Lie algebras – Examples sheet 1

All Lie algebras and modules are finite dimensional, over an arbitrary field k unless otherwise stated.

Definition chasing questions

1. Let L be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) and verify that this makes L into a Lie algebra. Write down the multiplication table for L relative to the usual basis for $\mathbb{R}^3$.

2. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}_2(k)$. Write down the multiplication table for $\mathfrak{sl}_2(k)$ relative to this basis. Hence, compute the matrices of $\operatorname{ad} e, \operatorname{ad} h$ and $\operatorname{ad} f$ with respect to this basis.

3. (a) Show that there is precisely one Lie algebra of dimension 1 (up to isomorphism).

(b) Show that there are precisely two non-isomorphic Lie algebras of dimension 2 (one is abelian, the other is not).

(c) Let L be the Lie algebra over k with basis $\{x, y, z\}$ and relations $[xy] = z, [yz] = x, [zx] = y$ (compare with question 1). If $k = \mathbb{C}$, show that L is isomorphic to $\mathfrak{sl}_2(k)$, but that this is *false* if $k = \mathbb{R}$.

(So, the classification of 3-dimensional Lie algebras depends on the ground field k.)

4. Prove that the centre of $\mathfrak{gl}_n(k)$ is the set of scalar matrices. Prove that $\mathfrak{sl}_n(k)$ has centre 0, unless $\operatorname{char} k$ divides n, in which case the centre is again the set of scalar matrices.

5. Show $\mathfrak{sl}_2(k)$ is simple if $\operatorname{char} k \neq 2$. What happens if $\operatorname{char} k = 2$?

(*Hints:* Work in the basis of question 2. By applying $\operatorname{ad} e$ twice or $\operatorname{ad} f$ twice, show that if $0 \neq ae + bf + ch$ lies in an ideal I of $\mathfrak{sl}_2(k)$ then one of e, f or h lies in I, hence $I = \mathfrak{sl}_2(k)$.)

Derivations

6. Prove that the set of all inner derivations $\operatorname{ad} x, x \in L$ is an ideal of $\operatorname{Der} L$.

7. Verify that the commutator of two derivations of a k-algebra is again a derivation. Is the ordinary product always a derivation?

The PBW theorem

8. If L is a free Lie algebra on a set X, show that $U(L)$ is isomorphic to $T(V)$, where V is the vector space with X as basis.

9. Describe the free Lie algebra on the set $X = \{x\}$.

10. Let L be an arbitrary finite dimensional Lie algebra. Use the PBW theorem to show that $U(L)$ has no zero divisors.

Soluble and nilpotent Lie algebras

11. Let $\mathfrak{d}_n(k), \mathfrak{n}_n(k)$ and $\mathfrak{t}_n(k)$ be the set of all diagonal, strictly upper triangular (ie zeros on the diagonal) and upper triangular (ie anything on the diagonal) $n \times n$ matrices over k respectively. Show that these are Lie subalgebras of $\mathfrak{gl}_n(k)$.

12. Let $L = \mathfrak{n}_n(k)$ as in question 11. Show that the lower central series of L is

$$L = L^0 > L^1 > L_2 > \cdots > L^r = 0$$

where L^s equals $\{M \in \mathfrak{gl}_n(k) \mid M_{i,j} = 0 \text{ for all } 1 \leq i, j \leq n \text{ with } j - s \leq i \leq n\}$. Deduce that $\mathfrak{n}_n(k)$ is nilpotent.

13. Using question 12, show that $\mathfrak{t}_n(k)$ is soluble.

14. Show $\mathfrak{sl}_2(k)$ is nilpotent if $\operatorname{char} k = 2$.

15. Let L be nilpotent and K be a proper subalgebra of L. Show that $N_L(K)$ is strictly larger than K.

16. Let k be a field of characteristic $p > 0$. Let $x, y \in \mathfrak{gl}_p(k)$ be the following $p \times p$ matrices:

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, y = \operatorname{diag}(0, 1, \ldots, p-1).$$

Show that x, y generate a 2-dimensional soluble subalgebra of $\mathfrak{gl}_p(k)$ but that they have no common eigenvector. Hence, Lie's theorem is false in general in non-zero characteristic.

