Combinatorics

Lectured by I. B. Leader

Michaelmas Term 2008, 2010 & 2012

Note. Chapter 3 was lectured in 2008 and 2012. Chapter 3′ was lectured in 2010.

Books (for chapter 1)

– Bollobás, *Combinatorics* (CUP, 1986) ("Bedtime reading")

– Anderson, *Combinatorics and Finite Sets* (OUP, 1987) ("Simple and clear")

Pre-requisites

– basic concepts of graph theory (graph, path, Hall's Theorem)
– integers mod p
– vector spaces

Last updated: Tue 23rd Apr, 2013
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Course description (from the Part III booklet)

The flavour of the course is similar to that of the Part II Graph Theory course, although we shall not rely on many of the results from that course.

We shall study collections of subsets of a finite set, with special emphasis on size, intersection and containment. There are many very natural and fundamental questions to ask about families of subsets; although many of these remain unsolved, several have been answered using a great variety of elegant techniques.

We shall cover a number of 'classical' extremal theorems, such as those of Erdős-Ko-Rado and Kruskal-Katona, together with more recent results concerning isoperimetric inequalities and intersecting families. The aim of the course is to give an introduction to a very active area of mathematics.

We hope to cover the following material.

Set Systems

Definitions. Antichains; Sperner's lemma and related results. Shadows. Compression operators and the Kruskal-Katona theorem. Intersecting families; the Erdős-Ko-Rado theorem.

Isoperimetric Inequalities

Harper's theorem and the edge-isoperimetric inequality in the cube. Inequalities in the grid. The classical isoperimetric inequality on the sphere. The 'concentration of measure' phenomenon. Applications.

Intersecting Families (2008 & 2012)

Katona's t-intersecting theorem. The Ahlswede-Khachatrian theorem. Restricted intersections. The Kahn-Kalai counterexample to Borsuk's conjecture.

Projections (2010)

The trace of a set system; the Sauer-Shelah lemma. Bounds on projections: the Bollobás-Thomason box theorem. Hereditary properties. Intersecting families of graphs.

Desirable Previous Knowledge

The only prerequisites are the very basic concepts of graph theory.

Introductory Reading

1. Bollobás, B., Combinatorics, C.U.P. 1986.

Chapter 1 : Set Systems

Let X be a set. A **set system** on X (or **family of subsets** of X) is a family $\mathcal{A} \subset \mathbb{P}(X)$.

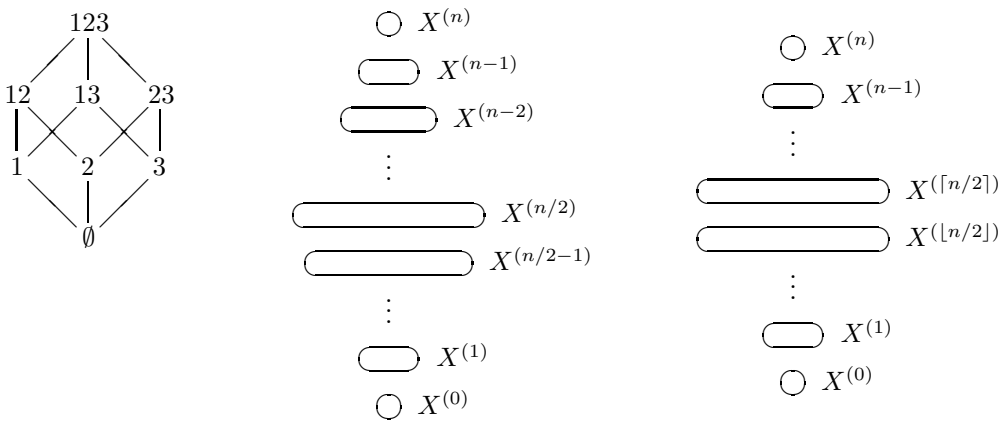
E.g., $X^{(r)} = \{A \subset X : |A| = r\}$.

Unless otherwise stated, $X = [n] = \{1, \ldots, n\}$. So, e.g., $\left|X^{(r)}\right| = \binom{n}{r}$.

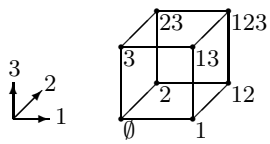
For example, $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$, where "12" $= \{1, 2\}$.

Often, we make $\mathbb{P}(X)$ into a graph by joining A to B if $|A \triangle B| = 1$. I.e., if $A = B \cup \{i\}$ for some $i \notin B$, or vice versa. This graph is the **discrete cube** Q_n.

E.g., Q_3 $\qquad$ Q_n (n even) $\qquad$ Q_n (n odd)



If we identify a point $A \in \mathbb{P}(X)$ with a 0–1 sequence of length n (e.g., $\{1, 3\} \mapsto 10100\ldots00$, via the indicator function $A \leftrightarrow \mathbb{I}_A$), then Q_n is precisely the unit cube in $\mathbb{R}^n$.



Chains and Antichains

$\mathcal{A} \subset \mathbb{P}(X)$ is a **chain** if for all $A, B \in \mathcal{A}$, either $A \subset B$ or $B \subset A$.

$\mathcal{A} \subset \mathbb{P}(X)$ is an **antichain** if for all $A, B \in \mathcal{A}$, $A \neq B$, we have $A \not\subset B$.

E.g., $\{12, 1257, 12357\}$ is a chain, and $\{1, 346, 2589\}$ is an antichain.

How large can a chain be? We can have $|\mathcal{A}| = n + 1$, e.g. $\mathcal{A} = \{\emptyset, 1, 12, 123, \ldots, [n]\}$. We cannot beat this, as a chain can meet a "level" $X^{(r)}$ in at most one point.

How large can an antichain be? We can have $|\mathcal{A}| = n$, e.g. $\mathcal{A} = \{1, 2, 3, \ldots, n\}$. Indeed, $X^{(r)}$ is always an antichain, so we can achieve size $\binom{n}{n/2}$ (for n even) or $\binom{n}{\lfloor n/2 \rfloor}$ (for n odd).

Can we beat that?

Theorem 1 (Sperner's Lemma). Let $\mathcal{A} \subset \mathbb{P}(X)$ be an antichain. Then $|\mathcal{A}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$.

Idea. Inspired by "chains having $\leqslant 1$ in each layer", let's try to decompose cube into chains.

Proof. We'll decompose $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, and then we're done.

To do this, it's sufficient to show that:

(i) for each $r < n/2$, there is a matching (a set of disjoint edges) from $X^{(r)}$ to $X^{(r+1)}$,
(ii) for each $r > n/2$, there is a matching from $X^{(r)}$ to $X^{(r-1)}$.

Then just put together these matchings to form our chains. Taking complements, it's sufficient to prove (i).

Consider the induced subgraph of Q_n spanned by $X^{(r)} \cup X^{(r+1)}$. This is bipartite, and has $d(A) = n - r$ for all $A \in X^{(r)}$ (the things that can be added to the r-set), and $d(A) = r + 1$ for all $A \in X^{(r+1)}$ (the ways to throw out an element of an $(r+1)$-set).

Given $S \subset X^{(r)}$, the number of S–$\Gamma(S)$ edges equals $|S|(n-r)$ (counting from below), and is $\leqslant |\Gamma(S)|(r+1)$ (counting from above).

Thus, $|\Gamma(S)| \geqslant |S| \left(\dfrac{n-r}{r+1}\right) \geqslant |S|$, as $r < n/2$.

So there exists a matching, by Hall's Theorem. $\square$

Remarks. 1. We can achieve $|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}$. E.g., $\mathcal{A} = X^{(\lfloor n/2 \rfloor)}$.

2. Uniqueness? When can $|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}$? The above proof tells us nothing.

Aim. If $\mathcal{A}$ is an antichain, then $\displaystyle\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leqslant 1$.

"Sum of percentages of levels filled is at most 1."

This trivially implies Sperner. (Exercise: make sure this is clear.)

For $\mathcal{A} \subset X^{(r)}$, $1 \leqslant r \leqslant n$, the **shadow** or **lower shadow** of $\mathcal{A}$ is the set system:

$$\partial\mathcal{A} = \partial^{-}\mathcal{A} = \{B \in X^{(r-1)} : B \cup \{i\} \in \mathcal{A} \text{ for some } i \notin B\} \subset X^{(r-1)}.$$

E.g., if $\mathcal{A} = \{123, 124, 234, 135\}$, then $\partial\mathcal{A} = \{12, 13, 23, 14, 24, 34, 15, 35\}$.

Proposition 2 (Local LYM). Let $1 \leqslant r \leqslant n$, and let $\mathcal{A} \subset X^{(r)}$. Then $\dfrac{|\partial\mathcal{A}|}{\binom{n}{r-1}} \geqslant \dfrac{|\mathcal{A}|}{\binom{n}{r}}$.

"The fraction occupied by $\partial\mathcal{A}$ is $\geqslant$ that for $\mathcal{A}$."

("LYM" = Lubell, Yamamoto, Meshalkin.)

Proof. The number of edges from $\mathcal{A}$ to $\partial\mathcal{A}$ is equal to $|\mathcal{A}|r$ (counting from $\mathcal{A}$), and is $\leqslant |\partial\mathcal{A}|(n-r+1)$ (counting from $\partial\mathcal{A}$).

Thus $|\mathcal{A}|r \leqslant |\partial\mathcal{A}|(n-r+1)$, so $\dfrac{|\partial\mathcal{A}|}{|\mathcal{A}|} \geqslant \dfrac{r}{n-r+1}$. But $\dfrac{\binom{n}{r-1}}{\binom{n}{r}} = \dfrac{r}{n-r+1}$. $\square$

Equality in Local LYM? Must have that: $\forall A \in \mathcal{A}$, $\forall i \in A$, $\forall j \notin A$, we have $(A-i)\cup j \in \mathcal{A}$. ("Add something and remove something, and you're still in $\mathcal{A}$.") Thus $\mathcal{A} = \emptyset$ or $X^{(r)}$.

Theorem 3 (LYM inequality). Let $\mathcal{A} \subset \mathbb{P}(X)$ be an antichain. Then $\displaystyle\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leqslant 1$.

Proof 1. "Bubble down using Local LYM."

Write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$. Firstly, we have $\dfrac{|\mathcal{A}_n|}{\binom{n}{n}} \leqslant 1$.

Now, $\partial\mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are disjoint subsets of $X^{(n-1)}$ (as we are in an antichain).

So $\dfrac{|\partial\mathcal{A}_n|}{\binom{n}{n-1}} + \dfrac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \dfrac{|\partial\mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leqslant 1$. So by Local LYM, $\dfrac{|\mathcal{A}_n|}{\binom{n}{n}} + \dfrac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \leqslant 1$.

Also, $\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})$ and $\mathcal{A}_{n-2}$ are disjoint, so $\dfrac{|\partial(\partial\mathcal{A}_n \cup \mathcal{A}_{n-1})|}{\binom{n}{n-2}} + \dfrac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leqslant 1$.

So by Local LYM, $\dfrac{|\partial\mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \dfrac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leqslant 1$, and thus $\dfrac{|\mathcal{A}_n|}{\binom{n}{n}} + \dfrac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \dfrac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \leqslant 1$.

Keep going. We get $\dfrac{|\mathcal{A}_n|}{\binom{n}{n}} + \ldots + \dfrac{|\mathcal{A}_0|}{\binom{n}{0}} \leqslant 1$. $\square$

Equality in LYM? We must have had equality in each application of Local LYM. Hence, for the greatest r with $\mathcal{A}_r \neq \emptyset$, we must have $\mathcal{A}_r = X^{(r)}$, whence $\mathcal{A} = X^{(r)}$ as $\mathcal{A}$ is an antichain.

Conclusion: equality in LYM $\Leftrightarrow \mathcal{A} = X^{(r)}$, some r.

In particular, equality in Sperner $\Leftrightarrow \begin{cases} \mathcal{A} = X^{(n/2)} \ (n \text{ even}) \\ \mathcal{A} = X^{(\lfloor n/2 \rfloor)} \text{ or } X^{(\lceil n/2 \rceil)} \ (n \text{ odd}) \end{cases}$.

Proof 2. Choose, uniformly at random, a maximal chain $\mathcal{C}$ (i.e., $C_0 \subset C_1 \subset \ldots \subset C_n$, with $|C_i| = i$ for all i).

For a fixed r-set A, we have $P(A \in \mathcal{C}) = \dfrac{1}{\binom{n}{r}}$ (as all r-sets equally likely to be in $\mathcal{C}$).

So $P(\mathcal{C} \text{ meets } \mathcal{A}_r) = \dfrac{|\mathcal{A}_r|}{\binom{n}{r}}$ (as events are disjoint).

So $P(\mathcal{C} \text{ meets } \mathcal{A}) = \displaystyle\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}}$ (as events are disjoint). Thus $\displaystyle\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leqslant 1$. $\square$

Equivalently, the number of maximal chains $= n!$, and the number of maximal chains containing a given r-set $= r!(n-r)!$. So $\sum_{r=0}^{n} |\mathcal{A}_r|\, r!(n-r)! \leqslant n!$.

Shadows

If $\mathcal{A} \subset X^{(r)}$, we know $|\partial\mathcal{A}| \geqslant |\mathcal{A}|\dfrac{r}{n-r+1}$, but equality is rare (only for $\mathcal{A} = \emptyset$ or $X^{(r)}$).

What happens in between?

How would we choose $\mathcal{A} \subset X^{(r)}$, with $|\mathcal{A}|$ given, to minimise $|\partial\mathcal{A}|$? It's believable that for $|\mathcal{A}| = \binom{k}{r}$ for some k, we should take $\mathcal{A} = [k]^{(r)}$, yielding $\partial\mathcal{A} = [k]^{(r-1)}$.

What if $\binom{k}{r} < |\mathcal{A}| < \binom{k+1}{r}$? It's believable that we would take $\mathcal{A}$ to be $[k]^{(r)}$ and some extra sets from $[k+1]^{(r)}$.

E.g., for $|\mathcal{A}| = \dbinom{7}{3} + \dbinom{4}{2}$ in $X^{(3)}$, we would try $\mathcal{A} = [7]^{(3)} \cup \big\{A \cup \{8\} : A \in [4]^{(2)}\big\}$.

Two total orderings on $X^{(r)}$

We're given $A, B \in X^{(r)}$, say $A = \{a_1, \ldots, a_r\}$ where $a_1 < \ldots < a_r$, and $B = \{b_1, \ldots, b_r\}$ where $b_1 < \ldots < b_r$.

Say that $A < B$ in the **lexicographic** or **lex** order if there exists i with $a_i < b_i$ and $a_j = b_j$ for all $j < i$. Equivalently, $A < B$ if $a_i < b_i$ where $i = \min\{j : a_j \neq b_j\}$.

"Use small elements if possible (like a dictionary)."

E.g., lex on $[4]^{(2)}$: $12, 13, 14, 23, 24, 34$.

lex on $[6]^{(3)}$: $123, 124, 125, 126, 134, 135, 136, 145, 146, 156,$
$234, 235, 236, 245, 246, 256, 345, 346, 356, 456.$

Say that $A < B$ in the **colexicographic** or **colex** order if there exists i with $a_i < b_i$ and $a_j = b_j$ for all $j > i$. Equivalently, $A < B$ is $a_i < b_i$ where $i = \max\{j : a_j \neq b_j\}$.

"Don't use large elements."

Equivalently, $A < B$ if $\sum_{i \in A} 2^i < \sum_{i \in B} 2^i$.

