Part IA

Algebra and Geometry

Year

1/I/1A **Algebra and Geometry**

(i) The spherical polar unit basis vectors $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$ in $\mathbb{R}^3$ are given in terms of the Cartesian unit basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ by

$$\begin{aligned} \mathbf{e}_r &= \mathbf{i}\cos\phi\sin\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\theta\,, \\ \mathbf{e}_\theta &= \mathbf{i}\cos\phi\cos\theta + \mathbf{j}\sin\phi\cos\theta - \mathbf{k}\sin\theta\,, \\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\,. \end{aligned}$$

Express $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ in terms of $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$.

(ii) Use suffix notation to prove the following identity for the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $\mathbb{R}^3$:

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\,\mathbf{A}\,.$$

1/I/2B **Algebra and Geometry**

For the equations

$$\begin{aligned} px + y + z &= 1, \\ x + 2y + 4z &= t, \\ x + 4y + 10z &= t^2, \end{aligned}$$

find the values of p and t for which

(i) there is a unique solution;

(ii) there are infinitely many solutions;

(iii) there is no solution.

1/II/5B **Algebra and Geometry**

(i) Describe geometrically the following surfaces in three-dimensional space:

(a) $\mathbf{r}\cdot\mathbf{u} = \alpha|\mathbf{r}|$, where $0 < |\alpha| < 1$;

(b) $|\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}| = \beta$, where $\beta > 0$.

Here α and β are fixed scalars and $\mathbf{u}$ is a fixed unit vector. You should identify the meaning of α, β and $\mathbf{u}$ for these surfaces.

(ii) The plane $\mathbf{n}\cdot\mathbf{r} = p$, where $\mathbf{n}$ is a fixed unit vector, and the sphere with centre $\mathbf{c}$ and radius a intersect in a circle with centre $\mathbf{b}$ and radius ρ.

(a) Show that $\mathbf{b} - \mathbf{c} = \lambda\mathbf{n}$, where you should give λ in terms of a and ρ.

(b) Find ρ in terms of $\mathbf{c}, \mathbf{n}, a$ and p.

1/II/6C Algebra and Geometry

Let $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\mathbf{x} \mapsto \mathbf{x}' = a\mathbf{x} + b(\mathbf{n} \times \mathbf{x}),$$

where a and b are positive scalar constants, and $\mathbf{n}$ is a unit vector.

(i) By considering the effect of $\mathcal{M}$ on $\mathbf{n}$ and on a vector orthogonal to $\mathbf{n}$, describe geometrically the action of $\mathcal{M}$.

(ii) Express the map $\mathcal{M}$ as a matrix M using suffix notation. Find a, b and $\mathbf{n}$ in the case

$$\mathrm{M} = \begin{pmatrix} 2 & -2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & 2 \end{pmatrix}.$$

(iii) Find, in the general case, the inverse map (i.e. express $\mathbf{x}$ in terms of $\mathbf{x}'$ in vector form).

1/II/7C Algebra and Geometry

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in a real vector space with scalar product denoted by $\mathbf{x} \cdot \mathbf{y}$. Prove that $(\mathbf{x} \cdot \mathbf{y})^{\mathbf{2}} \leq (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$, and prove also that $(\mathbf{x} \cdot \mathbf{y})^{\mathbf{2}} = (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ if and only if $\mathbf{x} = \lambda\mathbf{y}$ for some scalar λ.

(i) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, prove that the inequality $x^2 + y^2 + z^2 \geq yz + zx + xy$ holds for any real numbers x, y and z.

(ii) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that only one choice of real numbers x, y, z satisfies $3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0$, and find these numbers.

1/II/8A Algebra and Geometry

(i) Show that any line in the complex plane $\mathbb{C}$ can be represented in the form

$$\bar{c}z + c\bar{z} + r = 0\,,$$

where $c \in \mathbb{C}$ and $r \in \mathbb{R}$.

(ii) If z and u are two complex numbers for which

$$\left|\frac{z+u}{z+\bar{u}}\right| = 1\,,$$

show that either z or u is real.

(iii) Show that any Möbius transformation

$$w = \frac{az+b}{cz+d} \quad (bc - ad \neq 0)$$

that maps the real axis $z = \bar{z}$ into the unit circle $|w| = 1$ can be expressed in the form

$$w = \lambda \frac{z+k}{z+\bar{k}}\,,$$

where $\lambda, k \in \mathbb{C}$ and $|\lambda| = 1$.

3/I/1D Algebra and Geometry

Prove that every permutation of $\{1, \ldots, n\}$ may be expressed as a product of disjoint cycles.

Let $\sigma = (1234)$ and let $\tau = (345)(678)$. Write $\sigma\tau$ as a product of disjoint cycles. What is the order of $\sigma\tau$?

3/I/2D Algebra and Geometry

What does it mean to say that groups G and H are *isomorphic*?

Prove that no two of C_8, $C_4 \times C_2$ and $C_2 \times C_2 \times C_2$ are isomorphic. [Here C_n denotes the cyclic group of order n.]

Give, with justification, a group of order 8 that is not isomorphic to any of those three groups.

3/II/5D **Algebra and Geometry**

Let x be an element of a finite group G. What is meant by the *order* of x? Prove that the order of x must divide the order of G. [*No version of Lagrange's theorem or the Orbit-Stabilizer theorem may be used without proof.*]

If G is a group of order n, and d is a divisor of n with $d < n$, is it always true that G must contain an element of order d? Justify your answer.

Prove that if m and n are coprime then the group $C_m \times C_n$ is cyclic.

If m and n are not coprime, can it happen that $C_m \times C_n$ is cyclic?

[Here C_n denotes the cyclic group of order n.]

3/II/6D **Algebra and Geometry**

What does it mean to say that a subgroup H of a group G is *normal*? Give, with justification, an example of a subgroup of a group that is normal, and also an example of a subgroup of a group that is not normal.

If H is a normal subgroup of G, explain carefully how to make the set of (left) cosets of H into a group.

Let H be a normal subgroup of a finite group G. Which of the following are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) If G is cyclic then H and G/H are cyclic.

(ii) If H and G/H are cyclic then G is cyclic.

(iii) If G is abelian then H and G/H are abelian.

(iv) If H and G/H are abelian then G is abelian.

3/II/7D Algebra and Geometry

Let A be a real symmetric $n \times n$ matrix. Prove that every eigenvalue of A is real, and that eigenvectors corresponding to distinct eigenvalues are orthogonal. Indicate clearly where in your argument you have used the fact that A is real.

What does it mean to say that a real $n \times n$ matrix P is *orthogonal* ? Show that if P is orthogonal and A is as above then $P^{-1}AP$ is symmetric. If P is any real invertible matrix, must $P^{-1}AP$ be symmetric? Justify your answer.

Give, with justification, real 2×2 matrices B, C, D, E with the following properties:

(i) B has no real eigenvalues;

(ii) C is not diagonalisable over $\mathbb{C}$;

(iii) D is diagonalisable over $\mathbb{C}$, but not over $\mathbb{R}$;

(iv) E is diagonalisable over $\mathbb{R}$, but does not have an orthonormal basis of eigenvectors.

