, Sam quickly adds some hot water to the soup, so the temperature increases instantaneously by $\beta$, where $\beta>\alpha T_{0} / 2$. Find $t_{1}$ and $T(t)$ for $t>t_{1}$.
(iii) Sketch $T(t)$ for $t>0$.
(iv) Sam wants the soup to be at temperature $\alpha T_{0}$ at time $t_{2}$, where $t_{2}>t_{1}$. What value of $\beta$ should Sam choose to achieve this? Give your answer in terms of $\alpha$, $k, t_{2}$ and $T_{0}$.

## Paper 2, Section II

## 6A Differential Equations

(a) The function $y(x)$ satisfies

$$
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0
$$

(i) Define the Wronskian $W(x)$ of two linearly independent solutions $y_{1}(x)$ and $y_{2}(x)$. Derive a linear first-order differential equation satisfied by $W(x)$.
(ii) Suppose that $y_{1}(x)$ is known. Use the Wronskian to write down a first-order differential equation for $y_{2}(x)$. Hence express $y_{2}(x)$ in terms of $y_{1}(x)$ and $W(x)$.
(b) Verify that $y_{1}(x)=\cos \left(x^{\gamma}\right)$ is a solution of

$$
a x^{\alpha} y^{\prime \prime}+b x^{\alpha-1} y^{\prime}+y=0
$$

where $a, b, \alpha$ and $\gamma$ are constants, provided that these constants satisfy certain conditions which you should determine.

Use the method that you described in part (a) to find a solution which is linearly independent of $y_{1}(x)$.

## Paper 2, Section II

## 7A Differential Equations

The function $y(x)$ satisfies

$$
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0
$$

What does it mean to say that the point $x=0$ is (i) an ordinary point and (ii) a regular singular point of this differential equation? Explain what is meant by the indicial equation at a regular singular point. What can be said about the nature of the solutions in the neighbourhood of a regular singular point in the different cases that arise according to the values of the roots of the indicial equation?

State the nature of the point $x=0$ of the equation

$$
\begin{equation*}
x y^{\prime \prime}+(x-m+1) y^{\prime}-(m-1) y=0 . \tag{*}
\end{equation*}
$$

Set $y(x)=x^{\sigma} \sum_{n=0}^{\infty} a_{n} x^{n}$, where $a_{0} \neq 0$, and find the roots of the indicial equation.
(a) Show that one solution of $(*)$ with $m \neq 0,-1,-2, \cdots$ is

$$
y(x)=x^{m}\left(1+\sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n}}{(m+n)(m+n-1) \cdots(m+1)}\right),
$$

and find a linearly independent solution in the case when $m$ is not an integer.
(b) If $m$ is a positive integer, show that $(*)$ has a polynomial solution.
(c) What is the form of the general solution of ( $*$ ) in the case $m=0$ ? [You do not need to find the general solution explicitly.]

## Paper 2, Section II

## 8A Differential Equations

(a) By considering eigenvectors, find the general solution of the equations

$$
\begin{align*}
& \frac{d x}{d t}=2 x+5 y \\
& \frac{d y}{d t}=-x-2 y
\end{align*}
$$

and show that it can be written in the form

$$
\binom{x}{y}=\alpha\binom{5 \cos t}{-2 \cos t-\sin t}+\beta\binom{5 \sin t}{\cos t-2 \sin t},
$$

where $\alpha$ and $\beta$ are constants.
(b) For any square matrix $M, \exp (M)$ is defined by

$$
\exp (M)=\sum_{n=0}^{\infty} \frac{M^{n}}{n!}
$$

Show that if $M$ has constant elements, the vector equation $\frac{d \mathbf{x}}{d t}=M \mathbf{x}$ has a solution $\mathbf{x}=\exp (M t) \mathbf{x}_{0}$, where $\mathbf{x}_{0}$ is a constant vector. Hence solve $(\dagger)$ and show that your solution is consistent with the result of part (a).

## Paper 2, Section I

## 1B Differential Equations

Find the general solution of the equation

$$
\begin{equation*}
\frac{d y}{d x}-2 y=e^{\lambda x} \tag{*}
\end{equation*}
$$

where $\lambda$ is a constant not equal to 2 .
By subtracting from the particular integral an appropriate multiple of the complementary function, obtain the limit as $\lambda \rightarrow 2$ of the general solution of $(*)$ and confirm that it yields the general solution for $\lambda=2$.

Solve equation $(*)$ with $\lambda=2$ and $y(1)=2$.

## Paper 2, Section I

## 2B Differential Equations

Find the general solution of the equation

$$
2 \frac{d y}{d t}=y-y^{3} .
$$

Compute all possible limiting values of $y$ as $t \rightarrow \infty$.
Find a non-zero value of $y(0)$ such that $y(t)=y(0)$ for all $t$.

## Paper 2, Section II

## 5B Differential Equations

Write as a system of two first-order equations the second-order equation

$$
\begin{equation*}
\frac{d^{2} \theta}{d t^{2}}+c \frac{d \theta}{d t}\left|\frac{d \theta}{d t}\right|+\sin \theta=0, \tag{*}
\end{equation*}
$$

where $c$ is a small, positive constant, and find its equilibrium points. What is the nature of these points?

Draw the trajectories in the $(\theta, \omega)$ plane, where $\omega=d \theta / d t$, in the neighbourhood of two typical equilibrium points.

By considering the cases of $\omega>0$ and $\omega<0$ separately, find explicit expressions for $\omega^{2}$ as a function of $\theta$. Discuss how the second term in (*) affects the nature of the equilibrium points.

## Paper 2, Section II

## 6B Differential Equations

Consider the equation

$$
\begin{equation*}
2 \frac{\partial^{2} u}{\partial x^{2}}+3 \frac{\partial^{2} u}{\partial y^{2}}-7 \frac{\partial^{2} u}{\partial x \partial y}=0 \tag{*}
\end{equation*}
$$

for the function $u(x, y)$, where $x$ and $y$ are real variables. By using the change of variables

$$
\xi=x+\alpha y, \quad \eta=\beta x+y,
$$

where $\alpha$ and $\beta$ are appropriately chosen integers, transform $(*)$ into the equation

$$
\frac{\partial^{2} u}{\partial \xi \partial \eta}=0 .
$$

Hence, solve equation (*) supplemented with the boundary conditions

$$
u(0, y)=4 y^{2}, \quad u(-2 y, y)=0, \quad \text { for all } y
$$

## Paper 2, Section II

## 7B Differential Equations

Suppose that $u(x)$ satisfies the equation

$$
\frac{d^{2} u}{d x^{2}}-f(x) u=0,
$$

where $f(x)$ is a given non-zero function. Show that under the change of coordinates $x=x(t)$,

$$
\frac{d^{2} u}{d t^{2}}-\frac{\ddot{x}}{\dot{x}} \frac{d u}{d t}-\dot{x}^{2} f(x) u=0,
$$

where a dot denotes differentiation with respect to $t$. Furthermore, show that the function

$$
U(t)=\dot{x}^{-\frac{1}{2}} u(x)
$$

satisfies

$$
\frac{d^{2} U}{d t^{2}}-\left[\dot{x}^{2} f(x)+\dot{x}^{-\frac{1}{2}}\left(\frac{\ddot{x}}{\dot{x}} \frac{d}{d t}\left(\dot{x}^{\frac{1}{2}}\right)-\frac{d^{2}}{d t^{2}}\left(\dot{x}^{\frac{1}{2}}\right)\right)\right] U=0 .
$$

Choosing $\dot{x}=(f(x))^{-\frac{1}{2}}$, deduce that

$$
\frac{d^{2} U}{d t^{2}}-(1+F(t)) U=0
$$

for some appropriate function $F(t)$. Assuming that $F$ may be neglected, deduce that $u(x)$ can be approximated by

$$
u(x) \approx A(x)\left(c_{+} e^{G(x)}+c_{-} e^{-G(x)}\right),
$$

where $c_{+}, c_{-}$are constants and $A, G$ are functions that you should determine in terms of $f(x)$.

## Paper 2, Section II

## 8B Differential Equations

Suppose that $\mathbf{x}(t) \in \mathbb{R}^{3}$ obeys the differential equation

$$
\begin{equation*}
\frac{d \mathbf{x}}{d t}=M \mathbf{x} \tag{*}
\end{equation*}
$$

where $M$ is a constant $3 \times 3$ real matrix.
(i) Suppose that $M$ has distinct eigenvalues $\lambda_{1}, \lambda_{2}, \lambda_{3}$ with corresponding eigenvectors $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$. Explain why $\mathbf{x}$ may be expressed in the form $\sum_{i=1}^{3} a_{i}(t) \mathbf{e}_{i}$ and deduce by substitution that the general solution of $(*)$ is

$$
\mathbf{x}=\sum_{i=1}^{3} A_{i} e^{\lambda_{i} t} \mathbf{e}_{i}
$$

where $A_{1}, A_{2}, A_{3}$ are constants.
(ii) What is the general solution of $(*)$ if $\lambda_{2}=\lambda_{3} \neq \lambda_{1}$, but there are still three linearly independent eigenvectors?
(iii) Suppose again that $\lambda_{2}=\lambda_{3} \neq \lambda_{1}$, but now there are only two linearly independent eigenvectors: $\mathbf{e}_{1}$ corresponding to $\lambda_{1}$ and $\mathbf{e}_{2}$ corresponding to $\lambda_{2}$. Suppose that a vector $\mathbf{v}$ satisfying the equation $\left(M-\lambda_{2} I\right) \mathbf{v}=\mathbf{e}_{2}$ exists, where $I$ denotes the identity matrix. Show that $\mathbf{v}$ is linearly independent of $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$, and hence or otherwise find the general solution of $(*)$.