E.g., colex on $[4]^{(2)}$: $12, 13, 23, 14, 24, 34$.

colex on $[6]^{(3)}$: $123, 124, 134, 234, 125, 135, 235, 145, 245, 345,$
$126, 136, 236, 146, 246, 346, 156, 256, 356, 456.$

Note that $[m]^{(r)}$ is an initial segment (i.e., first t elements, for some t) of $[m+1]^{(r)}$ in colex. Thus we could view colex as an enumeration of $\mathbb{N}^{(r)}$. (This is false for lex.)

Aim. Initial segments of colex have smallest shadow ∂.

I.e., if $\mathcal{A} \subset X^{(r)}$ and $\mathcal{C} \subset X^{(r)}$ is the initial segment of colex with $|\mathcal{C}| = |\mathcal{A}|$, then $|\partial\mathcal{C}| \leqslant |\partial\mathcal{A}|$.

In particular, $|\mathcal{A}| = \dbinom{k}{r} \;\Rightarrow\; |\partial\mathcal{A}| \geqslant \dbinom{k}{r-1}$.

Compressions

Given $\mathcal{A} \subset X^{(r)}$, we would like to replace $\mathcal{A}$ by some $\mathcal{A}' \subset X^{(r)}$, where:

$$\text{(i) } |\mathcal{A}'| = |\mathcal{A}|, \quad \text{(ii) } |\partial\mathcal{A}'| \leqslant |\partial\mathcal{A}|, \quad \text{(iii) } \mathcal{A}' \text{ “looks more like” } \mathcal{C} \text{ than } \mathcal{A} \text{ did.}$$

We would like to find several such “compression” operations: $\mathcal{A} \to \mathcal{A}' \to \mathcal{A}'' \to \cdots \to \mathcal{B}$, for which either $\mathcal{B} = \mathcal{C}$, or $\mathcal{B}$ is so similar to $\mathcal{C}$ that we can see directly that $|\partial\mathcal{B}| \geqslant |\partial\mathcal{C}|$.

“Colex prefers 1 to 2” inspires the following.

Fix $1 \leqslant i < j \leqslant n$. The ij**-compression** is defined as follows.

For $A \in X^{(r)}$, let $C_{ij}(A) = \begin{cases} A \cup i - j & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}$.

And for $\mathcal{A} \subset X^{(r)}$, let $C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$.

I.e., “replace j by i if possible”.

E.g., if $\mathcal{A} = \{123, 124, 135, 235, 245, 367\}$ then $C_{12}(\mathcal{A}) = \{123, 124, 135, 235, 145, 367\}$.

We clearly have $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$. Say that $\mathcal{A}$ is ij**-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}$.

Lemma 4. Let $\mathcal{A} \subset X^{(r)}$. Then for any $1 \leqslant i < j \leqslant n$, we have $|\partial C_{ij}(\mathcal{A})| \leqslant |\partial\mathcal{A}|$.

Proof. Write $\mathcal{A}'$ for $C_{ij}(\mathcal{A})$. We'll show that for each $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we have $j \notin B$, $i \in B$, and $B \cup j - i \in \partial\mathcal{A} - \partial\mathcal{A}'$. (Then done.)

We have $B \cup x \in \mathcal{A}'$ for some x, and $B \cup x \notin \mathcal{A}$, so $i \in B \cup x$, $j \notin B \cup x$. We cannot have $i = x$, as then $B \cup i \in \mathcal{A}'$, so either $B \cup i$ or $B \cup j$ is in $\mathcal{A}$, so $B \in \partial\mathcal{A}$, contradiction. Thus we know $i \in B$ and $j \notin B$. Since $B \cup x \cup j - i \in \mathcal{A}$, we certainly have $B \cup j - i \in \partial\mathcal{A}$.

Claim. $B \cup j - i \notin \partial\mathcal{A}'$.

Proof of claim. Suppose $(B \cup j - i) \cup y \in \mathcal{A}'$ for some y. Then we cannot have $y = i$, or else $B \cup j \in \mathcal{A}'$, whence $B \cup j \in \mathcal{A}$ and so $B \in \partial\mathcal{A}$, contradiction.

So $i \notin (B \cup j - i) \cup y$ and $j \in (B \cup j - i) \cup y$. And so $(B \cup j - i) \cup y$ and $B \cup y$ are both in $\mathcal{A}$ (by definition of $\mathcal{A}'$). So $B \in \partial\mathcal{A}$, contradiction $\square$

Remark. We actually showed $\partial C_{ij}(\mathcal{A}) \subset C_{ij}(\partial\mathcal{A})$.

Say that $\mathcal{A}$ is **left-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}$ for all $i < j$.

Corollary 5. Let $\mathcal{A} \subset X^{(r)}$. Then there exists $\mathcal{B} \subset X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$, $|\partial\mathcal{B}| \leqslant |\partial\mathcal{A}|$, and $\mathcal{B}$ is left-compressed.

Proof. Define a sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots \subset X^{(r)}$ as follows. Set $\mathcal{A}_0 = \mathcal{A}$. Having chosen $\mathcal{A}_0, \ldots, \mathcal{A}_k$, if $\mathcal{A}_k$ is left-compressed, stop the sequence with $\mathcal{A}_k$. If not, choose $i < j$ with $\mathcal{A}_k$ not ij-compressed, and set $\mathcal{A}_{k+1} = C_{ij}(\mathcal{A}_k)$. This must terminate – e.g., as the function $\sum_{A \in \mathcal{A}_k} \sum_{x \in A} x$ is decreasing in k. The system $\mathcal{B} = \mathcal{A}_k$ has $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leqslant |\partial\mathcal{A}|$ by Lemma 4. $\square$

Remarks. 1. Alternatively, among all $\mathcal{B} \subset X^{(r)}$ with $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leqslant |\partial\mathcal{A}|$, choose one with minimal $\sum_{A\in\mathcal{B}} \sum_{x\in A} x$.

2. It is possible to apply each C_{ij} at most once if we choose the order sensibly.

Any initial segment of colex is left-compressed. However, the converse is false, for example $\mathcal{A} = \{123, 124, 125, 126, 127\}$.

"Colex prefers 23 to 14" inspires the following.

For $U, V \subset X$ with $|U| = |V|$ and $U \cap V = \emptyset$, define the UV**-compression** as follows.

For $A \in X^{(r)}$, let $C_{UV}(A) = \begin{cases} A \cup U - V & \text{if } V \subset A,\ A \cap U = \emptyset \\ A & \text{otherwise} \end{cases}$.

"Replace Vs with Us."

And for $\mathcal{A} \subset X^{(r)}$, let $C_{UV}(\mathcal{A}) = \{C_{UV}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{UV}(A) \in \mathcal{A}\}$.

E.g., if $\mathcal{A} = \{123, 145, 147, 234, 235, 267\}$ then $C_{23,14}(\mathcal{A}) = \{123, 145, 237, 234, 235, 267\}$.

Note that $|C_{UV}(\mathcal{A})| = |\mathcal{A}|$, and that $C_{\{i\},\{j\}}(\mathcal{A}) = C_{ij}(\mathcal{A})$.

Unfortunately, we can have $|\partial C_{UV}(\mathcal{A})| > |\partial\mathcal{A}|$. E.g., let $\mathcal{A} = \{147, 478\}$. Then $|\partial\mathcal{A}| = 5$, but $C_{23,14}(\mathcal{A}) = \{237, 478\}$ with $|\partial C_{23,14}(\mathcal{A})| = 6$.

Say $\mathcal{A}$ is UV**-compressed** if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Lemma 6. Let $U, V \subset X$ be disjoint, with $|U| = |V|$. Let $\mathcal{A} \subset X^{(r)}$. Suppose that for all $u \in U$, there is $v \in V$ such that $\mathcal{A}$ is $(U-u, V-v)$-compressed. Then $|\partial C_{UV}(\mathcal{A})| \leqslant |\partial\mathcal{A}|$.

Proof. Write $\mathcal{A}'$ for $C_{UV}(\mathcal{A})$. For any $B \in \partial\mathcal{A}' - \partial\mathcal{A}$, we'll show that $U \subset B$, $V \cap B = \emptyset$, and $B \cup V - U \in \partial\mathcal{A} - \partial\mathcal{A}'$. (Then done.)

We have $B \cup x \in \mathcal{A}'$ for some x, and $B \cup x \notin \mathcal{A}$. Thus $U \subset B \cup x$, $V \cap (B \cup x) = \emptyset$, and $B \cup x \cup V - U \in \mathcal{A}$. So certainly $V \cap B = \emptyset$.

Also, $x \notin U$. For if $x \in U$, then there is $y \in V$ with $\mathcal{A}$ being $(U - x, V - y)$-compressed. But $B \cup x \cup V - U \in \mathcal{A}$, so then $B \cup y \in \mathcal{A}$, contradicting $B \notin \partial\mathcal{A}$. Thus $U \subset B$.

And we have $B \cup V - U \in \partial\mathcal{A}$, because $B \cup x \cup V - U \in \mathcal{A}$.

Suppose $B \cup V - U \in \partial\mathcal{A}'$. So $(B \cup V - U) \cup w \in \mathcal{A}'$ for some w.

If $w \notin U$, then because $(B \cup V - U) \cup w \in \mathcal{A}'$, we must have $(B \cup V - U) \cup w$ and $B \cup w$ in $\mathcal{A}$ (by definition of C_{UV}), contradicting $B \notin \partial\mathcal{A}$.

If $w \in U$, then we have that $\mathcal{A}$ is $(U - w, V - z)$-compressed for some $z \in V$. So from $(B \cup V - U) \cup w \in \mathcal{A}$ (which is true as it contains V, so cannot have moved), we obtain $B \cup z \in \mathcal{A}$, a contradiction. $\square$

Remark. We actually showed that $\partial C_{UV}(\mathcal{A}) \subset C_{UV}(\partial\mathcal{A})$.

Theorem 7 (Kruskal-Katona Theorem). Let $\mathcal{A} \subset X^{(r)}$ $(1 \leqslant r \leqslant n)$, and let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial\mathcal{A}| \geqslant |\partial\mathcal{C}|$.

In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial\mathcal{A}| \geqslant \binom{k}{r-1}$.

Proof. Let $\Gamma = \{\, (U,V) : U,V \subset X,\ |U| = |V| > 0,\ U \cap V = \emptyset,\ \max V > \max U \,\}$.

Define a sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ as follows. Set $\mathcal{A}_0 = \mathcal{A}$. Having chosen $\mathcal{A}_0, \ldots, \mathcal{A}_k$, if $\mathcal{A}_k$ is UV-compressed for all $(U,V) \in \Gamma$, stop the sequence with $\mathcal{A}_k$. If not, choose $(U,V) \in \Gamma$, with $|U|$ minimal, such that $\mathcal{A}_k$ is not UV-compressed, and set $\mathcal{A}_{k+1} = C_{UV}(\mathcal{A}_k)$.

For each $u \in U$, setting $v = \min V$, we have $(U-u, V-v) \in \Gamma \cup \{(\emptyset,\emptyset)\}$. So $\mathcal{A}_k$ is $(U-u, V-v)$-compressed. Thus by Lemma 6, we have $|\partial\mathcal{A}_{k+1}| \leqslant |\partial\mathcal{A}_k|$. Continue.

This sequence must terminate, as $\sum_{A \in \mathcal{A}_k} \sum_{i \in A} 2^i$ is decreasing. The final term, $\mathcal{B} = \mathcal{A}_k$, satisfies: $|\mathcal{B}| = |\mathcal{A}|$, $|\partial\mathcal{B}| \leqslant |\partial\mathcal{A}|$, and $\mathcal{B}$ is UV-compressed for all $(U,V) \in \Gamma$.

Claim. $\mathcal{B} = \mathcal{C}$.

Proof of claim. Suppose not. Then there exists $A, B \in X^{(r)}$ with $A < B$ in colex, and $A \notin \mathcal{B}$, $B \in \mathcal{B}$. Set $U = A - B$, $V = B - A$. We have $\max V > \max U$ (as $A < B$ in colex). Thus $(U,V) \in \Gamma$. So $B \in \mathcal{B} \Rightarrow A \in \mathcal{B}$. Contradiction. □

Remarks. 1. Equivalently, if $|\mathcal{A}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \ldots + \binom{k_s}{s}$, where $k_r > k_{r-1} > \ldots > k_s$ and $s > 0$, then $|\partial\mathcal{A}| \geqslant \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \ldots + \binom{k_s}{s-1}$.

2. The proof used only Lemma 6, and not Lemma 4 or Corollary 5.

3. Equality? We can check that if $|\mathcal{A}| = \binom{k}{r}$ and we have equality in Kruskal-Katona (i.e., $|\partial\mathcal{A}| = \binom{k}{r-1}$), then $\mathcal{A} = Y^{(r)}$ for some $Y \subset X$ with $|Y| = k$. But it is false in general that if equality holds (i.e., $|\partial\mathcal{A}| = |\partial\mathcal{C}|$) then $\mathcal{A}$ is isomorphic to $\mathcal{C}$.

($\mathcal{A}$, $\mathcal{B}$ are **isomorphic** if there is a bijection $f : X \to X$ sending $\mathcal{A}$ to $\mathcal{B}$.)

For $\mathcal{A} \subset X^{(r)}$ $(0 \leqslant r \leqslant n-1)$, the **upper shadow** of $\mathcal{A}$ is $\partial^+\mathcal{A} \subset X^{(r+1)}$, given by $\partial^+\mathcal{A} = \{A \cup x : A \in \mathcal{A}, x \in X, x \notin A\}$.

Note $A < B$ in colex on $X^{(r)} \Leftrightarrow A^c < B^c$ in lex on $X^{(n-r)}$ with the ground set order reversed.

Corollary 8. Let $\mathcal{A} \subset X^{(r)}$ $(0 \leqslant r \leqslant n-1)$ and let $\mathcal{C}$ be the initial segment of lex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^+\mathcal{C}| \leqslant |\partial^+\mathcal{A}|$.

Proof. Take complements. □

The shadow of an initial segment of colex on $X^{(r)}$ is an initial segment of colex on $X^{(r-1)}$ (for if $\mathcal{C} = \{A \in X^{(r)} : A \leqslant a_1 \ldots a_r\}$ then $\partial\mathcal{C} = \{B \in X^{(r-1)} : B \leqslant a_2 \ldots a_r\}$), so we have:

Corollary 9. Let $\mathcal{A} \subset X^{(r)}$, and let $\mathcal{C}$ be the initial segment of colex on $X^{(r)}$ with $|\mathcal{C}| = |\mathcal{A}|$. Then $|\partial^t\mathcal{A}| \geqslant |\partial^t\mathcal{C}|$ for all $1 \leqslant t \leqslant r$. In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial^t\mathcal{A}| \geqslant \binom{k}{r-t}$.

Proof. If $|\partial^t\mathcal{A}| \geqslant |\partial^t\mathcal{C}|$ then $|\partial^{t+1}\mathcal{A}| \geqslant |\partial^{t+1}\mathcal{C}|$ by Kruskal-Katona. □

Intersecting Families

Say that $\mathcal{A} \subset \mathbb{P}(X)$ is **intersecting** if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$.

How large can an intersecting family be?

We can have $|\mathcal{A}| = 2^{n-1}$, e.g. $\mathcal{A} = \{A \subset X : 1 \in A\}$.

Proposition 10. Let $\mathcal{A} \subset \mathbb{P}(X)$ be intersecting. Then $|\mathcal{A}| \leqslant 2^{n-1}$.