3/II/8D Algebra and Geometry

In the group of Möbius maps, what is the order of the Möbius map $z \mapsto \dfrac{1}{z}$? What is the order of the Möbius map $z \mapsto \dfrac{1}{1-z}$?

Prove that every Möbius map is conjugate either to a map of the form $z \mapsto \mu z$ (some $\mu \in \mathbb{C}$) or to the map $z \mapsto z+1$. Is $z \mapsto z+1$ conjugate to a map of the form $z \mapsto \mu z$?

Let f be a Möbius map of order n, for some positive integer n. Under the action on $\mathbb{C} \cup \{\infty\}$ of the group generated by f, what are the various sizes of the orbits? Justify your answer.

1/I/1B **Algebra and Geometry**

Consider the cone K in $\mathbb{R}^3$ defined by

$$x_3^2 = x_1^2 + x_2^2, \quad x_3 > 0.$$

Find a unit normal $\mathbf{n} = (n_1, n_2, n_3)$ to K at the point $\mathbf{x} = (x_1, x_2, x_3)$ such that $n_3 \geqslant 0$. Show that if $\mathbf{p} = (p_1, p_2, p_3)$ satisfies

$$p_3^2 \geqslant p_1^2 + p_2^2$$

and $p_3 \geqslant 0$ then

$$\mathbf{p} \cdot \mathbf{n} \geqslant 0.$$

1/I/2A **Algebra and Geometry**

Express the unit vector $\mathbf{e}_r$ of spherical polar coordinates in terms of the orthonormal Cartesian basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Express the equation for the paraboloid $z = x^2 + y^2$ in (i) cylindrical polar coordinates (ρ, ϕ, z) and (ii) spherical polar coordinates (r, θ, ϕ).

In spherical polar coordinates, a surface is defined by $r^2 \cos 2\theta = a$, where a is a real non-zero constant. Find the corresponding equation for this surface in Cartesian coordinates and sketch the surfaces in the two cases $a > 0$ and $a < 0$.

1/II/5C **Algebra and Geometry**

Prove the Cauchy–Schwarz inequality,

$$|\mathbf{x} \cdot \mathbf{y}| \leqslant |\mathbf{x}||\mathbf{y}|,$$

for two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. Under what condition does equality hold?

Consider a pyramid in $\mathbb{R}^n$ with vertices at the origin O and at $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_1 = (1, 0, 0, \ldots)$, $\mathbf{e}_2 = (0, 1, 0, \ldots)$, and so on. The "base" of the pyramid is the $(n-1)$-dimensional object B specified by $(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n) \cdot \mathbf{x} = 1$, $\mathbf{e}_i \cdot \mathbf{x} \geqslant 0$ for $i = 1, \ldots, n$.

Find the point C in B equidistant from each vertex of B and find the length of OC. (C is the centroid of B.)

Show, using the Cauchy–Schwarz inequality, that this is the closest point in B to the origin O.

Calculate the angle between OC and any edge of the pyramid connected to O. What happens to this angle and to the length of OC as n tends to infinity?

1/II/6C Algebra and Geometry

Given a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, write down the vector $\mathbf{x}'$ obtained by rotating $\mathbf{x}$ through an angle θ.

Given a unit vector $\mathbf{n} \in \mathbb{R}^3$, any vector $\mathbf{x} \in \mathbb{R}^3$ may be written as $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$ where $\mathbf{x}_{\parallel}$ is parallel to $\mathbf{n}$ and $\mathbf{x}_{\perp}$ is perpendicular to $\mathbf{n}$. Write down explicit formulae for $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$, in terms of $\mathbf{n}$ and $\mathbf{x}$. Hence, or otherwise, show that the linear map

$$\mathbf{x} \mapsto \mathbf{x}' = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta(\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}) + \sin\theta(\mathbf{n} \times \mathbf{x}) \qquad (*)$$

describes a rotation about $\mathbf{n}$ through an angle θ, in the positive sense defined by the right hand rule.

Write equation $(*)$ in matrix form, $x_i' = R_{ij}x_j$. Show that the trace $R_{ii} = 1+2\cos\theta$.

Given the rotation matrix

$$R = \frac{1}{2}\begin{pmatrix} 1+r & 1-r & 1 \\ 1-r & 1+r & -1 \\ -1 & 1 & 2r \end{pmatrix},$$

where $r = 1/\sqrt{2}$, find the two pairs $(\theta, \mathbf{n})$, with $-\pi \leqslant \theta < \pi$, giving rise to R. Explain why both represent the same rotation.

1/II/7B Algebra and Geometry

(i) Let $\mathbf{u}, \mathbf{v}$ be unit vectors in $\mathbb{R}^3$. Write the transformation on vectors $\mathbf{x} \in \mathbb{R}^3$

$$\mathbf{x} \mapsto (\mathbf{u} \cdot \mathbf{x})\mathbf{u} + \mathbf{v} \times \mathbf{x}$$

in matrix form as $\mathbf{x} \mapsto A\mathbf{x}$ for a matrix A. Find the eigenvalues in the two cases (a) when $\mathbf{u} \cdot \mathbf{v} = 0$, and (b) when $\mathbf{u}, \mathbf{v}$ are parallel.

(ii) Let $\mathcal{M}$ be the set of 2×2 complex hermitian matrices with trace zero. Show that if $A \in \mathcal{M}$ there is a unique vector $\mathbf{x} \in \mathbb{R}^3$ such that

$$A = \mathcal{R}(\mathbf{x}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Show that if U is a 2×2 unitary matrix, the transformation

$$A \mapsto U^{-1}AU$$

maps $\mathcal{M}$ to $\mathcal{M}$, and that if $U^{-1}\mathcal{R}(\mathbf{x})U = \mathcal{R}(\mathbf{y})$, then $\|\mathbf{x}\| = \|\mathbf{y}\|$ where $\|\cdot\|$ means ordinary Euclidean length. [*Hint: Consider determinants.*]

1/II/8A **Algebra and Geometry**

(i) State de Moivre's theorem. Use it to express $\cos 5\theta$ as a polynomial in $\cos\theta$.

(ii) Find the two fixed points of the Möbius transformation

$$z \longmapsto \omega = \frac{3z+1}{z+3}\,,$$

that is, find the two values of z for which $\omega = z$.

Given that $c \neq 0$ and $(a-d)^2 + 4bc \neq 0$, show that a general Möbius transformation

$$z \longmapsto \omega = \frac{az+b}{cz+d}\,, \qquad ad - bc \neq 0\,,$$

has two fixed points α, β given by

$$\alpha = \frac{a-d+m}{2c}\,, \qquad \beta = \frac{a-d-m}{2c}\,,$$

where $\pm m$ are the square roots of $(a-d)^2 + 4bc$.

Show that such a transformation can be expressed in the form

$$\frac{\omega-\alpha}{\omega-\beta} = k\frac{z-\alpha}{z-\beta}\,,$$

where k is a constant that you should determine.

3/I/1D **Algebra and Geometry**

Give an example of a real 3×3 matrix A with eigenvalues -1, $(1\pm i)/\sqrt{2}$. Prove or give a counterexample to the following statements:

(i) any such A is diagonalisable over $\mathbb{C}$;

(ii) any such A is orthogonal;

(iii) any such A is diagonalisable over $\mathbb{R}$.