## Paper 2, Section I

## 1B Differential Equations

The following equation arises in the theory of elastic beams:

$$
t^{4} \frac{d^{2} u}{d t^{2}}+\lambda^{2} u=0, \quad \lambda>0, t>0
$$

where $u(t)$ is a real valued function.
By using the change of variables

$$
t=\frac{1}{\tau}, \quad u(t)=\frac{v(\tau)}{\tau},
$$

find the general solution of the above equation.

## Paper 2, Section I

## 2B Differential Equations

Consider the ordinary differential equation

$$
\begin{equation*}
P(x, y)+Q(x, y) \frac{d y}{d x}=0 . \tag{*}
\end{equation*}
$$

State an equation to be satisfied by $P$ and $Q$ that ensures that equation (*) is exact. In this case, express the general solution of equation $(*)$ in terms of a function $F(x, y)$ which should be defined in terms of $P$ and $Q$.

Consider the equation

$$
\frac{d y}{d x}=-\frac{4 x+3 y}{3 x+3 y^{2}}
$$

satisfying the boundary condition $y(1)=2$. Find an explicit relation between $y$ and $x$.

## Paper 2, Section II

## 5B Differential Equations

Use the transformation

$$
y(t)=\frac{1}{c x(t)} \frac{d x(t)}{d t}
$$

where $c$ is a constant, to map the Ricatti equation

$$
\frac{d y}{d t}+c y^{2}+a(t) y+b(t)=0, \quad t>0
$$

to a linear equation.
Using the above result, as well as the change of variables $\tau=\ln t$, solve the boundary value problem

$$
\begin{gathered}
\frac{d y}{d t}+y^{2}+\frac{y}{t}-\frac{\lambda^{2}}{t^{2}}=0, \quad t>0 \\
y(1)=2 \lambda
\end{gathered}
$$

where $\lambda$ is a positive constant. What is the value of $t>0$ for which the solution is singular?

## Paper 2, Section II

## 6B Differential Equations

The so-called "shallow water theory" is characterised by the equations

$$
\begin{aligned}
& \frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}[(h+\zeta) u]=0 \\
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+g \frac{\partial \zeta}{\partial x}=0
\end{aligned}
$$

where $g$ denotes the gravitational constant, the constant $h$ denotes the undisturbed depth of the water, $u(x, t)$ denotes the speed in the $x$-direction, and $\zeta(x, t)$ denotes the elevation of the water.
(i) Assuming that $|u|$ and $|\zeta|$ and their gradients are small in some appropriate dimensional considerations, show that $\zeta$ satisfies the wave equation

$$
\begin{equation*}
\frac{\partial^{2} \zeta}{\partial t^{2}}=c^{2} \frac{\partial^{2} \zeta}{\partial x^{2}} \tag{*}
\end{equation*}
$$

where the constant $c$ should be determined in terms of $h$ and $g$.
(ii) Using the change of variables

$$
\xi=x+c t, \quad \eta=x-c t
$$

show that the general solution of $(*)$ satisfying the initial conditions

$$
\zeta(x, 0)=u_{0}(x), \quad \frac{\partial \zeta}{\partial t}(x, 0)=v_{0}(x)
$$

is given by

$$
\zeta(x, t)=f(x+c t)+g(x-c t)
$$

where

$$
\begin{aligned}
& \frac{d f(x)}{d x}=\frac{1}{2}\left[\frac{d u_{0}(x)}{d x}+\frac{1}{c} v_{0}(x)\right] \\
& \frac{d g(x)}{d x}=\frac{1}{2}\left[\frac{d u_{0}(x)}{d x}-\frac{1}{c} v_{0}(x)\right] .
\end{aligned}
$$

Simplify the above to find $\zeta$ in terms of $u_{0}$ and $v_{0}$.
(iii) Find $\zeta(x, t)$ in the particular case that

$$
u_{0}(x)=H(x+1)-H(x-1), \quad v_{0}(x)=0, \quad-\infty<x<\infty
$$

where $H(\cdot)$ denotes the Heaviside step function.
Describe in words this solution.

## Paper 2, Section II

## 7B Differential Equations

(a) Let $y_{1}(x)$ be a solution of the equation

$$
\frac{d^{2} y}{d x^{2}}+p(x) \frac{d y}{d x}+q(x) y=0
$$

Assuming that the second linearly independent solution takes the form $y_{2}(x)=$ $v(x) y_{1}(x)$, derive an ordinary differential equation for $v(x)$.
(b) Consider the equation

$$
\left(1-x^{2}\right) \frac{d^{2} y}{d x^{2}}-2 x \frac{d y}{d x}+2 y=0, \quad-1<x<1
$$

By inspection or otherwise, find an explicit solution of this equation. Use the result in (a) to find the solution $y(x)$ satisfying the conditions

$$
y(0)=\frac{d y}{d x}(0)=1
$$

## Paper 2, Section II

## 8B Differential Equations

Consider the damped pendulum equation

$$
\begin{equation*}
\frac{d^{2} \theta}{d t^{2}}+c \frac{d \theta}{d t}+\sin \theta=0 \tag{*}
\end{equation*}
$$

where $c$ is a positive constant. The energy $E$, which is the sum of the kinetic energy and the potential energy, is defined by

$$
E(t)=\frac{1}{2}\left(\frac{d \theta}{d t}\right)^{2}+1-\cos \theta
$$

(i) Verify that $E(t)$ is a decreasing function.
(ii) Assuming that $\theta$ is sufficiently small, so that terms of order $\theta^{3}$ can be neglected, find an approximation for the general solution of $(*)$ in terms of two arbitrary constants. Discuss the dependence of this approximate solution on $c$.
(iii) By rewriting $(*)$ as a system of equations for $x(t)=\theta$ and $y(t)=\dot{\theta}$, find all stationary points of $(*)$ and discuss their nature for all $c$, except $c=2$.
(iv) Draw the phase plane curves for the particular case $c=1$.

## Paper 2, Section I

## 1A Differential Equations

Solve the equation

$$
\ddot{y}-\dot{y}-2 y=3 e^{2 t}+3 e^{-t}+3+6 t
$$

subject to the conditions $y=\dot{y}=0$ at $t=0$.

## Paper 2, Section I

## 2A Differential Equations

Use the transformation $z=\ln x$ to solve

$$
\ddot{z}=-\dot{z}^{2}-1-e^{-z}
$$

subject to the conditions $z=0$ and $\dot{z}=V$ at $t=0$, where $V$ is a positive constant.
Show that when $\dot{z}(t)=0$

$$
z=\ln \left(\sqrt{V^{2}+4}-1\right)
$$

## Paper 2, Section II

## 5A Differential Equations

The function $y(x)$ satisfies the equation

$$
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0
$$

Give the definitions of the terms ordinary point, singular point, and regular singular point for this equation.

For the equation

$$
x y^{\prime \prime}+y=0
$$

classify the point $x=0$ according to your definitions. Find the series solution about $x=0$ which satisfies

$$
y=0 \quad \text { and } \quad y^{\prime}=1 \quad \text { at } x=0
$$

For a second solution with $y=1$ at $x=0$, consider an expansion

$$
y(x)=y_{0}(x)+y_{1}(x)+y_{2}(x)+\ldots
$$

where $y_{0}=1$ and $x y_{n+1}^{\prime \prime}=-y_{n}$. Find $y_{1}$ and $y_{2}$ which have $y_{n}(0)=0$ and $y_{n}^{\prime}(1)=0$. Comment on $y^{\prime}$ near $x=0$ for this second solution.

## Paper 2, Section II

## 6A Differential Equations

Consider the function

$$
f(x, y)=\left(x^{2}-y^{4}\right)\left(1-x^{2}-y^{4}\right) .
$$

Determine the type of each of the nine critical points.
Sketch contours of constant $f(x, y)$.

## Paper 2, Section II

## 7A Differential Equations

Find $x(t)$ and $y(t)$ which satisfy

$$
\begin{aligned}
& 3 \dot{x}+\dot{y}+5 x-y=2 e^{-t}+4 e^{-3 t}, \\
& \dot{x}+4 \dot{y}-2 x+7 y=-3 e^{-t}+5 e^{-3 t}
\end{aligned}
$$

subject to $x=y=0$ at $t=0$.

## Paper 2, Section II

## 8A Differential Equations

Medical equipment is sterilised by placing it in a hot oven for a time $T$ and then removing it and letting it cool for the same time. The equipment at temperature $\theta(t)$ warms and cools at a rate equal to the product of a constant $\alpha$ and the difference between its temperature and its surroundings, $\theta_{1}$ when warming in the oven and $\theta_{0}$ when cooling outside. The equipment starts the sterilisation process at temperature $\theta_{0}$.

Bacteria are killed by the heat treatment. Their number $N(t)$ decreases at a rate equal to the product of the current number and a destruction factor $\beta$. This destruction factor varies linearly with temperature, vanishing at $\theta_{0}$ and having a maximum $\beta_{\max }$ at $\theta_{1}$.

Find an implicit equation for $T$ such that the number of bacteria is reduced by a factor of $10^{-20}$ by the sterilisation process.

A second hardier species of bacteria requires the oven temperature to be increased to achieve the same destruction factor $\beta_{\max }$. How is the sterilisation time $T$ affected?

## Paper 2, Section I

## 1A Differential Equations

Find two linearly independent solutions of

$$
y^{\prime \prime}+4 y^{\prime}+4 y=0 .
$$

Find the solution in $x \geqslant 0$ of

$$
y^{\prime \prime}+4 y^{\prime}+4 y=e^{-2 x}
$$

subject to $y=y^{\prime}=0$ at $x=0$.

## Paper 2, Section I

## 2A Differential Equations

Find the constant solutions (those with $u_{n+1}=u_{n}$ ) of the discrete equation

$$
u_{n+1}=\frac{1}{2} u_{n}\left(1+u_{n}\right),
$$

and determine their stability.