Proof. For any $A \in \mathbb{P}(X)$, at most one of A, A^c can belong to $\mathcal{A}$. $\square$

Remark. There are many extremal systems. E.g., $\{A \in \mathbb{P}(X) : |A| > n/2\}$ for n odd.

What if we restrict to $\mathcal{A} \subset X^{(r)}$?

If $r > n/2$, silly: we can take $\mathcal{A} = X^{(r)}$.

If $r = n/2$, take one of each complementary pair A, A^c – gives $\frac{1}{2}\binom{n}{r}$. (Optimal, as we can never take both of A and A^c.)

So assume $r < n/2$.

Obvious guess: $\mathcal{A} = \{A \in X^{(r)} : 1 \in A\}$. This has $|\mathcal{A}| = \dfrac{\binom{n-1}{r-1}}{} = \dfrac{r}{n}\binom{n}{r}$.

We could also try, e.g., $\mathcal{B} = \left\{A \in X^{(r)} : |A \cap \{1,2,3\}| \geqslant 2\right\}$.

E.g., in $[8]^{(3)}$, $|\mathcal{A}| = 21$ and $|\mathcal{B}| = 1 + 3.5 = 16 < 21$.

Theorem 11 (Erdős-Ko-Rado Theorem). Let $r < n/2$, and $\mathcal{A} \subset X^{(r)}$ be intersecting.

Then $|\mathcal{A}| \leqslant \dbinom{n-1}{r-1}$.

Proof 1. "Bubble down with Kruskal-Katona."

For any $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$. I.e., $A \not\subset B^c$. Thus, writing $\overline{\mathcal{A}} = \{B^c : B \in \mathcal{A}\} \subset X^{(n-r)}$, we have that $\mathcal{A}$ and $\partial^{n-2r}\overline{\mathcal{A}}$ are *disjoint* subsets of $X^{(r)}$.

Suppose that $|\mathcal{A}| > \dbinom{n-1}{r-1}$.

Then $|\overline{\mathcal{A}}| > \dbinom{n-1}{r-1} = \dbinom{n-1}{n-r}$. So $|\partial^{n-2r}\overline{\mathcal{A}}| \geqslant \dbinom{n-1}{r}$, by Corollary 9.

Thus $|\mathcal{A}| + |\partial^{n-2r}\overline{\mathcal{A}}| > \dbinom{n-1}{r-1} + \dbinom{n-1}{r} = \dbinom{n}{r}$. Contradiction. $\square$

Note. Numbers *had* to work out, given that if $\mathcal{A} = \{A \in X^{(r)} : 1 \in A\}$, we have equality, and $\mathcal{A}$, $\partial^{n-2r}\overline{\mathcal{A}}$ partition $X^{(r)}$.

Proof 2. Consider a cyclic ordering of X, i.e. a bijection $c : \mathbb{Z}_n \to X$. How many $A \in \mathcal{A}$ are *intervals* (blocks of r consecutive elements) in this ordering? Answer: at most r. Indeed, suppose $\{c_1, c_2, \ldots, c_r\} \in \mathcal{A}$. Then for each $1 \leqslant i \leqslant r-1$, at most one of $\{c_{i-r+1}, \ldots, c_i\}$ and $\{c_{i+1}, \ldots, c_{i+r}\}$ belongs to $\mathcal{A}$.

Also, each r-set is an interval in precisely $n\, r!\, (n-r)!$ of the $n!$ cyclic orderings.

where inside interval outside interval

Thus, $|\mathcal{A}|\, n\, r!\, (n-r)! \leqslant n!\, r$, and so $|\mathcal{A}| \leqslant \dfrac{r}{n}\dbinom{n}{r}$. $\square$.

Notes. 1. Equivalently, we are double-counting the edges in the bipartite graph with vertex classes $\mathcal{A}$ and the cyclic orderings, where we join $A \in \mathcal{A}$ to a cyclic ordering c if A is an interval in c.

2. This method is called **averaging**, or **Katona's method**.

Equality in Erdős-Ko-Rado

Want: if $\mathcal{A} \subset X^{(r)}$ is intersecting ($r < n/2$) and $|\mathcal{A}| = \binom{n-1}{r-1}$, then $\mathcal{A} = \{A \in X^{(r)} : i \in A\}$, some i.

From proof 2, for each cyclic ordering c, we have r intervals in $\mathcal{A}$. These must be all of the r intervals containing a point $x(c)$, say. Our task is to show that $x(c) = x(c')$ for all c, c'.

Fix a cyclic ordering c, and say $x(c) = c_0$.

Thus $\{c_0, \ldots, c_{r-1}\}$, $\{c_{-r+1}, \ldots, c_0\} \in \mathcal{A}$, and $\{c_1, \ldots, c_r\}$, $\{c_{-r}, \ldots, c_{-1}\} \notin \mathcal{A}$.

Let c' be obtained from c by swapping two adjacent elements $\neq c_0$. Say we swap c_i and c_{i+1}. Wlog (else reflect) $i \geqslant (n-1)/2$. (Not $n/2$, in case $i = (n-1)/2$, $i+1 = (n+1)/2$.)

Then $\{c_0, \ldots, c_{r-1}\}$ is an interval of c', and $\{c_1, \ldots, c_r\}$ is an interval of c' (unless $r = (n-1)/2$ and $i = (n-1)/2$, in which case $\{c_1, \ldots, c_{r-1}, c_{r+1}\}$ is an interval of c').

But $\{c_0, \ldots, c_{r-1}\} \in \mathcal{A}$ and $\{c_1, \ldots, c_r\} \notin \mathcal{A}$ (and $\{c_1, \ldots, c_{r-1}, c_{r+1}\} \notin \mathcal{A}$, as disjoint from $\{c_{-r+1}, \ldots, c_0\} \in \mathcal{A}$).

Thus $x(c') = c_0$. Hence $x(c) = x(c')$ for all c, c', as required. $\square$

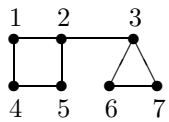
Chapter 2 : Isoperimetric Inequalities

"How small can the boundary of a set of given size be?"

E.g., – among subsets of $\mathbb{R}^2$ of given area, the disc has smallest perimeter.
– among subsets of $\mathbb{R}^3$ of given volume, solid sphere has smallest surface area.
– among subsets of S^2 of given area, circular cap has smallest perimeter.

For a graph G and $A \subset V(G)$, the **boundary** of A is:

$$b(A) = \{x \in G : x \notin A, xy \in E(G) \text{ for some } y \in A\}.$$

E.g., 

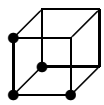
– if $A = \{1, 2, 5\}$ then $b(A) = \{3, 4\}$.

An **isoperimetric inequality** on G is an inequality of the form: $|b(A)| \geqslant f(|A|)$ for all $A \subset V(G)$.

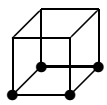
Equivalently, minimise the **neighbourhood** of A, $N(A) = A \cup b(A) = \{x : d(x, A) \leqslant 1\}$, where $d =$ usual graph distance. Often a good guess is $B(x, r) = \{y : d(x, y) \leqslant r\}$.

What happens in Q_n?

E.g., $|A| = 4$ in Q_3.



$|b(A)| = 3$



$|b(A)| = 4$

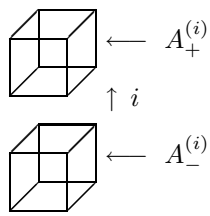
Guess: $B(\emptyset, r) = X^{(\leqslant r)} = X^{(0)} \cup \ldots \cup X^{(r)}$ are best.

What if $|X^{(\leqslant r)}| < |A| < |X^{(\leqslant r+1)}|$? Guess: take $A = X^{(\leqslant r)} \cup B$, some $B \subset X^{(r+1)}$. Then $b(A) = (X^{(r+1)} - B) \cup \partial^+ B$, so we'd take B an initial segment of lex, by Kruskal-Katona.

This suggests the following. The **simplicial ordering** on $\mathbb{P}(X)$ is defined by: $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x < y$ in lex.

Aim. Initial segments of the simplicial order are best.

Let $A \subset Q_n$, and let $1 \leqslant i \leqslant n$.



The i**-sections** of A are the sets $A_+^{(i)}, A_-^{(i)} \subset \mathbb{P}(X - i)$ given by

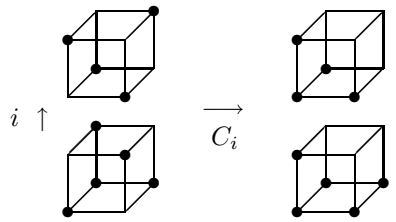
$$A_-^{(i)} = \{x \in A : i \notin x\}, \quad A_+^{(i)} = \{x - i : x \in A, i \in x\}.$$

The i**-compression** $C_i(A)$ of A is defined by giving its i-sections:

$\big(C_i(A)\big)_+^{(i)}$ is the initial segment of the simplicial order on $\mathbb{P}(X - i)$ of size $|A_+^{(i)}|$.

$\big(C_i(A)\big)_-^{(i)}$ is the initial segment of the simplicial order on $\mathbb{P}(X - i)$ of size $|A_-^{(i)}|$.

E.g.,



Note that $|C_i(A)| = |A|$, and also that $C_i(A)$ "looks more like" a Hamming ball than A did. (A **Hamming ball** is a set $A \subset Q_n$ with $X^{(\leqslant r)} \subset A \subset X^{(\leqslant r+1)}$, some r.)

Say A is i**-compressed** if $C_i(A) = A$.

Theorem 1 (Harper's Theorem). Let $A \subset Q_n$ and let C be the initial segment of the simplicial order with $|C| = |A|$. Then $|N(A)| \geqslant |N(C)|$.

In particular, if $|A| \geqslant \sum_{i=0}^{r} \binom{n}{i}$ then $|N(A)| \geqslant \sum_{i=0}^{r+1} \binom{n}{i}$.

Remarks. 1. If we knew that A was a Hamming ball, we would be done by Kruskal-Katona.

2. Conversely, Theorem 1 implies Kruskal-Katona: given some $B \subset X^{(r)}$, apply Theorem 1 to $A = X^{(<r)} \cup B$.

Proof. Induction on n. Done if $n = 1$. Given $A \subset Q_n$ $(n > 1)$, $1 \leqslant i \leqslant n$, claim:

Claim. $|N(C_i(A))| \leqslant |N(A)|$.

Proof of claim. Write B for $C_i(A)$. We have $|N(A)| = |N(A_-) \cup A_+| + |N(A_+) \cup A_-|$ and $|N(B)| = |N(B_-) \cup B_+| + |N(B_+) \cup B_-|$.

Now, $|B_+| = |A_+|$, by definition of B, and $|N(B_-)| \leqslant |N(A_-)|$, by induction. But B_+ is an initial segment of the simplicial order, and so is $N(B_-)$ (as the neighbourhood of an initial segment is an initial segment). Thus B_+ and $N(B_-)$ are nested (i.e., one is contained in the other).

And so certainly $|N(B_-) \cup B_+| \leqslant |N(A_-) \cup A_+|$.

Similarly, we have $|N(B_+) \cup B_-| \leqslant |N(A_+) \cup A_-|$. Thus $|N(B)| \leqslant |N(A)|$.

Define $A_0, A_1, \ldots$ as follows. Set $A_0 = A$. Having chosen $A_0, \ldots, A_k$, if A_k is i-compressed for all i then stop the sequence with A_k.

If not, choose i with $C_i(A_k) \neq A_k$, set $A_{k+1} = C_i(A_k)$ and continue. This must terminate, as $\sum_{x \in A_k} f(x)$ is decreasing, where $f(x)$ denotes the position of x in the simplicial order.

Then $B = A_k$ satisfies: $|B| = |A|$, $|N(B)| \leqslant |N(A)|$, and B is i-compressed for all i.

Does B being i-compressed for all i imply that B is an initial segment of the simplicial order? (If so then we are done, and $B = C$.)

Answer: no, e.g. $\{\emptyset, 1, 2, 12\} \subset Q_3$. However, we are done by Lemma 2 below.

Lemma 2. Let $B \subset Q_n$ be i-compressed for all i, but not an initial segment of the simplicial order. Then for n odd, say $n = 2k+1$, we have

$$B = X^{(\leqslant k)} - \{k+2, k+3, \ldots, 2k+1\} \cup \{1, 2, 3, \ldots, k+1\},$$

and for n even, say $n = 2k$, we have

$$B = X^{(<k)} \cup \{x \in X^{(k)} : 1 \in x\} - \{1, k+2, k+3, \ldots, 2k\} \cup \{2, 3, \ldots, k+1\}.$$

(Then done, as in each case we have $|N(B)| \geqslant |N(C)|$.)

Proof. We have some $x < y$ with $x \notin B$, $y \in B$. For each i, we cannot have $i \in x$ and $i \in y$, as B is i-compressed. Similarly, we cannot have $i \notin x$ and $i \notin y$. Thus $x = y^c$.

So for each $x \notin B$, we have at most one later point $y \in B$ (namely x^c), and for each $y \in B$, we have at most one earlier point $x \notin B$ (namely y^c).

So $B = \{z : z \leqslant y\} - \{x\}$, where x is the predecessor of y, and $x = y^c$.

If n is odd, we must have x the last $\frac{n-1}{2}$ set.
If n is even, we must have x the last $\frac{n}{2}$ set containing 1. So done. □

Notes. 1. Can also prove Harper's Theorem by UV-compressions.

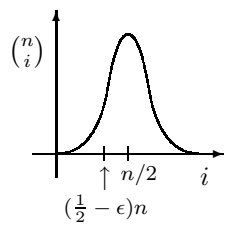
2. Can also use these "codimension-1" compressions to prove Kruskal-Katona directly.

For $A \subset Q_n$, the **t-neighbourhood** of A is $N^t(A) = A_{(t)} = \{x \in Q_n : d(x, A) \leqslant t\}$.

Corollary 3. Let $A \subset Q_n$ with $|A| \geqslant \sum_{i=0}^{r} \binom{n}{i}$. Then for $1 \leqslant t \leqslant n-r$, $|N^t(A)| \geqslant \sum_{i=0}^{r+t} \binom{n}{i}$.

Proof. Theorem 1, plus induction. □

To get a feel for the strength of Corollary 3, we'll need some estimates on $\sum_{i=0}^{r} \binom{n}{i}$, etc.



Proposition 4. Let $0 < \epsilon < 1/4$. Then $\displaystyle\sum_{i=0}^{\lfloor(\frac{1}{2}-\epsilon)n\rfloor} \binom{n}{i} \leqslant \frac{1}{\epsilon} e^{-\epsilon^2 n/2}\, 2^n$.

Note. This is an *exponentially small* fraction of 2^n (for ϵ fixed, $n \to \infty$).

("This is $\sim \epsilon\sqrt{n}$ standard deviations from the mean $n/2$.")

Proof. $\binom{n}{i-1} = \binom{n}{i}\frac{i}{n-i+1}$. So for $i \leqslant \lfloor(\frac{1}{2}-\epsilon)n\rfloor$, we have

$$\frac{\binom{n}{i-1}}{\binom{n}{i}} = \frac{i}{n-i+1} \leqslant \frac{(\frac{1}{2}-\epsilon)n}{(\frac{1}{2}+\epsilon)n} = \frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon} = 1 - \frac{2\epsilon}{\frac{1}{2}+\epsilon} \leqslant 1-2\epsilon.$$

So, $\displaystyle\sum_{i=0}^{\lfloor(\frac{1}{2}-\epsilon)n\rfloor}\binom{n}{i} \leqslant \frac{1}{2\epsilon}\binom{n}{\lfloor(\frac{1}{2}-\epsilon)n\rfloor}$ (sum of a GP).