3/I/2D **Algebra and Geometry**

Show that if H and K are subgroups of a group G, then $H \cap K$ is also a subgroup of G. Show also that if H and K have orders p and q respectively, where p and q are coprime, then $H \cap K$ contains only the identity element of G. [*You may use Lagrange's theorem provided it is clearly stated.*]

3/II/5D **Algebra and Geometry**

Let G be a group and let A be a non-empty subset of G. Show that

$$C(A) = \{g \in G : \; gh = hg \quad \text{for all} \;\; h \in A\}$$

is a subgroup of G.

Show that $\rho : G \times G \to G$ given by

$$\rho(g, h) = ghg^{-1}$$

defines an action of G on itself.

Suppose G is finite, let $O_1, \ldots, O_n$ be the orbits of the action ρ and let $h_i \in O_i$ for $i = 1, \ldots, n$. Using the Orbit–Stabilizer Theorem, or otherwise, show that

$$|G| = |C(G)| + \sum_i |G|/|C(\{h_i\})|$$

where the sum runs over all values of i such that $|O_i| > 1$.

Let G be a finite group of order p^r, where p is a prime and r is a positive integer. Show that $C(G)$ contains more than one element.

3/II/6D **Algebra and Geometry**

Let $\theta : G \to H$ be a homomorphism between two groups G and H. Show that the image of θ, $\theta(G)$, is a subgroup of H; show also that the kernel of θ, $\ker(\theta)$, is a normal subgroup of G.

Show that $G/\ker(\theta)$ is isomorphic to $\theta(G)$.

Let $O(3)$ be the group of 3×3 real orthogonal matrices and let $SO(3) \subset O(3)$ be the set of orthogonal matrices with determinant 1. Show that $SO(3)$ is a normal subgroup of $O(3)$ and that $O(3)/SO(3)$ is isomorphic to the cyclic group of order 2.

Give an example of a homomorphism from $O(3)$ to $SO(3)$ with kernel of order 2.

3/II/7D Algebra and Geometry

Let $SL(2,\mathbb{R})$ be the group of 2×2 real matrices with determinant 1 and let $\sigma : \mathbb{R} \to SL(2,\mathbb{R})$ be a homomorphism. On $K = \mathbb{R} \times \mathbb{R}^2$ consider the product

$$(x, \mathbf{v}) * (y, \mathbf{w}) = (x + y, \mathbf{v} + \sigma(x)\mathbf{w}).$$

Show that K with this product is a group.

Find the homomorphism or homomorphisms σ for which K is a commutative group.

Show that the homomorphisms σ for which the elements of the form $(0, \mathbf{v})$ with $\mathbf{v} = (a, 0)$, $a \in \mathbb{R}$, commute with every element of K are precisely those such that

$$\sigma(x) = \begin{pmatrix} 1 & r(x) \\ 0 & 1 \end{pmatrix},$$

with $r : (\mathbb{R}, +) \to (\mathbb{R}, +)$ an arbitrary homomorphism.

3/II/8D Algebra and Geometry

Show that every Möbius transformation can be expressed as a composition of maps of the forms: $S_1(z) = z + \alpha$, $S_2(z) = \lambda\, z$ and $S_3(z) = 1/z$, where $\alpha, \lambda \in \mathbb{C}$.

Show that if z_1, z_2, z_3 and w_1, w_2, w_3 are two triples of distinct points in $\mathbb{C} \cup \{\infty\}$, there exists a unique Möbius transformation that takes z_j to w_j $(j = 1, 2, 3)$.

Let G be the group of those Möbius transformations which map the set $\{0, 1, \infty\}$ to itself. Find all the elements of G. To which standard group is G isomorphic?

1/I/1C **Algebra and Geometry**

Convert the following expressions from suffix notation (assuming the summation convention in three dimensions) into standard notation using vectors and/or matrices, where possible, identifying the one expression that is incorrectly formed:

(i) δ_{ij},

(ii) $\delta_{ii}\,\delta_{ij}$,

(iii) $\delta_{ll}\,a_i\,b_j\,C_{ij}\,d_k - C_{ik}\,d_i$,

(iv) $\epsilon_{ijk}\,a_k\,b_j$,

(v) $\epsilon_{ijk}\,a_j\,a_k$.

Write the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ in suffix notation and derive an equivalent expression that utilises scalar products. Express the result both in suffix notation and in standard vector notation. Hence or otherwise determine $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and $\mathbf{c} = \mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}$.

1/I/2B **Algebra and Geometry**

Let $\mathbf{n}\in\mathbb{R}^3$ be a unit vector. Consider the operation

$$\mathbf{x}\mapsto\mathbf{n}\times\mathbf{x}\,.$$

Write this in matrix form, i.e., find a 3×3 matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{x} = \mathbf{n}\times\mathbf{x}$ for all $\mathbf{x}$, and compute the eigenvalues of $\mathbf{A}$. In the case when $\mathbf{n} = (0,0,1)$, compute $\mathbf{A}^2$ and its eigenvalues and eigenvectors.

1/II/5C **Algebra and Geometry**

Give the real and imaginary parts of each of the following functions of $z = x + iy$, with x, y real,

(i) e^z,

(ii) $\cos z$,

(iii) $\log z$,

(iv) $\dfrac{1}{z} + \dfrac{1}{\bar{z}}$,

(v) $z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3 - \bar{z}$,

where $\bar{z}$ is the complex conjugate of z.

An ant lives in the complex region R given by $|z-1| \leq 1$. Food is found at z such that

$$(\log z)^2 = -\frac{\pi^2}{16}.$$

Drink is found at z such that

$$\frac{z + \frac{1}{2}\bar{z}}{\left(z - \frac{1}{2}\bar{z}\right)^2} = 3, \quad z \neq 0.$$

Identify the places within R where the ant will find the food or drink.

1/II/6B **Algebra and Geometry**

Let $\mathbf{A}$ be a real 3×3 matrix. Define the rank of $\mathbf{A}$. Describe the space of solutions of the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,, \tag{†}$$

organizing your discussion with reference to the rank of $\mathbf{A}$.

Write down the equation of the tangent plane at $(0,1,1)$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 2$ and the equation of a general line in $\mathbb{R}^3$ passing through the origin $(0,0,0)$.

Express the problem of finding points on the intersection of the tangent plane and the line in the form (†). Find, and give geometrical interpretations of, the solutions.

1/II/7A Algebra and Geometry

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Show that $\mathbf{a}$ may be written as the sum of two vectors: one parallel (or anti-parallel) to $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$. By setting the former equal to $\cos\theta|\mathbf{a}|\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is a unit vector along $\mathbf{b}$, show that

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\,.$$

Explain why this is a sensible definition of the angle θ between $\mathbf{a}$ and $\mathbf{b}$.

Consider the 2^n vertices of a cube of side 2 in $\mathbb{R}^n$, centered on the origin. Each vertex is joined by a straight line through the origin to another vertex: the lines are the 2^{n-1} diagonals of the cube. Show that no two diagonals can be perpendicular if n is odd.

For $n = 4$, what is the greatest number of mutually perpendicular diagonals? List all the possible angles between the diagonals.