## Paper 2, Section II

## 5A Differential Equations

Find the first three non-zero terms in the series solutions $y_{1}(x)$ and $y_{2}(x)$ for the differential equation

$$
x^{2} y^{\prime \prime}-2 x y^{\prime}+\left(2-x^{2}\right) y=0,
$$

that satisfy

$$
\begin{aligned}
y_{1}^{\prime}(0)=a \quad & \text { and } \quad y_{1}^{\prime \prime}(0)=0, \\
y_{2}^{\prime}(0)=0 & \text { and } \quad y_{2}^{\prime \prime}(0)=2 b .
\end{aligned}
$$

Identify these solutions in closed form.

## Paper 2, Section II

## 6A Differential Equations

Consider the function

$$
V(x, y)=x^{4}-x^{2}+2 x y+y^{2} .
$$

Find the critical (stationary) points of $V(x, y)$. Determine the type of each critical point. Sketch the contours of $V(x, y)=$ constant.

Now consider the coupled differential equations

$$
\frac{d x}{d t}=-\frac{\partial V}{\partial x}, \quad \frac{d y}{d t}=-\frac{\partial V}{\partial y} .
$$

Show that $V(x(t), y(t))$ is a non-increasing function of $t$. If $x=1$ and $y=-\frac{1}{2}$ at $t=0$, where does the solution tend to as $t \rightarrow \infty$ ?

## Paper 2, Section II

## 7A Differential Equations

Find the solution to the system of equations

$$
\begin{aligned}
\frac{d x}{d t}+\frac{-4 x+2 y}{t} & =-9 \\
\frac{d y}{d t}+\frac{x-5 y}{t} & =3
\end{aligned}
$$

in $t \geqslant 1$ subject to

$$
x=0 \quad \text { and } \quad y=0 \quad \text { at } \quad t=1 .
$$

[Hint: powers of t.]

## Paper 2, Section II

## 8A Differential Equations

Consider the second-order differential equation for $y(t)$ in $t \geqslant 0$

$$
\begin{equation*}
\ddot{y}+2 k \dot{y}+\left(k^{2}+\omega^{2}\right) y=f(t) . \tag{*}
\end{equation*}
$$

(i) For $f(t)=0$, find the general solution $y_{1}(t)$ of $(*)$.
(ii) For $f(t)=\delta(t-a)$ with $a>0$, find the solution $y_{2}(t, a)$ of $(*)$ that satisfies $y=0$ and $\dot{y}=0$ at $t=0$.
(iii) For $f(t)=H(t-b)$ with $b>0$, find the solution $y_{3}(t, b)$ of $(*)$ that satisfies $y=0$ and $\dot{y}=0$ at $t=0$.
(iv) Show that

$$
y_{2}(t, b)=-\frac{\partial y_{3}}{\partial b}
$$

## Paper 2, Section I

## 1A Differential Equations

(a) Consider the homogeneous $k$ th-order difference equation

$$
\begin{equation*}
a_{k} y_{n+k}+a_{k-1} y_{n+k-1}+\ldots+a_{2} y_{n+2}+a_{1} y_{n+1}+a_{0} y_{n}=0 \tag{*}
\end{equation*}
$$

where the coefficients $a_{k}, \ldots, a_{0}$ are constants. Show that for $\lambda \neq 0$ the sequence $y_{n}=\lambda^{n}$ is a solution if and only if $p(\lambda)=0$, where

$$
p(\lambda)=a_{k} \lambda^{k}+a_{k-1} \lambda^{k-1}+\ldots+a_{2} \lambda^{2}+a_{1} \lambda+a_{0}
$$

State the general solution of $(*)$ if $k=3$ and $p(\lambda)=(\lambda-\mu)^{3}$ for some constant $\mu$.
(b) Find an inhomogeneous difference equation that has the general solution

$$
y_{n}=a 2^{n}-n, \quad a \in \mathbb{R}
$$

## Paper 2, Section I

## 2A Differential Equations

(a) For a differential equation of the form $\frac{\mathrm{d} y}{\mathrm{~d} x}=f(y)$, explain how $f^{\prime}(y)$ can be used to determine the stability of any equilibrium solutions and justify your answer.
(b) Find the equilibrium solutions of the differential equation

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=y^{3}-y^{2}-2 y
$$

and determine their stability. Sketch representative solution curves in the $(x, y)$-plane.

# UNIVERSITY OF 

CAMBRIDGE

## Paper 2, Section II

## 5A Differential Equations

(a) Find the general real solution of the system of first-order differential equations

$$
\begin{aligned}
\dot{x} & =x+\mu y \\
\dot{y} & =-\mu x+y,
\end{aligned}
$$

where $\mu$ is a real constant.
(b) Find the fixed points of the non-linear system of first-order differential equations

$$
\begin{aligned}
\dot{x} & =x+y \\
\dot{y} & =-x+y-2 x^{2} y
\end{aligned}
$$

and determine their nature. Sketch the phase portrait indicating the direction of motion along trajectories.

## Paper 2, Section II

## 6A Differential Equations

(a) A surface in $\mathbb{R}^{3}$ is defined by the equation $f(x, y, z)=c$, where $c$ is a constant. Show that the partial derivatives on this surface satisfy

$$
\begin{equation*}
\left.\left.\left.\frac{\partial x}{\partial y}\right|_{z} \frac{\partial y}{\partial z}\right|_{x} \frac{\partial z}{\partial x}\right|_{y}=-1 \tag{*}
\end{equation*}
$$

(b) Now let $f(x, y, z)=x^{2}-y^{4}+2 a y^{2}+z^{2}$, where $a$ is a constant.
(i) Find expressions for the three partial derivatives $\left.\frac{\partial x}{\partial y}\right|_{z},\left.\frac{\partial y}{\partial z}\right|_{x}$ and $\left.\frac{\partial z}{\partial x}\right|_{y}$ on the surface $f(x, y, z)=c$, and verify the identity $(*)$.
(ii) Find the rate of change of $f$ in the radial direction at the point $(x, 0, z)$.
(iii) Find and classify the stationary points of $f$.
(iv) Sketch contour plots of $f$ in the $(x, y)$-plane for the cases $a=1$ and $a=-1$.

## Paper 2, Section II

## 7A Differential Equations

(a) Define the Wronskian $W$ of two solutions $y_{1}(x)$ and $y_{2}(x)$ of the differential equation

$$
\begin{equation*}
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0 \tag{*}
\end{equation*}
$$

and state a necessary and sufficient condition for $y_{1}(x)$ and $y_{2}(x)$ to be linearly independent. Show that $W(x)$ satisfies the differential equation

$$
W^{\prime}(x)=-p(x) W(x)
$$

(b) By evaluating the Wronskian, or otherwise, find functions $p(x)$ and $q(x)$ such that $(*)$ has solutions $y_{1}(x)=1+\cos x$ and $y_{2}(x)=\sin x$. What is the value of $W(\pi)$ ? Is there a unique solution to the differential equation for $0 \leqslant x<\infty$ with initial conditions $y(0)=0, y^{\prime}(0)=1$ ? Why or why not?
(c) Write down a third-order differential equation with constant coefficients, such that $y_{1}(x)=1+\cos x$ and $y_{2}(x)=\sin x$ are both solutions. Is the solution to this equation for $0 \leqslant x<\infty$ with initial conditions $y(0)=y^{\prime \prime}(0)=0, y^{\prime}(0)=1$ unique? Why or why not?

## Paper 2, Section II

## 8A Differential Equations

(a) The circumference $y$ of an ellipse with semi-axes 1 and $x$ is given by

$$
\begin{equation*}
y(x)=\int_{0}^{2 \pi} \sqrt{\sin ^{2} \theta+x^{2} \cos ^{2} \theta} \mathrm{~d} \theta \tag{*}
\end{equation*}
$$

Setting $t=1-x^{2}$, find the first three terms in a series expansion of $(*)$ around $t=0$.
(b) Euler proved that $y$ also satisfies the differential equation

$$
x\left(1-x^{2}\right) y^{\prime \prime}-\left(1+x^{2}\right) y^{\prime}+x y=0
$$

Use the substitution $t=1-x^{2}$ for $x \geqslant 0$ to find a differential equation for $u(t)$, where $u(t)=y(x)$. Show that this differential equation has regular singular points at $t=0$ and $t=1$.

Show that the indicial equation at $t=0$ has a repeated root, and find the recurrence relation for the coefficients of the corresponding power-series solution. State the form of a second, independent solution.

Verify that the power-series solution is consistent with your answer in (a).

# UNIVERSITY OF 

CAMBRIDGE

## Paper 2, Section I

## 1A Differential Equations

Find the general solutions to the following difference equations for $y_{n}, n \in \mathbb{N}$.
(i) $\quad y_{n+3}-3 y_{n+1}+2 y_{n}=0$,
(ii) $\quad y_{n+3}-3 y_{n+1}+2 y_{n}=2^{n}$,
(iii) $y_{n+3}-3 y_{n+1}+2 y_{n}=(-2)^{n}$,
(iv) $\quad y_{n+3}-3 y_{n+1}+2 y_{n}=(-2)^{n}+2^{n}$.