The Killing form

17. Using question 2, compute the Killing form explicitly for $\mathfrak{sl}_2(\mathbb{C})$ and hence verify directly that it is non-degenerate on $\mathfrak{sl}_2(\mathbb{C})$.

18. Let k have characteristic 3. Show $\mathfrak{sl}_3(k)$ modulo its centre is semisimple but has degenerate Killing form.

Jon Brundan, 5/1/97.

Lie algebras – Examples sheet 2

L denotes a finite dimensional, semisimple Lie algebra over $\mathbb{C}$, unless otherwise stated.

Representations

1. Show an L-module V is completely reducible if and only if every L-submodule W of V has an L-stable complement W' such that $V = W \oplus W'$.

2. If V and W are L-modules, we made $\mathrm{Hom}(V, W)$ into an L-module by setting $(x.f)(v) = x.f(v) - f(x.v)$ for all $x \in L, f \in \mathrm{Hom}(V, W), v \in V$. Verify directly that this gives a well-defined L-module structure on $\mathrm{Hom}(V, W)$.

3. Show that if L is a nilpotent Lie algebra, the only irreducible L-module is the trivial module. Show that if L is a soluble Lie algebra, the irreducible L-modules are all 1-dimensional. Describe the irreducible modules for $\mathfrak{t}_n(\mathbb{C})$ explicitly.

4. Using the fact that the Lie algebra $L = \mathfrak{sl}_n(\mathbb{C})$ is simple, show that the Killing form $(x, y) := \mathrm{tr}_L(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ is related to the form $\langle x, y\rangle := \mathrm{tr}(xy)$ by $(x, y) = 2n\langle x, y\rangle$.

Representations of $\mathfrak{sl}_2(\mathbb{C})$

In these exercises, $V(m)$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $m + 1$.

5. Embed $\mathfrak{sl}_2(\mathbb{C})$ into $\mathfrak{sl}_3(\mathbb{C})$ in its upper left hand 2×2 position. The restriction of the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$ defines an 8-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V. Show that $V \cong V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

6. Let $V = V(1)$ denote the natural $\mathfrak{sl}_2(\mathbb{C})$-module, with its usual basis x_1, x_2. Let $P = \mathbb{C}[x_1, x_2]$ be the polynomial algebra in two variables, and extend the action of L on $\langle x_1, x_2\rangle < P$ to all of P by the rule

$$z.fg = (z.f)g + f(z.g)$$

for all $z \in L, f, g \in P$. Show that this makes P into an infinite dimensional L-module, and that the subspace $P_m < P$ of homogeneous polynomials of degree m is an L-submodule of P. Show $P_m \cong V(m)$.

7. Let $0 \le m \le n$. Prove the *Clebsch-Gordan formula:*

$$V(m) \otimes V(n) \cong V(n-m) \oplus V(n-m+2) \oplus \cdots \oplus V(n+m-2) \oplus V(n+m).$$

The Jordan decomposition

8. Show that L is nilpotent if and only if every element x of L is nilpotent (ie $\mathrm{ad}_L\, x$ is a nilpotent endomorphism of L). Give an example to show that the statement "L is semisimple if and only if every element x of L is semisimple" (ie $\mathrm{ad}_L\, x$ is diagonalisable) is false.

9. Let L be the Lie algebra $\mathbb{C}$ with multiplication $[xy] = 0$ for all $x, y \in \mathbb{C}$. Every element of L is both semisimple and nilpotent. Verify that the following maps $L \to \mathfrak{gl}_2(\mathbb{C})$ are representations of L:

(a) $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$; (b) $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$; (c) $x \mapsto \begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix}$.