1/II/8A Algebra and Geometry

Given a non-zero vector v_i, any 3×3 symmetric matrix T_{ij} can be expressed as

$$T_{ij} = A\delta_{ij} + Bv_iv_j + (C_iv_j + C_jv_i) + D_{ij}$$

for some numbers A and B, some vector C_i and a symmetric matrix D_{ij}, where

$$C_iv_i = 0, \qquad D_{ii} = 0, \qquad D_{ij}v_j = 0\,,$$

and the summation convention is implicit.

Show that the above statement is true by finding A, B, C_i and D_{ij} explicitly in terms of T_{ij} and v_j, or otherwise. Explain why A, B, C_i and D_{ij} together provide a space of the correct dimension to parameterise an arbitrary symmetric 3×3 matrix T_{ij}.

3/I/1D Algebra and Geometry

Let A be a real 3×3 symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Consider the surface S in $\mathbb{R}^3$ given by

$$x^T A x = 1.$$

Find the minimum distance between the origin and S. How many points on S realize this minimum distance? Justify your answer.

3/I/2D Algebra and Geometry

Define what it means for a group to be cyclic. If p is a prime number, show that a finite group G of order p must be cyclic. Find all homomorphisms $\varphi : C_{11} \to C_{14}$, where C_n denotes the cyclic group of order n. [*You may use Lagrange's theorem.*]

3/II/5D Algebra and Geometry

Define the notion of an action of a group G on a set X. Assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Let G be a finite group and X the set of its subgroups. Show that $g(K) = gKg^{-1}$ ($g \in G$, $K \in X$) defines an action of G on X. If H is a subgroup of G, show that the orbit of H has at most $|G|/|H|$ elements.

Suppose H is a subgroup of G and $H \neq G$. Show that there is an element of G which does not belong to any subgroup of the form gHg^{-1} for $g \in G$.

3/II/6D Algebra and Geometry

Let $\mathcal{M}$ be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $SL(2, \mathbb{C})$ be the group of all 2×2 complex matrices with determinant 1.

Show that the map $\theta : SL(2, \mathbb{C}) \to \mathcal{M}$ given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

is a surjective homomorphism. Find its kernel.

Show that every $T \in \mathcal{M}$ not equal to the identity is conjugate to a Möbius map S where either $Sz = \mu z$ with $\mu \neq 0, 1$, or $Sz = z \pm 1$. [*You may use results about matrices in $SL(2, \mathbb{C})$, provided they are clearly stated.*]

Show that if $T \in \mathcal{M}$, then T is the identity, or T has one, or two, fixed points. Also show that if $T \in \mathcal{M}$ has only one fixed point z_0 then $T^n z \to z_0$ as $n \to \infty$ for any $z \in \mathbb{C} \cup \{\infty\}$.

3/II/7D Algebra and Geometry

Let G be a group and let $Z(G) = \{h \in G : \; gh = hg \text{ for all } \; g \in G\}$. Show that $Z(G)$ is a normal subgroup of G.

Let H be the set of all 3×3 real matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that H is a subgroup of the group of invertible real matrices under multiplication.

Find $Z(H)$ and show that $H/Z(H)$ is isomorphic to $\mathbb{R}^2$ with vector addition.

3/II/8D Algebra and Geometry

Let A be a 3×3 real matrix such that $\det(A) = -1$, $A \neq -I$, and $A^T A = I$, where A^T is the transpose of A and I is the identity.

Show that the set E of vectors x for which $Ax = -x$ forms a 1-dimensional subspace.

Consider the plane Π through the origin which is orthogonal to E. Show that A maps Π to itself and induces a rotation of Π by angle θ, where $\cos\theta = \frac{1}{2}(\text{trace}\,(A) + 1)$. Show that A is a reflection in Π if and only if A has trace 1. [*You may use the fact that* $\text{trace}(BAB^{-1}) = \text{trace}(A)$ *for any invertible matrix* B.]

Prove that $\det(A - I) = 4(\cos\theta - 1)$.

1/I/1B Algebra and Geometry

The linear map $H : \mathbb{R}^3 \to \mathbb{R}^3$ represents reflection in the plane through the origin with normal $\mathbf{n}$, where $|\mathbf{n}| = 1$, and $\mathbf{n} = (n_1, n_2, n_3)$ referred to the standard basis. The map is given by $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is a (3×3) matrix.

Show that

$$M_{ij} = \delta_{ij} - 2n_i n_j \, .$$

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors such that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is an orthonormal set. Show that

$$\mathbf{M}\mathbf{n} = -\mathbf{n}, \quad \mathbf{M}\mathbf{u} = \mathbf{u}, \quad \mathbf{M}\mathbf{v} = \mathbf{v} \, ,$$

and find the matrix $\mathbf{N}$ which gives the mapping relative to the basis $(\mathbf{u}, \mathbf{v}, \mathbf{n})$.

1/I/2C Algebra and Geometry

Show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}$$

for any real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. Using this inequality, show that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of unit length in $\mathbb{R}^n$ then $|\mathbf{a} \cdot \mathbf{b}| \leqslant 1$.

1/II/5B Algebra and Geometry

The vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b} \, ,$$

where $\mathbf{A}$ is a (3×3) matrix and $\mathbf{b}$ is a (3×1) column vector. State the conditions under which this equation has (a) a unique solution, (b) an infinity of solutions, (c) no solution for $\mathbf{x}$.

Find all possible solutions for the unknowns x, y and z which satisfy the following equations:

$$\begin{aligned} x + y + z &= 1 \\ x + y + \lambda z &= 2 \\ x + 2y + \lambda z &= 4 \, , \end{aligned}$$

in the cases (a) $\lambda = 0$, and (b) $\lambda = 1$.

1/II/6A **Algebra and Geometry**

Express the product $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ in suffix notation and thence prove that the result is zero.

Silver Beard the space pirate believed people relied so much on space-age navigation techniques that he could safely write down the location of his treasure using the ancient art of vector algebra. Spikey the space jockey thought he could follow the instructions, by moving by the sequence of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ one stage at a time. The vectors (expressed in 1000 parsec units) were defined as follows:

1. Start at the centre of the galaxy, which has coordinates $(0, 0, 0)$.

2. Vector $\mathbf{a}$ has length $\sqrt{3}$, is normal to the plane $x + y + z = 1$ and is directed into the positive quadrant.

3. Vector $\mathbf{b}$ is given by $\mathbf{b} = (\mathbf{a} \cdot \mathbf{m})\mathbf{a} \times \mathbf{m}$, where $\mathbf{m} = (2, 0, 1)$.

4. Vector $\mathbf{c}$ has length $2\sqrt{2}$, is normal to $\mathbf{a}$ and $\mathbf{b}$, and moves you closer to the x axis.

5. Vector $\mathbf{d} = (1, -2, 2)$.

6. Vector $\mathbf{e}$ has length $\mathbf{a} \cdot \mathbf{b}$. Spikey was initially a little confused with this one, but then realised the orientation of the vector did not matter.

7. Vector $\mathbf{f}$ has unknown length but is parallel to $\mathbf{m}$ and takes you to the treasure located somewhere on the plane $2x - y + 4z = 10$.