## Paper 2, Section I

## 2A Differential Equations

Let $f(x, y)=g(u, v)$ where the variables $\{x, y\}$ and $\{u, v\}$ are related by a smooth, invertible transformation. State the chain rule expressing the derivatives $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and use this to deduce that

$$
\frac{\partial^{2} g}{\partial u \partial v}=\frac{\partial x}{\partial u} \frac{\partial x}{\partial v} \frac{\partial^{2} f}{\partial x^{2}}+\left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v}+\frac{\partial x}{\partial v} \frac{\partial y}{\partial u}\right) \frac{\partial^{2} f}{\partial x \partial y}+\frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \frac{\partial^{2} f}{\partial y^{2}}+H \frac{\partial f}{\partial x}+K \frac{\partial f}{\partial y}
$$

where $H$ and $K$ are second-order partial derivatives, to be determined.
Using the transformation $x=u v$ and $y=u / v$ in the above identity, or otherwise, find the general solution of

$$
x \frac{\partial^{2} f}{\partial x^{2}}-\frac{y^{2}}{x} \frac{\partial^{2} f}{\partial y^{2}}+\frac{\partial f}{\partial x}-\frac{y}{x} \frac{\partial f}{\partial y}=0
$$

## Paper 2, Section II

## 5A Differential Equations

(a) Consider the differential equation

$$
\begin{equation*}
a_{n} \frac{d^{n} y}{d x^{n}}+a_{n-1} \frac{d^{n-1} y}{d x^{n-1}}+\ldots+a_{2} \frac{d^{2} y}{d x^{2}}+a_{1} \frac{d y}{d x}+a_{0} y=0 \tag{1}
\end{equation*}
$$

with $n \in \mathbb{N}$ and $a_{0}, \ldots, a_{n} \in \mathbb{R}$. Show that $y(x)=e^{\lambda x}$ is a solution if and only if $p(\lambda)=0$ where

$$
p(\lambda)=a_{n} \lambda^{n}+a_{n-1} \lambda^{n-1}+\ldots+a_{2} \lambda^{2}+a_{1} \lambda+a_{0}
$$

Show further that $y(x)=x e^{\mu x}$ is also a solution of (1) if $\mu$ is a root of the polynomial $p(\lambda)$ of multiplicity at least 2 .
(b) By considering $v(t)=\frac{d^{2} u}{d t^{2}}$, or otherwise, find the general real solution for $u(t)$ satisfying

$$
\begin{equation*}
\frac{d^{4} u}{d t^{4}}+2 \frac{d^{2} u}{d t^{2}}=4 t^{2} \tag{2}
\end{equation*}
$$

By using a substitution of the form $u(t)=y\left(t^{2}\right)$ in (2), or otherwise, find the general real solution for $y(x)$, with $x$ positive, where

$$
4 x^{2} \frac{d^{4} y}{d x^{4}}+12 x \frac{d^{3} y}{d x^{3}}+(3+2 x) \frac{d^{2} y}{d x^{2}}+\frac{d y}{d x}=x
$$

# UNIVERSITY OF 

CAMBRIDGE

## Paper 2, Section II

## 6A Differential Equations

(a) By using a power series of the form

$$
y(x)=\sum_{k=0}^{\infty} a_{k} x^{k}
$$

or otherwise, find the general solution of the differential equation

$$
\begin{equation*}
x y^{\prime \prime}-(1-x) y^{\prime}-y=0 \tag{1}
\end{equation*}
$$

(b) Define the Wronskian $W(x)$ for a second order linear differential equation

$$
\begin{equation*}
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0 \tag{2}
\end{equation*}
$$

and show that $W^{\prime}+p(x) W=0$. Given a non-trivial solution $y_{1}(x)$ of (2) show that $W(x)$ can be used to find a second solution $y_{2}(x)$ of $(2)$ and give an expression for $y_{2}(x)$ in the form of an integral.
(c) Consider the equation (2) with

$$
p(x)=-\frac{P(x)}{x} \quad \text { and } \quad q(x)=-\frac{Q(x)}{x}
$$

where $P$ and $Q$ have Taylor expansions

$$
P(x)=P_{0}+P_{1} x+\ldots, \quad Q(x)=Q_{0}+Q_{1} x+\ldots
$$

with $P_{0}$ a positive integer. Find the roots of the indicial equation for (2) with these assumptions. If $y_{1}(x)=1+\beta x+\ldots$ is a solution, use the method of part (b) to find the first two terms in a power series expansion of a linearly independent solution $y_{2}(x)$, expressing the coefficients in terms of $P_{0}, P_{1}$ and $\beta$.

## Paper 2, Section II

## 7A Differential Equations

(a) Find the general solution of the system of differential equations

$$
\left(\begin{array}{c}
\dot{x}  \tag{1}\\
\dot{y} \\
\dot{z}
\end{array}\right)=\left(\begin{array}{rrr}
-1 & 2 & -1 \\
1 & 0 & -1 \\
1 & -2 & 1
\end{array}\right)\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right) .
$$

(b) Depending on the parameter $\lambda \in \mathbb{R}$, find the general solution of the system of differential equations

$$
\left(\begin{array}{c}
\dot{x}  \tag{2}\\
\dot{y} \\
\dot{z}
\end{array}\right)=\left(\begin{array}{rrr}
-1 & 2 & -1 \\
1 & 0 & -1 \\
1 & -2 & 1
\end{array}\right)\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)+2\left(\begin{array}{r}
-\lambda \\
1 \\
\lambda
\end{array}\right) e^{2 t},
$$

and explain why (2) has a particular solution of the form $\mathbf{c} e^{2 t}$ with constant vector $\mathbf{c} \in \mathbb{R}^{3}$ for $\lambda=1$ but not for $\lambda \neq 1$.
[Hint: decompose $\left(\begin{array}{r}-\lambda \\ 1 \\ \lambda\end{array}\right)$ in terms of the eigenbasis of the matrix in (1).]
(c) For $\lambda=-1$, find the solution of (2) which goes through the point $(0,1,0)$ at $t=0$.

## Paper 2, Section II

## 8A Differential Equations

(a) State how the nature of a critical (or stationary) point of a function $f(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^{n}$ can be determined by consideration of the eigenvalues of the Hessian matrix $H$ of $f(\mathbf{x})$, assuming $H$ is non-singular.
(b) Let $f(x, y)=x y(1-x-y)$. Find all the critical points of the function $f(x, y)$ and determine their nature. Determine the zero contour of $f(x, y)$ and sketch a contour plot showing the behaviour of the contours in the neighbourhood of the critical points.
(c) Now let $g(x, y)=x^{3} y^{2}(1-x-y)$. Show that $(0,1)$ is a critical point of $g(x, y)$ for which the Hessian matrix of $g$ is singular. Find an approximation for $g(x, y)$ to lowest non-trivial order in the neighbourhood of the point $(0,1)$. Does $g$ have a maximum or a minimum at $(0,1)$ ? Justify your answer.

## Paper 2, Section I

## 1C Differential Equations

The size of the population of ducks living on the pond of a certain Cambridge college is governed by the equation

$$
\frac{\mathrm{d} N}{\mathrm{~d} t}=\alpha N-N^{2},
$$

where $N=N(t)$ is the number of ducks at time $t$ and $\alpha$ is a positive constant. Given that $N(0)=2 \alpha$, find $N(t)$. What happens as $t \rightarrow \infty$ ?

## Paper 2, Section I

## 2C Differential Equations

Solve the differential equation

$$
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}-5 \frac{\mathrm{~d} y}{\mathrm{~d} x}+6 y=\mathrm{e}^{3 x}
$$

subject to the conditions $y=\mathrm{d} y / \mathrm{d} x=0$ when $x=0$.

## Paper 2, Section II

## 5C Differential Equations

Consider the first-order ordinary differential equation

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} x}=f_{1}(x) y+f_{2}(x) y^{p}, \tag{*}
\end{equation*}
$$

where $y \geqslant 0$ and $p$ is a positive constant with $p \neq 1$. Let $u=y^{1-p}$. Show that $u$ satisfies

$$
\frac{\mathrm{d} u}{\mathrm{~d} x}=(1-p)\left[f_{1}(x) u+f_{2}(x)\right] .
$$

Hence, find the general solution of equation $(*)$ when $f_{1}(x)=1, f_{2}(x)=x$.
Now consider the case $f_{1}(x)=1, f_{2}(x)=-\alpha^{2}$, where $\alpha$ is a non-zero constant. For $p>1$ find the two equilibrium points of equation (*), and determine their stability. What happens when $0<p<1$ ?

## Paper 2, Section II

## 6C Differential Equations

Consider the second-order ordinary differential equation

$$
\ddot{x}+2 k \dot{x}+\omega^{2} x=0,
$$

where $x=x(t)$ and $k$ and $\omega$ are constants with $k>0$. Calculate the general solution in the cases (i) $k<\omega$, (ii) $k=\omega$, (iii) $k>\omega$.

Now consider the system

$$
\ddot{x}+2 k \dot{x}+\omega^{2} x= \begin{cases}a & \text { when } \dot{x}>0 \\ 0 & \text { when } \dot{x} \leqslant 0\end{cases}
$$

with $x(0)=x_{1}, \dot{x}(0)=0$, where $a$ and $x_{1}$ are positive constants. In the case $k<\omega$ find $x(t)$ in the ranges $0 \leqslant t \leqslant \pi / p$ and $\pi / p \leqslant t \leqslant 2 \pi / p$, where $p=\sqrt{\omega^{2}-k^{2}}$. Hence, determine the value of $x_{1}$ for which $x(t)$ is periodic. For $k>\omega$ can $x(t)$ ever be periodic? Justify your answer.

## Paper 2, Section II

## 7C Differential Equations

Consider the differential equation

$$
x \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}+(c-x) \frac{\mathrm{d} y}{\mathrm{~d} x}-y=0
$$

where $c$ is a constant with $0<c<1$. Determine two linearly independent series solutions about $x=0$, giving an explicit expression for the coefficient of the general term in each series.

Determine the solution of

$$
x \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}+(c-x) \frac{\mathrm{d} y}{\mathrm{~d} x}-y=x
$$

for which $y(0)=0$ and $\mathrm{d} y / \mathrm{d} x$ is finite at $x=0$.

## Paper 2, Section II

## 8C Differential Equations

(a) The function $y(x, t)$ satisfies the forced wave equation

$$
\frac{\partial^{2} y}{\partial x^{2}}-\frac{\partial^{2} y}{\partial t^{2}}=4
$$

with initial conditions $y(x, 0)=\sin x$ and $\partial y / \partial t(x, 0)=0$. By making the change of variables $u=x+t$ and $v=x-t$, show that

$$
\frac{\partial^{2} y}{\partial u \partial v}=1
$$

Hence, find $y(x, t)$.
(b) The thickness of an axisymmetric drop of liquid spreading on a flat surface satisfies

$$
\frac{\partial h}{\partial t}=\frac{1}{r} \frac{\partial}{\partial r}\left(r h^{3} \frac{\partial h}{\partial r}\right)
$$

where $h=h(r, t)$ is the thickness of the drop, $r$ is the radial coordinate on the surface and $t$ is time. The drop has radius $R(t)$. The boundary conditions are that $\partial h / \partial r=0$ at $r=0$ and $h(r, t) \propto(R(t)-r)^{1 / 3}$ as $r \rightarrow R(t)$.