Similarly, $\displaystyle\binom{n}{\lfloor(\frac{1}{2}-\epsilon)n\rfloor} \leqslant \binom{n}{\lfloor(\frac{1}{2}-\frac{\epsilon}{2})n\rfloor}\left(1-2\,\frac{\epsilon}{2}\right)^{\epsilon n/2-1}$ (same argument, $\epsilon \to \epsilon/2$)

which is $\leqslant 2^n\,2\,(1-\epsilon)^{\epsilon n/2} \leqslant 2^n\,2\,e^{-\epsilon^2 n/2}$, as $1-\epsilon \leqslant e^{-\epsilon}$.

Thus, $\displaystyle\sum_{i=0}^{\lfloor(\frac{1}{2}-\epsilon)n\rfloor}\binom{n}{i} \leqslant \frac{1}{2\epsilon}\,2\,e^{-\epsilon^2 n/2}\,2^n$. □

Theorem 5. Let $A \subset Q_n$, $0 < \epsilon < 1/4$. Then $\dfrac{|A|}{2^n} \geqslant \dfrac{1}{2} \;\Rightarrow\; \dfrac{|A_{(\epsilon n)}|}{2^n} \geqslant 1 - \dfrac{2}{\epsilon}\,e^{-\epsilon^2 n/2}$.

"$\frac{1}{2}$-sized sets have exponentially large ϵn-neighbourhoods."

Proof. It is enough to show that, assuming ϵn is an integer, we have $\dfrac{|A_{(\epsilon n)}|}{2^n} \geqslant 1 - \dfrac{1}{\epsilon}\,e^{-\epsilon^2 n/2}$.

We have $|A| \geqslant \displaystyle\sum_{i=0}^{\lceil n/2-1\rceil}\binom{n}{i}$, so by Harper we have $|A_{(\epsilon n)}| \geqslant \displaystyle\sum_{i=0}^{\lceil n/2+\epsilon n-1\rceil}\binom{n}{i}$.

So $|(A_{(\epsilon n)})^c| \leqslant \displaystyle\sum_{i=\lceil n/2+\epsilon n\rceil}^{n}\binom{n}{i} = \sum_{i=0}^{\lfloor n/2-\epsilon n\rfloor}\binom{n}{i} \leqslant \frac{1}{\epsilon}\,e^{-\epsilon^2 n/2}\,2^n$. □

Remark. The above is concerned with $\frac{1}{2}$-sized sets, but the same argument would show that

$$\frac{|A|}{2^n} \geqslant \frac{1}{\epsilon}\,e^{-\epsilon^2 n/2} \;\Rightarrow\; \frac{|A_{(2\epsilon n)}|}{2^n} \geqslant 1 - \frac{1}{\epsilon}\,e^{-\epsilon^2 n/2}.$$

Concentration of Measure

Say $f : Q_n \to \mathbb{R}$ is **Lipschitz** if $|f(x) - f(y)| \leqslant 1$ for all x, y adjacent.

A real number M is a **median** or **Lévy mean** for f if $|\{x : f(x) \leqslant M\}| \geqslant 2^{n-1}$ and $|\{x : f(x) \geqslant M\}| \geqslant 2^{n-1}$.

We are now ready to show that "every well-behaved function on Q_n is roughly constant nearly everywhere".

Theorem 6. Let f be a Lipschitz function on Q_n, with median M. Then

$$\frac{|\{x : |f(x) - M| \leqslant \epsilon n\}|}{2^n} \geqslant 1 - \frac{4}{\epsilon} e^{-\epsilon^2 n/2} \quad (0 < \epsilon < 1/4).$$

"If you look at the cube, all you see is the 'middle' layer."

Note. This is the "concentration of measure" phenomenon.

Proof. Let $A = \{x : f(x) \leqslant M\}$. Then $\frac{|A|}{2^n} \geqslant \frac{1}{2}$, so $\frac{|A_{(\epsilon n)}|}{2^n} \geqslant 1 - \frac{2}{\epsilon}\, e^{-\epsilon^2 n/2}$.

But $x \in A_{(\epsilon n)} \Rightarrow f(x) \leqslant M + \epsilon n$ (as f Lipschitz).

And so $\frac{|\{x : f(x) \leqslant M + \epsilon n\}|}{2^n} \geqslant 1 - \frac{2}{\epsilon}\, e^{-\epsilon^2 n/2}$.

Similarly, $\frac{|\{x : f(x) \geqslant M - \epsilon n\}|}{2^n} \geqslant 1 - \frac{2}{\epsilon}\, e^{-\epsilon^2 n/2}$. □

Let G be a graph of diameter D. (I.e., $D = \max\{d(x, y) : x, y \in G\}$.)

Define $\alpha(G, \epsilon) = \max\left\{1 - \frac{|A_{(\epsilon D)}|}{|G|} : A \subset G, \frac{|A|}{|G|} \geqslant \frac{1}{2}\right\}$.

So "$\alpha(G, \epsilon)$ small" says: $\frac{1}{2}$-sized sets have big ϵD-neighbourhoods.

A sequence $G_1, G_2, \ldots$ of graphs is a **Lévy family** if $\alpha(G_n, \epsilon) \to 0$ as $n \to \infty$, for each $\epsilon > 0$.

So, e.g., Theorem 5 tells us that $(Q_n)_{n=1}^{\infty}$ is a Lévy family, and even a **normal** Lévy family (meaning $\alpha(G_n, \epsilon)$ is exponentially small, for any ϵ).

So we again have concentration of measure ("Lipschitz functions on G_n are almost constant nearly everywhere") for any Lévy family. It turns out that many natural families of graphs are Lévy families. For example, the permutation groups S_n (made into a graph by: σ adjacent to τ if $\sigma^{-1}\tau$ is a transposition) form a Lévy family.

Similarly, we can define $\alpha(S, \epsilon)$ for any metric measure space S (of finite diameter and finite measure), so we can again define Lévy families. It turns out that many natural families of metric spaces form Lévy families.

Example. The sphere S^n. Two ingredients.

1. Isoperimetric inequality in S^n: $|A| = |C| \;\Rightarrow\; |A_{(\epsilon)}| \geqslant |C_{(\epsilon)}|$ where C is a circular cap. To prove this, we can use compressions, e.g. analogue of ij-compressions.

"Stamp on your set" – i.e., replace by the bottom point, if possible. (**2-point symmetrisation**)

2. Estimate: $\frac{1}{2}$-sized circular cap has angle $\frac{\pi}{2}$, so its ϵ-neighbourhood is a circular cap of angle $\frac{\pi}{2} + \epsilon$. But the surface area of everything else, $\int_{\epsilon}^{\pi/2} \cos^n t\, dt \to 0$ as $n \to \infty$, for any fixed ϵ.

We deduced concentration of measure from isoperimetric estimates. Conversely:

Proposition 7. Let G be a graph such that for every Lipschitz $f : G \to \mathbb{R}$ of median M, we have

$$\frac{|\{x \in G : |f(x) - M| > t\}|}{|G|} \leqslant \alpha \quad \text{(some fixed } t, \alpha\text{)}.$$

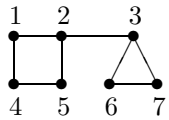
Then $\dfrac{|A|}{|G|} \geqslant \dfrac{1}{2} \;\Rightarrow\; \dfrac{|A_{(t)}|}{|G|} \geqslant 1 - \alpha.$

Proof. Let $f(x) = d(x, A)$. Then f is Lipschitz, and has 0 as a median (as $|A| \geqslant \frac{1}{2}|G|$). $\square$

Edge-Isoperimetric Inequalities

For a graph G, $A \subset V(G)$, the **edge-boundary** of A is

$$\partial_e A = \partial A = \{xy \in E(G) : x \in A, y \notin A\}.$$

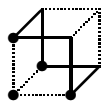
E.g., 

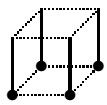
– if $A = \{1, 2, 5\}$ then $\partial A = \{14, 23, 45\}$.

An **edge-isoperimetric inequality** on G is an inequality of the form $A \subset G$, $|A| = m \Rightarrow |\partial A| \geqslant |f(m)|$.

In the cube

E.g., $|A| = 4$ in Q_3.


$|\partial A| = 6$


$|\partial A| = 4$

This suggests that subcubes are best.

The **binary** ordering on Q_n is given by: $x < y$ if $\max(x \triangle y) \in y$.

Equivalently, if $\sum_{i \in x} 2^i < \sum_{i \in y} 2^i$. "Go up in subcubes."

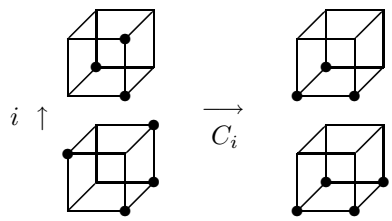
Aim. Initial segments of binary minimise ∂.

For $A \subset Q_n$ and $1 \leqslant i \leqslant n$, the i**-binary-compression** $B_i(A)$ is defined by giving its i-sections:

$$\big(B_i(A)\big)_+^{(i)} \text{ is the initial segment of binary on } \mathbb{P}(X - i) \text{ of size } A_+^{(i)}.$$

$$\big(B_i(A)\big)_-^{(i)} \text{ is the initial segment of binary on } \mathbb{P}(X - i) \text{ of size } A_-^{(i)}.$$

E.g.,



Clearly $|B_i(A)| = |A|$. Say A is **i-binary-compressed** if $B_i(A) = A$.

Theorem 8 (Edge-isoperimetric inequality in the cube). Let $A \subset Q_n$, and let C be the initial segment of binary with $|C| = |A|$. Then $|\partial \mathcal{C}| \leqslant |\partial A|$. In particular, $|A| = 2^k \;\Rightarrow\; |\partial A| \geqslant (n-k)2^k$.

Remark. Sometimes called "the theorem of Harper, Lindsey, Bernstein and Hart".

Proof. Induction on n. Done if $n = 1$. Given $A \subset Q_n$, $1 \leqslant i \leqslant n$.

Claim. $|\partial B_i(A)| \leqslant |\partial A|$.

Proof of claim. Write B for $B_i(A)$.

Have $|\partial A| = \underset{\substack{\uparrow \\ \text{edges in} \\ \text{the bottom}}}{|\partial(A_-)|} + \underset{\substack{\uparrow \\ \text{edges in} \\ \text{the top}}}{|\partial(A_+)|} + \underset{\substack{\uparrow \\ \text{edges} \\ \text{across}}}{|A_+ \triangle A_-|}$.

Similarly, $|\partial B| = |\partial(B_-)| + |\partial(B_+)| + |B_+ \triangle B_-|$.

Now, $|\partial(B_-)| \leqslant |\partial(A_-)|$ and $|\partial(B_+)| \leqslant |\partial(A_+)|$, by induction.

Also, $|B_+ \triangle B_-| \leqslant |A_+ \triangle A_-|$, because $|B_+| = |A_+|$, $|B_-| = |A_-|$, and the sets B_+, B_- are nested, as each is an initial segment of the binary order. Thus $|\partial B| \leqslant |\partial A|$.

Define $A_0, A_1, \ldots$ as follows. Set $A_0 = A$. Having chosen $A_0, \ldots, A_k$, if A_k is i-binary-compressed for all i, then stop. If not, choose i with $B_i(A_k) \neq A_k$ and set $A_{k+1} = B_i(A_k)$. This must terminate, e.g. because $\sum_{x \in A_k}$ (position of x in binary) is decreasing.

The final set $B = A_k$ satisfies: $|B| = |A|$, $|\partial B| \leqslant |\partial A|$, and B is i-binary-compressed for all i. As before, B need not be an initial segment, e.g. $\{\emptyset, 1, 2, 3\} \subset Q_3$. But we are done by Lemma 9 below.

Lemma 9. Let $B \subset Q_n$ be i-binary-compressed for all i, but not an initial segment of binary. Then $B = \mathbb{P}(X - n) \cup \{n\} - \{1, 2, 3, \ldots, n-1\}$.

(Then done, as certainly $|\partial B| \geqslant |\partial C|$ in this case.)

Proof. Have some $x < y$ with $x \notin B$, $y \in B$. Then for each i, cannot have $i \in x$, $i \in y$, or $i \notin x$, $i \notin y$, as B is i-binary-compressed. So $x = y^c$.

So for each $x \notin B$, at most one $y > x$ has $y \in B$ (namely x^c), and for each $y \in B$, at most one $x < y$ has $x \notin B$ (namely y^c).

Thus $B = \{z : z \leqslant y\} - \{x\}$, where x is the predecessor of y, and $x = y^c$.

Hence $y = \{n\}$, as required. ("As n changes hands only once.") $\square$

Remark. Vital, in the above proof, that the extremal sets in dimension $n-1$ were *nested* (i.e., given by initial segments of some ordering).

For a graph G, the **isoperimetric number** of G is: $i(G) = \min\left\{\frac{|\partial A|}{|A|} : A \subset G, \frac{|A|}{|G|} \leqslant \frac{1}{2}\right\}$.

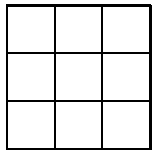
"How small can the average out-degree be?"

Corollary 10. $i(Q_n) = 1$.

Proof. The set $A = \mathbb{P}([n-1])$ show $i(Q_n) \leqslant 1$. To show $i(Q_n) \geqslant 1$, let C be any initial segment of binary with $|C| \leqslant 2^{n-1}$. Then $C \subset \mathbb{P}([n-1])$, so certainly $|\partial C| \geqslant |C|$. $\square$

Inequalities in the Grid

The **grid** is the graph on $[k]^n = \{(x_1, \ldots, x_n) : x_i \in \{1, \ldots, k\}\ \forall i\}$, in which $x = (x_1, \ldots, x_n)$ is joined to $y = (y_1, \ldots, y_n)$ if for some i we have $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. ("l_1-norm")

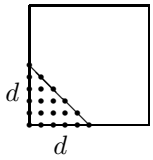


$[4]^2$

Note. For $k = 2$, this is exactly the graph Q_n.

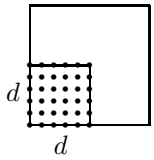
Do Theorem 1 and Theorem 8 extend to the grid?

Vertex-isoperimetric inequality in the grid



$b(A) \sim d$
$\sim \sqrt{2|A|}$

versus

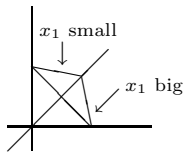


$b(A) \sim 2d$
$\sim 2\sqrt{|A|}$

Good guess: sets of the form $\{x : |x| \leqslant r\}$ are best, where $|x| = x_1 + \ldots + x_n$.

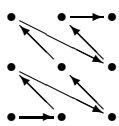
What if $|\{x : |x| \leqslant r\}| < |A| < |\{x : |x| \leqslant r+1\}|$? We'd take A of the form $\{x : |x| \leqslant r\} \cup B$, some $B \subset \{x : |x| = r+1\}$.

Define the **simplicial ordering** on $[k]^n$ by: $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x_i > y_i$ where $i = \min\{j : x_j \neq y_j\}$.



E.g., on $[3]^2$: $(1,1)$, $(2,1)$, $(1,2)$, $(3,1)$, $(2,2)$, $(1,3)$, $(3,2)$, $(2,3)$, $(3,3)$.

on $[4]^3$: $(1,1,1)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,2)$, $(1,3,1)$, $(1,2,2)$, $(1,1,3)$, $(4,1,1)$, $\ldots$



Note. This agrees with the previous definition of the simplicial ordering for $k = 2$.