Determine the location of the way-points Spikey will use and thence the location of the treasure.

1/II/7A Algebra and Geometry

Simplify the fraction

$$\zeta = \cfrac{1}{\bar{z} + \cfrac{1}{z + \cfrac{1}{\bar{z}}}},$$

where $\bar{z}$ is the complex conjugate of z. Determine the geometric form that satisfies

$$\mathrm{Re}(\zeta) = \mathrm{Re}\left(\frac{z + \frac{1}{4}}{|z|^2}\right).$$

Find solutions to

$$\mathrm{Im}(\log z) = \frac{\pi}{3}$$

and

$$z^2 = x^2 - y^2 + 2ix,$$

where $z = x + iy$ is a complex variable. Sketch these solutions in the complex plane and describe the region they enclose. Derive complex equations for the circumscribed and inscribed circles for the region. [The circumscribed circle is the circle that passes through the vertices of the region and the inscribed circle is the largest circle that fits within the region.]

1/II/8C Algebra and Geometry

(i) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfy $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 \neq 0$. Are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(ii) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$ have the property that every subset comprising $(n-1)$ of the vectors is linearly independent. Are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(iii) For each value of t in the range $0 \leqslant t < 1$, give a construction of a linearly independent set of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfying

$$\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij} + t(1 - \delta_{ij}),$$

where δ_{ij} is the Kronecker delta.

3/I/1D Algebra and Geometry

State Lagrange's Theorem.

Show that there are precisely two non-isomorphic groups of order 10. [You may assume that a group whose elements are all of order 1 or 2 has order 2^k.]

3/I/2D Algebra and Geometry

Define the Möbius group, and describe how it acts on $\mathbb{C} \cup \{\infty\}$.

Show that the subgroup of the Möbius group consisting of transformations which fix 0 and ∞ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now show that the subgroup of the Möbius group consisting of transformations which fix 0 and 1 is also isomorphic to $\mathbb{C}^*$.

3/II/5D Algebra and Geometry

Let $G = \langle g, h \mid h^2 = 1, g^6 = 1, hgh^{-1} = g^{-1} \rangle$ be the dihedral group of order 12.

i) List all the subgroups of G of order 2. Which of them are normal?

ii) Now list all the remaining proper subgroups of G. [There are 6+3 of them.]

iii) For each proper normal subgroup N of G, describe the quotient group G/N.

iv) Show that G is not isomorphic to the alternating group A_4.

3/II/6D Algebra and Geometry

State the conditions on a matrix A that ensure it represents a rotation of $\mathbb{R}^3$ with respect to the standard basis.

Check that the matrix

$$A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

represents a rotation. Find its axis of rotation $\mathbf{n}$.

Let Π be the plane perpendicular to the axis $\mathbf{n}$. The matrix A induces a rotation of Π by an angle θ. Find $\cos\theta$.

3/II/7D **Algebra and Geometry**

Let A be a real symmetric matrix. Show that all the eigenvalues of A are real, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Give an example of a non-zero *complex* (2×2) symmetric matrix whose only eigenvalues are zero. Is it diagonalisable?

3/II/8D **Algebra and Geometry**

Compute the characteristic polynomial of

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4-s & 2s-2 \\ 0 & -2s+2 & 4s-1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A for all values of s.

For which values of s is A diagonalisable?

1/I/1B **Algebra and Geometry**

(a) Write the permutation

$$(123)(234)$$

as a product of disjoint cycles. Determine its order. Compute its sign.

(b) Elements x and y of a group G are *conjugate* if there exists a $g \in G$ such that $gxg^{-1} = y$.

Show that if permutations x and y are conjugate, then they have the same sign and the same order. Is the converse true? (Justify your answer with a proof or counter-example.)

1/I/2D **Algebra and Geometry**

Find the characteristic equation, the eigenvectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and the corresponding eigenvalues $\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}, \lambda_{\mathbf{d}}$ of the matrix

$$A = \begin{pmatrix} i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{pmatrix} .$$

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ spans the complex vector space $\mathbb{C}^4$.

Consider the four subspaces of $\mathbb{C}^4$ defined parametrically by

$$\mathbf{z} = s\mathbf{a}, \quad \mathbf{z} = s\mathbf{b}, \quad \mathbf{z} = s\mathbf{c}, \quad \mathbf{z} = s\mathbf{d} \qquad (\mathbf{z} \in \mathbb{C}^4, \ s \in \mathbb{C}) .$$

Show that multiplication by A maps three of these subspaces onto themselves, and the remaining subspace into a smaller subspace to be specified.

1/II/5B **Algebra and Geometry**

(a) In the standard basis of $\mathbb{R}^2$, write down the matrix for a rotation through an angle θ about the origin.

(b) Let A be a real 3×3 matrix such that $\det A = 1$ and $AA^{\mathrm{T}} = I$, where A^{T} is the transpose of A.

(i) Suppose that A has an eigenvector $\mathbf{v}$ with eigenvalue 1. Show that A is a rotation through an angle θ about the line through the origin in the direction of $\mathbf{v}$, where $\cos\theta = \frac{1}{2}(\operatorname{trace} A - 1)$.

(ii) Show that A must have an eigenvector $\mathbf{v}$ with eigenvalue 1.

1/II/6A **Algebra and Geometry**

Let α be a linear map

$$\alpha : \mathbb{R}^3 \to \mathbb{R}^3 .$$

Define the kernel K and image I of α.

Let $\mathbf{y} \in \mathbb{R}^3$. Show that the equation $\alpha\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x} \in \mathbb{R}^3$ if and only if $\mathbf{y} \in I$.

Let α have the matrix

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t & -2b \\ 1 & t & 0 \end{pmatrix}$$

with respect to the standard basis, where $b \in \mathbb{R}$ and t is a real variable. Find K and I for α. Hence, or by evaluating the determinant, show that if $0 < b < 2$ and $\mathbf{y} \in I$ then the equation $\alpha\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x} \in \mathbb{R}^3$ for all values of t.

1/II/7B **Algebra and Geometry**

(i) State the orbit-stabilizer theorem for a group G acting on a set X.

(ii) Denote the group of *all* symmetries of the cube by G. Using the orbit-stabilizer theorem, show that G has 48 elements.

Does G have any non-trivial normal subgroups?

Let L denote the line between two diagonally opposite vertices of the cube, and let

$$H = \{g \in G \mid gL = L\}$$

be the subgroup of symmetries that preserve the line. Show that H is isomorphic to the direct product $S_3 \times C_2$, where S_3 is the symmetric group on 3 letters and C_2 is the cyclic group of order 2.

1/II/8D Algebra and Geometry

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy–Schwarz inequality, and deduce the triangle inequality.

By means of a sketch, give a geometric interpretation of the scalar product $\mathbf{x} \cdot \mathbf{y}$ in the case $n = 3$, relating the value of $\mathbf{x} \cdot \mathbf{y}$ to the angle α between the directions of $\mathbf{x}$ and $\mathbf{y}$.