Show that in (a) every non-zero element of the image of L is semisimple but not nilpotent and that in (b) every non-zero element of the image of L is nilpotent but not semisimple. In (c), show that the semisimple and nilpotent parts of every non-zero element of the image of L are not even elements of the image of L. Thus, the Jordan decomposition is false in general if L is not a semisimple Lie algebra.

10. Let $L' < L$ be two semisimple Lie algebras. For $x \in L'$, show that its abstract Jordan decomposition regarded as an element of L' agrees with its abstract Jordan decomposition regarded as an element of L.

The Cartan decomposition

11. Compute explicitly the Cartan decomposition of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ taking H to be the set of all diagonal matrices (ie verify all the details from the lectures). This is the most important example on this sheet!!

12. Compute the restriction of the Killing form on $\mathfrak{sl}_n(\mathbb{C})$ to the set H of all diagonal matrices directly (without using question 4). Hence verify directly that the restriction of the Killing form to H is non-degenerate.

13. Calculate explicitly the *Cartan integers* for $\mathfrak{sl}_n(\mathbb{C})$: the numbers $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ for all α, β in the root system Φ (they should all be $0, 2$ or ± 1!).

13. If L is semisimple, H a maximal toral subalgebra, prove that $H = N_L(H)$.

14. Prove that every maximal toral subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ is one dimensional.

15. Prove that every three dimensional semisimple Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

16. Using just the Cartan decomposition, prove that no $4, 5$ or 7 dimensional semisimple Lie algebras exist.

Jon Brundan, 9/2/97.

Lie algebras – Examples sheet 3

1. Let $(a_{i,j})_{1\leq i,j\leq l}$ be the Cartan matrix of an abstract root system. Prove directly from the definition that

(i) $a_{i,i} = 2$;

(ii) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$;

(iii) if $i \neq j$ then $a_{i,j} \leq 0$.

2. Prove that the only abstract root systems of rank two are

$$\underset{A_1A_1}{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} \quad \underset{A_2}{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}} \quad \underset{B_2}{\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}} \quad \underset{G_2}{\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}}$$

3. Let $\Phi \subset E$ be an abstract root system with base Δ. For $\alpha \in \Delta$, show that $s_\alpha \in W$ stabilises $\Phi^+ \setminus \{\alpha\}$. Deduce that $s_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in \Delta$, where

$$\rho = \frac{1}{2}\sum_{\beta\in\Phi^+} \beta.$$

4. Let $\Phi \subset E$ be an abstract root system.

(a) Show that

$$\Phi^\vee = \left\{ \frac{2\alpha}{(\alpha,\alpha)} \,\middle|\, \alpha \in \Phi \right\}$$

is also an abstract root system in E, known as the *dual root system*.

(b) Show that the Weyl group of $\Phi^\vee$ is isomorphic to the Weyl group of Φ.

(c) Show that $\Phi^\vee$ is irreducible if and only if Φ is irreducible, and that the double dual of Φ is isomorphic to Φ.

(d) Show that the dual root system to A_l, B_l, C_l or D_l is A_l, C_l, B_l or D_l respectively.

5. Let $\Phi \subset E$ be an abstract root system.

(a) Let $\Phi' \subset \Phi$ be a subset such that if $\alpha_1, \dots, \alpha_n \in \Phi'$ and $\alpha = \sum a_i\alpha_i \in \Phi$ for certain coefficients $a_i \in \mathbb{Z}$, then $\alpha \in \Phi'$. Show that Φ' is a root system in the subspace $E' < E$ that it spans. Such subsystems of the root system Φ are called *closed subsystems*.

(b) Verify that the set of long roots in the root system of type G_2 is a closed subsystem of type A_2, whereas the set of short roots in the root system of type G_2 is *not* a closed subsystem.

(c) More generally, show that the set of long roots in any irreducible root system is a closed subsystem.

(d) What subsystem does one obtain from the long roots in type B_l? Type C_l?

6. Let $\mathrm{Aut}\,\Phi$ be the set of all automorphisms of the abstract root system Φ, that is, all bijections $\theta : \Phi \to \Phi$ such that $\langle \theta(\alpha), \theta(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$.