Show that

$$
M=\int_{0}^{R(t)} r h \mathrm{~d} r
$$

is independent of time. Given that $h(r, t)=f\left(r / t^{\alpha}\right) t^{-1 / 4}$ for some function $f$ (which need not be determined) and that $R(t)$ is proportional to $t^{\alpha}$, find $\alpha$.

## 2/I/1A Differential Equations

Let $a$ be a positive constant. Find the solution to the differential equation

$$
\frac{d^{4} y}{d x^{4}}-a^{4} y=\mathrm{e}^{-a x}
$$

that satisfies $y(0)=1$ and $y \rightarrow 0$ as $x \rightarrow \infty$.

## 2/I/2A Differential Equations

Find the fixed points of the difference equation

$$
u_{n+1}=\lambda u_{n}\left(1-u_{n}^{2}\right) .
$$

Show that a stable fixed point exists when $-1<\lambda<1$ and also when $1<\lambda<2$.

## 2/II/5A Differential Equations

Two cups of hot tea at temperatures $T_{1}(t)$ and $T_{2}(t)$ cool in a room at ambient constant temperature $T_{\infty}$. Initially $T_{1}(0)=T_{2}(0)=T_{0}>T_{\infty}$.

Cup 1 has cool milk added instantaneously at $t=1$; in contrast, cup 2 has cool milk added at a constant rate for $1 \leqslant t \leqslant 2$. Briefly explain the use of the differential equations

$$
\begin{aligned}
& \frac{d T_{1}}{d t}=-a\left(T_{1}-T_{\infty}\right)-\delta(t-1) \\
& \frac{d T_{2}}{d t}=-a\left(T_{2}-T_{\infty}\right)-H(t-1)+H(t-2)
\end{aligned}
$$

where $\delta(t)$ and $H(t)$ are the Dirac delta and Heaviside functions respectively, and $a$ is a positive constant.
(i) Show that for $0 \leqslant t<1$

$$
T_{1}(t)=T_{2}(t)=T_{\infty}+\left(T_{0}-T_{\infty}\right) \mathrm{e}^{-a t}
$$

(ii) Determine the jump (discontinuity) condition for $T_{1}$ at $t=1$ and hence find $T_{1}(t)$ for $t>1$.
(iii) Using continuity of $T_{2}(t)$ at $t=1$ show that for $1<t<2$

$$
T_{2}(t)=T_{\infty}-\frac{1}{a}+\mathrm{e}^{-a t}\left(T_{0}-T_{\infty}+\frac{1}{a} \mathrm{e}^{a}\right) .
$$

(iv) Compute $T_{2}(t)$ for $t>2$ and show that for $t>2$

$$
T_{1}(t)-T_{2}(t)=\left(\frac{1}{a} \mathrm{e}^{a}-1-\frac{1}{a}\right) \mathrm{e}^{(1-t) a} .
$$

(v) Find the time $t^{*}$, after $t=1$, at which $T_{1}=T_{2}$.

## 2/II/6A Differential Equations

The linear second-order differential equation

$$
\frac{d^{2} y}{d x^{2}}+p(x) \frac{d y}{d x}+q(x) y=0
$$

has linearly independent solutions $y_{1}(x)$ and $y_{2}(x)$. Define the Wronskian $W$ of $y_{1}(x)$ and $y_{2}(x)$.

Suppose that $y_{1}(x)$ is known. Use the Wronskian to write down a first-order differential equation for $y_{2}(x)$. Hence express $y_{2}(x)$ in terms of $y_{1}(x)$ and $W$.

Show further that $W$ satisfies the differential equation

$$
\frac{d W}{d x}+p(x) W=0
$$

Verify that $y_{1}(x)=x^{2}-2 x+1$ is a solution of

$$
\begin{equation*}
(x-1)^{2} \frac{d^{2} y}{d x^{2}}+(x-1) \frac{d y}{d x}-4 y=0 . \tag{*}
\end{equation*}
$$

Compute the Wronskian and hence determine a second, linearly independent, solution of $(*)$.

## 2/II/7A Differential Equations

Find the first three non-zero terms in series solutions $y_{1}(x)$ and $y_{2}(x)$ for the differential equation

$$
\begin{equation*}
x \frac{d^{2} y}{d x^{2}}-\frac{d y}{d x}+4 x^{3} y=0 \tag{*}
\end{equation*}
$$

that satisfy the boundary conditions

$$
\begin{aligned}
& y_{1}(0)=a, \quad y_{1}^{\prime \prime}(0)=0, \\
& y_{2}(0)=0, \quad y_{2}^{\prime \prime}(0)=b,
\end{aligned}
$$

where $a$ and $b$ are constants.
Determine the value of $\alpha$ such that the change of variable $u=x^{\alpha}$ transforms $(*)$ into a differential equation with constant coefficients. Hence find the general solution of $(*)$.

## 2/II/8A Differential Equations

Consider the function

$$
f(x, y)=x^{2}+y^{2}-\frac{1}{2} x^{4}-b x^{2} y^{2}-\frac{1}{2} y^{4}
$$

where $b$ is a positive constant.
Find the critical points of $f(x, y)$, assuming $b \neq 1$. Determine the type of each critical point and sketch contours of constant $f(x, y)$ in the two cases (i) $b<1$ and (ii) $b>1$.

For $b=1$ describe the subset of the $(x, y)$ plane on which $f(x, y)$ attains its maximum value.

9

## 2/I/1B Differential Equations

Find the solution $y(x)$ of the equation

$$
y^{\prime \prime}-6 y^{\prime}+9 y=\cos (2 x) \mathrm{e}^{3 x}
$$

that satisfies $y(0)=0$ and $y^{\prime}(0)=1$.

## 2/I/2B Differential Equations

Investigate the stability of:
(i) the equilibrium points of the equation

$$
\frac{d y}{d t}=\left(y^{2}-4\right) \tan ^{-1}(y)
$$

(ii) the constant solutions $\left(u_{n+1}=u_{n}\right)$ of the discrete equation

$$
u_{n+1}=\frac{1}{2} u_{n}^{2}\left(1+u_{n}\right) .
$$

## 2/II/5B Differential Equations

(i) The function $y(z)$ satisfies the equation

$$
y^{\prime \prime}+p(z) y^{\prime}+q(z) y=0 .
$$

Give the definitions of the terms ordinary point, singular point, and regular singular point for this equation.
(ii) For the equation

$$
4 z y^{\prime \prime}+2 y^{\prime}+y=0
$$

classify the point $z=0$ according to the definitions you gave in (i), and find the series solutions about $z=0$. Identify these solutions in closed form.

## 2/II/6B Differential Equations

Find the most general solution of the equation

$$
6 \frac{\partial^{2} u}{\partial x^{2}}-5 \frac{\partial^{2} u}{\partial x \partial y}+\frac{\partial^{2} u}{\partial y^{2}}=1
$$

by making the change of variables

$$
\xi=x+2 y, \quad \eta=x+3 y
$$

Find the solution that satisfies $u=0$ and $\partial u / \partial y=x$ when $y=0$.

## 2/II/7B Differential Equations

(i) Find, in the form of an integral, the solution of the equation

$$
\alpha \frac{d y}{d t}+y=f(t)
$$

that satisfies $y \rightarrow 0$ as $t \rightarrow-\infty$. Here $f(t)$ is a general function and $\alpha$ is a positive constant.

Hence find the solution in each of the cases:
(a) $f(t)=\delta(t)$;
(b) $f(t)=H(t)$, where $H(t)$ is the Heaviside step function.
(ii) Find and sketch the solution of the equation

$$
\frac{d y}{d t}+y=H(t)-H(t-1)
$$

given that $y(0)=0$ and $y(t)$ is continuous.

## 2/II/8B Differential Equations

(i) Find the general solution of the difference equation

$$
u_{k+1}+5 u_{k}+6 u_{k-1}=12 k+1
$$

(ii) Find the solution of the equation

$$
y_{k+1}+5 y_{k}+6 y_{k-1}=2^{k}
$$

that satisfies $y_{0}=y_{1}=1$. Hence show that, for any positive integer $n$, the quantity $2^{n}-26(-3)^{n}$ is divisible by 10 .

## 2/I/1B Differential Equations

Solve the initial value problem

$$
\frac{d x}{d t}=x(1-x), \quad x(0)=x_{0}
$$

and sketch the phase portrait. Describe the behaviour as $t \rightarrow+\infty$ and as $t \rightarrow-\infty$ of solutions with initial value satisfying $0<x_{0}<1$.

## 2/I/2B Differential Equations

Consider the first order system

$$
\frac{d \mathbf{x}}{d t}-A \mathbf{x}=e^{\lambda t} \mathbf{v}
$$

to be solved for $\mathbf{x}(t)=\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t)\right) \in \mathbb{R}^{n}$, where $A$ is an $n \times n$ matrix, $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^{n}$. Show that if $\lambda$ is not an eigenvalue of $A$ there is a solution of the form $\mathbf{x}(t)=e^{\lambda t} \mathbf{u}$. For $n=2$, given

$$
A=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right), \quad \lambda=1, \quad \text { and } \quad \mathbf{v}=\binom{1}{1}
$$

find this solution.