Aim. Initial segments of simplicial are best for vertex-isoperimetric.

Let $A \subset [k]^n$. For $1 \leqslant i \leqslant n$, the i**-sections** of A are the sets $A_1, \ldots, A_k$ (or $A_1^{(i)}, \ldots, A_k^{(i)}$) in $[k]^{n-1}$ given by:

$$A_t = \{x = (x_1, \ldots, x_{n-1}) \in [k]^{n-1} : (x_1, \ldots, x_{i-1}, t, x_i, x_{i+1}, \ldots, x_{n-1}) \in A\}.$$

The i-**compression** of A is the set $C_i(A) \subset [k]^n$ defined by:

$$\big(C_i(A)\big)_t \text{ is the initial segment of simplicial on } [k]^{n-1} \text{ of size } |A_t|.$$

Say A is i-**compressed** if $C_i(A) = A$.

Theorem 11. Let $A \subset [k]^n$, and let C be the initial segment of simplicial with $|C| = |A|$.

Then $|N(C)| \leqslant |N(A)|$.

In particular, $|A| \geqslant \big|\{x : |x| \leqslant r\}\big| \;\Rightarrow\; |N(A)| \geqslant \big|\{x : |x| \leqslant r+1\}\big|$.

(Called the "vertex-isoperimetric inequality in the grid".)

Proof. Induction on n. For $n = 1$: for any $A \subset [k]^1$ with $A \neq \emptyset, [k]^1$, we have $|N(A)| \geqslant |A| + 1 = |N(C)|$.

Given $A \subset [k]^n$, fix $1 \leqslant i \leqslant n$.

Claim. $|N(C_i(A))| \leqslant |N(A)|$.

Proof of claim. Write B for $C_i(A)$.

For any $1 \leqslant t \leqslant k$, have $N(A)_t = \underset{\substack{\uparrow \\ \text{own} \\ \text{layer}}}{N(A_t)} \cup \underset{\substack{\uparrow \\ \text{below}}}{A_{t-1}} \cup \underset{\substack{\uparrow \\ \text{above}}}{A_{t+1}}$ (taking $A_0 = A_{k+1} = \emptyset$).

So $|N(A)| = \sum_t \big|N(A_t) \cup A_{t-1} \cup A_{t+1}\big|$.

Similarly, $|N(B)| = \sum_t \big|N(B_t) \cup B_{t-1} \cup B_{t+1}\big|$.

But $|B_{t-1}| = |A_{t-1}|$, $|B_{t+1}| = |A_{t+1}|$, and $|N(B_t)| \leqslant |N(A_t)|$, by induction.

Also, the sets $N(B_t)$, B_{t-1}, B_{t+1} are nested, as each is an initial segment of simplicial on $[k]^{n-1}$.

Thus $\big|N(A_t) \cup A_{t-1} \cup A_{t+1}\big| \geqslant \big|N(B_t) \cup B_{t-1} \cup B_{t+1}\big|$, and so $|N(A)| \geqslant |N(B)|$.

Among all $B \subset [k]^n$ with $|B| = |A|$ and $|N(B)| \leqslant |N(A)|$, choose one for which $\sum_{x \in B}$ (position of x in simplicial) is minimal. Then B is i-compressed for all i (else $C_i(B)$ contradicts the minimality of B).

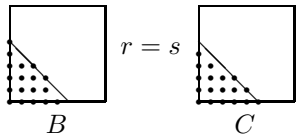
So it remains to show that $|N(B)| \geqslant |N(C)|$.

Case 1: $n = 2$**.**

B is i-compressed for all i iff B is a **down-set**. (I.e., if $x \in B$ and y has $y_i \leqslant x_i$ for all i, then $y \in B$. "Closed under going left and going down.")

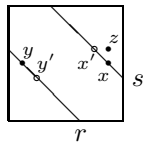
Let $s = \max\{|x| : x \in B\}$ and $r = \min\{|x| : x \notin B\}$.

Then $r \leqslant s$ (else $B = C$).



If $r = s$, then $\{x : |x| < r\} \subset B \subset \{x : |x| \leqslant r\}$, so certainly $|N(B)| \geqslant |N(C)|$.

If $r < s$, cannot have $\{x : |x| = s\} \subset B$, as B a down-set (and $\exists\, x$, $|x| = r$, $x \notin B$). Hence there are x, x' such that $|x| = |x'| = s$, $x \in B$, $x' \notin B$, and $x = x' \pm (e_1 - e_2)$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. (I.e., x, x' are adjacent.)



Similarly, cannot have $\{y : |y| = r\} \subset B^c$, so again there are y, y' with $|y| = |y'| = r$, $y \in B$, $y' \notin B$, and $y = y' \pm (e_1 - e_2)$.

But now let $B' = B \cup y' - x$. Then $|N(B')| \leqslant |N(B)|$ (as we have lost $\geqslant 1$ point from the neighbourhood in level $s + 1$ (e.g. z) and gained $\leqslant 1$ point in level $r + 1$), contradicting the choice of B.

Case 2: $n \geqslant 3$.

For $x \in B$ with $x_n \geqslant 2$, have $x - e_n + e_i \in B$ for all i with $x_i < k$ (as B is j-compressed for any $j \neq n, i$). Thus $N(B_t) \subset B_{t-1}$ for all $t = 2, \ldots, k$. We had $N(B)_t = N(B_t) \cup B_{t+1} \cup B_{t-1}$. So we actually have $N(B)_t = B_{t-1}$.

$$\text{So } |N(B)| = \underset{\substack{\uparrow \\ \text{height } k}}{|B_{k-1}|} + \underset{\substack{\uparrow \\ \text{height } k-1}}{|B_{k-2}|} + \ldots + \underset{\substack{\uparrow \\ \text{height } 2}}{|B_1|} + \underset{\substack{\uparrow \\ \text{height } 1}}{|N(B_1)|} = |B| - |B_k| + |N(B_1)|.$$

Similarly, have $|N(C)| = |C| - |C_k| + |N(C_1)|$.

Thus, to complete the proof, it is sufficient to show that $|B_k| \leqslant |C_k|$ and $|B_1| \geqslant |C_1|$.

- $|B_k| \leqslant |C_k|$

Define a set $D \subset [k]^n$ by: $D_k = B_k$, and $D_t = N(D_{t+1})$ for $t = k-1, k-2, \ldots, 1$.

Then D is an initial segment of simplicial, and $D \subset B$. So $|D| \leqslant |B| = |C|$, so $D \subset C$ (as D, C are initial segments of simplicial). Thus $D_k \subset C_k$, so $|D_k| \leqslant |C_k|$.

- $|B_1| \geqslant |C_1|$

Define a set $E \subset [k]^n$ by: $E_1 = B_1$, and $E_t = \{x \in [k]^{n-1} : N(\{x\}) \subset E_{t-1}\}$ for $t = 2, 3, \ldots, k$.

Then E is an initial segment of simplicial, and $E \supset B$. So $|E| \geqslant |B| = |C|$, so $E \supset C$. Thus $E_1 \supset C_1$, so $|E_1| \geqslant |C_1|$. $\square$

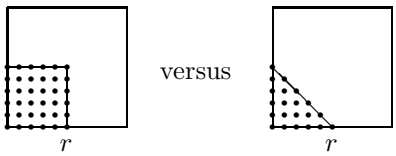
Corollary 12. Let $A \subset [k]^n$ with $|A| \geqslant \big|\{x : |x| \leqslant r\}\big|$. Then $|A_{(t)}| \geqslant \big|\{x : |x| \leqslant r + t\}\big|$. $\square$

Remark. Can check from this that for any fixed k, the sequence $([k]^n)_{n=1}^{\infty}$ is a normal Lévy family.

Edge-isoperimetric inequality in the grid

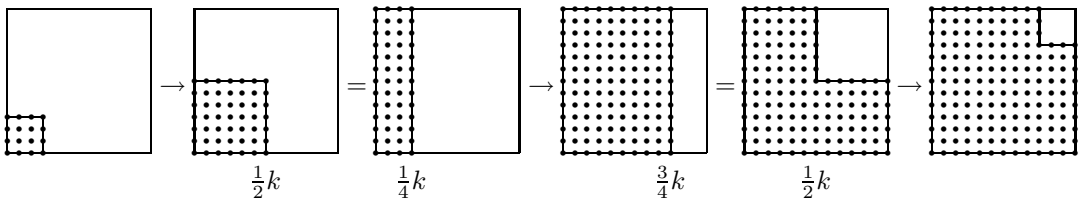
To minimise $|\partial A|$ for $|A|$ given.

<u>$n = 2$.</u>



First has size r^2, second has size $\sim \frac{1}{2}r^2$, and both have boundary $2r$.

This suggests that squares are best. But:



Sadly, the extremal sets are *not* nested: there are "phase transitions" at size $k^2/4$ and $3k^2/4$. (So we can't order them, which seems to rule out compressions.)

<u>$n = 3$.</u> Get: cube $[a]^3 \longrightarrow$ square column $[a]^2 \times [k] \longrightarrow$ halfspace $[a] \times [k]^2$
$\longrightarrow$ complement of square column $\longrightarrow$ complement of cube.

Aim. Best to take $[a]^d \times [k]^{n-d}$ (some d) or complements.

Observe that if $A = [a]^d \times [k]^{n-d}$ then $|\partial A| = da^{d-1}k^{n-d} = d|A|^{1-1/d}k^{n/d-1}$.

Theorem 13. Let $A \subset [k]^n$ with $|A| \leqslant k^n/2$.

Then $|\partial A| \geqslant \min\left\{d|A|^{1-1/d}k^{n/d-1} : d = 1, 2, \ldots, n\right\}$.

(The "edge-isoperimetric inequality in the grid".)

**** Non-examinable section ****

Proof (sketch). Wlog, A is a down-set in $[k]^n$. (Stamp on A – i.e., apply 1-dimensional compressions.)

For $1 \leqslant i \leqslant n$, define $C_i(A)$ by giving its i-sections:

$$C_i(A)_t \text{ is extremal in } [k]^{n-1}, \text{ with } |C_i(A)_t| = |A_t|.$$
$$\uparrow$$
$$\text{i.e., of form } [a]^d \times [k]^{n-1-d}, \text{ or complement}$$

Write B for $C_i(A)$.

Then $|\partial A| = \sum |\partial A_t| + |A_1| - |A_k|$.

horizontal vertical, as A is a down-set

And $|\partial B| = \sum |\partial B_t| + ?$ – as extremal sets in $[k]^{n-1}$ are *not* nested.

Indeed, we can clearly have $|\partial B| > |\partial A|$.

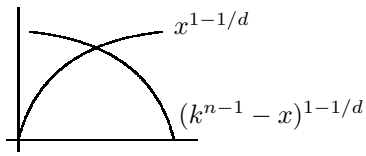
Try to introduce a 'fake boundary', ∂'. We would want $\partial' \leqslant \partial$, with equality for our extremal sets, and $\partial' C_i(A) \leqslant \partial' A$ for all A.

Obvious guess: $\partial' A = \sum |\partial A_t| + |A_1| - |A_k|$. (Note $\partial' = \partial$ on down-sets.) So we do have $\partial' B \leqslant \partial' A$, where $B = C_i(A)$. But it fails for C_j, $j \neq i$.

(Could try $\partial'' A = \sum_{i=1}^n |A_1^{(i)}| - |A_k^{(i)}|$ – cannot work, e.g. because the 'outside shell' of $[k]^n$ would have $\partial'' = 0$.)

We know: $|\partial A| = \partial' A \geqslant \partial' B = \sum |\partial B_t| + |B_1| - |B_k| = \sum f(|B_t|) + |B_1| - |B_k|$, where f is the extremal function in dimension $n-1$.

Now, $f(x)$ is minimum of some functions of the form $c.x^{1-1/d}$ and $c(k^{n-1} - x)^{1/1-d}$. Each of these is concave, so their pointwise minimum f is also concave.



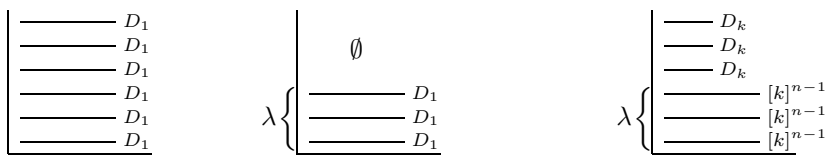
Consider varying $|B_2|, \ldots, |B_{k-1}|$, keeping $|B_2| + \ldots + |B_{k-1}|$ fixed, and such that $|B_1| \geqslant |B_2| \geqslant \ldots \geqslant |B_{k-1}| \geqslant |B_k|$.

By concavity of f, have $\partial' B \geqslant \partial' C$, where for some λ we have $C_t = \begin{cases} B_1 & \text{if } t \leqslant \lambda \\ B_k & \text{it } t > \lambda \end{cases}$.

We have $|\partial A| = \partial' A \geqslant \partial' B \geqslant \partial' C$, with $\partial' C = \lambda f(|B_1|) + (k-\lambda) f(|B_k|) + |B_1| - |B_k|$ (but C still not a down-set).

Now consider varying $|B_1|$ and $|B_k|$ (λ fixed), keeping $\lambda|B_1| + (k-\lambda)|B_k|$ fixed, and keeping $|B_1| \geqslant |B_k|$. Again, this is a concave function of $|B_1|$ (as it is concave+concave+linear), so by concavity of f, we have $\partial' C \geqslant \partial' D$, where either

$$D_t = D_1 \ \forall t, \quad \text{or} \quad D_t = \begin{cases} D_1 & \forall t \leqslant \lambda \\ \emptyset & \forall t > \lambda \end{cases}, \quad \text{or} \quad D_t = \begin{cases} [k]^{t-1} & \forall t \leqslant \lambda \\ D_k & \forall t > \lambda \end{cases}.$$



Thus $|\partial A| = \partial' A \geqslant \partial' B \geqslant \partial' C \geqslant \partial' D = |\partial D|$. (Miraculously, D is a down-set.)

So, take as our i-compression the map $A \mapsto D$.

Then finish proof as usual. $\square$

Remark. To make the argument precise, work in the continuous cube $[0,1]^n$ instead, and then pass to discrete at the end.

**** End of non-examinable section ****

Very few isoperimetric inequalities are known (even approximately).

For example, what about r-sets? Take the graph on $X^{(r)}$ with x, y joined if $|x \cap y| = r - 1$ (i.e., $d(x, y) = 2$ in Q_n). No good isoperimetric inequality is known.

Most interesting case is $r = n/2$. It's conjectured that balls are best – i.e., sets of the form $\{x \in X^{(r)} : d(x, x_0) \leqslant d\}$, i.e., of the form $\big\{x \subset [2r] : |x \cap [r]| \geqslant k\big\}$.

Unknown!

Chapter 3 : Intersecting Families

t-intersecting families

Say $A \subset \mathbb{P}(X)$ is **t-intersecting** if $|x \cap y| \geqslant t$ for all $x, y \in A$.

How large can A be?

E.g., $t = 2$. Could take $\{x : 1, 2 \in x\}$ – size $\frac{1}{4}.2^n$. Beaten by $\{x : |x| \geqslant \frac{n}{2} + 1\}$ – size $\sim \frac{1}{2}.2^n$.

Theorem 1 (Katona's t-intersecting theorem). Let $A \subset \mathbb{P}(X)$ be t-intersecting, where $n + t$ is even. Then $|A| \leqslant \left|X^{\left(\geqslant \frac{n+t}{2}\right)}\right|$.