(a) Show that W is a normal subgroup of $\mathrm{Aut}\,\Phi$.

(b) Let Γ be the set of all $\theta \in \mathrm{Aut}\,\Phi$ such that $\theta(\Delta)+ = \Delta$, where Δ is a fixed base of Φ. Show that $\mathrm{Aut}\,\Phi$ is the semidirect product of Γ and W, that is, $\mathrm{Aut}\,\Phi = W\Gamma, W \cap \Gamma = 1$.

(c) Show that Γ can be identified with the set of all automorphisms (of directed graphs) of the Dynkin diagram of Φ.

(d) Prove that the map $\alpha \mapsto -\alpha$ ($\alpha \in \Phi$) is an automorphism of Φ. For which irreducible root systems Φ is this map an element of the Weyl group W?

Jon Brundan, 24/2/97.

Lie algebras – Examples sheet 4

Notation Always, L is a semisimple Lie algebra. Fix a maximal toral subalgebra H of L, with corresponding root system Φ. Let Δ be a base for Φ, Φ^+ the corresponding positive roots, W the Weyl group and $\varepsilon : W \to \{\pm 1\}$ be the sign representation of W relative to the simple reflections $\{s_\alpha \mid \alpha \in \Delta\}$. Writing $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, let $\omega_1, \ldots, \omega_l$ denote the corresponding fundamental dominant weights, so that $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$. Let X and X^+ denote the integral and dominant integral weights respectively.

1. Let $L = \mathfrak{sl}_2(\mathbb{C})$ with standard basis e, f, h. Let τ be the endomorphism of L defined by $\tau := \exp(\operatorname{ad} e)\exp(\operatorname{ad}(-f))\exp(\operatorname{ad} e)$. Verify explicitly that the automorphism τ acts on L as conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Deduce that $\tau(e) = -f, \tau(f) = -e, \tau(h) = -h$.

2. If V and W are finite dimensional L-modules, show that $\operatorname{ch}(V \oplus W) = \operatorname{ch}(V) + \operatorname{ch}(W)$ and $\operatorname{ch}(V \otimes W) = \operatorname{ch}(V).\operatorname{ch}(W)$.

3. Show that $\rho := \frac{1}{2}\sum_{\alpha > 0} \alpha$ can also be written as $\sum_{\alpha \in \Delta} \alpha$.

4. For $\lambda \in X$, prove that $\operatorname{ch} M(\lambda) = \dfrac{e(\lambda)}{\prod_{\alpha > 0}(1 - e(-\alpha))} = \dfrac{e(\lambda)}{\sum_{w \in W} \varepsilon(w\rho - \rho)}$.

5. For the root system of type B_2, order the base $\Delta = \{\alpha_1, \alpha_2\}$ so that the Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. Compute the corresponding fundamental dominant weights $\{\omega_1, \omega_2\}$ in terms of α_1 and α_2. Do the same for A_2, verifying the calculation in section 8.1 in the lectures.

6. Apply Weyl's character formula to compute the dimension of all weight spaces of the module $V(2\omega_1 + \omega_2)$ where $L = B_2$ and ω_1, ω_2 are as in question 5. Verify that the sum of the dimensions of all weight spaces equal the dimension as computed by Weyl's dimension formula.

7. Let $L = \mathfrak{sl}_3(\mathbb{C})$ with fundamental dominant weights ω_1, ω_2. Abbreviate $V(m_1\omega_1 + m_2\omega_2)$ by $V(m_1, m_2)$. Use Weyl's dimension formula to show

$$\dim V(m_1, m_2) = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2).$$

8. With notation as in question 7, show that $V(1,1) \otimes V(1,2) \cong V(2,3) \oplus V(3,1) \oplus V(0,4) \oplus V(1,2) \oplus V(1,2) \oplus V(2,0) \oplus V(0,1)$.