## 2/II/5B Differential Equations

Find the general solution of the system

$$
\begin{aligned}
& \frac{d x}{d t}=5 x+3 y+e^{2 t} \\
& \frac{d y}{d t}=2 x+2 e^{t}, \\
& \frac{d z}{d t}=x+y+e^{t} .
\end{aligned}
$$

## 2/II/6B Differential Equations

(i) Consider the equation

$$
\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\frac{\partial^{2} u}{\partial x^{2}}+f(t, x)
$$

and, using the change of variables $(t, x) \mapsto(s, y)=(t, x-t)$, show that it can be transformed into an equation of the form

$$
\frac{\partial U}{\partial s}=\frac{\partial^{2} U}{\partial y^{2}}+F(s, y)
$$

where $U(s, y)=u(s, y+s)$ and you should determine $F(s, y)$.
(ii) Let $H(y)$ be the Heaviside function. Find the general continuously differentiable solution of the equation

$$
w^{\prime \prime}(y)+H(y)=0 .
$$

(iii) Using (i) and (ii), find a continuously differentiable solution of

$$
\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\frac{\partial^{2} u}{\partial x^{2}}+H(x-t)
$$

such that $u(t, x) \rightarrow 0$ as $x \rightarrow-\infty$ and $u(t, x) \rightarrow-\infty$ as $x \rightarrow+\infty$.

## 2/II/7B Differential Equations

Let $p, q$ be continuous functions and let $y_{1}(x)$ and $y_{2}(x)$ be, respectively, the solutions of the initial value problems

$$
\begin{array}{ll}
y_{1}^{\prime \prime}+p(x) y_{1}^{\prime}+q(x) y_{1}=0, & y_{1}(0)=0, y_{1}^{\prime}(0)=1, \\
y_{2}^{\prime \prime}+p(x) y_{2}^{\prime}+q(x) y_{2}=0, & y_{2}(0)=1, y_{2}^{\prime}(0)=0 .
\end{array}
$$

If $f$ is any continuous function show that the solution of

$$
y^{\prime \prime}+p(x) y^{\prime}+q(x) y=f(x), \quad y(0)=0, y^{\prime}(0)=0
$$

is

$$
y(x)=\int_{0}^{x} \frac{y_{1}(s) y_{2}(x)-y_{1}(x) y_{2}(s)}{W(s)} f(s) d s
$$

where $W(x)=y_{1}(x) y_{2}^{\prime}(x)-y_{1}^{\prime}(x) y_{2}(x)$ is the Wronskian. Use this method to find $y=y(x)$ such that

$$
y^{\prime \prime}+y=\sin x, \quad y(0)=0, y^{\prime}(0)=0 .
$$

## 2/II/8B Differential Equations

Obtain a power series solution of the problem

$$
x y^{\prime \prime}+y=0, \quad y(0)=0, y^{\prime}(0)=1 .
$$

[You need not find the general power series solution.]
Let $y_{0}(x), y_{1}(x), y_{2}(x), \ldots$ be defined recursively as follows: $y_{0}(x)=x$. Given $y_{n-1}(x)$, define $y_{n}(x)$ to be the solution of

$$
x y_{n}^{\prime \prime}(x)=-y_{n-1}, \quad y_{n}(0)=0, y_{n}^{\prime}(0)=1 .
$$

By calculating $y_{1}, y_{2}, y_{3}$, or otherwise, obtain and prove a general formula for $y_{n}(x)$. Comment on the relation to the power series solution obtained previously.

## 2/I/1B Differential Equations

Solve the equation

$$
\frac{d y}{d x}+3 x^{2} y=x^{2}
$$

with $y(0)=a$, by use of an integrating factor or otherwise. Find $\lim _{x \rightarrow+\infty} y(x)$.

## 2/I/2B Differential Equations

Obtain the general solution of

$$
\begin{equation*}
x^{2} \frac{d^{2} y}{d x^{2}}+x \frac{d y}{d x}+y=0 \tag{*}
\end{equation*}
$$

by using the indicial equation.
Introduce $z=\log x$ as a new independent variable and find an equivalent second order differential equation with constant coefficients. Determine the general solution of this new equation, and show that it is equivalent to the general solution of $(*)$ found previously.

## 2/II/5B Differential Equations

Find two linearly independent solutions of the difference equation

$$
X_{n+2}-2 \cos \theta X_{n+1}+X_{n}=0
$$

for all values of $\theta \in(0, \pi)$. What happens when $\theta=0$ ? Find two linearly independent solutions in this case.

Find $X_{n}(\theta)$ which satisfy the initial conditions

$$
X_{1}=1, \quad X_{2}=2
$$

for $\theta=0$ and for $\theta \in(0, \pi)$. For every $n$, show that $X_{n}(\theta) \rightarrow X_{n}(0)$ as $\theta \rightarrow 0$.

## 2/II/6B Differential Equations

Find all power series solutions of the form $W=\sum_{n=0}^{\infty} a_{n} x^{n}$ to the equation

$$
-W^{\prime \prime}+2 x W^{\prime}=E W,
$$

for $E$ a real constant.
Impose the condition $W(0)=0$ and determine those values of $E$ for which your power series gives polynomial solutions (i.e., $a_{n}=0$ for $n$ sufficiently large). Give the values of $E$ for which the corresponding polynomials have degree less than 6 , and compute these polynomials.

Hence, or otherwise, find a polynomial solution of

$$
-W^{\prime \prime}+2 x W^{\prime}=x-\frac{4}{3} x^{3}+\frac{4}{15} x^{5}
$$

satisfying $W(0)=0$.

## 2/II/7B Differential Equations

The Cartesian coordinates $(x, y)$ of a point moving in $\mathbb{R}^{2}$ are governed by the system

$$
\begin{aligned}
& \frac{d x}{d t}=-y+x\left(1-x^{2}-y^{2}\right) \\
& \frac{d y}{d t}=x+y\left(1-x^{2}-y^{2}\right)
\end{aligned}
$$

Transform this system of equations to polar coordinates $(r, \theta)$ and hence find all periodic solutions (i.e., closed trajectories) which satisfy $r=$ constant.

Discuss the large time behaviour of an arbitrary solution starting at initial point $\left(x_{0}, y_{0}\right)=\left(r_{0} \cos \theta_{0}, r_{0} \sin \theta_{0}\right)$. Summarize the motion using a phase plane diagram, and comment on the nature of any critical points.

## 2/II/8B Differential Equations

Define the Wronskian $W\left[u_{1}, u_{2}\right]$ for two solutions $u_{1}, u_{2}$ of the equation

$$
\frac{d^{2} u}{d x^{2}}+p(x) \frac{d u}{d x}+q(x) u=0
$$

and obtain a differential equation which exhibits its dependence on $x$. Explain the relevance of the Wronskian to the linear independence of $u_{1}$ and $u_{2}$.

Consider the equation

$$
\begin{equation*}
x^{2} \frac{d^{2} y}{d x^{2}}-2 y=0 \tag{*}
\end{equation*}
$$

and determine the dependence on $x$ of the Wronskian $W\left[y_{1}, y_{2}\right]$ of two solutions $y_{1}$ and $y_{2}$. Verify that $y_{1}(x)=x^{2}$ is a solution of $(*)$ and use the Wronskian to obtain a second linearly independent solution.

## 2/I/1B Differential Equations

By writing $y(x)=m x$ where $m$ is a constant, solve the differential equation

$$
\frac{d y}{d x}=\frac{x-2 y}{2 x+y}
$$

and find the possible values of $m$.
Describe the isoclines of this differential equation and sketch the flow vectors. Use these to sketch at least two characteristically different solution curves.

Now, by making the substitution $y(x)=x u(x)$ or otherwise, find the solution of the differential equation which satisfies $y(0)=1$.

## 2/I/2B Differential Equations

Find two linearly independent solutions of the differential equation

$$
\frac{d^{2} y}{d x^{2}}+2 p \frac{d y}{d x}+p^{2} y=0
$$

Find also the solution of

$$
\frac{d^{2} y}{d x^{2}}+2 p \frac{d y}{d x}+p^{2} y=e^{-p x}
$$

that satisfies

$$
y=0, \frac{d y}{d x}=0 \quad \text { at } x=0
$$

## 2/II/5B Differential Equations

Construct a series solution $y=y_{1}(x)$ valid in the neighbourhood of $x=0$, for the differential equation

$$
\frac{d^{2} y}{d x^{2}}+4 x^{3} \frac{d y}{d x}+x^{2} y=0
$$

satisfying

$$
y_{1}=1, \frac{d y_{1}}{d x}=0 \quad \text { at } x=0 .
$$

Find also a second solution $y=y_{2}(x)$ which satisfies

$$
y_{2}=0, \frac{d y_{2}}{d x}=1 \quad \text { at } x=0 .
$$

Obtain an expression for the Wronskian of these two solutions and show that

$$
y_{2}(x)=y_{1}(x) \int_{0}^{x} \frac{e^{-\xi^{4}}}{y_{1}^{2}(\xi)} d \xi
$$

## 2/II/6B Differential Equations

Two solutions of the recurrence relation

$$
x_{n+2}+b(n) x_{n+1}+c(n) x_{n}=0
$$

are given as $p_{n}$ and $q_{n}$, and their Wronskian is defined to be

$$
W_{n}=p_{n} q_{n+1}-p_{n+1} q_{n}
$$

Show that

$$
\begin{equation*}
W_{n+1}=W_{1} \prod_{m=1}^{n} c(m) \tag{*}
\end{equation*}
$$

Suppose that $b(n)=\alpha$, where $\alpha$ is a real constant lying in the range $[-2,2]$, and that $c(n)=1$. Show that two solutions are $x_{n}=e^{i n \theta}$ and $x_{n}=e^{-i n \theta}$, where $\cos \theta=-\alpha / 2$. Evaluate the Wronskian of these two solutions and verify (*).

## 2/II/7B Differential Equations

Show how a second-order differential equation $\ddot{x}=f(x, \dot{x})$ may be transformed into a pair of coupled first-order equations. Explain what is meant by a critical point on the phase diagram for a pair of first-order equations. Hence find the critical points of the following equations. Describe their stability type, sketching their behaviour near the critical points on a phase diagram.
(i) $\ddot{x}+\cos x=0$
(ii) $\ddot{x}+x\left(x^{2}+\lambda x+1\right)=0, \quad$ for $\lambda=1,5 / 2$.