Proof. For any $x, y \in A$, have $d(x, y^c) \geqslant t$. Thus writing $\overline{A} = \{x^c : x \in A\}$, we have $d(A, \overline{A}) \geqslant t$. I.e., $A_{(t-1)}$ is disjoint from $\overline{A}$.

Suppose $|A| > \left|X^{\left(\geqslant \frac{n+t}{2}\right)}\right|$. Then $|A_{(t-1)}| \underset{\substack{\uparrow \\ \text{by Harper}}}{\geqslant} \left|X^{\left(\geqslant \frac{n+t}{2} - (t-1)\right)}\right| = \left|X^{\left(\geqslant \frac{n-t}{2} + 1\right)}\right|$.

But $\left|\overline{A}\right| > \left|X^{\left(\leqslant \frac{n-t}{2}\right)}\right|$, contradicting $A_{(t-1)} \cap \overline{A} = \emptyset$. $\square$

What about $A \subset X^{(r)}$ being t-intersecting? $(1 \leqslant t \leqslant r)$

Could guess $A_0 = \{x \in X^{(r)} : 1, 2, \ldots, t \in x\}$.

Could also try, for $1 \leqslant \alpha \leqslant r - t$, the set $A_\alpha = \left\{x \in X^{(r)} : |x \cap \{1, 2, \ldots, t + 2\alpha\}| \geqslant t + \alpha\right\}$.

E.g., $t = 2$, in $[7]^{(4)}$: $|A_0| = \binom{5}{2} = 10$, $|A_1| = 1 + \binom{4}{3}\binom{3}{1} = 13$, $|A_2| = \binom{6}{4} = 15$.

in $[8]^{(4)}$: $|A_0| = \binom{6}{2} = 15$, $|A_1| = 1 + \binom{4}{3}\binom{4}{1} = 17$, $|A_2| = \binom{6}{4} = 15$.

in $[9]^{(4)}$: $|A_0| = \binom{7}{2} = 21$, $|A_1| = 1 + \binom{4}{3}\binom{5}{1} = 21$, $|A_2| = \binom{6}{4} = 15$.

As $|A_0|$ is quadratic (in n), $|A_1|$ is linear, and $|A_2|$ is constant, we will have A_0 winning if n is large.

Theorem 2. Let $A \subset X^{(r)}$ be t-intersecting, where r, t are fixed, $1 \leqslant t \leqslant r$.

Then $|A| \leqslant |A_0| = \binom{n-t}{r-t}$, if n is sufficiently large.

Remarks. 1. Sometimes called the "second Erdős-Ko-Rado Theorem".

2. Bound on n is $(16r)^r$ (crude) or $2tr^3$ (careful).

Proof. "Idea: A_0 has $r-t$ degrees of freedom."

Wlog A is maximal t-intersecting, and so there are $x, y \in A$ with $|x \cap y| = t$ (or else for all $x \in A$ and all $i \in x$, $j \notin x$, have $x \cup j - i \in A$ by maximality, hence $A = X^{(r)}$ – contradiction).

Cannot have $x \cap y \subset z$ for all $z \in A$ (else $|A| \leqslant |A_0|$, so done). So choose $z \in A$ with $z \not\supset x \cap y$. Then for any $w \in A$, we must have $|w \cap (x \cup y \cup z)| \geqslant t+1$.

$$\text{So } |A| \leqslant \underset{\substack{\uparrow \\ w \text{ on } x \cup y \cup z}}{2^{3r}} \left(\binom{n}{r-t-1} + \binom{n}{r-t-2} + \ldots + \binom{n}{0} \right),$$

$$\nwarrow \qquad \underset{\substack{\uparrow \\ w \text{ off } x \cup y \cup z}}{} \qquad \nearrow$$

which is a polynomial of degree $r-t-1$, hence $< |A_0|$ for n large. □

Remark. Frankl Conjecture was: $A \subset X^{(r)}$ t-intersecting $\Rightarrow |A| \leqslant \max(|A_0|, \ldots, |A_{r-t}|)$ (any n). Proved by Ahslwede and Khachatrian in 1998.

Modular intersections

So far, we have banned $|x \cap y| = 0$. What if we instead banned $|x \cap y| \equiv 0$ mod something?

E.g., suppose r is odd, and we want $A \subset X^{(r)}$ with $|x \cap y|$ odd for all $x, y \in A$. We can achieve $|A| = \binom{\lfloor \frac{n-1}{2} \rfloor}{\frac{r-1}{2}}$ by taking all sets x of the form: 1, together with $\frac{r-1}{2}$ of the pairs 23, 45, ...

What if (still for r odd) we wanted $|x \cap y|$ even for all $x, y \in A$ ($x \neq y$). We can achieve $n-r+1$ by taking all x containing $1, 2, \ldots r-1$. Amazingly:

Proposition 3. Let r be odd and let $A \subset X^{(r)}$ have $|x \cap y|$ even for all $x, y \in A$ ($x \neq y$). Then $|A| \leqslant n$.

Idea. Try to find $|A|$ linearly independent vectors in an n-dimensional vector space.

Proof. View each point $x \in Q_n$ as a point $\overline{x}$ in $\mathbb{Z}_2^n$, the n-dimensional vector space over the field $\mathbb{Z}_2$. (E.g., $\{1, 3, 4\} \leftrightarrow 10110\ldots0$.) Consider $\{\overline{x} : x \in A\}$.

Have $\langle \overline{x}, \overline{x} \rangle = 1$ for all $x \in A$ (where $\langle , \rangle$ is the usual dot product), and $\langle \overline{x}, \overline{y} \rangle = 0$ for all $x, y \in A$, $x \neq y$. Hence $\{\overline{x} : x \in A\}$ is linearly independent. (If $\sum \lambda_i x_i = 0$, dot with x_i to get $\lambda_i = 0$.)

So $\left|\{\overline{x} : x \in A\}\right| \leqslant n$. □

For r even? If $|x \cap y|$ even for all distinct $x, y \in A \subset X^{(r)}$, we can have $|A| \geqslant \binom{\lfloor n/2 \rfloor}{r/2}$, by taking $r/2$ of the pairs 12, 34, ...

If $|x \cap y|$ is odd for all distinct $x, y \in A \subset X^{(r)}$, then we must have $|A| \leqslant n+1$ – else add the point $n+1$ to each $x \in A$ to contradict Proposition 3.

So banning $|x \cap y| \equiv r \pmod 2$ forces $|A|$ to be small. Does this generalise?

We now show that "s allowed intersections mod $p \Rightarrow |A| \leqslant \binom{n}{s}$".

Theorem 4 (Frankl-Wilson Theorem). Let p be prime, and let $A \subset X^{(r)}$ be such that for all distinct $x, y \in A$ we have $|x \cap y| \equiv \lambda_i \pmod p$, some i, where $\lambda_1, \ldots, \lambda_s$ $(s \leqslant r)$ are integers, with none $\equiv r \pmod p$.

Then $|A| \leqslant \dbinom{n}{s}$.

Remarks. 1. The bound is a polynomial in n, independent of r (!)

2. Bound is essentially best possible. Take $A = \{x \in X^{(r)} : [r-s] \subset x\}$.

This has $|A| = \dbinom{n-r+s}{s} \sim \dbinom{n}{s}$. (Ratio $\to 1$ as $n \to \infty$.)

3. No polynomial bound if we allow $|x \cap y| \equiv r \pmod p$.

Indeed, write $r = a + \lambda p$ $(0 \leqslant a \leqslant p-1)$. $\underbrace{(\leftarrow a \rightarrow)}_{\text{take this}}\underbrace{(\leftarrow p \rightarrow)(\leftarrow p \rightarrow)(\leftarrow p \rightarrow)\ldots}_{\text{take } \lambda \text{ of these}}$

Can have $|x \cap y| \equiv r \pmod p$ for all $x, y \in A$,
with $|A| = \dbinom{\lfloor \frac{n-a}{p} \rfloor}{\lambda}$. But the degree grows as λ grows.

Idea. Try to find $|A|$ linearly independent points in a vector space of dimension $\binom{n}{s}$, by somehow "applying the polynomial $(t-\lambda_1)\ldots(t-\lambda_s)$ to the values of $|x \cap y|$".

Proof. For $i \leqslant j$, let $M(i,j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix, with rows indexed by $X^{(i)}$ and columns indexed by $X^{(j)}$, with $M(i,j)_{xy} = \begin{cases} 1 & \text{if } x \subset y \\ 0 & \text{if } x \not\subset y \end{cases}$.

The matrix $M(s,r)$ has $\binom{n}{s}$ rows. Let V be the vector space (over $\mathbb{R}$) spanned by those rows. Thus $\dim V \leqslant \binom{n}{s}$.

Consider the matrix $M(i,s)M(s,r)$, where $i \leqslant s$. For $x \in X^{(i)}$ and $y \in X^{(r)}$,

$$\begin{aligned} \big(M(i,s)M(s,r)\big)_{xy} &= \text{the number of } z \in X^{(s)} \text{ with } x \subset z \subset y \\ &= \begin{cases} \binom{r-i}{s-i} & \text{if } x \subset y \\ 0 & \text{otherwise} \end{cases} \\ &= \binom{r-i}{s-i} M(i,r)_{xy}. \end{aligned}$$

Thus $M(i,s)M(s,r) = \binom{r-i}{s-i} M(i,r)$, and the rows of $M(i,r)$ belong to V (as pre-multiplying is just taking linear combinations of the rows).

Consider $M(i) = M(i,r)^T M(i,r)$ – again, must have all rows in V. For $x, y \in X^{(r)}$,

$$\begin{aligned} M(i)_{xy} &= \text{the number of } z \in X^{(i)} \text{ with } z \subset x, z \subset y \\ &= \binom{|x \cap y|}{i}. \end{aligned}$$

Write the integer polynomial $(t-\lambda_1)\ldots(t-\lambda_s)$ as $\sum_{i=0}^{s} a_i\binom{t}{i}$, for some integers a_i. (This is possible, as $t(t-1)\ldots(t-i+1) = i!\binom{t}{i}$.)

Let $M = \sum_{i=0}^{s} a_i M(i)$. All rows are in V. Then

$$M_{xy} \;=\; \sum_{i=0}^{s} a_i \binom{|x\cap y|}{i} \;=\; (|x\cap y|-\lambda_1)\ldots(|x\cap y|-\lambda_s)$$

$$\begin{cases} \equiv 0 \pmod p & \text{if } |x\cap y| \equiv \text{ some } \lambda_i \pmod p \\ \not\equiv 0 \pmod p & \text{otherwise} \end{cases}.$$

Now look at the submatrix whose rows and columns are indexed by A.

It is $\begin{pmatrix} \not\equiv 0 & & O \\ & \ddots & \\ O & & \not\equiv 0 \end{pmatrix}$.

Rows are integer-valued, and are linearly independent over $\mathbb{Z}_p$, so also over $\mathbb{Z}$, so also over $\mathbb{Q}$, so also over $\mathbb{R}$.

Thus $|A| \leqslant \dim V \leqslant \binom{n}{s}$. □

Remark. We do need p prime. Grolmusz constructed, for each n, a value of $r \equiv 0 \pmod 6$ and $A \subset X^{(r)}$ with $|x\cap y| \not\equiv 0 \pmod 6$ for all distinct $x, y \in A$, with $|A| > n^{c \log n/\log\log n}$, some $c > 0$. (Not a polynomial in n.)

Corollary 5. Let $A \subset X^{(r)}$ with $|x\cap y| \not\equiv r \pmod p$ for all distinct $x, y \in A$ (some prime $p < r$). Then $|A| \leqslant \binom{n}{p-1}$.

Proof. We are allowed $p-1$ values of $|x\cap y| \pmod p$. □

Two $\frac{n}{2}$-sets from $[n]$ intersect (on average) in about $n/4$ points. But an intersection size of exactly $n/4$ is very unlikely. However, amazingly:

Corollary 6. Let p be prime, and let $A \subset [4p]^{(2p)}$ be such that $|x\cap y| \neq p$ for all $x, y \in A$. Then $|A| \leqslant 2\binom{4p}{p-1}$.

("Not much of a constraint.")

Note. $\binom{4p}{p-1}$ is extremely small: have $\binom{n}{n/4} \leqslant 2\,e^{-n/32}\,2^n$, so $\binom{n}{n/4}$ is an exponentially small fraction of $\binom{n}{n/2} \sim \frac{c}{\sqrt{2}}\,2^n$.

Proof. Halving $|A|$ if necessary, we may assume that we never have $x, x^c \in A$. Hence, for distinct $x, y \in A$, we have $|x\cap y| \neq 0, p$, so $|x\cap y| \not\equiv 0 \pmod p$.

Thus $|A| \leqslant \binom{4p}{p-1}$ by Corollary 5. □

Borsuk's Conjecture

Suppose S is a bounded set in $\mathbb{R}^n$. How many pieces do we need to break S into such that each piece has smaller diameter than S? Taking $S \subset \mathbb{R}^n$ to be a regular simplex (i.e., $n+1$ equally-spaced points), it's clear that we may need as many as $n+1$ pieces.

Borsuk's Conjecture. $n+1$ pieces is always possible.

Known for $n = 1, 2, 3$, and also for $S \subset \mathbb{R}^n$ being a smooth convex body or a symmetric convex body (i.e., $x \in S \Rightarrow -x \in S$).

In fact, Borsuk's Conjecture is massively false.

Theorem 7 (Kahn, Kalai). For all n, there exists a bounded set $S \subset \mathbb{R}^n$ such that to break S into pieces of smaller diameter, we need $\geqslant c^{\sqrt{n}}$ pieces, for some constant $c > 1$.

Notes. 1. Our proof will give Borsuk false for $n \geqslant 2000$.

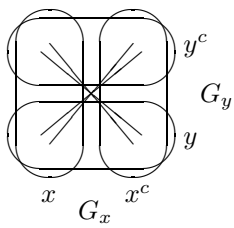
2. We'll prove Theorem 7 for n of the form $\binom{4p}{2}$, p prime – then it follows (with a different c) for all n, e.g. because there exists p with $n/2 \leqslant p \leqslant n$.

Proof. We'll consider $S \subset Q^n \subset \mathbb{R}^n$. In fact $S \subset [n]^{(r)}$. ("A genuine idea.")

For $x, y \in S$, we have $||x-y||^2 = 2(r-|x \cap y|)$ (where $||\cdot||$ is the usual Euclidean distance). Thus we seek S with $\min |x \cap y| = k$ say, but any subset of S with $\min |x \cap y| > k$ is much smaller than S.

Let $n = \binom{4p}{2}$, where p is prime. Identify $[n]$ with $E(K_{4p})$ – the edge-set of the complete graph on $\{1, \ldots, 4p\}$. For each $x \in [4p]^{(2p)}$, let G_x be the complete bipartite graph on vertex classes x, x^c.

Let $S = \{G_x : x \in [4p]^{(2p)}\}$. So $S \subset [n]^{(4p^2)}$, and $|S| = \frac{1}{2}\binom{4p}{2p}$.



We have $|G_x \cap G_y| = |x \cap y|^2 + (2p - |x \cap y|)^2$
$= d^2 + (2p-d)^2$, where $d = |x \cap y|$.

This is minimal when $d = p$, i.e. when $|x \cap y| = p$.

Now let $S' \subset S$ have smaller diameter than S. Say $S' = \{G_x : x \in A\}$.

Then we must have $|x \cap y| \neq p$ for all $x, y \in A$ (else diameter of S' = diameter of S).