What is a unit vector? Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u}\,.$$

Show that

(i) for any fixed, linearly independent $\mathbf{u}$ and $\mathbf{v}$, the minimum of S over $\mathbf{w}$ is attained when $\mathbf{w} = \lambda(\mathbf{u} + \mathbf{v})$ for some $\lambda \in \mathbb{R}$;

(ii) $\lambda \leqslant -\frac{1}{2}$ in all cases;

(iii) $\lambda = -1$ and $S = -3/2$ in the case where $\mathbf{u} \cdot \mathbf{v} = \cos(2\pi/3)$.

3/I/1A Algebra and Geometry

The mapping α of $\mathbb{R}^3$ into itself is a reflection in the plane $x_2 = x_3$. Find the matrix A of α with respect to any basis of your choice, which should be specified.

The mapping β of $\mathbb{R}^3$ into itself is a rotation about the line $x_1 = x_2 = x_3$ through $2\pi/3$, followed by a dilatation by a factor of 2. Find the matrix B of β with respect to a choice of basis that should again be specified.

Show explicitly that

$$B^3 = 8A^2$$

and explain why this must hold, irrespective of your choices of bases.

3/I/2B Algebra and Geometry

Show that if a group G contains a normal subgroup of order 3, and a normal subgroup of order 5, then G contains an element of order 15.

Give an example of a group of order 10 with no element of order 10.

3/II/5E **Algebra and Geometry**

(a) Show, using vector methods, that the distances from the centroid of a tetrahedron to the centres of opposite pairs of edges are equal. If the three distances are u, v, w and if a, b, c, d are the distances from the centroid to the vertices, show that

$$u^2 + v^2 + w^2 = \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2).$$

[*The centroid of k points in $\mathbb{R}^3$ with position vectors $\mathbf{x}_i$ is the point with position vector* $\dfrac{1}{k}\sum \mathbf{x}_i$.]

(b) Show that

$$|\mathbf{x} - \mathbf{a}|^2 \cos^2 \alpha = [(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n}]^2,$$

with $\mathbf{n}^2 = 1$, is the equation of a right circular double cone whose vertex has position vector $\mathbf{a}$, axis of symmetry $\mathbf{n}$ and opening angle α.

Two such double cones, with vertices $\mathbf{a}_1$ and $\mathbf{a}_2$, have parallel axes and the same opening angle. Show that if $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 \neq \mathbf{0}$, then the intersection of the cones lies in a plane with unit normal

$$\mathbf{N} = \frac{\mathbf{b}\cos^2 \alpha - \mathbf{n}(\mathbf{n} \cdot \mathbf{b})}{\sqrt{\mathbf{b}^2 \cos^4 \alpha + (\mathbf{b} \cdot \mathbf{n})^2 (1 - 2\cos^2 \alpha)}}\,.$$

3/II/6E **Algebra and Geometry**

Derive an expression for the triple scalar product $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3$ in terms of the determinant of the matrix E whose rows are given by the components of the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Use the geometrical interpretation of the cross product to show that $\mathbf{e}_a$, $a = 1, 2, 3$, will be a *not necessarily orthogonal* basis for $\mathbb{R}^3$ as long as $\det E \neq 0$.

The rows of another matrix $\hat{E}$ are given by the components of three other vectors $\hat{\mathbf{e}}_b$, $b = 1, 2, 3$. By considering the matrix $E\hat{E}^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes the transpose, show that there is a unique choice of $\hat{E}$ such that $\hat{\mathbf{e}}_b$ is also a basis and

$$\mathbf{e}_a \cdot \hat{\mathbf{e}}_b = \delta_{ab}.$$

Show that the new basis is given by

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3} \qquad \text{etc.}$$

Show that if either $\mathbf{e}_a$ or $\hat{\mathbf{e}}_b$ is an orthonormal basis then E is a rotation matrix.

3/II/7B Algebra and Geometry

Let G be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $X = \{\alpha, \beta, \gamma\}$ be a set of three distinct points in $\mathbb{C} \cup \{\infty\}$.

(i) Show that there exists a $g \in G$ sending α to 0, β to 1, and γ to ∞.

(ii) Hence show that if $H = \{g \in G \mid gX = X\}$, then H is isomorphic to S_3, the symmetric group on 3 letters.

3/II/8B Algebra and Geometry

(a) Determine the characteristic polynomial and the eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

Is it diagonalizable?

(b) Show that an $n \times n$ matrix A with characteristic polynomial $f(t) = (t - \mu)^n$ is diagonalizable if and only if $A = \mu I$.

1/I/1B **Algebra and Geometry**

(a) State the Orbit-Stabilizer Theorem for a finite group G acting on a set X.

(b) Suppose that G is the group of rotational symmetries of a cube C. Two regular tetrahedra T and T' are inscribed in C, each using half the vertices of C. What is the order of the stabilizer in G of T?

1/I/2D **Algebra and Geometry**

State the Fundamental Theorem of Algebra. Define the characteristic equation for an arbitrary 3×3 matrix A whose entries are complex numbers. Explain why the matrix must have three eigenvalues, not necessarily distinct.

Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and hence find the three eigenvalues of A. Find a set of linearly independent eigenvectors, specifying which eigenvector belongs to which eigenvalue.

1/II/5B **Algebra and Geometry**

(a) Find a subset T of the Euclidean plane $\mathbb{R}^2$ that is not fixed by any isometry (rigid motion) except the identity.

Let G be a subgroup of the group of isometries of $\mathbb{R}^2$, T a subset of $\mathbb{R}^2$ not fixed by any isometry except the identity, and let S denote the union $\bigcup_{g \in G} g(T)$. Does the group H of isometries of S contain G? Justify your answer.

(b) Find an example of such a G and T with $H \neq G$.

1/II/6B **Algebra and Geometry**

(a) Suppose that g is a Möbius transformation, acting on the extended complex plane. What are the possible numbers of fixed points that g can have? Justify your answer.

(b) Show that the operation c of complex conjugation, defined by $c(z) = \bar{z}$, is not a Möbius transformation.

1/II/7B **Algebra and Geometry**

(a) Find, with justification, the matrix, with respect to the standard basis of $\mathbb{R}^2$, of the rotation through an angle α about the origin.

(b) Find the matrix, with respect to the standard basis of $\mathbb{R}^3$, of the rotation through an angle α about the axis containing the point $(\frac{3}{5}, \frac{4}{5}, 0)$ and the origin. You may express your answer in the form of a product of matrices.

1/II/8D **Algebra and Geometry**

Define what is meant by a vector space V over the real numbers $\mathbb{R}$. Define subspace, proper subspace, spanning set, basis, and dimension.

Define the sum $U + W$ and intersection $U \cap W$ of two subspaces U and W of a vector space V. Why is the intersection never empty?

Let $V = \mathbb{R}^4$ and let $U = \{\mathbf{x} \in V : x_1 - x_2 + x_3 - x_4 = 0\}$, where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, and let $W = \{\mathbf{x} \in V : x_1 - x_2 - x_3 + x_4 = 0\}$. Show that $U \cap W$ has the orthogonal basis $\mathbf{b}_1$, $\mathbf{b}_2$ where $\mathbf{b}_1 = (1, 1, 0, 0)$ and $\mathbf{b}_2 = (0, 0, 1, 1)$. Extend this basis to find orthogonal bases of U, W, and $U + W$. Show that $U + W = V$ and hence verify that, in this case,

$$\dim U + \dim W \;=\; \dim(U + W) + \dim(U \cap W)\;.$$

3/I/1A **Algebra and Geometry**

Given two real non-zero 2×2 matrices A and B, with $AB = 0$, show that A maps $\mathbb{R}^2$ onto a line. Is it always true that $BA = 0$? Show that there is always a non-zero matrix C with $CA = 0 = AC$. Justify your answers.