Sketch the phase portraits of these equations marking clearly the direction of flow.

## 2/II/8B Differential Equations

Construct the general solution of the system of equations

$$
\begin{gathered}
\dot{x}+4 x+3 y=0 \\
\dot{y}+4 y-3 x=0
\end{gathered}
$$

in the form

$$
\binom{x(t)}{y(t)}=\mathbf{x}=\sum_{j=1}^{2} a_{j} \mathbf{x}^{(j)} e^{\lambda_{j} t}
$$

and find the eigenvectors $\mathbf{x}^{(j)}$ and eigenvalues $\lambda_{j}$.
Explain what is meant by resonance in a forced system of linear differential equations.

Consider the forced system

$$
\begin{aligned}
& \dot{x}+4 x+3 y=\sum_{j=1}^{2} p_{j} e^{\lambda_{j} t} \\
& \dot{y}+4 y-3 x=\sum_{j=1}^{2} q_{j} e^{\lambda_{j} t} .
\end{aligned}
$$

Find conditions on $p_{j}$ and $q_{j}(j=1,2)$ such that there is no resonant response to the forcing.

## 2/I/1D Differential Equations

Consider the equation

$$
\begin{equation*}
\frac{d y}{d x}=1-y^{2} \tag{*}
\end{equation*}
$$

Using small line segments, sketch the flow directions in $x \geqslant 0,-2 \leqslant y \leqslant 2$ implied by the right-hand side of $(*)$. Find the general solution
(i) in $|y|<1$,
(ii) in $|y|>1$.

Sketch a solution curve in each of the three regions $y>1,|y|<1$, and $y<-1$.

## 2/I/2D Differential Equations

Consider the differential equation

$$
\frac{d x}{d t}+K x=0
$$

where $K$ is a positive constant. By using the approximate finite-difference formula

$$
\frac{d x_{n}}{d t}=\frac{x_{n+1}-x_{n-1}}{2 \delta t}
$$

where $\delta t$ is a positive constant, and where $x_{n}$ denotes the function $x(t)$ evaluated at $t=n \delta t$ for integer $n$, convert the differential equation to a difference equation for $x_{n}$.

Solve both the differential equation and the difference equation for general initial conditions. Identify those solutions of the difference equation that agree with solutions of the differential equation over a finite interval $0 \leqslant t \leqslant T$ in the limit $\delta t \rightarrow 0$, and demonstrate the agreement. Demonstrate that the remaining solutions of the difference equation cannot agree with the solution of the differential equation in the same limit.
[You may use the fact that, for bounded $|u|, \quad \lim _{\epsilon \rightarrow 0}(1+\epsilon u)^{1 / \epsilon}=e^{u}$. ]

## 2/II/5D Differential Equations

(a) Show that if $\mu(x, y)$ is an integrating factor for an equation of the form

$$
f(x, y) d y+g(x, y) d x=0
$$

then $\partial(\mu f) / \partial x=\partial(\mu g) / \partial y$.
Consider the equation

$$
\cot x d y-\tan y d x=0
$$

in the domain $0 \leqslant x \leqslant \frac{1}{2} \pi, \quad 0 \leqslant y \leqslant \frac{1}{2} \pi$. Using small line segments, sketch the flow directions in that domain. Show that $\sin x \cos y$ is an integrating factor for the equation. Find the general solution of the equation, and sketch the family of solutions that occupies the larger domain $-\frac{1}{2} \pi \leqslant x \leqslant \frac{1}{2} \pi,-\frac{1}{2} \pi \leqslant y \leqslant \frac{1}{2} \pi$.
(b) The following example illustrates that the concept of integrating factor extends to higher-order equations. Multiply the equation

$$
\left[y \frac{d^{2} y}{d x^{2}}+\left(\frac{d y}{d x}\right)^{2}\right] \cos ^{2} x=1
$$

by $\sec ^{2} x$, and show that the result takes the form $\frac{d}{d x} h(x, y)=0$, for some function $h(x, y)$ to be determined. Find a particular solution $y=y(x)$ such that $y(0)=0$ with $d y / d x$ finite at $x=0$, and sketch its graph in $0 \leqslant x<\frac{1}{2} \pi$.

## 2/II/6D Differential Equations

Define the Wronskian $W(x)$ associated with solutions of the equation

$$
\frac{d^{2} y}{d x^{2}}+p(x) \frac{d y}{d x}+q(x) y=0
$$

and show that

$$
W(x) \propto \exp \left(-\int^{x} p(\xi) d \xi\right)
$$

Evaluate the expression on the right when $p(x)=-2 / x$.
Given that $p(x)=-2 / x$ and that $q(x)=-1$, show that solutions in the form of power series,

$$
y=x^{\lambda} \sum_{n=0}^{\infty} a_{n} x^{n} \quad\left(a_{0} \neq 0\right),
$$

can be found if and only if $\lambda=0$ or 3 . By constructing and solving the appropriate recurrence relations, find the coefficients $a_{n}$ for each power series.

You may assume that the equation is satisfied by $y=\cosh x-x \sinh x$ and by $y=\sinh x-x \cosh x$. Verify that these two solutions agree with the two power series found previously, and that they give the $W(x)$ found previously, up to multiplicative constants.

$$
\left[\text { Hint: } \cosh x=1+\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\ldots, \quad \sinh x=x+\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\ldots .\right]
$$

2/II/7D Differential Equations
Consider the linear system

$$
\dot{\mathbf{x}}(t)-A \mathbf{x}(t)=\mathbf{z}(t)
$$

where the $n$-vector $\mathbf{z}(t)$ and the $n \times n$ matrix $A$ are given; $A$ has constant real entries, and has $n$ distinct eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ and $n$ linearly independent eigenvectors $\mathbf{a}_{1}, \mathbf{a}_{2}, \ldots, \mathbf{a}_{n}$. Find the complementary function. Given a particular integral $\mathbf{x}_{\mathrm{p}}(t)$, write down the general solution. In the case $n=2$ show that the complementary function is purely oscillatory, with no growth or decay, if and only if

$$
\operatorname{trace} A=0 \quad \text { and } \quad \operatorname{det} A>0
$$

Consider the same case $n=2$ with trace $A=0$ and $\operatorname{det} A>0$ and with

$$
\mathbf{z}(t)=\mathbf{a}_{1} \exp \left(i \omega_{1} t\right)+\mathbf{a}_{2} \exp \left(i \omega_{2} t\right)
$$

where $\omega_{1}, \omega_{2}$ are given real constants. Find a particular integral when
(i) $i \omega_{1} \neq \lambda_{1}$ and $i \omega_{2} \neq \lambda_{2}$;
(ii) $i \omega_{1} \neq \lambda_{1}$ but $i \omega_{2}=\lambda_{2}$.

In the case

$$
A=\left(\begin{array}{cc}
1 & 2 \\
-5 & -1
\end{array}\right)
$$

with $\mathbf{z}(t)=\binom{2}{3 i-1} \exp (3 i t)$, find the solution subject to the initial condition $\mathbf{x}=\binom{1}{0}$ at $t=0$.

## 2/II/8D Differential Equations

For all solutions of

$$
\begin{aligned}
\dot{x} & =\frac{1}{2} \alpha x+y-2 y^{3} \\
\dot{y} & =-x
\end{aligned}
$$

show that $d K / d t=\alpha x^{2}$ where

$$
K=K(x, y)=x^{2}+y^{2}-y^{4} .
$$

In the case $\alpha=0$, analyse the properties of the critical points and sketch the phase portrait, including the special contours for which $K(x, y)=\frac{1}{4}$. Comment on the asymptotic behaviour, as $t \rightarrow \infty$, of solution trajectories that pass near each critical point, indicating whether or not any such solution trajectories approach from, or recede to, infinity.

Briefly discuss how the picture changes when $\alpha$ is made small and positive, using your result for $d K / d t$ to describe, in qualitative terms, how solution trajectories cross $K$-contours.

2/I/1D Differential Equations
Solve the equation

$$
\ddot{y}+\dot{y}-2 y=e^{-t}
$$

subject to the conditions $y(t)=\dot{y}(t)=0$ at $t=0$. Solve the equation

$$
\ddot{y}+\dot{y}-2 y=e^{t}
$$

subject to the same conditions $y(t)=\dot{y}(t)=0$ at $t=0$.

## 2/I/2D Differential Equations

Consider the equation

$$
\begin{equation*}
\frac{d y}{d x}=x\left(\frac{1-y^{2}}{1-x^{2}}\right)^{1 / 2} \tag{*}
\end{equation*}
$$

where the positive square root is taken, within the square $\mathcal{S}$ : $0 \leqslant x<1,0 \leqslant y \leqslant 1$. Find the solution that begins at $x=y=0$. Sketch the corresponding solution curve, commenting on how its tangent behaves near each extremity. By inspection of the righthand side of $(*)$, or otherwise, roughly sketch, using small line segments, the directions of flow throughout the square $\mathcal{S}$.

## 2/II/5D Differential Equations

Explain what is meant by an integrating factor for an equation of the form

$$
\frac{d y}{d x}+f(x, y)=0
$$

Show that $2 y e^{x}$ is an integrating factor for

$$
\frac{d y}{d x}+\frac{2 x+x^{2}+y^{2}}{2 y}=0
$$

and find the solution $y=y(x)$ such that $y(0)=a$, for given $a>0$.
Show that $2 x+x^{2} \geqslant-1$ for all $x$ and hence that

$$
\frac{d y}{d x} \leqslant \frac{1-y^{2}}{2 y}
$$

For a solution with $a \geqslant 1$, show graphically, by considering the sign of $d y / d x$ first for $x=0$ and then for $x<0$, that $d y / d x<0$ for all $x \leqslant 0$.