So $|A| \leqslant 2\binom{4p}{2p-1}$, by Corollary 6. Thus the number of pieces needed is

$$\geqslant \frac{\frac{1}{2}\binom{4p}{2p}}{2\binom{4p}{p-1}} \geqslant \frac{c\,2^{4p}/\sqrt{p}}{4e^{-p/8}\,2^{4p}} \geqslant (c')^p \quad (\text{some } c' > 1)$$

$$\geqslant (c'')^{\sqrt{n}} \quad (\text{some } c'' > 1). \qquad \square$$

Chapter 3′ : Projections

"If a set has small projections, must it be small?"

Let $A \subset \mathbb{P}(X)$ and let $Y \subset X$. The **projection** or **trace** of A on Y is $A|_Y = \{x \cap Y : x \in A\}$. Thus $A|_Y \subset \mathbb{P}(Y)$ – 'project A onto the coordinates corresponding to Y'.

E.g., if $A = \{1, 12, 13, 124, 34, 345\}$ and $Y = \{1, 3\}$, then $A|_Y = \{1, 13, 3\}$.

Say A **covers** or **shatters** Y if $A|_Y = \mathbb{P}(Y)$.

The **trace number** (or **VC-dimension**) of A is $\operatorname{tr} A = \max\{|Y| : Y \text{ shattered by } A\}$.

Given $|A|$, how small can $\operatorname{tr} A$ be? Equivalently, how large can $|A|$ be given $\operatorname{tr} A < k$?

We could take $A = X^{(<k)}$. This clearly does not shatter any k-set (as if $|Y| = k$ then $Y \notin A|_Y$). Our aim is to show that we cannot do better than $|X^{(<k)}|$.

The main idea is that this is trivial if A is a down-set (i.e., if whenever $x \in A$ and $y \subset x$ then also $y \in A$), because then if $|A| > |X^{(<k)}|$, A must contain a set of size $\geqslant k$, and we're now done because A is a down-set.

Next idea: try to compress an arbitrary A into a down-set.

For $1 \leqslant i \leqslant n$, the **$i$-down-compression** of A is defined as follows. For $x \in \mathbb{P}(X)$, let

$$D_i(x) = \begin{cases} x - \{i\} & \text{if } i \in x \\ x & \text{otherwise} \end{cases},$$

and for $A \subset \mathbb{P}(X)$, let

$$D_i(A) = \{D_i(x) : x \in A\} \cup \{x \in A : D_i(x) \in A\}.$$

I.e., we 'compress A downwards in direction i'. Note that $|D_i(A)| = |A|$. We say that A is **i-down-compressed** if $D_i(A) = A$.

Remark. D_i is a 1-dimensional compression.

Theorem 1 (Sauer-Shelah Lemma). If $A \subset \mathbb{P}(X)$ with $|A| > |X^{(<k)}|$ then $\operatorname{tr} A \geqslant k$.

Proof. Claim. For any $A \subset \mathbb{P}(X)$ and $1 \leqslant i \leqslant n$, we have $\operatorname{tr} D_i(A) \leqslant \operatorname{tr} A$.

> **Proof of claim.** Write A' for $D_i(A)$. Suppose A' shatters Y. We shall show that A also shatters Y. If $i \notin Y$ then $A'|_Y = A|_Y$, and so we are done.
>
> So suppose $i \in Y$. Then for $z \subset Y$ with $i \notin z$ we have $z \cup \{i\} \in A'|_Y$, so there exists $x \in A'$ with $x \cap Y = z \cup \{i\}$. But then $i \in x$, so x, $x - \{i\} \in A$ (by definition of A'). Thus z, $z \cup \{i\} \in A|_Y$. Hence $A|_Y = \mathbb{P}(Y)$.

Set $B = D_n(D_{n-1}(D_{n-2}(\ldots(D_1(A))\ldots)))$. Then $|B| = |A|$, $\operatorname{tr} B \leqslant \operatorname{tr} A$, and B is a down-set. But $|B| > |X^{(<k)}|$, so B contains some k-set, so $\operatorname{tr} B \geqslant k$. □

In general, do upper bounds on some projections $\left|A|_{Y_i}\right|$ give us upper bounds on $|A|$?

For example, Sauer-Shelah says that if $\left|A|_Y\right| \leqslant 2^k - 1$ for all k-sets Y, then $|A| \leqslant |X^{(<k)}|$.

A **brick** or **box** in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $a_i \leqslant b_i$ for all i. A **body** $S \subset \mathbb{R}^n$ is a finite union of bricks. The volume of S is written $|S|$ or $m(S)$.

Remarks. 1. In fact, everything will go through for a general compact $S \subset \mathbb{R}^n$.

2. A set system $A \subset Q_n$ gives a body $\widehat{A} \subset \mathbb{R}^n$, namely $\widehat{A} = \bigcup_{x \in A} B_x$, where $B_x = [x_1, x_1 + 1] \times \cdots \times [x_n, x_n + 1]$. We have $|A| = m(\widehat{A})$, where for a body S, $m(S) = |S|$ is the volume of S.

For a body $S \subset \mathbb{R}^n$ and $Y \subset [n]$, the projection of S onto the span of $\{e_i : i \in Y\}$ is denoted by S_Y. For example, if $S \subset \mathbb{R}^3$, then S_1 is the projection of S onto the x-axis:

$$S_1 = \{x_1 \in \mathbb{R} : (x_1, x_2, x_3) \in S \text{ for some } x_2, x_3 \in \mathbb{R}\},$$

and S_{12} is the projection of S onto the xy-plane:

$$S_{12} = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2, x_3) \in S \text{ for some } x_3 \in \mathbb{R}\}.$$

We have that $S_A \subset \mathbb{R}^{|A|}$.

What bounds on $|S|$ do we get given bounds on some S_Y?

For example, let S be a body in $\mathbb{R}^3$. Then trivially $|S| \leqslant |S_1||S_2||S_3|$ as $S \subset S_1 \times S_2 \times S_3$. Similarly, $|S| \leqslant |S_{12}||S_3|$ as $S \subset S_{12} \times S_3$.

What if $|S_{12}|$ and $|S_{13}|$ are known? This tells us nothing – e.g., $S = [0, 1/n] \times [0, n] \times [0, n]$.

What if $|S_{12}|$, $|S_{13}|$ and $|S_{23}|$ are known?

Proposition 2. Let S be a body in $\mathbb{R}^3$. Then $|S|^2 \leqslant |S_{12}||S_{13}||S_{23}|$.

Remark. We have equality if S is a brick.

For $S \subset \mathbb{R}^n$, the n**-sections** are the sets $S(x) \subset \mathbb{R}^{n-1}$ for each $x \in \mathbb{R}$ defined by

$$S(x) = \{(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, x_2, \ldots, x_{n-1}, x) \in S\}.$$

Proof of Proposition 2. Consider first the case when each 3-section is a square, i.e. when $S(x) = [0, f(x)] \times [0, f(x)]$. Then $|S_{12}| = M^2$, where $M = \max_{x \in \mathbb{R}} f(x)$. Also $|S_{13}| = |S_{23}| = \int f(x)\,dx$, and $|S| = \int f(x)^2\,dx$. Thus we want:

$$\left(\int f(x)^2\,dx\right)^2 \leqslant M^2 \left(\int f(x)\,dx\right)^2.$$

But $\int f(x)^2\,dx \leqslant M \int f(x)\,dx$ as $f(x) \leqslant M$ for all x, so this indeed holds.

For the general case, define a body $T \subset \mathbb{R}^3$ by

$$T(x) = \left[0, \sqrt{|S(x)|}\right] \times \left[0, \sqrt{|S(x)|}\right].$$

Then $|T| = |S|$ and $|T_{12}| \leqslant |S_{12}|$, as $|T_{12}| = \max_{x \in \mathbb{R}} |T(x)|$.

Let $g(x) = |S(x)_1|$ and $h(x) = |S(x)_2|$, so $|S(x)| \leqslant g(x)h(x)$. Then

$$|T_{13}| = |T_{23}| = \int \sqrt{|S(x)|}\, dx \leqslant \int \sqrt{g(x)h(x)}\, dx.$$

Also, $|S_{13}| = \int g(x)\, dx$ and $|S_{23}| = \int h(x)\, dx$. So we will be done (by reduction to the case of T) if

$$\left(\int \sqrt{g(x)h(x)}\, dx\right)^2 \leqslant \left(\int g(x)\, dx\right) \left(\int h(x)\, dx\right),$$

i.e.

$$\int \sqrt{g(x)h(x)}\, dx \leqslant \left(\int g(x)\, dx\right)^{1/2} \left(\int h(x)\, dx\right)^{1/2},$$

which is just the Cauchy-Schwarz inequality on the functions $\sqrt{g(x)}$ and $\sqrt{h(x)}$. $\square$

We say that sets $Y_1, Y_2, \ldots, Y_r$ **cover** $[n]$ if $\bigcup_{j=1}^r Y_j = [n]$. They are a **$k$-uniform cover** if each $i \in [n]$ belong to exactly k of the Y_j.

For example, for $n = 3$: $\{1\}, \{2\}, \{3\}$ is a 1-uniform cover; $\{1\}, \{2, 3\}$ is a 1-uniform cover; $\{1, 2\}, \{1, 3\}, \{2, 3\}$ is a 2-uniform cover; $\{1, 2\}, \{1, 3\}$ is not uniform.

Our aim is to show that if $Y_1, Y_2, \ldots, Y_r$ form a k-uniform cover then $|S|^k \leqslant |S_{Y_1}||S_{Y_2}| \cdots |S_{Y_r}|$.

Let $\mathcal{C} = \{Y_1, Y_2, \ldots, Y_r\}$ be a k-uniform cover of $[r]$. Note that $\mathcal{C}$ is a *multiset*, i.e. repetitions are allowed – for example, $\{1, 1, 2, 3, 23\}$ is a 2-uniform cover of [3].

Put $\mathcal{C}_- = \{Y_i : n \notin Y_i\}$ and $C_+ = \{Y_i - n : n \in Y_i\}$ as usual, so $|\mathcal{C}_+| = k$, and $\mathcal{C}_- \cup \mathcal{C}_+$ is a k-uniform cover of $[n-1]$.

Note that if $n \in Y$ then $|S_Y| = \int |S(x)_{Y-n}|\, dx$ (e.g. if $S \subset \mathbb{R}^3$ then $|S_{13}| = \int |S(x)_1|\, dx$), and this holds even if $Y = [n]$. Also, if $n \notin Y$ then $|S_Y| \geqslant |S(x)_Y|$ for all x (e.g. $|S_1| \geqslant |S(x)_1|$ for all x).

In the proof of Proposition 2, we used Cauchy-Schwarz: $\displaystyle\int fg \leqslant \left(\int f^2\right)^{1/2} \left(\int g^2\right)^{1/2}$.

Here, we will need Hölder's inequality: $\displaystyle\int fg \leqslant \left(\int f^p\right)^{1/p} \left(\int g^q\right)^{1/q}$, for $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Whence, iterating, we get: $\displaystyle\int f_1 \ldots f_k \leqslant \left(\int f_1^k\right)^{1/k} \cdots \left(\int f_k^k\right)^{1/k}$.

Theorem 3 (Uniform Covers Theorem). Let S be a body in $\mathbb{R}^n$, and let $\mathcal{C}$ be a k-uniform cover of $[n]$.

Then $|S|^k \leqslant \displaystyle\prod_{Y \in \mathcal{C}} |S_Y|$.

Proof. The proof is by induction on n. The case $n = 1$ is trivial. Given a body $S \subset \mathbb{R}^n$ for $n \geqslant 2$, we have

$$
\begin{aligned}
|S| &= \int_x |S(x)|\,dx \\
&\leqslant \int_x \prod_{Y\in\mathcal{C}_+} |S(x)_Y|^{1/k} \prod_{Y\in\mathcal{C}_-} |S(x)_Y|^{1/k}\,dx \qquad \text{(induction)} \\
&\leqslant \prod_{Y\in\mathcal{C}_-} |S_Y|^{1/k} \int_x \prod_{Y\in\mathcal{C}_+} |S(x)_Y|^{1/k}\,dx \\
&\leqslant \prod_{Y\in\mathcal{C}_-} |S_Y|^{1/k} \prod_{Y\in\mathcal{C}_+} \left(\int_x |S(x)_Y|\,dx\right)^{1/k} \qquad \text{(Hölder)} \\
&= \prod_{Y\in\mathcal{C}_-} |S_Y|^{1/k} \prod_{Y\in\mathcal{C}_+} |S_{Y\cup n}|^{1/k} \\
&= \prod_{Y\in\mathcal{C}} |S_Y|^{1/k}\,.
\end{aligned}
$$

□

Corollary 4 (Loomis-Whitney Theorem). Let S be a body in $\mathbb{R}^n$.

Then $|S|^{n-1} \leqslant \prod_{i=1}^{n} |S_{[n]-i}|$.

Proof. The family $[n]-1$, $[n]-2$, …, $[n]-n$ is an $(n-1)$-uniform cover of $[n]$. □

Remark. The case $n=3$ of the Loomis-Whitney theorem is Proposition 2.

Corollary 5. Let $A \subset Q_n$, and let $\mathcal{C}$ be a k-uniform cover of $[n]$.

Then $|A|^k \leqslant \prod_{Y\in\mathcal{C}} \big|A|_Y\big|$.

In particular, if $\mathcal{C}$ is a uniform cover with $\big|A|_Y\big| \leqslant \left(2^{|Y|}\right)^c$ for all $y \in \mathcal{C}$ then $|A| \leqslant (2^n)^c$.

Proof. For the first part, apply Theorem 3 to the body

$$\widehat{A} = \bigcup_{x\in A} [x_1, x_1+1] \times \cdots \times [x_n, x_n+1].$$

Then $m(\widehat{A}) = |A|$ and $m(\widehat{A}|_Y) = \big|A|_Y\big|$ for all Y.

For the second part, suppose that $\mathcal{C}$ is a k-cover. Then

$$|A|^k \leqslant \prod_{Y\in\mathcal{C}} \big|A|_Y\big| \leqslant \prod_{Y\in\mathcal{C}} \left(2^{|Y|}\right)^c = \left(2^{\sum_{Y\in\mathcal{C}}|Y|}\right)^c = \left(2^{kn}\right)^c.$$

□

Our next aim is the 'Bollobás-Thomason box theorem', that for any body $S \subset \mathbb{R}^n$ there is a box $B \subset \mathbb{R}^n$ with $|B| = |S|$ and $|B_Y| \leqslant |S_Y|$ for all $Y \subset [n]$. This has no right to be true! For example, we can then read off all possible projection theorems – just check them for boxes.

A uniform cover $\mathcal{C}$ of $[n]$ is **irreducible** if we cannot write $\mathcal{C} = \mathcal{C}' \sqcup \mathcal{C}''$ where $\mathcal{C}'$ and $\mathcal{C}''$ are uniform covers. For example, if $n=3$ then 12, 13, 23 form an irreducible cover, but 1, 23, 13, 2 do not.

Lemma 6. There are only finitely many irreducible uniform covers of $[n]$.