3/I/2B **Algebra and Geometry**

(a) What does it mean for a group to be cyclic? Give an example of a finite abelian group that is not cyclic, and justify your assertion.

(b) Suppose that G is a finite group of rotations of $\mathbb{R}^2$ about the origin. Is G necessarily cyclic? Justify your answer.

3/II/5E **Algebra and Geometry**

Prove, using the standard formula connecting δ_{ij} and ϵ_{ijk}, that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \,.$$

Define, in terms of the dot and cross product, the triple scalar product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ and show that it is invariant under cyclic permutation of the vectors.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a *not necessarily orthonormal* basis for $\mathbb{R}^3$, and define

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,, \quad \hat{\mathbf{e}}_2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,, \quad \hat{\mathbf{e}}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,.$$

By calculating $[\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3]$, show that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ is also a basis for $\mathbb{R}^3$.

The vectors $\hat{\hat{\mathbf{e}}}_1, \hat{\hat{\mathbf{e}}}_2, \hat{\hat{\mathbf{e}}}_3$ are constructed from $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in the same way that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are constructed from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Show that

$$\hat{\hat{\mathbf{e}}}_1 = \mathbf{e}_1, \;\; \hat{\hat{\mathbf{e}}}_2 = \mathbf{e}_2, \;\; \hat{\hat{\mathbf{e}}}_3 = \mathbf{e}_3,$$

Show that a vector $\mathbf{V}$ has components $\mathbf{V} \cdot \hat{\mathbf{e}}_1, \; \mathbf{V} \cdot \hat{\mathbf{e}}_2, \; \mathbf{V} \cdot \hat{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. What are the components of the vector $\mathbf{V}$ with respect to the basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$?

3/II/6E **Algebra and Geometry**

(a) Give the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the equations

$$\mathbf{x} + \mathbf{y} = 2\mathbf{a} \,, \qquad \mathbf{x} \cdot \mathbf{y} = c \qquad (c < \mathbf{a} \cdot \mathbf{a}) \,.$$

Show in particular that $\mathbf{x}$ and $\mathbf{y}$ must lie at opposite ends of a diameter of a sphere whose centre and radius should be specified.

(b) If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair are also perpendicular to each other. Show also that the sum of the lengths squared of two opposite edges is the same for each pair.

3/II/7A **Algebra and Geometry**

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is $0, 1$ or infinite, where A is a real 3×3 matrix and $\mathbf{b} \in \mathbb{R}^3$. Let α be the mapping which A represents. State necessary and sufficient conditions on $\mathbf{b}$ and α for each of these possibilities to hold.

Let A and B be 3×3 matrices representing linear mappings α and β. Give necessary and sufficient conditions on α and β for the existence of a 3×3 matrix X with $AX = B$. When is X unique?

Find X when

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

3/II/8B **Algebra and Geometry**

Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of a regular tetrahedron T in $\mathbb{R}^3$ and that $\mathbf{a} = (1,1,1)$, $\mathbf{b} = (-1,-1,1)$, $\mathbf{c} = (-1,1,-1)$, $\mathbf{d} = (1,x,y)$.

(a) Find x and y.

(b) Find a matrix M that is a rotation leaving T invariant such that $M\mathbf{a} = \mathbf{b}$ and $M\mathbf{b} = \mathbf{a}$.

1/I/1C **Algebra and Geometry**

Show, using the summation convention or otherwise, that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$, for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

The function $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\Pi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n})$ where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. Show that Π is linear and find the elements of a matrix P such that $\Pi(\mathbf{x}) = P\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Find all solutions to the equation $\Pi(\mathbf{x}) = \mathbf{x}$. Evaluate $\Pi(\mathbf{n})$. Describe the function Π geometrically. Justify your answer.

1/I/2C **Algebra and Geometry**

Define what is meant by the statement that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ are linearly independent. Determine whether the following vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ are linearly independent and justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

For the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ taken from a real vector space V consider the statements

A) $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent,
B) $\exists\, \alpha, \beta, \gamma \in \mathbb{R} : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
C) $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$, not all $= 0 : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
D) $\exists\, \alpha, \beta \in \mathbb{R}$, not both $= 0 : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
E) $\exists\, \alpha, \beta \in \mathbb{R} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
F) $\nexists$ basis of V that contains all 3 vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

State if the following implications are true or false (no justification is required):

i)	A $\Rightarrow$ B,	vi)	B $\Rightarrow$ A,
ii)	A $\Rightarrow$ C,	vii)	C $\Rightarrow$ A,
iii)	A $\Rightarrow$ D,	viii)	D $\Rightarrow$ A,
iv)	A $\Rightarrow$ E,	ix)	E $\Rightarrow$ A,
v)	A $\Rightarrow$ F,	x)	F $\Rightarrow$ A.

1/II/5C Algebra and Geometry

The matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 2\alpha+1 \\ 1 & \alpha-1 & 1 \\ 1+\alpha & -1 & \alpha^2+4\alpha+1 \end{pmatrix}$$

defines a linear map $\Phi_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_\alpha(\mathbf{x}) = A_\alpha \mathbf{x}$. Find a basis for the kernel of Φ_α for all values of $\alpha \in \mathbb{R}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases of $\mathbb{R}^3$. Show that there exists a matrix S, to be determined in terms of $\mathcal{B}$ and $\mathcal{C}$, such that, for every linear mapping Φ, if Φ has matrix A with respect to $\mathcal{B}$ and matrix A' with respect to $\mathcal{C}$, then $A' = S^{-1}AS$.

For the bases

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad , \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\} \quad ,$$

find the basis transformation matrix S and calculate $S^{-1}A_0S$.

1/II/6C Algebra and Geometry

Assume that $\mathbf{x}_p$ is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x}, \mathbf{b} \in \mathbb{R}^3$ and a real 3×3 matrix A. Explain why the general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = \mathbf{x}_p + \mathbf{h}$ where $\mathbf{h}$ is any vector such that $A\mathbf{h} = \mathbf{0}$.

Now assume that A is a real symmetric 3×3 matrix with three different eigenvalues λ_1, λ_2 and λ_3. Show that eigenvectors of A with respect to different eigenvalues are orthogonal. Let $\mathbf{x}_k$ be a normalised eigenvector of A with respect to the eigenvalue λ_k, $k = 1, 2, 3$. Show that the linear system

$$(A - \lambda_k I)\mathbf{x} \;=\; \mathbf{b} \;,$$

where I denotes the 3×3 unit matrix, is solvable if and only if $\mathbf{x}_k.\mathbf{b} = 0$. Show that the general solution is given by

$$\mathbf{x} \;=\; \sum_{i \neq k} \frac{\mathbf{b}.\mathbf{x}_i}{\lambda_i - \lambda_k} \mathbf{x}_i + \beta \mathbf{x}_k, \quad \beta \in \mathbb{R}.$$

[*Hint: consider the components of* $\mathbf{x}$ *and* $\mathbf{b}$ *with respect to a basis of eigenvectors of* A.]