9. Use Weyl's dimension formula to show that a faithful, irreducible, finite dimensional L-module of smallest possible dimension has highest weight equal to ω_i for some i. Hence verify that the smallest dimension of a faithful, irreducible G_2-module is 7.

Jon Brundan, 7/3/97.

Part III Lie algebras – tripos-like questions

Attempt TWO questions from section A and TWO questions from section B. Questions in section B are worth twice as many marks as questions in section A.

Throughout, L denotes a finite dimensional Lie algebra over the field $\mathbb{C}$ of complex numbers.

Section A

1. Let $L = \mathfrak{sl}_2(\mathbb{C})$. Show that L contains elements e, f, h such that $[ef] = h, [he] = 2e, [hf] = -2f$.

Let V and W be two irreducible, finite dimensional L-modules. Show that $V \cong W$ if and only if $\dim V = \dim W$.

(*You may assume any results you need about the Jordan decomposition of L providing you state them clearly.*)

2. Let L be semisimple. Show that all associative bilinear forms on L are symmetric.

Let V be a faithful L-module and suppose that the bilinear form $(.,.)$ on L defined by $(x, y) = \mathrm{tr}_V(xy)$ is non-degenerate. Let $x_1, \ldots, x_n$ and $y_1, \ldots y_n$ be bases for L such that $(x_i, y_j) = \delta_{i,j}$. Show that the operator $\sum_{i=1}^n x_i y_i$ commutes with the action of L on V.

Deduce that if V contains an irreducible L-submodule W such that $\dim V = \dim W + 1$, then there is an L-stable submodule W' of V such that $V = W \oplus W'$.

3. Let L be a semisimple Lie algebra with Cartan decomposition

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where H is a fixed maximal toral subalgebra of L. Show that the restriction of the Killing form on L to H is non-degenerate.

Stating any results that you use from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$, show that

(i) $\dim L_\alpha = 1$ for all $\alpha \in \Phi$;

(ii) if $\alpha \in \Phi$, the only other scalar multiple of α which is a root is $-\alpha$.

Section B

4. (a) What does it mean to say that $\Phi \subset E$ is an abstract root system? What is a base Δ of Φ? Define the Dynkin diagram of a root system Φ, and show that Φ is determined (up to an isomorphism of abstract root systems) by its Dynkin diagram.

(b) Now let L be the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$, that is, the set of all 4×4 matrices of the form $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ as A, B, C, D run over all 2×2 matrices satisfying $A = -D^T, B = B^T, C = C^T$. Show that the set H of all diagonal matrices in L is a maximal toral subalgebra, and describe the Cartan decomposition of L with respect to H.

(c) By computing the Killing form, show that L is a semisimple Lie algebra.

(d) Compute the Dynkin diagram of L. Hence, show that L is the simple Lie algebra of type ~~B_2~~. C_2.

5. Let $L = \mathfrak{sl}_3(\mathbb{C})$.

(a) Give an explicit Cartan decomposition of L. Show that the Dynkin diagram of L is of type A_2. List all the roots in terms opf the simple roots α_1 and α_2. Show that its Weyl group is isomorphic to D_6, the dihedral group of order 6 with presentation $\langle x, y \mid x^2 = y^3 = 1, x^{-1}yx = y^2 \rangle$.

(b) Let H denote the maximal toral subalgebra of L in (a). Let $t : H^* \to H$ be the bijection induced by the Killing form. Put $t_i = t(\alpha_i)$ for $i = 1, 2$. Let V_{m_1,m_2} denote the irreducible L-module of highest weight λ, where $\lambda(t_i) = m_i$ for $i = 1, 2$. Assuming $m_1 \geq m_2$, prove

$$V_{m_1,0} \otimes V_{0,m_2} \cong V_{m_1,m_2} \oplus V_{m_1-1,m_2-1} \oplus \cdots \oplus V_{0,m_2-m_1}.$$

(Hint: consider symmetric powers of the obvious modules!)

6. Write an essay on Weyl's character formula.