Proof. Suppose $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$ are distinct irreducible covers. List $\mathbb{P}([n])$ as $E_1, E_2, \ldots, E_{2^n}$. Choose a subsequence $\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \ldots$ on which the number of copies of E_1 is increasing (not necessarily strictly). Then choose a subsequence of $C_{i_1}, \mathcal{C}_{i_2}, \ldots$ on which the number of copies of E_2 is increasing. Repeating for E_3, then E_4, then $\ldots$, then E_{2^n}, we obtain a subsequence $\mathcal{C}_{j_1}, \mathcal{C}_{j_2}, \mathcal{C}_{j_3}, \ldots$, on which the number of copies of E_i is increasing for all i. But then $\mathcal{C}_{j_2}$ is not irreducible, as $\mathcal{C}_{j_2} \supset \mathcal{C}_{j_1}$. Contradiction. $\square$

Theorem 7 (Bollobás-Thomason Box Theorem). Let S be a (non-empty) body in $\mathbb{R}^n$. Then there is a box $B \in \mathbb{R}^n$ with $|B| = |S|$ and $|B_Y| \leqslant |S_Y|$ for all $Y \subset [n]$.

Proof. We may assume without loss of generality that $|S| > 0$ and $n \geqslant 2$. Take real variables x_Y for each $Y \in \mathbb{P}([n])$ with $Y \neq \emptyset$ or $[n]$, with constraints:

(i) $0 \leqslant x_Y \leqslant |S_Y|$ for all Y;

(ii) $x_Y \leqslant \prod_{i \in Y} x_i$ for all Y with $|Y| \geqslant 2$; and

(iii) $|S|^k \leqslant \prod_{Y \in \mathcal{C}} x_Y$ for each k-uniform irreducible cover $\mathcal{C} \neq \{[n]\}$.

If (iii) is satisfied for all irreducible covers, then it is satisfied for *all* uniform covers. We denote the condition (iii) for all uniform covers by (iii)$'$. We 'want a minimal solution'.

We have a solution, namely $x_Y = |S_Y|$ for all Y. The solution set is compact, so there exists a solution with minimal $\sum_Y x_Y$. We must have $x_Y > 0$ for all Y, because every Y occurs in some uniform cover, whence (iii)$'$ gives $|x_Y| > 0$ (as $|S| > 0$).

Claim. For $1 \leqslant i \leqslant n$, x_i appears on the RHS of an inequality from (iii) in which equality holds.

Proof of claim. We must have x_i on the RHS of some constraint for which equality holds, or else we could decrease x_i (as the set of constraints is finite). It is not an inequality from (i), as $x_i > 0$. If it is an inequality from (iii) then we are done.

If it is an inequality from (ii), then $x_Y = \prod_{j \in Y} x_j$ for some Y with $\{i\} \in Y$. We must have x_Y on the RHS of an inequality that is an equality (by minimality of x_Y), which must be of type (iii). So $|S|^k = \prod_{Z \in \mathcal{C}} x_Z$, for some irreducible cover $\mathcal{C}$ with $Y \in \mathcal{C}$.

Then $\mathcal{C} - \{Y\} \cup \big\{\{j\} : j \in Y\big\}$ is also a uniform cover with equality in (iii)$'$, and $\{i\}$ belongs to the cover. Now take any irreducible cover $\mathcal{C}'$ from this cover which includes $\{i\}$.

Thus for each i, we have a uniform cover $\mathcal{C}_i$ with equality in (iii) and with $\{i\} \in \mathcal{C}_i$. Consider $\mathcal{C} = \bigcup_{i=1}^n \mathcal{C}_i$. Then $\mathcal{C}$ is a uniform cover with equality in (iii)', and $\{1\}, \{2\}, \ldots, \{n\} \in \mathcal{C}$. Put $\mathcal{C}' = \mathcal{C} - \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$. Then $\mathcal{C}'$ is also a uniform cover, say a k-cover, and we have $|S|^k \leqslant \prod_{Y \in \mathcal{C}'} x_Y$ and $|S|^{k+1} = \prod_{Y \in \mathcal{C}'} x_Y \prod_{i=1}^n x_i$. Thus $|S| = \prod_{i=1}^n x_i$. Now for any Y, consider the uniform cover $\{Y, Y^c\}$ of $[n]$. We have

$$|S| \;\leqslant\; x_Y x_{Y^c} \;\leqslant\; \left(\prod_{i \in Y} x_i\right)\left(\prod_{i \in Y^c} x_i\right) \;=\; |S|,$$

so $x_Y = \prod_{i \in Y} x_i$. Thus $B = [0, x_1] \times [0, x_2] \times \cdots \times [0, x_n]$. $\square$

Intersecting families of graphs

What happens to intersecting families if we have more structure in our ground set?

One natural example is to take our ground set to be $[n]^{(2)}$, the edges of the complete graph on $[n]$. There are a total of $2^{\binom{n}{2}}$ graphs on $[n]$.

How many graphs can we find such that any two intersect in something containing P_2, the path of length 2? We want to find $\max|A|$ subject to $G, H \in A \Rightarrow G \cap H \supset P_2$.

Clearly $|A| \leqslant \frac{1}{2}\,2^{\binom{n}{2}}$, as we cannot have both $G \in A$ and $G^c \in A$ for any graph G.

We can get $|A| \sim \frac{1}{2}\,2^{\binom{n}{2}}$ by fixing $x \in [n]$ and taking

$$A = \left\{G : d_G(x) \geqslant \frac{n}{2} + 1\right\}.$$

This has

$$|A| \sim \left(\frac{1}{2} - \frac{c}{\sqrt{n}}\right) 2^{\binom{n}{2}}.$$

Similarly, we can get $|A| \sim \frac{1}{2}\,2^{\binom{n}{2}}$ for $G \cap H$ containing a star.

Conjecture 8. If $G, H \in A \Rightarrow G \cap H$ contains a triangle, then $|A| \leqslant \frac{1}{8}\,2^{\binom{n}{2}}$.

Note that we can obtain $|A| = \frac{1}{8}\,2^{\binom{n}{2}}$ by taking A to consist of all graphs G which contain some fixed triangle.

Theorem 9. Let $A \subset \mathbb{P}\left([n]^{(2)}\right)$ be such that if $G, H \in A$ then $G \cap H$ contains a triangle. Then $|A| \leqslant \frac{1}{4}\,2^{\binom{n}{2}}$.

Proof. We want $|A| \leqslant 2^{\binom{n}{2}-2} = 2^{\binom{n}{2}\left(1-2/\binom{n}{2}\right)}$, so it is enough to find a uniform cover $\mathcal{C}$ of $[n]^{(2)}$ such that for all $Y \in \mathcal{C}$ we have $\left|A|_Y\right| \leqslant 2^{c|Y|}$, where $c = 1 - 4/n(n-1)$.

For n even, take all Y of the form $B^{(2)} \cup (B^c)^{(2)}$ with $|B| = \frac{1}{2}|A|$. This is clearly a uniform cover. Now for any such Y, $G \cap H$ is not bipartite and so G and H meet on Y. Thus $A|_Y$ is intersecting, whence

$$\left|A|_Y\right| \;\leqslant\; \tfrac{1}{2}\,2^{|Y|} \;=\; 2^{2\binom{n/2}{2}-1} \;=\; 2^{2\binom{n/2}{2}\left(1-\frac{1}{2\binom{n/2}{2}}\right)},$$

so we need

$$1 - \frac{1}{2\binom{n/2}{2}} \;\leqslant\; 1 - \frac{4}{n(n-1)}.$$

For n odd, we do the same thing but with $|B| = (n-1)/2$. $\square$

1. Write down two antichains of size 10 in $\mathcal{P}([5])$. Write down an antichain of size 8 in $\mathcal{P}([5])$ whose members do not all have the same size.

2. What are the 99th, 100th and 101st elements in the colex order on $\mathbb{N}^{(4)}$? For which $A \in \mathbb{N}^{(4)}$ is it true that A and the successor of A (in colex) have the same sum?

3. Let $\mathcal{A} \subset [9]^{(3)}$ with $|\mathcal{A}| = 28$. How small can the lower shadow of $\mathcal{A}$ be? And the upper shadow?

4. Let n be even, and let $\mathcal{A} \subset \mathcal{P}(X)$ be a set system that contains no chain of length 3. Prove that $|\mathcal{A}| \leq \binom{n}{n/2} + \binom{n}{n/2-1}$.

5. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an antichain not of the form $X^{(r)}$, $0 \leq r \leq n$. Must there exist a maximal chain that is disjoint from $\mathcal{A}$?

6. A set system $\mathcal{A} \subset \mathcal{P}(X)$ is called a *cross-cut* if for every $B \in \mathcal{P}(X)$ there exists $A \in \mathcal{A}$ with $B \subset A$ or $A \subset B$. Prove that every cross-cut contains a cross-cut of size at most $\binom{n}{\lfloor n/2 \rfloor}$. Does every cross-cut contain a cross-cut that is an antichain?

7. Let $\mathcal{A} \subset X^{(r)}$, and let $U, V \subset X$ with $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. If $\mathcal{A}$ is left-compressed, can we have $|\partial C_{UV}(\mathcal{A})| > |\partial \mathcal{A}|$?

8. Find a set system $\mathcal{A}$ for which equality holds in the Kruskal-Katona theorem but which is not isomorphic to an initial segment of colex.

9. Let $x_1, \ldots, x_n$ be non-zero real numbers, and let a be real. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\sum_{i \in A} x_i$, $A \subset [n]$, can be equal to a.

$^+$10. For $n = 2r + 1$, give an explicit bijection $f : X^{(r)} \to X^{(r+1)}$ such that $A \subset f(A)$ for every $A \in X^{(r)}$.

$^+$11. Let $\mathcal{A} \subset X^{(r)}$ and $\mathcal{B} \subset X^{(r+1)}$ be initial segments of colex with $|\mathcal{A}| = |\mathcal{B}|$. Do we always have $|\partial \mathcal{A}| \leq |\partial \mathcal{B}|$?

1. Let $r < n/2$. What is the largest intersecting family contained in $X^{(\leq r)}$?

2. A set system $A \subset \mathcal{P}(X)$ is an *up-set* if whenever $x \in A$ and $x \subset y$ then also $y \in A$. Explain why every maximal intersecting family is an up-set of size 2^{n-1}. Conversely, if A is an up-set with $|A| = 2^{n-1}$, must A be intersecting?

3. Let $A \subset \mathcal{P}(X)$ with $|A| = 2^{n-1} + 1$, so that some pair $x, y \in A$ must be disjoint. What is the smallest number of such disjoint pairs that A can have? And what if $|A| = 2^{n-1} + 2$?

4. Let $A \subset Q_6$ with $|A| = 26$. How small can the vertex-boundary of A be? And the edge-boundary?

5. Suppose that we try to use codimension-1 compressions to prove the (false) result that initial segments of the simplicial order on Q_n minimise the edge-boundary. Where does the proof go wrong?

6. Write down a direct proof of the Kruskal-Katona theorem using codimension-1 compressions.

7. Does the sequence of paths $P_1, P_2, P_3, \ldots$ form a Lévy family? What about the sequence of two-dimensional grids $[1]^2, [2]^2, [3]^2, \ldots$?

8. Give an example of a connected graph G for which the extremal sets for the vertex-isoperimetric inequality do not form a nested family: in other words, there is no ordering of the vertices of G such that to minimise the neighbourhood of a set of m vertices it is best to take the first m vertices in that ordering.

9. Let $f_1, \ldots, f_{n^2}$ be Lipschitz functions on Q_n. Prove that, for n sufficiently large, there exists a point $x \in Q_n$ such that $|f_i(x) - f_i(x^c)| < n/100$ for all i.

10. Let $A \subset Q_n$ with $|A| = \left|X^{(\leq r)}\right|$, where $r < n/2$. Prove that we can always find a set of $\binom{n}{r}$ disjoint edges between A and A^c.

11. Let $A_1, A_2, \ldots, A_d \subset \mathcal{P}(X)$ be intersecting families. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_d| \leq 2^n - 2^{n-d}$.

12. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ be an intersecting family of finite sets. Must there exist a finite set $F \subset \mathbb{N}$ such that the family $\{A \cap F : A \in \mathcal{A}\}$ is intersecting? And what if $\mathcal{A} \subset \mathbb{N}^{(r)}$?

1. Let $A \subset [4]^3$ with $|A| = 23$. How small can the vertex-boundary of A be?

2. The *distance* between two (non-empty) subsets A and B of $[k]^n$ is $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. Show that if $d(A, B) \geq t$ then the distance between the first $|A|$ and the last $|B|$ points of $[k]^n$ in the simplicial order is also at least t.

3. Suppose that we wish to remove edges from $[k]^n$ in such a way that the resulting graph has all components of size at most $k^n/2$. Show that we must remove at least k^{n-1} edges.

4. The *diameter* of a set $A \subset Q_n$ is $\max\{d(x, y) : x, y \in A\}$. Deduce from Harper's theorem that if $A \subset Q_n$ has diameter d, where $d < n$ and d is even, then $|A| \leq \left|X^{(\leq d/2)}\right|$.

5. Let r be an odd positive integer. Show that there are infinitely many values of n for which there exists a set system $A \subset [n]^{(r)}$ satisfying $|x \cap y|$ even for all distinct $x, y \in A$ and having size $|A| = n$.

6. Does there exist an even positive integer r and a value of n greater than $r + 1$ for which there exists a set system $A \subset [n]^{(r)}$ satisfying $|x \cap y|$ odd for all distinct $x, y \in A$ and having size $|A| = n$?

7. Let $A \subset [n]^{(r)}$ be such that the number of values taken by $|x \cap y|$, over all distinct $x, y \in A$, is s. Prove that $|A| \leq \binom{n}{s}$.

8. Give a constructive proof that the Ramsey number $R(t)$ grows faster than any polynomial in t, by considering the graph on $[p^3]^{(p^2-1)}$, where p is prime, in which x is joined to y if $|x \cap y|$ is congruent to -1 mod p.

9. Consider the graph on $\mathbb{R}^n$ in which x is joined to y if $\|x - y\| = 1$, where $\|.\|$ denotes the usual Euclidean distance. Show that this graph has chromatic number at least c^n, for some constant $c > 1$.

$^+$10. Let A be a subset of $[k]^n$ of diameter d, where k is odd and d is even. How large can A be?

1. Write down two antichains of size 10 in $\mathcal{P}([5])$. Write down an antichain of size 8 in $\mathcal{P}([5])$ whose members do not all have the same size.

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4. Prove that, for any $A \subset \mathcal{P}(X)$, the number of sets shattered by A is at least $|A|$.

5. What is wrong with the following 'proof' that if S is a body in $\mathbb{R}^3$ then $|S|^2 \leq |S_{12}||S_{23}||S_{31}|$: if we let $T = S \times S \subset \mathbb{R}^6$, then $|S|^2 = |T| \leq |T_{12}||T_{34}||T_{56}| = |S_{12}||S_{31}||S_{23}|$.

6. Let $A \subset \mathcal{P}(X)$ be a set system that does not shatter any of the sets $\{1, 2, 3\}, \{2, 3, 4\}, \ldots, \{n-2, n-1, n\}, \{n-1, n, 1\}, \{n, 1, 2\}$. Explain why when n is a multiple of 3 it is trivial that $|A| \leq 7^{n/3}$, and then prove that in fact this holds for every value of n.

7. Let $Y_1, Y_2, \ldots, Y_r \subset [n]$ be sets that do *not* form a uniform cover of $[n]$. Show that knowledge of $|S_{Y_1}||S_{Y_2}| \ldots |S_{Y_r}|$ does not imply any upper bound on $|S|$.

8. Let S and T be bodies in $\mathbb{R}^2$, and let B and C be boxes verifying the Bollobás-Thomason box theorem for S and T respectively. Show that if $S \subset T$ then it is always possible to choose B and C such that $B \subset C$. What happens in $\mathbb{R}^3$?

$^+$9. Let A be a subset of $[k]^n$ of diameter d, where k is odd and d is even. How large can A be?