Sketch the solution for the case $a=1$, and show that property that $d y / d x \rightarrow-\infty$ both as $x \rightarrow-\infty$ and as $x \rightarrow b$ from below, where $b \approx 0.7035$ is the positive number that satisfies $b^{2}=e^{-b}$.
[Do not consider the range $x \geqslant b$.]

## 2/II/6D Differential Equations

Solve the differential equation

$$
\frac{d y}{d t}=r y(1-a y)
$$

for the general initial condition $y=y_{0}$ at $t=0$, where $r, a$, and $y_{0}$ are positive constants. Deduce that the equilibria at $y=a^{-1}$ and $y=0$ are stable and unstable, respectively.

By using the approximate finite-difference formula

$$
\frac{d y}{d t}=\frac{y_{n+1}-y_{n}}{\delta t}
$$

for the derivative of $y$ at $t=n \delta t$, where $\delta t$ is a positive constant and $y_{n}=y(n \delta t)$, show that the differential equation when thus approximated becomes the difference equation

$$
u_{n+1}=\lambda\left(1-u_{n}\right) u_{n}
$$

where $\lambda=1+r \delta t>1$ and where $u_{n}=\lambda^{-1} a(\lambda-1) y_{n}$. Find the two equilibria and, by linearizing the equation about them or otherwise, show that one is always unstable (given that $\lambda>1$ ) and that the other is stable or unstable according as $\lambda<3$ or $\lambda>3$. Show that this last instability is oscillatory with period $2 \delta t$. Why does this last instability have no counterpart for the differential equation? Show graphically how this instability can equilibrate to a periodic, finite-amplitude oscillation when $\lambda=3.2$.

## 2/II/7D Differential Equations

The homogeneous equation

$$
\ddot{y}+p(t) \dot{y}+q(t) y=0
$$

has non-constant, non-singular coefficients $p(t)$ and $q(t)$. Two solutions of the equation, $y(t)=y_{1}(t)$ and $y(t)=y_{2}(t)$, are given. The solutions are known to be such that the determinant

$$
W(t)=\left|\begin{array}{ll}
y_{1} & y_{2} \\
\dot{y}_{1} & \dot{y}_{2}
\end{array}\right|
$$

is non-zero for all $t$. Define what is meant by linear dependence, and show that the two given solutions are linearly independent. Show also that

$$
W(t) \propto \exp \left(-\int^{t} p(s) d s\right)
$$

In the corresponding inhomogeneous equation

$$
\ddot{y}+p(t) \dot{y}+q(t) y=f(t)
$$

the right-hand side $f(t)$ is a prescribed forcing function. Construct a particular integral of this inhomogeneous equation in the form

$$
y(t)=a_{1}(t) y_{1}(t)+a_{2}(t) y_{2}(t)
$$

where the two functions $a_{i}(t)$ are to be determined such that

$$
y_{1}(t) \dot{a}_{1}(t)+y_{2}(t) \dot{a}_{2}(t)=0
$$

for all $t$. Express your result for the functions $a_{i}(t)$ in terms of integrals of the functions $f(t) y_{1}(t) / W(t)$ and $f(t) y_{2}(t) / W(t)$.

Consider the case in which $p(t)=0$ for all $t$ and $q(t)$ is a positive constant, $q=\omega^{2}$ say, and in which the forcing $f(t)=\sin (\omega t)$. Show that in this case $y_{1}(t)$ and $y_{2}(t)$ can be taken as $\cos (\omega t)$ and $\sin (\omega t)$ respectively. Evaluate $f(t) y_{1}(t) / W(t)$ and $f(t) y_{2}(t) / W(t)$ and show that, as $t \rightarrow \infty$, one of the $a_{i}(t)$ increases in magnitude like a power of $t$ to be determined.

## 2/II/8D Differential Equations

For any solution of the equations

$$
\begin{aligned}
& \dot{x}=\alpha x-y+y^{3} \quad(\alpha \text { constant }) \\
& \dot{y}=-x
\end{aligned}
$$

show that

$$
\frac{d}{d t}\left(x^{2}-y^{2}+\frac{1}{2} y^{4}\right)=2 \alpha x^{2}
$$

What does this imply about the behaviour of phase-plane trajectories at large distances from the origin as $t \rightarrow \infty$, in the case $\alpha=0$ ? Give brief reasoning but do not try to find explicit solutions.

Analyse the properties of the critical points and sketch the phase portrait (a) in the case $\alpha=0$, (b) in the case $\alpha=0.1$, and (c) in the case $\alpha=-0.1$.

2/I/1B Differential Equations
Find the solution to

$$
\frac{d y(x)}{d x}+\tanh (x) y(x)=H(x)
$$

in the range $-\infty<x<\infty$ subject to $y(0)=1$, where $H(x)$ is the Heavyside function defined by

$$
H(x)=\left\{\begin{array}{ll}
0 & x<0 \\
1 & x>0
\end{array} .\right.
$$

Sketch the solution.

## 2/I/2B Differential Equations

The function $y(x)$ satisfies the inhomogeneous second-order linear differential equation

$$
y^{\prime \prime}-y^{\prime}-2 y=18 x e^{-x}
$$

Find the solution that satisfies the conditions that $y(0)=1$ and $y(x)$ is bounded as $x \rightarrow \infty$.

## 2/II/5B Differential Equations

The real sequence $y_{k}, k=1,2, \ldots$ satisfies the difference equation

$$
y_{k+2}-y_{k+1}+y_{k}=0
$$

Show that the general solution can be written

$$
y_{k}=a \cos \frac{\pi k}{3}+b \sin \frac{\pi k}{3}
$$

where $a$ and $b$ are arbitrary real constants.
Now let $y_{k}$ satisfy

$$
\begin{equation*}
y_{k+2}-y_{k+1}+y_{k}=\frac{1}{k+2} . \tag{*}
\end{equation*}
$$

Show that a particular solution of $(*)$ can be written in the form

$$
y_{k}=\sum_{n=1}^{k} \frac{a_{n}}{k-n+1},
$$

where

$$
a_{n+2}-a_{n+1}+a_{n}=0, \quad n \geq 1
$$

and $a_{1}=1, a_{2}=1$.
Hence, find the general solution to $(*)$.

## 2/II/6B Differential Equations

The function $y(x)$ satisfies the linear equation

$$
y^{\prime \prime}(x)+p(x) y^{\prime}(x)+q(x) y(x)=0 .
$$

The Wronskian, $W(x)$, of two independent solutions denoted $y_{1}(x)$ and $y_{2}(x)$ is defined to be

$$
W(x)=\left|\begin{array}{cc}
y_{1} & y_{2} \\
y_{1}{ }^{\prime} & y_{2}{ }^{\prime}
\end{array}\right|
$$

Let $y_{1}(x)$ be given. In this case, show that the expression for $W(x)$ can be interpreted as a first-order inhomogeneous differential equation for $y_{2}(x)$. Hence, by explicit derivation, show that $y_{2}(x)$ may be expressed as

$$
\begin{equation*}
y_{2}(x)=y_{1}(x) \int_{x_{0}}^{x} \frac{W(t)}{y_{1}(t)^{2}} d t \tag{*}
\end{equation*}
$$

where the rôle of $x_{0}$ should be briefly elucidated.
Show that $W(x)$ satisfies

$$
\frac{d W(x)}{d x}+p(x) W(x)=0
$$

Verify that $y_{1}(x)=1-x$ is a solution of

$$
x y^{\prime \prime}(x)-\left(1-x^{2}\right) y^{\prime}(x)-(1+x) y(x)=0
$$

Hence, using ( $*$ ) with $x_{0}=0$ and expanding the integrand in powers of $t$ to order $t^{3}$, find the first three non-zero terms in the power series expansion for a solution, $y_{2}(x)$, of $(\dagger)$ that is independent of $y_{1}(x)$ and satisfies $y_{2}(0)=0, y_{2}{ }^{\prime \prime}(0)=1$.

## 2/II/7B Differential Equations

Consider the linear system

$$
\begin{equation*}
\dot{\mathbf{z}}+A \mathbf{z}=\mathbf{h}, \tag{*}
\end{equation*}
$$

where

$$
\mathbf{z}(t)=\binom{x(t)}{y(t)}, \quad A=\left(\begin{array}{cc}
1+a & -2 \\
1 & -1+a
\end{array}\right), \quad \mathbf{h}(t)=\binom{2 \cos t}{\cos t-\sin t}
$$

where $\mathbf{z}(t)$ is real and $a$ is a real constant, $a \geq 0$.
Find a (complex) eigenvector, e, of $A$ and its corresponding (complex) eigenvalue, $l$. Show that the second eigenvector and corresponding eigenvalue are respectively $\overline{\mathbf{e}}$ and $\bar{l}$, where the bar over the symbols signifies complex conjugation. Hence explain how the general solution to $(*)$ can be written as

$$
\mathbf{z}(t)=\alpha(t) \mathbf{e}+\bar{\alpha}(t) \overline{\mathbf{e}}
$$

where $\alpha(t)$ is complex.
Write down a differential equation for $\alpha(t)$ and hence, for $a>0$, deduce the solution to $(*)$ which satisfies the initial condition $\mathbf{z}(0)=\underline{0}$.

Is the linear system resonant?
By taking the limit $a \rightarrow 0$ of the solution already found deduce the solution satisfying $\mathbf{z}(0)=\underline{0}$ when $a=0$.

## 2/II/8B Differential Equations

Carnivorous hunters of population $h$ prey on vegetarians of population $p$. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$
\begin{align*}
\dot{p} & =p\left(1-\frac{p}{a}\right)-\frac{p h}{a},  \tag{*}\\
\dot{h} & =\frac{h}{8}\left(\frac{p}{b}-1\right)
\end{align*}
$$

where $a$ and $b$ are positive constants, and $h(t)$ and $p(t)$ are non-negative functions of time, $t$. By giving an interpretation of each term explain briefly how these equations model the system described.

Consider these equations for $a=1$. In the two cases $0<b<1 / 2$ and $b>1$ determine the location and the stability properties of the critical points of $(*)$. In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