Consider the matrix A and the vector $\mathbf{b}$

$$A = \begin{pmatrix} -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \sqrt{2} + \sqrt{3} \\ -\sqrt{2} + \sqrt{3} \\ -2\sqrt{3} \end{pmatrix}.$$

Verify that $\frac{1}{\sqrt{3}}(1,1,1)^T$ and $\frac{1}{\sqrt{2}}(1,-1,0)^T$ are eigenvectors of A. Show that $A\mathbf{x} = \mathbf{b}$ is solvable and find its general solution.

1/II/7C Algebra and Geometry

For $\alpha, \gamma \in \mathbb{R}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $\beta\overline{\beta} \geqslant \alpha\gamma$ the equation $\alpha z\overline{z} - \beta\overline{z} - \overline{\beta}z + \gamma = 0$ describes a circle $C_{\alpha\beta\gamma}$ in the complex plane. Find its centre and radius. What does the equation describe if $\beta\overline{\beta} < \alpha\gamma$? Sketch the circles $C_{\alpha\beta\gamma}$ for $\beta = \gamma = 1$ and $\alpha = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1$.

Show that the complex function $f(z) = \beta\overline{z}/\overline{\beta}$ for $\beta \neq 0$ satisfies $f(C_{\alpha\beta\gamma}) = C_{\alpha\beta\gamma}$.

[*Hint:* $f(C) = C$ *means that* $f(z) \in C$ $\forall z \in C$ *and* $\forall w \in C$ $\exists z \in C$ *such that* $f(z) = w$.]

For two circles C_1 and C_2 a function $m(C_1, C_2)$ is defined by

$$m(C_1, C_2) = \max_{z \in C_1, w \in C_2} \left| z - w \right| .$$

Prove that $m(C_1, C_2) \leqslant m(C_1, C_3) + m(C_2, C_3)$. Show that

$$m(C_{\alpha_1\beta_1\gamma_1}, C_{\alpha_2\beta_2\gamma_2}) = \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{|\alpha_1\alpha_2|} + \frac{\sqrt{\beta_1\overline{\beta_1} - \alpha_1\gamma_1}}{|\alpha_1|} + \frac{\sqrt{\beta_2\overline{\beta_2} - \alpha_2\gamma_2}}{|\alpha_2|} .$$

1/II/8C Algebra and Geometry

Let $l_{\mathbf{x}}$ denote the straight line through $\mathbf{x}$ with directional vector $\mathbf{u} \neq \mathbf{0}$

$$l_{\mathbf{x}} = \left\{ \mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = \mathbf{x} + \lambda\mathbf{u}, \lambda \in \mathbb{R} \right\} .$$

Show that $l_{\mathbf{0}}$ is a subspace of $\mathbb{R}^3$ and show that $l_{\mathbf{x}_1} = l_{\mathbf{x}_2} \Leftrightarrow \mathbf{x_1} = \mathbf{x}_2 + \lambda\mathbf{u}$ for some $\lambda \in \mathbb{R}$.

For fixed $\mathbf{u} \neq \mathbf{0}$ let $\mathcal{L}$ be the set of all the parallel straight lines $l_{\mathbf{x}}$ ($\mathbf{x} \in \mathbb{R}^3$) with directional vector $\mathbf{u}$. On $\mathcal{L}$ an addition and a scalar multiplication are defined by

$$l_{\mathbf{x}} + l_{\mathbf{y}} = l_{\mathbf{x}+\mathbf{y}}, \ \alpha l_{\mathbf{x}} = l_{\alpha\mathbf{x}}, \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \ \alpha \in \mathbb{R} .$$

Explain why these operations are well-defined. Show that the addition is associative and that there exists a zero vector which should be identified.

You may now assume that $\mathcal{L}$ is a vector space. If $\{\mathbf{u}, \mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^3$ show that $\{l_{\mathbf{b}_1}, l_{\mathbf{b}_2}\}$ is a basis for $\mathcal{L}$.

For $\mathbf{u} = (1, 3, -1)^T$ a linear map $\Phi : \mathcal{L} \to \mathcal{L}$ is defined by

$$\Phi\left(l_{(1,-1,0)^T}\right) = l_{(2,4,-1)^T} \ , \ \Phi\left(l_{(1,1,0)^T}\right) = l_{(-4,-2,1)^T} .$$

Find the matrix A of Φ with respect to the basis $\{l_{(1,0,0)^T}, l_{(0,1,0)^T}\}$.

3/I/1F **Algebra and Geometry**

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 = 0$ if and only if $a = -d$ and $bc = -a^2$. Prove that $A^3 = 0$ if and only if $A^2 = 0$.

[*Hint: it is easy to check that* $A^2 - (a+d)A + (ad - bc)I = 0$.]

3/I/2D **Algebra and Geometry**

Show that the set of Möbius transformations of the extended complex plane $\mathbb{C} \cup \{\infty\}$ form a group. Show further that an arbitrary Möbius transformation can be expressed as the composition of maps of the form

$$f(z) = z + a, \quad g(z) = kz \quad \text{and} \quad h(z) = 1/z\ .$$

3/II/5F **Algebra and Geometry**

Let A, B, C be 2×2 matrices, real or complex. Define the trace $\operatorname{tr} C$ to be the sum of diagonal entries $C_{11} + C_{22}$. Define the commutator $[A, B]$ to be the difference $AB - BA$. Give the definition of the eigenvalues of a 2×2 matrix and prove that it can have at most two distinct eigenvalues. Prove that

a) $\operatorname{tr}[A, B] = 0$,

b) $\operatorname{tr} C$ equals the sum of the eigenvalues of C,

c) if all eigenvalues of C are equal to 0 then $C^2 = 0$,

d) either $[A, B]$ is a diagonalisable matrix or the square $[A, B]^2 = 0$,

e) $[A, B]^2 = \alpha I$ where $\alpha \in \mathbb{C}$ and I is the unit matrix.

3/II/6E **Algebra and Geometry**

Define the notion of an *action* of a group G on a set X. Define *orbit* and *stabilizer*, and then, assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Show that the group of rotations of a cube has order 24.

3/II/7E **Algebra and Geometry**

State Lagrange's theorem. Use it to describe all groups of order p, where p is a fixed prime number.

Find all the subgroups of a fixed cyclic group $\langle x \rangle$ of order n.

3/II/8D **Algebra and Geometry**

(i) Let A_4 denote the alternating group of even permutations of four symbols. Let X be the 3-cycle (123) and P, Q be the pairs of transpositions (12)(34) and (13)(24). Find $X^3, P^2, Q^2, X^{-1}PX, X^{-1}QX$, and show that A_4 is generated by X, P and Q.

(ii) Let G and H be groups and let

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

Show how to make $G \times H$ into a group in such a way that $G \times H$ contains subgroups isomorphic to G and H.

If D_n is the dihedral group of order n and C_2 is the cyclic group of order 2, show that D_{12} is isomorphic to $D_6 \times C_2$. Is the group D_{12} isomorphic to A_4?